

Maximal Regularity for Stochastic Integral Equations

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We show that, in the maximal regularity case, where

$$\beta - \alpha\theta - \eta = \frac{1}{2},$$

one has the estimate

$$\|A^\theta D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

where c is independent of G .

Here D_t^η denotes fractional integration if $\eta \in (-1, 0)$, and fractional differentiation if $\eta \in (0, 1)$, both with respect to the t -variable.

The proof relies on recent work on stochastic differential equations by v. Neerven, Veraar and Weis, and extends their maximal regularity result to the integral equation case.

Keywords stochastic integral equations, maximal regularity, bounded H-infinity calculus, Volterra equations, singular kernels

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MAXIMAL REGULARITY FOR STOCHASTIC INTEGRAL EQUATIONS

W. Desch and S-O. Londen

Abstract We examine the stochastic parabolic integral equation of convolution type

$$U(t) + A \int_0^t k_1(t-s)U(s) ds = \int_0^t k_2(t-s)G(s) dW_H(s), \quad t \geq 0,$$

where $U(t)$ takes values in $L_q(\mathcal{O}; \mathbb{R})$ with \mathcal{O} a σ -finite measure space, and $q \in [2, \infty)$. The linear operator A maps $\mathcal{D}(A) \subset L_q(\mathcal{O}; \mathbb{R})$ into $L_q(\mathcal{O}; \mathbb{R})$, is nonnegative and admits a bounded \mathbf{H}^∞ -calculus on $L_q(\mathcal{O}; \mathbb{R})$. The kernels are powers of t , with $k_1(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $k_2(t) = \frac{1}{\Gamma(\beta)}t^{\beta-1}$, and $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$.

We show that, in the maximal regularity case, where

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one has the estimate

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where c is independent of G .

Here D_t^η denotes fractional integration if $\eta \in (-1, 0)$, and fractional differentiation if $\eta \in (0, 1)$, both with respect to the t -variable.

The proof relies on recent work on stochastic differential equations by v. Neerven, Veraar and Weis, and extends their maximal regularity result to the integral equation case.

1. INTRODUCTION

We investigate the stochastic integral equation

$$(1.1) \quad U(t) + A \int_0^t k_1(t-s)U(s) ds = \int_0^t k_2(t-s)G(s) dW_H(s), \quad t \geq 0,$$

where we require $U(t)$ to take values in $L_q(\mathcal{O}; \mathbb{R})$ almost surely, with \mathcal{O} a σ -finite measure space, and $q \in [2, \infty)$. The linear operator A maps $\mathcal{D}(A) \subset L_q(\mathcal{O}; \mathbb{R})$ into $L_q(\mathcal{O}; \mathbb{R})$, is nonnegative and admits a bounded \mathbf{H}^∞ -calculus on $L_q(\mathcal{O}; \mathbb{R})$. The kernels are powers of t , with $k_1(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $k_2(t) = \frac{1}{\Gamma(\beta)}t^{\beta-1}$, and $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$. The integral on the left side of (1.1) is a standard Bochner integral.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, and take $W_H(t)$, $t \geq 0$, to be a cylindrical \mathcal{F} -Brownian motion on the separable Hilbert space H . The stochastic integral with respect to W_H on the right side of (1.1) is an $L_q(\mathcal{O}; \mathbb{R})$ -valued random variable as

defined, e.g., in [7, Sec. 2.1]. Assume $G \in L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))$ for some $p \in (2, \infty)$ and that G is \mathcal{F} -adapted.

Define the stochastic convolution process $U(t)$ by

$$(1.2) \quad U(t) = \int_0^t S_{\alpha\beta}(t-s)G(s) dW_H(s), \quad t \geq 0.$$

Here $S_{\alpha\beta}(t) \in \mathcal{L}(L_q(\mathcal{O}; \mathbb{R}))$, $t \geq 0$, is the resolvent (see Section 2 below) associated with A , k_1 , and k_2 , i.e., the solution of

$$(1.3) \quad S_{\alpha\beta} + A(k_1 * S_{\alpha\beta}) = k_2 I.$$

The process $U(t)$ solves (1.1), at least formally.

Let D_t^η denote fractional integration (if $\eta \in (-1, 0)$) and fractional differentiation (if $\eta \in (0, 1)$); both with respect to the t -variable. For $\theta \in (0, 1)$, let A^θ stand for the fractional power of A .

Our main result concerns the maximal regularity case

$$\beta - \alpha\theta - \eta = \frac{1}{2}.$$

We show, assuming this and making an appropriate spectral assumption on A , that $A^\theta D_t^\eta U$ is welldefined and satisfies

$$(1.4) \quad \|A^\theta D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

with the constant c independent of G , but depending on $p, q, \alpha, \beta, \eta, \theta$. If $q = 2$, the estimate (1.4) also holds for $p = 2$.

Essentially the same equation as (1.1) has previously been considered in [4] under the stronger condition

$$(1.5) \quad \beta - \alpha\theta - \eta > \frac{1}{2},$$

with $q = p \in [2, \infty)$ and t in compact sets. However, the assumption that A admits a bounded functional calculus was not made in [4], only the assumption that A be nonnegative, including the appropriate spectral condition. Estimates analogous to (1.4) were obtained.

Technically the proofs of [4] rely on an approach due to Krylov making use of the Burkholder-Davis-Gundy inequality and developed for stochastic differential equations, i.e., for the case $\alpha = \beta = 1$ in (1.1), and with $-A$ a second order elliptic differential operator on \mathbb{R}^n . In this case, Krylov's approach can be used to obtain maximal regularity, however, it relies on a deep Paley-Littlewood inequality [5], [6]. For general sectorial A in the integral equation case (i.e., $\alpha \in (0, 2), \beta \in (\frac{1}{2}, 2)$), this inequality may be replaced by estimates on the Dunford integral - but then an infinitesimal loss of regularity must be allowed, as illustrated by (1.5).

To obtain maximal regularity in the integral equation case - which in [4] was done only for the Laplacian on \mathbb{R}^n - a highly nontrivial generalization of Krylov's lemma was required [3].

Our result in this paper relies strongly on the deep approach of the recent paper [7], concerned with maximal regularity of (1.1) in the differential equation case, i.e., $\alpha = \beta = 1$. In fact, the major part of the proof of [7] can be adapted almost without changes to apply to (1.1). The only necessary major alteration is the construction of new, different k_j -functions (see Proposition 3.3 below). Note that our k_j -functions are constructed by a different approach and do not reduce to those of [7] in case $\alpha = \beta = 1$.

As already stated, our main result (Theorem 3.1) gives the estimate (1.4). Observe that this estimate may be interpreted as an estimate on $D_t^\eta(A^\theta U)$, i.e., on the fractional time-derivative of $A^\theta U$. Hence this estimate combined with the additional assumption $0 \in \rho(A)$ directly gives maximal space-time L_p -regularity, i.e., maximal regularity in Bessel potential spaces. (See Corollary 3.2 below). Therefore, no modification or extension of the Da Prato-Kwapień-Zabczyk factorization argument used in [7] to obtain Bessel potential space regularity is required here. Obviously, this simpler proof also applies in case $\alpha = \beta = 1$.

In Section 2 we formulate some technical preliminaries. Our results are given in Section 3. In Section 4 we show how Corollary 3.2 follows from Theorem 3.1. Section 5 is devoted to the construction of k_j -functions representing the resolvent. Section 6 contains the proof of Theorem 3.1.

2. TECHNICAL PRELIMINARIES

First, we consider fractional integration and differentiation in time.

Definition 2.1. Let X be a Banach space and $\eta \in (0, 1)$, let $u \in L_1((0, T); X)$ for some $T > 0$.

1) Fractional integration in time is defined by

$$D_t^{-\eta}u = \frac{1}{\Gamma(\eta)}t^{\eta-1} * u.$$

2) We say that u has a fractional derivative of order $\eta > 0$ provided $u = D_t^{-\eta}f$, for some $f \in L_1((0, T); X)$. If this is the case, we write $D_t^\eta u = f$.

Obviously, $D_t^0 f = f$

For the equivalence of fractional derivatives in L_p and fractional powers of the realization of the derivative in L_p we have:

Lemma 2.2. [2, Proposition 2] *Let $p \in [1, \infty)$, X a Banach space and define*

$$\mathcal{D}(L) = \{u \in W_p^1((0, T); X) \mid u(0) = 0\}, \quad Lu = u', \quad u \in \mathcal{D}(L).$$

Then, with $\beta \in (0, 1)$

$$L^\beta u = D_t^\beta u, \quad u \in \mathcal{D}(L^\beta),$$

where $\mathcal{D}(L^\beta)$ coincides with the set of functions u having a fractional derivative of order β in L_p , i.e.,

$$\mathcal{D}(L^\beta) = \{u \in L_p((0, T); X) \mid t^{-\beta} * u \in W_{0,p}^1((0, T); X)\}.$$

In particular, D_t^β is closed.

We then recall the definition of the resolvent associated with A , k_1 , and k_2 .

Definition 2.3. Let X be a Banach space, let A map $\mathcal{D}(A) \subset X \rightarrow X$ and let A be sectorial. Let k_1, k_2 be power functions as in the introduction. Let ϕ_A be the spectral angle of A and assume

$$(2.1) \quad \phi_A < \pi(1 - \frac{\alpha}{2}).$$

The resolvent $S_{\alpha\beta}$ associated with A , k_1 and k_2 is defined by

$$(2.2) \quad S_{\alpha\beta}(t)v = (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} \lambda^{\alpha-\beta} v \, d\lambda, \quad t > 0,$$

for $v \in X$; and where $\psi \in (\frac{\pi}{2}, \min\{\pi, \frac{\pi-\phi_A}{\alpha}\})$, and

$$(2.3) \quad \Gamma_{r,\psi} = \{r e^{it} \mid |t| \leq \psi\} \cup \{\rho e^{i\psi} \mid r < \rho < \infty\} \cup \{\rho e^{-i\psi} \mid r < \rho < \infty\}.$$

As to the properties of $S_{\alpha\beta}$ we have

Remark 2.4. [4] Let $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$, $\theta \in [0, 1]$, $\eta \in (-1, 1)$. Let $S_{\alpha\beta}(t)$ be the resolvent of Definition 2.3. Then one has

$$(2.4) \quad S_{\alpha\beta}(t) \in \mathcal{L}(X), \quad t > 0; \quad S_{\alpha\beta}(t)v \in \mathcal{D}(A), \quad t > 0, \quad v \in X,$$

$$(2.5) \quad \sup_{t>0} \|t^{1+\alpha\theta-\beta+\eta} D_t^\eta A^\theta S_{\alpha\beta}(t)\|_{\mathcal{L}(X)} < \infty,$$

$$(2.6) \quad S_{\alpha\beta}(t) + A \int_0^t k_1(t-s) S_{\alpha\beta}(s) \, ds = k_2(t)I, \quad t > 0.$$

For the proof of these properties, see [4].

3. RESULTS

Our main result is the following:

Theorem 3.1. Take $(\Omega, \mathcal{A}, \mathcal{P})$ to be a probability space with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Let $W_H(t)$, $t \geq 0$, be a cylindrical \mathcal{F} -Brownian motion on the Hilbert space H . Let $p \in [2, \infty)$.

Let $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$, $\theta \in [0, 1)$, and $\eta \in (-1, 1)$, and suppose that

$$(3.1) \quad \beta - \alpha\theta - \eta = \frac{1}{2}.$$

Let $q \in [2, \infty)$. Assume the operator A admits a bounded H^∞ -calculus on $L_q(\mathcal{O}; \mathbb{R})$ of angle ω satisfying

$$(3.2) \quad \omega < \pi(1 - \frac{\alpha}{2}).$$

Let $S_{\alpha\beta}(t) \in \mathcal{L}(L_q(\mathcal{O}; \mathbb{R}))$, $t \geq 0$, be the resolvent associated with A , k_1 , and k_2 , as defined in Definition 2.3, (where $X = L_q(\mathcal{O}; \mathbb{R})$).

Define, for any \mathcal{F} -adapted $G \in L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))$, the stochastic convolution $U(t)$ by

$$U(t) = \int_0^t S_{\alpha\beta}(t-s)G(s) dW_H(s), \quad t \geq 0.$$

Then $D_t^\eta U$ is welldefined in $L_q(\mathcal{O}; \mathbb{R})$, takes values in the domain $\mathcal{D}(A^\theta)$ almost surely and satisfies, for $2 < p < \infty$,

$$(3.3) \quad \|A^\theta D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

with the constant c independent of G . If $q = 2$, then this maximal regularity holds also for $p = 2$.

Note the necessary condition $\beta > \frac{1}{2}$. Thus, e.g., the case $\theta = 0$, $\eta \leq 0$, is not covered by Theorem 3.1. See however, the comments following Corollary 3.2.

If A is invertible, then we may, in the maximal regularity case, formulate a result in Bessel potential spaces.

Corollary 3.2. *Let, in addition to the assumptions of Theorem 3.1, $0 \in \rho(A)$, and let $\eta \in [0, 1)$ and assume*

$$(3.4) \quad \beta - \frac{1}{2} < \alpha.$$

Then

$$(3.5) \quad \|U\|_{H_p^\eta(\mathbb{R}^+; L_p(\Omega; \mathcal{D}(A^\theta)))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

with the constant c independent of G .

Note that with $\alpha = \beta = 1$ (the differential equation case of [7]), the condition (3.4) is trivially satisfied.

Also note that in the differential equation case (3.5) yields

$$\|U\|_{H^{\theta, p}(\mathbb{R}^+; L_p(\Omega; \mathcal{D}(A^{\frac{1}{2}-\theta})))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

for $\theta \in [0, \frac{1}{2}]$. (Cf. [7, Theorem 1.2(i)]).

Suppose next that

$$(3.6) \quad \beta - \alpha\theta - \eta > \frac{1}{2}.$$

In the case where $p = q$, and \mathbb{R}^+ is replaced by compact time-intervals we may apply results of [4]. These results do not require that A admits a bounded H^∞ -calculus but only that A is sectorial and satisfies the spectral condition (2.1).

If (3.6) holds and we wish to consider \mathbb{R}^+ , and with p not necessarily equal to q , then we may, for certain parameter values, argue as follows.

Suppose there exists $\theta_0 \in (\theta, 1)$ such that $\beta - \alpha\theta_0 - \eta = \frac{1}{2}$. Then, by Theorem 3.1,

$$\|A^{\theta_0} D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))},$$

with c independent of G . Arguing as in the proof of Corollary 3.2 we may then proceed to obtain norm estimates on U in Bessel potential spaces.

Finally observe that Theorem 3.1 and Corollary 3.2 allow us to interpolate analogously as in [7, Theorem 1.2 (2)] to obtain maximal estimates.

To prove Theorem 3.1, we need Proposition 3.3 below. This proposition contains the essential new element needed for the appropriate extension of the results of [7].

Below, we let $H^\infty(\Sigma_\sigma)$ denote the Banach space of all bounded analytic functions φ that map $\Sigma_\sigma = \{z \in \mathbb{C} \setminus 0 \mid |\arg z| < \sigma\} \rightarrow \mathbb{C}$ with the supremum norm. The linear subspace of $H^\infty(\Sigma_\sigma)$ consisting of all functions φ satisfying an estimate

$$|\varphi(z)| \leq \frac{c|z|^\epsilon}{(1 + |z|^2)^\epsilon}, \quad z \in \Sigma_\sigma,$$

for some $\epsilon > 0$, and some $c < \infty$, is denoted by $H_0^\infty(\Sigma_\sigma)$.

Proposition 3.3. *Let $\alpha \in (0, 2)$, $\beta \in (\frac{1}{2}, 2)$, $\theta \in [0, 1)$ and $\eta \in (-1, 1)$. Assume the operator A admits a bounded H^∞ -calculus on the Banach space X of angle ω satisfying*

$$(3.7) \quad \omega < \pi(1 - \frac{\alpha}{2}).$$

Let (3.1) be satisfied.

Let $S_{\alpha\beta}(t)$, $t \geq 0$, be the resolvent associated with A , $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $\frac{1}{\Gamma(\beta)}t^{\beta-1}$, as defined in Definition 2.3.

Then there exists $\omega_1 \in (\omega, \pi)$, and for $j = \pm 1$ functions $k_j : (0, \infty) \times (0, \infty) \rightarrow \mathbb{C}$, and $\varphi_j \in H_0^\infty(\Sigma_{\omega_1})$ such that, for $t > 0$, $x \in X$,

$$A^\theta D_t^\eta S_{\alpha\beta}(t)x = \sum_{j=\pm 1} \int_0^\infty k_j(u, t) [\varphi_j(uA)]^2 x \frac{du}{u}.$$

Moreover, for fixed u we have $k_j(u, \cdot) \in W_1^1((0, \infty); \mathbb{C})$, with

$$(3.8) \quad \int_0^\infty t^{\frac{1}{2}} \left| \frac{\partial}{\partial t} k_j(u, t) \right| dt \leq M,$$

where the constant M is independent of u , but depends on the parameters $\alpha, \beta, \theta, \eta$ and on the norm of the operator $H^\infty(\Sigma_{\omega_1}) \rightarrow \mathcal{L}(X)$ given by $\varphi \rightarrow \varphi(A)$.

Proposition 3.3 is proved in Section 5, via three lemmas.

In Section 6, we briefly outline the proof of Theorem 3.1, using the proof of [7], combined with Proposition 3.3 above. Our goal in this outline is only to demonstrate that Proposition 3.3, combined with the proof of [7] is sufficient to prove Theorem 3.1.

4. PROOF OF COROLLARY 3.2

Proof. First recall that by the assumption $0 \in \rho(A)$,

$$(4.1) \quad \|U\|_{H_p^\eta(\mathbb{R}^+; L_p(\Omega; \mathcal{D}(A^\theta)))} = \|A^\theta U\|_{H_p^\eta(\mathbb{R}^+; L_p(\Omega; L_q(\mathcal{O}; \mathbb{R})))}.$$

Then note that (see [9, p.29, Example 2.8.1] and recall that $U(t=0) = 0$) we may take

$$(4.2) \quad \|A^\theta U\|_{H_p^\eta(\mathbb{R}^+; L_p(\Omega; L_q(\mathcal{O}; \mathbb{R})))} = \|A^\theta D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} + \|A^\theta u\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))}.$$

By (3.3) of Theorem 3.1,

$$(4.3) \quad \|A^\theta D_t^\eta U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))}.$$

Then define θ_0 by $\beta - \alpha\theta_0 = \frac{1}{2}$. By the assumption $\beta > \frac{1}{2}$ we have $\theta_0 > 0$. By (3.4), $\theta_0 < 1$. Then, by Theorem 3.1,

$$\|A^{\theta_0} U\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))}.$$

By the assumption $0 \in \rho(A)$, and by the fact that $\theta_0 > \theta$, we have that $(A^{\theta_0 - \theta})^{-1}$ exists. See, e.g. [1, Lemma 5.3.5]. Thus

$$(4.4) \quad \|A^\theta u\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))} \leq c \|G\|_{L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))}.$$

From (4.1) - (4.4) we have (3.5). \square

5. PROOF OF PROPOSITION 3.3

Lemma 5.1. *Let the assumptions of Proposition 3.3 hold. Choose ρ, ω_1 such that*

$$(5.1) \quad \rho \in \left(\frac{\pi}{2}, \pi\right), \quad \omega_1 \in (\omega, \pi), \quad \alpha\rho + \omega_1 < \pi.$$

By (3.7) this is possible. Then, for $j = \pm 1$,

a) The function $s \rightarrow (se^{ij\rho})^\alpha ((se^{ij\rho})^\alpha + A)^{-1}$ is welldefined and uniformly bounded in $\mathcal{L}(X)$ for $s \in (0, \infty)$.

b) Define, for $j = \pm 1$,

$$(5.2) \quad \psi_j(z) = \frac{z^\theta}{1 + e^{-ij\rho\alpha} z}.$$

Then, for $j = \pm 1$, there exists a function $\varphi_j \in H_0^\infty(\Sigma_{\omega_1})$ such that

$$(5.3) \quad \psi_j(z) = \varphi_j(z)^2, \quad z \in \Sigma_{\omega_1}.$$

.

Proof. To prove (a), note first that for $s > 0$, by (5.1),

$$\mu = (se^{ij\rho})^\alpha \in \Sigma_{\pi-\omega_1}.$$

Consequently,

$$\sup_{z \in \Sigma_{\omega_1}} \left| \frac{\mu}{\mu + z} \right| = \sup_{z \in \Sigma_{\omega_1}} \left| \frac{1}{1 + \frac{z}{\mu}} \right| \leq c,$$

with c independent of μ . Now note that $\omega_1 > \omega$ and apply the bounded H^∞ -calculus of A to get a).

To prove (b), take $z \in \Sigma_{\omega_1}$ and notice that by (5.1) there exists a constant M such that

$$|1 + e^{-ij\rho\alpha}z| \geq M|z|, \quad z \in \Sigma_{\omega_1}.$$

Thus, since $\theta < 1$, $\psi_j(z) \in H_0^\infty(\Sigma_{\omega_1})$. Write

$$\psi_j(z) = [z^{-\theta} + e^{-ij\rho\alpha}z^{1-\theta}]^{-1}$$

to see that for $z \in \Sigma_{\omega_1}$,

$$|\arg(\psi_j(z))| \leq \max(\omega_1, \rho\alpha + \omega_1) < \pi.$$

Therefore, one may take $\phi_j = (\psi_j)^{\frac{1}{2}} \in H_0^\infty(\Sigma_{\omega_1})$, where $z^{\frac{1}{2}}$ denotes any branch of the square root in the slotted plane $\mathbb{C} \setminus (-\infty, 0]$. \square

Lemma 5.2. *Under the assumptions of Proposition 3.3, and with ρ as in Lemma 5.1; define for $j = \pm 1$; $t, u > 0$;*

$$(5.4) \quad k_j(u, t) = \frac{1}{2\pi i \alpha} e^{ij\rho(1+\eta-\beta)} u^{-\frac{1}{2\alpha}} e^{e^{ij\rho}tu^{-1/\alpha}}.$$

Then, for fixed $u > 0$, we have $k_j \in W_1^1((0, \infty); \mathbb{C})$ and

$$(5.5) \quad \int_0^\infty \sqrt{t} \left| \frac{\partial}{\partial t} k_j(u, t) \right| dt \leq M,$$

with a constant M independent of u .

Proof. By the fact that $\rho > \frac{\pi}{2}$, we have $\Re(e^{ij\rho}) < 0$, so that for fixed u the function $k_j(u, \cdot) \in W_1^1((0, \infty); \mathbb{C})$. Estimating the integral gives (with suitable constants K that may vary from line to line, and a change of variables $s = tu^{-\frac{1}{\alpha}}$),

$$\begin{aligned}
& \int_0^\infty \sqrt{t} \left| \frac{\partial}{\partial t} k_j(u, t) \right| dt \\
& \leq K \int_0^\infty \sqrt{t} u^{-1/(2\alpha)} \left| \frac{\partial}{\partial t} e^{e^{ij\rho}tu^{-1/\alpha}} \right| dt \\
& = K \int_0^\infty \sqrt{tu^{-1/\alpha}} |e^{ij\rho}u^{-1/\alpha} e^{e^{ij\rho}tu^{-1/\alpha}}| dt \\
& = K \int_0^\infty \sqrt{tu^{-1/\alpha}} |e^{e^{ij\rho}tu^{-1/\alpha}}| u^{-1/\alpha} dt \\
& = K \int_0^\infty \sqrt{s} |e^{e^{ij\rho}s}| ds \\
& = K.
\end{aligned}$$

□

Lemma 5.3. *With the conditions of Proposition 3.3, with ρ, ψ_j as in Lemma 5.1, and k_j as in Lemma 5.2, we have for $t > 0, x \in X$,*

$$(5.6) \quad A^\theta D_t^\eta S_{\alpha\beta}(t)x = \sum_{j=\pm 1} \int_0^\infty k_j(u, t) \psi_j(uA)x \frac{du}{u}.$$

Proof. The Laplace transform of $A^\theta D_t^\eta S_{\alpha\beta}(t)$ is given by (see [4, Definition 5.4],

$$\lambda^{\eta+\alpha-\beta} A^\theta (\lambda^\alpha I + A)^{-1}.$$

We let, for $r > 0$, Γ_r be the contour

$$\begin{aligned}
& e^{-i\rho}|t|, \quad t \leq -r, \\
& e^{i\rho}t, \quad t \geq r, \\
& e^{is}r, \quad |s| \leq \rho.
\end{aligned}$$

Then

$$A^\theta D_t^\eta S_{\alpha\beta}(t)x = \frac{1}{2\pi i} \int_{\Gamma_r} e^{\lambda t} \lambda^{\eta+\alpha-\beta} A^\theta (\lambda^\alpha I + A)^{-1} x d\lambda.$$

Note that

$$\sup_{\lambda \in \Gamma_r} \|A^\theta \lambda^{\alpha(1-\theta)} (\lambda^\alpha I + A)^{-1}\|_{\mathcal{L}(X)} < \infty,$$

and so, by (3.1),

$$\left\| \int_{\{e^{is}r \mid |s| \leq \rho\}} e^{\lambda t} \lambda^{\eta+\alpha-\beta} A^\theta (\lambda^\alpha I + A)^{-1} x d\lambda \right\|_X \leq Kr^{\frac{1}{2}},$$

for some constant K independent of r . Consequently, by analyticity we may without loss of generality take $r = 0$. Write $\Gamma = \Gamma_0$. We then have, for $t > 0$,

$$\begin{aligned}
& D_t^\eta A^\theta S_{\alpha\beta}(t)x \\
&= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\eta+\alpha-\beta} A^\theta (\lambda^\alpha + A)^{-1} x d\lambda \\
&= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\eta-\beta} A^\theta (1 + \lambda^{-\alpha} A)^{-1} x d\lambda \\
&= \frac{1}{2\pi i} \sum_{j=\pm 1} \int_0^\infty e^{se^{ij\rho}t} s^{\eta-\beta} e^{ij\rho(\eta-\beta)} A^\theta (1 + e^{-ij\alpha\rho} s^{-\alpha} A)^{-1} x e^{ij\rho} ds.
\end{aligned}$$

Continue with the change of variables $s = u^{-\frac{1}{\alpha}}$, $ds = -\frac{1}{\alpha} u^{-1-\frac{1}{\alpha}} du$ and later use (3.1), (5.4) and (5.2):

$$\begin{aligned}
& D_t^\eta A^\theta S_{\alpha\beta}(t)x \\
&= \frac{1}{2\pi i} \sum_{j=\pm 1} e^{ij\rho(1+\eta-\beta)}. \\
&\quad \int_0^\infty e^{\epsilon^{ij\rho}tu^{-1/\alpha}} u^{(\beta-\eta)/\alpha} u^{-\theta} (uA)^\theta (1 + e^{-ij\alpha\rho} uA)^{-1} x u^{-1/\alpha} \frac{1}{\alpha} \frac{du}{u} \\
&= \frac{1}{2\pi i\alpha} \sum_{j=\pm 1} e^{ij\rho(1+\eta-\beta)} \int_0^\infty e^{\epsilon^{ij\rho}tu^{-1/\alpha}} u^{-1/(2\alpha)} (uA)^\theta (1 + e^{-ij\alpha\rho} uA)^{-1} x \frac{du}{u} \\
&= \sum_{j=\pm 1} \int_0^\infty k_j(u, t) \psi_j(uA) x \frac{du}{u}.
\end{aligned}$$

□

By (5.5) and (5.6) we have Proposition 3.3.

6. PROOF OF THEOREM 3.1

Proof. As in [7] we note first that it suffices to consider \mathcal{F} -adapted finite step processes $G : \mathbb{R}^+ \times \Omega \rightarrow \mathcal{D}(A_H)$ with $A_H = A \otimes I_H$.

By Proposition 3.3, with $X = L_q(\mathcal{O}; H)$ we have the representation

$$(6.1) \quad A^\theta D_t^\eta S_{\alpha\beta}(t)x = \sum_{j=\pm 1} \int_0^\infty k_j(u, t) [\varphi_j(uA)]^2 x \frac{du}{u},$$

with

$$(6.2) \quad \varphi_j \in H_0^\infty(\Sigma_{\omega_1}), \quad \omega_1 \in (\omega, \pi).$$

(This corresponds to relation (4.2) of [7]). The functions k_j depend on α, β, η and, through (3.1), on θ . Below, we do not explicitly write out this dependence.

By the stochastic Fubini theorem, and by (6.1), for $t \geq 0$,

$$\begin{aligned} & A^\theta D_t^\eta \int_0^t S_{\alpha\beta}(t-s)G(s) dW_H(s) \\ &= \sum_{j=\pm 1} \int_0^t \int_0^\infty [\varphi_j(uA)]^2 k_j(u, t-s)G(s) \frac{du}{u} dW_H(s) \\ &= \sum_{j=\pm 1} \int_0^\infty [\varphi_j(uA)]^2 (I_j(u)G)(t) \frac{du}{u}, \end{aligned}$$

with

$$(I_j(u)G)(t) = \int_0^t k_j(u, t-s)G(s) dW_H(s).$$

The remainder of the proof follows that of [7, Theorem 4.3].

Our notation (as in [7]) resorts to the concept of Banach function spaces. Let E be a Banach function space over $(\mathcal{O}, \Sigma, \mu)$ and \mathcal{H} a Hilbert space. Then $E(\mathcal{H})$ denotes the space of all strongly μ -measurable functions $G: \mathcal{O} \rightarrow \mathcal{H}$, such that $\|G(\cdot)\|_{\mathcal{H}}$ belongs to E . In the sequel, we will use the following spaces:

$$\begin{aligned} (6.3) \quad E &= L_q(\mathcal{O}; \mathbb{R}), \\ \mathcal{H} &= L_2(\mathbb{R}^+; \frac{du}{u}), \\ E_1 &= L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H)), \\ E_2 &= L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R})). \end{aligned}$$

Let $p^{-1} + (p')^{-1} = 1$, $q^{-1} + (q')^{-1} = 1$. Then

$$\begin{aligned} E(\mathcal{H}) &= L_q(\mathcal{O}; \mathcal{H}) = L_q(\mathcal{O}; L_2(\mathbb{R}^+; \frac{du}{u})), \\ E_2(\mathcal{H}) &= L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; L_2(\mathbb{R}^+; \frac{du}{u}))), \\ E^* &= L_{q'}(\mathcal{O}; \mathbb{R}); \\ E_2^* &= L_{p'}(\mathbb{R}^+ \times \Omega; L_{q'}(\mathcal{O}; \mathbb{R})). \end{aligned}$$

Thus

$$E^*(\mathcal{H}) = L_{q'}(\mathcal{O}; L_2(\mathbb{R}^+; \frac{du}{u})),$$

$$E_2^*(\mathcal{H}) = L_{p'}(\mathbb{R}^+ \times \Omega; L_{q'}(\mathcal{O}; L_2(\mathbb{R}^+; \frac{du}{u}))).$$

Now take any $\xi^* \in E_2^*$. We wish to estimate $\langle A^\theta D_t^\eta U, \xi^* \rangle_{E_2, E_2^*}$. One has, using Fubini,

$$\begin{aligned}
(6.4) \quad & \langle A^\theta D_t^\eta U, \xi^* \rangle_{E_2, E_2^*} \\
&= \sum_{j=-1, +1} \mathcal{E} \left(\int_0^\infty \langle \int_0^\infty [\varphi_j(uA)]^2 (I_j(u)G)(t) \frac{du}{u}, \xi^*(t) \rangle_{E, E^*} dt \right) \\
&= \sum_{j=-1, +1} \mathcal{E} \int_0^\infty \int_{\mathcal{O}} \int_0^\infty [\varphi_j(uA)(I_j(u)G)(t)](x) \\
&\quad [\varphi_j(uA^*)\xi^*(t)](x) \frac{du}{u} dx dt.
\end{aligned}$$

So, using again Fubini and twice Hölders inequality,

$$\begin{aligned}
(6.5) \quad & | \langle A^\theta D_t^\eta U, \xi^* \rangle_{E_2, E_2^*} | \\
&\leq \sum_{j=-1, +1} \mathcal{E} \int_0^\infty \int_{\mathcal{O}} \| [\varphi_j(uA)(I_j(u)G)(t)](x) \|_{\mathcal{H}} \\
&\quad \| \varphi_j(uA^*)\xi^*(t) \|_{\mathcal{H}} dx dt \\
&\leq \sum_{j=\pm 1} \| \varphi_j(uA)I_j(u)G \|_{E_2(\mathcal{H})} \| \varphi_j(uA^*)\xi^* \|_{E_2^*(\mathcal{H})}.
\end{aligned}$$

The estimation of the last factor on the right side of (6.5) may be done by an application of the following Proposition ([7], [8]). (In [7] this proposition is stated under the assumption that the bounded functional calculus has angle $\omega < \frac{\pi}{2}$ but a simple rescaling argument shows that the proposition is true under the assumption $\omega < \pi$).

Proposition 6.1. *Take $q \in (1, \infty)$, and assume that A has a bounded $\mathbf{H}^\infty(\Sigma_\omega)$ -calculus on $L_q(\mathcal{O}; \mathbb{R})$ for some $\omega \in (0, \pi)$, and let $\omega_1 \in (\omega, \pi)$ be arbitrary. Then, for each $\varphi \in H_0^\infty(\Sigma_{\omega_1})$ there exists a constant c such that*

$$(6.6) \quad \| \varphi(uA)x \|_{E(\mathcal{H})} \leq c \| x \|_{L_q(\mathcal{O}; \mathbb{R})}, \quad x \in L_q(\mathcal{O}; \mathbb{R}),$$

$$(6.7) \quad \| \varphi(uA^*)x^* \|_{E^*(\mathcal{H})} \leq c \| x^* \|_{L_{q'}(\mathcal{O}; \mathbb{R})}, \quad x^* \in L_{q'}(\mathcal{O}; \mathbb{R}).$$

By (3.2) and (6.2) we may apply the above Proposition pointwise in $\mathbb{R}^+ \times \Omega$. Thus, by (6.7),

$$(6.8) \quad \| \varphi_j(uA^*)\xi^* \|_{E_2^*(\mathcal{H})} \leq c \| \xi^* \|_{E_2^*}.$$

The estimation of the terms

$$\| \varphi_j(uA)I_j(u)G \|_{E_2(\mathcal{H})}$$

requires a more elaborate machinery.

As in [7], define (M as in (3.8))

$$\mathcal{K} = \{ k \in W_1^1(\mathbb{R}^+) \mid \int_0^\infty \sqrt{t} |k'(t)| dt \leq M \},$$

and for $k \in \mathcal{K}$, and an \mathcal{F} -adapted finite rank step process $G : \mathbb{R}^+ \times \Omega \rightarrow L_q(\mathcal{O}; H)$, the process

$$I(k)G(t) = \int_0^t k(t-s)G(s) dW_H(s), \quad t \geq 0.$$

By the Ito isometry, these (scalarvalued) processes are welldefined. To extend $I(k)$ to a bounded operator from $L_{p,\mathcal{F}}(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))$ into $L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))$, use the inequality (see (2.1) of [7])

$$\mathcal{E} \left\| \int_0^\infty G dW_H \right\|_{L_q(\mathcal{O}; \mathbb{R})}^p \leq c\mathcal{E} \|G\|_{L_2(\mathbb{R}^+; L_q(\mathcal{O}; H))}^p.$$

Note that by Proposition 3.3, in particular by (3.8), the functions $k_j \in \mathcal{K}$.

The operators $I(k)$ are uniformly bounded, by the fact that

$$\sup_{k \in \mathcal{K}} \int_0^\infty k^2(t) dt < \infty.$$

To see this, write

$$\int_0^\infty k^2(t) dt = - \int_0^\infty k'(s) \int_0^s k(t) dt ds = - \int_0^\infty s^{\frac{1}{2}} k'(s) \left[s^{-\frac{1}{2}} \int_0^s k(t) dt \right] ds$$

and observe that

$$\begin{aligned} \left| s^{-\frac{1}{2}} \int_0^s k(t) dt \right| &= \left| \int_s^\infty s^{\frac{1}{2}} k'(\tau) d\tau + \int_0^s \tau s^{-\frac{1}{2}} k'(\tau) d\tau \right| \\ &\leq \int_0^\infty \tau^{\frac{1}{2}} |k'(\tau)| d\tau. \end{aligned}$$

But even more is true, the operator family I is \mathcal{R} -bounded, [7, Theorem 3.1]:

Proposition 6.2. *For all $p \in (2, \infty)$ and $q \in [2, \infty)$, the operator family I is \mathcal{R} -bounded from $L_{p,\mathcal{F}}(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; H))$ to $L_p(\mathbb{R}^+ \times \Omega; L_q(\mathcal{O}; \mathbb{R}))$. The same result is true if $p = q = 2$.*

For the definition of \mathcal{R} -boundedness and the proof of this proposition, see [7].

The following multiplier result makes use of the \mathcal{R} -boundedness. (See [7], [8].)

Proposition 6.3. *Let E_1, E_2 be Banach function spaces with finite cotype and let ν be a σ -finite Borel measure on \mathbb{R}^+ . Let $M : \mathbb{R}^+ \rightarrow \mathcal{L}(E_1, E_2)$ be a function with the following properties:*

- (1) *for all $x \in E_1$ the function $t \rightarrow M(t)x$ is strongly ν -measurable in E_2 ,*
- (2) *the range $\mathcal{M} = \{M(t) \mid t \in \mathbb{R}^+\}$ is \mathcal{R} -bounded in $\mathcal{L}(E_1, E_2)$.*

Then, for $G \in E_1(L_2(\mathbb{R}^+; \nu))$ we have $MG \in E_2(L_2(\mathbb{R}^+; \nu))$ and

$$\|MG\|_{E_2(L_2(\mathbb{R}^+; \nu))} \leq \mathcal{R}(\mathcal{M}) \|G\|_{E_1(L_2(\mathbb{R}^+; \nu))}.$$

Above, $\mathcal{R}(\mathcal{M})$ denotes the \mathcal{R} -bound of the operator family \mathcal{M} .

Take E_1, E_2 as in (6.3) and recall that $f \in E_1(\mathcal{H})$ is defined by requiring $\|f\|_{\mathcal{H}} \in E_1$. Then note that

$$\varphi_j(uA)I_j(u)G = I_j(u)\varphi_j(uA_H)G,$$

apply Proposition 6.3 and (6.6) of Proposition 6.1 pointwise in $\mathbb{R}^+ \times \Omega$ to obtain

$$(6.9) \quad \begin{aligned} \|\varphi_j(uA)I_j(u)G\|_{E_2(\mathcal{H})} &\leq \\ \mathcal{R}(I_j(u), u \in \mathbb{R}^+, j = \pm 1) \|\varphi_j(uA_H)G\|_{E_1(\mathcal{H})} &\leq \\ c\mathcal{R}(I_j(u), u \in \mathbb{R}^+, j = \pm 1) \|G\|_{E_1}. \end{aligned}$$

By (3.8) and by Proposition 6.2,

$$(6.10) \quad \mathcal{R}(I_j(u), u \in \mathbb{R}^+, j = \pm 1) \leq c < \infty.$$

By (6.8), (6.9), (6.10),

$$|\langle A^\theta D_t^\eta U, \xi^* \rangle_{E_2, E_2^*}| \leq c \|G\|_{E_1} \|\xi^*\|_{E_2^*}.$$

Taking the supremum over $\|\xi^*\|_{E_2^*} \leq 1$ gives

$$\|A^\theta D_t^\eta U\|_{E_2} \leq c \|G\|_{E_1}.$$

Recall the definitions of E_1 and E_2 to see that (3.3) has been obtained. \square

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