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RIEMANN-STIELTJES INTEGRALS WITH RESPECT TO FRACTIONAL BROWNIAN MOTION AND APPLICATIONS

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Abstract: In this dissertation we study Riemann-Stieltjes integrals with respect to (geometric) fractional Brownian motion, its financial counterpart and its application in estimation of quadratic variation process. From the point of view of financial mathematics, we study the fractional Black-Scholes model in continuous time.

We show that the classical change of variable formula with convex functions holds for the trajectories of fractional Brownian motion. Putting it simply, all European options with convex payoff can be hedged perfectly in such pricing model. This allows us to give new arbitrage examples in the geometric fractional Brownian motion case. Adding proportional transaction costs to the discretized version of the hedging strategy, we study an approximate hedging problem analogous to the corresponding discrete hedging problem in the classical Black-Scholes model. Using the change of variables formula result, one can see that fractional Brownian motion model shares some common properties with continuous functions of bounded variation. we also show a representation for running maximum of continuous functions of bounded variations such that fractional Brownian motion does not enjoy this property.

AMS subject classifications: 60G15, 60H05, 62M15, 91B28, 91B70

Keywords: fractional Brownian motion, pathwise stochastic integral, quadratic variation, functions of bounded variation, arbitrage, pricing by hedging, approximative hedging, proportional transaction costs

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List of included articles

The dissertation consists of an introduction to fractional Brownian motion, local time for Gaussian processes, pathwise stochastic integration in fractional Besov type spaces and the following articles.

I Azmoodeh, E., Mishura, Y., Valkeila, E. (2009). *On hedging European options in geometric fractional Brownian motion market model.* Statistics & Decisions, **27**, 129-143.

II Azmoodeh, E. (2010). *On the fractional Black-Scholes market with transaction costs.* `http://arxiv.org/pdf/1005.0211`.

III Azmoodeh, E., Tikanmäki, H., Valkeila, E. (2010). *When does fractional Brownian motion not behave as a continuous function with bounded variation?* Statistics & Probability Letters, **80**, Issues 19-20, 1543-1550.

IV Azmoodeh, E., Valkeila, E. (2010). *Spectral characterization of the quadratic variation of mixed Brownian fractional Brownian motion.* `http://arxiv.org/pdf/1005.4349v1`.

Author's contribution

I The work is a joint discussion with Yuliya Mishura from Kyiv university and Esko Valkeila from TKK. All main results of the article are formulated by the author and for the detail proofs too. The surprising result which is presented in section 4 is my independent study.

II This work completely represents my independent research studies.

III The theorem 3.1 in the main result section is a joint work with Heikki Tikanmäki from TKK. The subsection 3.2 of the main result section is my independent work. The study is initiated by Esko Valkeila.

IV The work is a joint discussion with Esko Valkeila. The general ideas are given by him. The author is formulated all the results and gives the detail proofs.

For all articles included in the thesis, the author is responsible for most of writting.

Contents

1 Introduction

The fractional Brownian motion (fBm) is a generalization of the more simple and more studied stochastic process of standard Brownian motion. More precisely, the fractional Brownian motion is a centered continuous Gaussian process with stationary increments and H*-self similar* properties. The *Hurst parameter* H, due to the British hydrologist H. E. Hurst, is between 0 and 1. The case $H = \frac{1}{2}$ corresponds to standard Brownian motion.

On the other hand, the increments of the fractional Brownian motion are not independent, except the in case of standard Brownian motion. The Hurst parameter H can be used to characterize the dependence of the increments and indicates the memory of the process. The increments over two disjoint time intervals are *positively correlated* when $H > \frac{1}{2}$ and are *negatively correlated* for $H < \frac{1}{2}$. In the first case, the dependence between two increments decay slowly so that it does sum to infinity as the time intervals grow apart, and exhibits *long range dependence* or the *long memory* property. For the latter case, the dependence is fast and is refered to as *short rage dependence* or *short memory*. Obviously, for $H = \frac{1}{2}$, the increments are independent.

The self-similarity and long range dependence properties allow us to use fractional Brownian motion as a model in different areas of applications e.g. hydrology, climatology, signal processing, network traffic analysis and mathematical finance. Besides such applications, it turns out that fractional Brownian motion is not a semimartingale nor a Markov process, except in the case when $H = \frac{1}{2}$. Hence, the classical stochastic integration theory for semimartingales is not at hand and so makes fractional Brownian motion more interesting from a purely mathematical point of view.

The financial pricing models with continuous trading, based on geometric fractional Brownian motion sometimes allow for existence of arbitrage. The existence of arbitrage essentially depends to the kind of stochastic integral in the definition of the wealth process. It can be shown that with Skorohod integration theory arbitrages disappear, but difficult to give economic interpretation, see Björk and Hult [17] and Sottinen and Valkeila [62]. On the other hand, Riemann - Stieltjes integrals seem more natural and sound better for economical interpretations. Using the Riemann - Stieltjes integration theory with adding *proportional transaction costs* lets us to construct a framework which acknowledges the pricing models with geometric fractional Brownian motion. First, Guasoni [29] showed that we have the absence of arbitrage in pricing model with proportional transaction costs based on geometric fractional Brownian motion with continuous trading. Moreover, in this setup, Guasoni, Rásonyi and Schachermayer [31] proved a fundamental theorem of asset pricing type result. The results by Guasoni, Rásonyi and Schachermayer open a new window to the pricing models based on fractional type processes such geometric fractional Brownian motion.

2 Fractional Brownian motion (fBm)

Definition 2.1. *The fractional Brownian motion* B^H *with Hurst parameter* $H \in (0,1)$*, is a centered Gaussian process with covariance function*

$$R_H(s,t) := \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big), \quad \text{for } s,t \in \mathbb{R}_+. \tag{1}$$

Remark 2.0.1. *Clearly* $R_{\frac{1}{2}}(s,t) = s \wedge t$*, and so* $B^{\frac{1}{2}} \overset{d}{=} W$*, where* W *is a standard Brownian motion, and* $\overset{d}{=}$ *stands for equality in finite dimensional distributions.*

Remark 2.0.2. *A general existence result for zero mean Gaussian processes with a given covariance function* (*see proposition 3.7 of [54]*) *implies that fractional Brownian motion exists. For a different construction of fractional Brownian motion using white noise theory, see [7].*

2.1 Primary properties of fBm

Here we list some properties of fBm that can be obtained directly from the covariance function R_H.
Stationary increments: For any $s \in \mathbb{R}_+$, we have that

$$\{B^H_{t+s} - B^H_s\}_{t\in\mathbb{R}_+} \overset{d}{=} \{B^H_t\}_{t\in\mathbb{R}_+}.$$

This follows from the fact that $\mathbb{E}|B^H_t - B^H_s|^2 = f(t-s)$, where $f(t) = |t|^{2H}$, and from proposition 3 of section 4 of [41].
Hölder continuity: Let $0 < \alpha < 1$. For a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, set

$$\|f\|_{C^\alpha(\mathbb{R}_+)} := \sup_{0\le s<t} \frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

If $\|f\|_{C^\alpha(\mathbb{R}_+)} < \infty$, we say that f is an α-Hölder continuous function. The class of all α-Hölder continuous functions on the real line (or interval $[0,T]$) is denoted by $C^\alpha(\mathbb{R}_+)$(or $C^\alpha[0,T]$). According to the Kolmogorov continuity criterion (see [54]), a stochastic process $X = \{X_t\}_{t\in\mathbb{R}_+}$ has a continuous modification $\tilde{X}$, if there exist constants $\alpha, \beta, c > 0$ such that

$$\mathbb{E}|X_t - X_s|^\alpha \le c|t-s|^{1+\beta}, \quad \text{for } s,t \in \mathbb{R}_+.$$

Moreover, the modification $\tilde{X}$ has locally Hölder continuous trajectories of any order $\lambda \in [0, \frac{\beta}{\alpha})$ almost surely, i.e. for any given compact set $K \subset \mathbb{R}_+$, there exists an almost surely finite and positive random variable $C = C(\lambda, K)$ such that

$$|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)| \le C(\omega)|t-s|^\lambda, \quad \text{for } s,t \in K,$$

almost surely.

Theorem 2.1. *The fractional Brownian motion B^H has a continuous modification whose trajectories are locally λ-Hölder continuous for any $\lambda < H$. Moreover for any $\lambda > H$, the fractional Brownian motion trajectories are nowhere λ-Hölder continuous on any interval almost surely.*

Proof. For any $\alpha > 0$, we have

$$\mathbb{E}|B_t^H - B_s^H|^\alpha = \mathbb{E}|B_1^H|^\alpha |t-s|^{\alpha H}.$$

So the claim follows by the Kolmogorov continuity criterion. Moreover by the *law of iterated logarithm* for fractional Brownian motion (see [1]): for all $t > 0$, almost surely

$$\limsup_{\epsilon \to 0^+} \frac{B_{t+\epsilon} - B_t}{\sqrt{2\epsilon^{2H} \log\log(\frac{1}{\epsilon})}} = 1.$$

Hence, the trajectories of fractional Brownian motion B^H cannot be Hölder continuous of any order greater than H on any interval with probability one. □

Markov property: According to lemma 5.1.9 of [45], a centered Gaussian process X with continuous covariance function R is a Gaussian Markov process iff R can be expressed in the form

$$R(s,t) = \begin{cases} p(s)q(t) & \text{if } s \le t, \\ p(t)q(s) & \text{if } t < s, \end{cases}$$

for some positive functions p and q. Hence,

Proposition 2.1. *The fractional Brownian motion B^H is a Gaussian Markov process iff $H = \frac{1}{2}$.*

2.2 Self-similarity and long-range dependence

Definition 2.2. *We say that a stochastic process $X = \{X_t\}_{t \in \mathbb{R}_+}$ is self-similar with index $H > 0$ or H - self-similar, if for any $a > 0$,*

$$\{X_{at}\}_{t \in \mathbb{R}_+} \stackrel{d}{=} \{a^H X_t\}_{t \in \mathbb{R}_+}.$$

Since the covariance function R_H is homogeneous of order $2H$, we have the following:

Proposition 2.2. *The fractional Brownian motion B^H is a H - self-similar process.*

Definition 2.3. *The stationary and H - self-similar sequence*

$$Z_n := B_{n+1}^H - B_n^H, \qquad n \in \mathbb{N}_0 \tag{2}$$

is called fractional Gaussian noise.

Proposition 2.3. *For fractional Gaussian noise Z defined in (2), let*

$$r_H(n) := \mathbb{E}(Z_{n+k}Z_k) = \frac{1}{2}\big((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\big), \quad n \in \mathbb{N}.$$

Then, we have that
(i) For $n \neq 0$,

$$r_H(n) \begin{cases} < 0 & \text{if } H \in (0, \frac{1}{2}), \ (\textit{negatively correlated increments}) \\ = 0 & \text{if } H = \frac{1}{2}, \ (\textit{independent increments}) \\ > 0 & \text{if } H \in (\frac{1}{2}, 1), \ (\textit{positively correlated increments}). \end{cases} \tag{3}$$

(ii) For $H \neq \frac{1}{2}$, as $n \to \infty$

$$r_H(n) \sim H(2H-1)n^{2H-2}, \ \textit{i.e.} \lim_{n\to\infty} \frac{r_H(n)}{H(2H-1)n^{2H-2}} = 1. \tag{4}$$

Proposition 2.4. *Let Z be fractional Gaussian noise defined in (2) with covariance function $r_H(n)$. Then we have that*

$$\begin{aligned} \sum_{n\in\mathbb{N}_0} r_H(n) = \infty, \quad & \textit{if } H > \frac{1}{2}, \\ \sum_{n\in\mathbb{N}_0} |r_H(n)| < \infty, \quad & \textit{if } H < \frac{1}{2}. \end{aligned} \tag{5}$$

Definition 2.4. *A stationary sequence $X = \{X_n\}_{n\in\mathbb{N}_0}$ with covariance function $r(n) :=$ Cov (X_{n+k}, X_k) exhibits long-range dependence (or long memory), if*

$$\lim_{n\to\infty} \frac{r(n)}{cn^{-\alpha}} = 1, \quad \textit{or} \quad \sum_{n\in\mathbb{N}_0} r(n) = \infty, \tag{6}$$

for some constants c and $\alpha \in (0,1)$.

Proposition 2.5. *The fractional Gaussian noise Z defined in (2) exhibits long-range dependence property iff $H > \frac{1}{2}$.*

The definition of long-range dependence can be given using the spectral density function, see [63] for equivalent definitions. For a survey on the theory of long-range dependence and its applications, consult [23] and [57].

2.3 p - variation

In this section we recall the concept of p - variation which gives information about the regularity of trajectories of stochastic processes. Young [68] noticed that p - variation can be useful in integration theory when one has integrators of unbounded variation (see section 4). For more details on p - variation, see Dudley and Norvaiša [24], [25] and Mikosch and Norvaiša [46].

Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing, continuous, unbounded, convex function and $\Phi(0) = 0$. For each partition $\pi := \{0 = t_1 < t_2 < \cdots <$

$t_n = T\}$ of the interval $[0,T]$, the mesh of π, denoted by $|\pi|$, is defined by $|\pi| := \max_{1\le k\le n}(t_k - t_{k-1})$. For a function $f : [0,T] \to \mathbb{R}$, set

$$v_\Phi(f;\pi) := \sum_{k=1}^{n} \Phi|f(t_k) - f(t_{k-1})|.$$

Definition 2.5. *We define Φ-variation of the function f over the interval $[0,T]$ by*

$$v_\Phi(f) := \sup_\pi v_\Phi(f;\pi), \tag{7}$$

where the supremum is taken over all partitions π of the interval $[0,T]$. If $v_\Phi(f) < \infty$, we say that f has the bounded Φ-variation property and we denote by $\mathcal{W}_\Phi$ the class of all functions f with bounded Φ-variation.

The function $\Phi(x) = x^p$ where $x \ge 0$ and $1 \le p < \infty$, serves as a special example. The case $p = 1$ corresponds to the classical case of bounded variation. For the function $\Phi(x) = x^p$, we denote $v_\Phi(f) = v_p(f)$ and $\mathcal{W}_\Phi = \mathcal{W}_p$. Moreover, we define the *index* of the function f by

$$v(f) := \inf\{p \ge 1; v_p(f) < \infty\}.$$

Let

$$\|f\|_{(p)} := (v_p(f))^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{[p]} := \|f\|_{(p)} + \|f\|_\infty,$$

where $\|f\|_\infty := \sup_{t\in[0,T]} |f(t)|$. Then we have that

Proposition 2.6. *The function $\|\cdot\|_{[p]}$ is a norm on the class $\mathcal{W}_p$ and the pair $(\mathcal{W}_p, \|\cdot\|_{[p]})$ is a Banach space.*

The next proposition (see Dudley and Norvaiša [24] for a proof) shows the link between Hölder continuous and bounded p-variation functions.

Proposition 2.7. *Let $1 \le p < \infty$. Then the function $f : [0,T] \to \mathbb{R}$ belongs to $\mathcal{W}_p$ if and only if $f = g \circ h$, where h is a bounded, non-negative and increasing function on $[0,T]$ and g is a Hölder continuous function of order $\frac{1}{p}$ on $[h(0), h(T)]$.*

Let $X = \{X_t\}_{t\in[0,T]}$ be a separable centered Gaussian process with incremental variance $\sigma_X^2(s,t) = \mathbb{E}(X_t - X_s)^2$. Then a result by Jain and Monrad [32] gives a sufficient condition for X belonging to $\mathcal{W}_p$ with probability one for $p \ge 1$ by using the function σ_X. See also Kawada and Kono [38] for a related study of p - variation for a general class of Gaussian processes.

Theorem 2.2. *Let X be as above. Set*

$$\kappa(p,\sigma) := \sup_\pi \sum_{t_k\in\pi} \left[\sigma_X(t_k,t_{k-1})\big(\log^*(\log^* \sigma_X(t_k,t_{k-1}))\big)^{\frac{1}{2}}\right]^p,$$

where the supremum is taken over all partitions π of the interval $[0,T]$ and $\log^*(x) = \max\{1, |\log(x)|\}$. Now condition $\kappa(p,\sigma) < \infty$ implies that $X \in \mathcal{W}_p$ with probability one.

Applying the result of Kawada and Kono to fractional Brownian motion, we have (see Mishura [47], page 93 for a proof):

Proposition 2.8. *Almost surely, $B^H \in \mathcal{W}_p$ for any $p > \frac{1}{H}$ and $v_p(B^H) = \infty$ for $p < \frac{1}{H}$. Moreover $v(B^H) = \frac{1}{H}$ with probability one.*

Remark 2.2.1. *Let us emphasize that there is a difference between bounded 2-variation and the concept of finite quadratic variation. Let $\{\pi_n\}$ be a sequence of partitions of $[0,T]$ such that $|\pi_n| \to 0$. For a stochastic process X, the quadratic variation $[X,X]_T$ along the sequence $\{\pi_n\}$ is defined by*

$$[X,X]_T := \mathbb{P} - \lim_{|\pi_n|\to 0} \sum_{t_k^n \in \pi_n} \left(X_{t_k^n} - X_{t_{k-1}^n}\right)^2, \tag{8}$$

if the limit exists. For example, for standard Brownian motion W, we have $[W,W]_t = t$; $t \in [0,T]$ along any refining sequence of partitions of $[0,T]$ with mesh goes to 0, but $v_2(W) = \infty$ with probability one.

Further, we have the following result for the quadratic variation of fractional Brownian motion.

Theorem 2.3. *For the fractional Brownian motion $B^H = \{B_t^H\}_{t\in[0,T]}$, we have that $[B^H, B^H]_T = 0$ if $H > \frac{1}{2}$ and that $[B^H, B^H]_T$ does not exists if $H < \frac{1}{2}$, where $[B^H, B^H]_T$ is defined by (8). Moreover, B^H is of unbounded variation almost surely.*

Proof. Consider the equidistant partitions $\hat{\pi}_n := \{t_k^n = \frac{kT}{n}; 0 \le k \le n\}$, $n \in \mathbb{N}$. Then by the self-similarity of B^H, we have

$$\begin{aligned} v_p(B^H, \hat{\pi}_n) = \sum_{k=1}^{n} |B^H_{t_k^n} - B^H_{t_{k-1}^n}|^p &\overset{d}{=} T^{pH} n^{1-pH} \frac{1}{n} \sum_{k=1}^{n} |B^H_k - B^H_{k-1}|^p \\ &\to \begin{cases} \infty & \text{if } p < \frac{1}{H}, \\ T\mathbb{E}|B_1^H|^{\frac{1}{H}} & \text{if } p = \frac{1}{H}, \\ 0 & \text{if } p > \frac{1}{H}, \end{cases} \end{aligned}$$

as n tends to infinity. The convergence can be shown to take place in $L^2(\Omega, \mathbb{P})$ by a result from ergodic theory. Also, when $H > \frac{1}{2}$, take $\alpha \in (\frac{1}{2}, H)$. Then using the Hölder continuity of trajectories of B^H, for any sequence π_n of partitions of the interval $[0,T]$ such that $|\pi_n| \to 0$, we can write

$$\begin{aligned} [B^H, B^H]_T &= \mathbb{P} - \lim_{|\pi_n|\to 0} \sum_{t_k \in \pi_n} \left(B^H_{t_k} - B^H_{t_{k-1}}\right)^2 \\ &\le C^2(\omega) \lim_{|\pi_n|\to 0} \sum_{t_k \in \pi_n} (t_k - t_{k-1})^{2\alpha} \\ &\le C^2(\omega) \lim_{|\pi_n|\to 0} |\pi_n|^{2\alpha-1} \sum_{t_k \in \pi_n} (t_k - t_{k-1}) \\ &= 0 \end{aligned}$$

almost surely as n tends to infinity. □

2.4 Non semimartingale property

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered probability space, satisfying the usual conditions. We denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$, the class of all almost surely finite random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.6. *A simple integrand is a stochastic process $H = \{H_t\}_{t\in[0,T]}$ with the representation*

$$H_t = \sum_{k=1}^{n} H_k \boldsymbol{1}_{(\tau_{k-1},\tau_k]}(t), \qquad t \in [0, T],$$

where $n \in \mathbb{N}$ is a finite number, $H_k \in L^\infty(\Omega, \mathcal{F}_{\tau_{k-1}}, \mathbb{P})$ and the τ_k's are an increasing sequence of stopping times such that $0 = \tau_0 \le \tau_1 \le \cdots \le \tau_n = T$.

We denote by $\mathbf{S}_u$, the class of all simple integrands on filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ and endow it with a uniform norm on (t, ω), i.e.

$$\|H\|_\infty = \sup_{t\in[0,T]} \|H_t\|_{L^\infty(\Omega,\mathbb{P})}.$$

Let $X = \{X_t\}_{t\in[0,T]}$ be a càdlàg and adapted stochastic process. We define the integration operator $I_X : \mathbf{S}_u \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ by

$$I_X\Big(\sum_{k=1}^{n} H_k \mathbf{1}_{(\tau_{k-1},\tau_k]}\Big) = \sum_{k=1}^{n} H_k (X_{\tau_k} - X_{\tau_{k-1}}). \tag{9}$$

Following Protter [53], we define a good integrator as follows:

Definition 2.7. *Assume $L^0(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the topology of convergence in probability. We call a real-valued, càdlàg and adapted stochastic process $X = \{X_t\}_{t\in[0,T]}$ a " good integrator " if the integration operator*

$$I_X : \boldsymbol{S}_u \to L^0(\Omega, \mathcal{F}, \mathbb{P})$$

is continuous.

The next theorem gives a complete characterization of the structure of stochastic processes X for which the integration operator I_X given by (9) is continuous.

Theorem 2.4. (*Bichteler - Dellacherie Theorem*) *[14], [15], [21] Let $X = \{X_t\}_{t\in[0,T]}$ be a real-valued, càdlàg and adapted stochastic process. Then we have the equivalent statements:*
(i) X is a good integrator. (*see definition 2.7.*)
(ii) X is a semimartingale, i.e. X can be decomposed as $X = M + A$, for some local martingale M and an adapted and finite variation process A.

Applying this to fractional Brownian motion, it turns out that the fractional Brownian motion B^H is not a good integrator in the sense of definition 2.7.

Theorem 2.5. *The fractional Brownian motion B^H is a semimartingale iff $H = \frac{1}{2}$.*

Proof. [5] Let $H \neq \frac{1}{2}$ and $t_k^n := \frac{Tk}{n}$. Define

$$H_t^n := n^{2H-1} \sum_{k=1}^{n-1} \left(B_{t_k^n}^H - B_{t_{k-1}^n}^H \right) \mathbf{1}_{(t_k^n, t_{k+1}^n]}(t).$$

Then we have that $\|H^n\|_\infty \to 0$ as n tends to infinity, but

$$\begin{aligned} \mathbb{P} - \lim_{n\to\infty} I_S(H^n) &= \mathbb{P} - \lim_{n\to\infty} n^{2H-1} \sum_{k=1}^{n-1} \left(B_{t_k^n}^H - B_{t_{k-1}^n}^H \right) \left(B_{t_{k+1}^n}^H - B_{t_k^n}^H \right) \\ &= T^{2H} (2^{2H-1} - 1) \neq 0 \end{aligned}$$

by theorem 9.5.2 of [37]. Therefore, the fractional Brownian motion B^H is not a semimartingale if $H \neq \frac{1}{2}$. □

3 Local time

In this section, we summarize some results on another characteristic which deals with the regularity of trajectories. For a nice survey article on the subject (non random and random functions), see Geman and Horowitz [43]. See also the book by Marcus and Rosen [45].

3.1 Local time of Gaussian processes

Let $X = \{X_t\}_{t\in[0,T]}$ be a real-valued continuous Gaussian process. The *occupation measure* of X is

$$\Gamma_X(A, I) := m(X^{-1}(A) \cap I) = m(t \in I;\ X_t \in A), \tag{10}$$

where $I \in \mathcal{B}([0,T]), A \in \mathcal{B}(\mathbb{R})$ and m is Lebesgue measure on the real line. Clearly, this is a random measure, depending on $\omega \in \Omega$. If the occupation measure Γ_X as a set function of A is absolutely continuous with respect to Lebesgue measure, then its density, $l(x, I)$, is called the *local time* of X with respect to I. The local time $l(x, I)$ can be interpreted as the amount of time spent at x by the process X during the time period I. Hence, by the definition, we have

$$\Gamma_X(A, I) = \int_A l(x, I) dx, \qquad \text{for all } A \text{ and } I. \tag{11}$$

Moreover, we define the two parameter stochastic process $l(x,t)$ of space parameter x and time parameter t as

$$l(x,t) := l(x, [0,t]), \quad t \in [0,T],\ x \in \mathbb{R}.$$

For the local time $l(x, .)$ as a set function on the Borel sets $\mathcal{B}([0,T])$, we have the following result.

Proposition 3.1. *For the local time l we have:*
(i) Almost surely, $l(x,.)$ is a finite measure on $\mathcal{B}([0,T])$ for every x.
(ii) Almost surely, the measure $l(x,.)$ is carried by the level set at x, i.e.

$$I_x = \{t \in [0,T];\ X_t = x\}.$$

Now, we state a general result on the existence of local time for Gaussian processes given by Berman.

Theorem 3.1. *Let $X = \{X_t\}_{t\in[0,T]}$ be a centered, continuous Gaussian process with $\sigma_X^2(s,t) = \mathbb{E}(X_t - X_s)^2$. There exists a local time $l \in L^2(m \times \mathbb{P})$ for the process X, if and only if*

$$\int_0^T \int_0^T \frac{1}{\sqrt{\sigma_X(s,t)}} dsdt < \infty. \tag{12}$$

It is clarified in a work by Berman [11], that the concept of *local nondeterminism* (LND), defined below, plays a central role in the study of local times of Gaussian processes. See, for example, the works by Berman [12], [13], for more information on local nondeterminism and the introduction of the work by Xiao [67] for more references. Also Cuzick [20] gives a generalization to local ϕ- nondeterminism. Now assume that $X = \{X_t\}_{t\in[0,T]}$ is a centered Gaussian process and there is a $\delta > 0$ such that

$$\mathbb{E}(X_t)^2 > 0, \text{ for } t > 0 \quad \text{and} \quad \mathbb{E}(X_t - X_s)^2 > 0$$

for all $s,t \in [0,T]$ and $0 < |t-s| < \delta$.

Definition 3.1. *Let $t_1 < t_2 < \cdots < t_m$ be chosen from the interval $[0,T]$ and $m \geq 2$. Set*

$$V_m := \frac{Var\big(X_{t_m} - X_{t_{m-1}} | X_{t_1}, \cdots, X_{t_{m-1}}\big)}{Var(X_{t_m} - X_{t_{m-1}})}.$$

We say that X is locally nondeterministic on the interval $[0,T]$, if for any integer $m \geq 2$

$$\lim_{\epsilon\to 0} \inf_{t_m - t_1 \leq \epsilon} V_m > 0.$$

Let $X = \{X_t\}_{t\in[0,T]}$ be a centered, stationary increments Gaussian process with $\sigma_X^2(t) = \mathbb{E}(X_{t+s} - X_s)^2$. Moreover, assume that $|t|^{-\beta}\sigma_X(t) \to c > 0$ as $t \to 0$ for some index $\beta \in (0,1)$.

Theorem 3.2. *Let X be as above and LND. Then X has a jointly continuous local time $l(x,t)$ such that for any compact set $K \subseteq \mathbb{R}$,*
(i) for any $\lambda < min\{1, \frac{1-\beta}{2\beta}\}$

$$\sup_{t\in[0,T]} \sup_{x,y\in K} \frac{|l(x,t) - l(y,t)|}{|x-y|^\lambda} < \infty \quad a.s. \quad and$$

(ii) for any $\theta < 1 - \beta$

$$\sup_{x\in K} \sup_{s,t\in[0,T]} \frac{|l(x,t) - l(x,s)|}{|t-s|^\theta} < \infty \quad a.s.$$

In 1978, Loren D. Pitt proved that fractional Brownian motion is LND, [52], lemma 7.1.

Proposition 3.2. *The fractional Brownian motion* $B^H = \{B_t^H\}_{t\in[0,T]}$ *has a jointly continuous local time* $l^H(x,t)$ *which is Hölder continuous in the time variable* t *of any order* $\theta < 1-H$ *and in the space variable* x *of any order* $\lambda < \frac{1-H}{2H}$.

3.2 Approximation of local time

As proposition 3.1 suggests, one may approximate the local time $l(x,t)$ of a stochastic process $X = \{X_t\}_{t\in[0,T]}$ with irregular trajectories by the *number of level* x *crossings*, i.e.

$$N^x(X,[0,T]) := \#\{t \in [0,T];\ X_t = x\}, \tag{13}$$

with a suitable normalization factor and an appropriate convergence. For general processes, this can be done in two different ways, (i) and (ii) below. Here we consider only Gaussian processes. For a more detailed account, see for example the works by Azaïs and Wschebor [2], [3], [4], [65], [66] and a survey article by Kratz [39] and the references therein.

(i) *Regularized approximation*: For a process X with irregular trajectories, the regularization of X is defined by

$$X_\epsilon := X * \psi_\epsilon, \quad \text{where} \quad \psi_\epsilon(t) = \frac{1}{\epsilon}\psi(\frac{t}{\epsilon}),$$

and ψ is a non-negative, C^∞ function with compact support and $*$ means the convolution of functions. Then Wschebor [65] showed that:

Theorem 3.3. *Let* $W = \{W_t\}_{t\in[0,T]}$ *be a Brownian motion. Then for any* $x \in \mathbb{R}$,

$$\sqrt{\frac{\pi}{2}}\frac{\epsilon^{\frac{1}{2}}}{\|\psi\|_2}N^x(W_\epsilon,[0,T]) \to l(x,[0,T]) \qquad \text{as } \epsilon \to 0,$$

where convergence is in $L^p(\Omega,\mathbb{P})$ *for any* $p \in \mathbb{N}$.

Later, Azaïs [2] studied more general stochastic processes including Gaussian processes. He provided the sufficient conditions for L^2 convergence of the number of level crossings of some regularized approximation process X_ϵ of X to the local time of X, using a suitable normalization factor.

(ii) *Polygonal approximation*: Following Azaïs, the polygonal approximation of size Δ of stochastic process X, X_Δ, is the polygonal line connecting the points $\{(k\Delta, X(k\Delta))\}$, i.e. for $k\Delta \le t \le (k+1)\Delta$,

$$X_\Delta(t) := \big(\frac{t}{\Delta} - k\big)X((k+1)\Delta) + \big(1 - \frac{t}{\Delta} + k\big)X(k\Delta).$$

For any $x \in \mathbb{R}$, put

$$C^x(X_\Delta, [0,T]) = \{t \in [0,T]; X_\Delta(t) = x \text{ and } t \neq k\Delta \text{ for each index } k\}$$

and

$$N^x(X_\Delta, [0,T]) = \# C^x(X_\Delta, [0,T]),$$

i.e. the number of level x crossings of X_Δ over the interval $[0,T]$. Then, Azaïs [2] proved that

Theorem 3.4. *Let $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then for any $x \in \mathbb{R}$,*

$$\sqrt{\frac{\pi}{2}}\Delta^{1-H} N^x(B_\Delta^H, [0,T]) \to l^H(x, [0,T]) \qquad \text{as } \Delta \to 0,$$

where convergence is in $L^2(\Omega, \mathbb{P})$.

Note that in [2], Azaïs showed convergence in L^2 of normalized level crossings of polygonal approximation to the local time for more general Gaussian processes. He did so by putting some technical assumptions on the covariance functions of the Gaussian process and its polygonal approximation. His results include some stationary Gaussian processes and fractional Brownian motion as examples.

4 Pathwise integration with respect to fractional Brownian motion

Fractional Brownian motion is not a semimartingale, and hence the stochastic integral with respect to fractional Brownian motion B^H becomes more challenging. It turns out that fractional calculus creates a path to defining a kind of integral with respect to paths of fractional Brownian motion. For a complete treatment of deterministic fractional calculus, see the book by Samko. et. al. [56].

4.1 Fractional calculus on a finite interval

Let $a < b$ be two real numbers and $f : [a,b] \to \mathbb{R}$ be a function. Then by a straightforward induction argument, a multiple integral of f can be expressed as

$$\int_a^{t_n} \cdots \int_a^{t_2} \int_a^{t_1} f(u) du dt_1 \cdots dt_{n-1} = \frac{1}{(n-1)!} \int_a^{t_n} f(u)(t_n - u)^{n-1} du, \quad (14)$$

where $t_n \in [a,b]$ and $n \geq 1$. (By convention, $(0)! = 1$ and $a^0 = 1$.) We know that $(n-1)! = \Gamma(n)$. So replacing n by a real number $\alpha > 0$ in (14), we are motivated to define the so-called *fractional integrals* as follows.

Definition 4.1. *Let $f \in L^1[a,b]$ and $\alpha > 0$. The integrals*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} \int_a^b f(s)(t-s)_+^{\alpha-1} ds, \qquad (15)$$

for $t \in (a,b)$, and

$$I^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(s)(s-t)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} \int_a^b f(s)(s-t)_+^{\alpha-1} ds, \qquad (16)$$

where $t \in (a,b)$, are called fractional integrals of order α.

The fractional integral $I^{\alpha}_{a^+}$ is called left-sided since the integration in (15) is over the left hand side of the interval $[a,t]$ of the interval $[a,b]$. Similarly, the fractional integral $I^{\alpha}_{b^-}$ is called right-sided. Both integral $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are also called *Riemann - Liouville fractional integrals.*

Remark 4.0.1. *The fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are well - defined for functions $f \in L^1[a,b]$, and so also for functions $f \in L^p[a,b]$, for $p > 1$ as well, i.e. the integrals in (15) and (16) converge for almost all $t \in (a,b)$ with respect to Lebesgue measure.*

Remark 4.0.2. *The left (right) - sided fractional integrals can be defined on the whole real line in a similar way.*

Proposition 4.1. *For $\alpha > 0$, the fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ have the following properties:*
(i) Semigroup property: for $f \in L^1[a,b]$ and $\alpha, \beta > 0$,

$$I^{\alpha}_{a^+} I^{\beta}_{a^+} f = I^{\alpha+\beta}_{a^+} f \text{ and } I^{\alpha}_{b^-} I^{\beta}_{b^-} f = I^{\alpha+\beta}_{b^-} f. \qquad (17)$$

(ii) Fractional integration by parts formula: let $f \in L^p[a,b]$ and $g \in L^q[a,b]$ either with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, or with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Then we have

$$\int_a^b f(t)(I^{\alpha}_{a^+} g)(t) dt = \int_a^b g(t)(I^{\alpha}_{b^-} f)(t) dt. \qquad (18)$$

(iii) If $I^{\alpha}_{a^+} f = 0$ or $I^{\alpha}_{b^-} f = 0$ then $f(u) = 0$ almost everywhere.

For $0 < \alpha < 1$, we define the operator $I^{-\alpha}_{a^+}(I^{-\alpha}_{b^-})$ as the inverse of the fractional integral operator in the following way.

Definition 4.2. *Let $0 < \alpha < 1$. The integrals*

$$\mathcal{D}^{\alpha}_{a^+} f(t) = (I^{-\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(s)(t-s)^{-\alpha} ds \text{ and} \qquad (19)$$

$$\mathcal{D}^{\alpha}_{b^-} f(t) = (I^{-\alpha}_{b^-} f)(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b f(s)(s-t)^{-\alpha} ds, \qquad (20)$$

for $t \in (a,b)$, are called fractional derivatives of order α. Both (19) and (20) are also called the Riemann - Liouville fractional derivatives.

Remark 4.0.3. *The fractional derivatives $\mathcal{D}^{\alpha}_{a+}f$ and $\mathcal{D}^{\alpha}_{b-}f$ are well defined if, for example, function f can be expressed as $f = I^{\alpha}_{a+}\phi$ or $f = I^{\alpha}_{b-}\phi$, for some $\phi \in L^p[a,b]$ and $p \geq 1$.*

The next property will become useful in the context of the generalized Lebesgue - Stieltjes integration, as shown in proposition 4.3.

Remark 4.0.4. *Let two functions f_1 and f_2 be such that $f_1 = f_2$ almost everywhere. This implies that $\mathcal{D}^{\alpha}_{a^+(b^-)}f_1 = \mathcal{D}^{\alpha}_{a^+(b^-)}f_2$.*

Proposition 4.2. *For $0 < \alpha < 1$, the fractional derivatives $\mathcal{D}^{\alpha}_{a+}$ and $\mathcal{D}^{\alpha}_{b-}$ have the following properties:*
(i) Semigroup property: let $\alpha, \beta \geq 0$ and $f \in I^{\alpha+\beta}_{a^+(b^-)}(L^1[a,b])$. Then we have

$$\mathcal{D}^{\alpha}_{a^+(b^-)}\mathcal{D}^{\beta}_{a^+(b^-)}f = \mathcal{D}^{\alpha+\beta}_{a^+(b^-)}f.$$

(ii) For any f such that $f = I^{\alpha}_{a^+(b^-)}\phi$, we have that $I^{\alpha}_{a^+(b^-)}\mathcal{D}^{\alpha}_{a^+(b^-)}f = f$.
(iii) Integration by parts formula: for $0 < \alpha < 1$, $f \in I^{\alpha}_{a+}(L^p[a,b])$ and $g \in I^{\alpha}_{b-}(L^q[a,b])$, where $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, we have that

$$\int_a^b (\mathcal{D}^{\alpha}_{a+}f)(t)g(t)dt = \int_a^b f(t)(\mathcal{D}^{\alpha}_{b-}g)(t)dt. \tag{21}$$

Now, we are ready to briefly mention an approach that extends Young integration theory for the Lebesgue - Stieltjes integral, (LS) $- \int_a^b fdg$, to a larger class of integrands f and allows for integrators g to be of unbounded variation. For more details, see Zähle [69] and Mishura [47].

Consider two deterministic functions $f, g : [a,b] \to \mathbb{R}$ such that the right limit $f(t^+) = \lim_{\delta\to 0} f(t+\delta)$ and left limit $g(t^-) = \lim_{\delta\to 0} g(t-\delta)$ exist for any $t \in [a,b)$ and $t \in (a,b]$ respectively. Put

$$f_{a^+}(t) := (f(t) - f(a^+))\mathbf{1}_{(a,b)}(t) \text{ and } g_{b^-}(t) := (g(b^-) - g(t))\mathbf{1}_{(a,b)}(t).$$

Suppose that $f_{a^+} \in I^{\alpha}_{a+}(L^p[a,b])$ and $g_{b^-} \in I^{1-\alpha}_{b^-}(L^q[a,b])$, for some $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1$ and $0 < \alpha < 1$.

Definition 4.3. *The generalized Lebesgue - Stieltjes integral $\int_a^b fdg$ is defined as*

$$\int_a^b fdg := \int_a^b (\mathcal{D}^{\alpha}_{a+}f_{a^+})(t)(\mathcal{D}^{1-\alpha}_{b^-}g_{b^-})(t)dt + f(a^+)\big(g(b^-) - g(a^+)\big). \tag{22}$$

Remark 4.0.5. *The definition of the generalized Lebesgue - Stieltjes integral in (22) does not depend on the choice of α.*

Proposition 4.3. *Some elementary properties of generalized Lebesgue - Stieltjes integrals are:*
(i) $\int_a^b \mathbf{1}_{(c,d)}fdg = \int_c^d fdg$, if both integrals exist in the sense of the definition

(22).
(ii) $\int_a^c fdg + \int_c^b fdg = \int_a^b fdg$, *if each of integrals exist in the sense of the definition (22) and g is continuous.*
(iii) $\int_a^b \mathbf{1}_{(c,d]} dg = g(d) - g(c)$, *if g is Hölder continuous on* $[a,b]$.
(iv) $\int_a^b f_1 dg = \int_a^b f_2 dg$, *if* $f_1 = f_2$ *almost everywhere and both integrals exist in the sense of the definition (22)*

We denote the class of bounded variation functions on the interval $[a, b]$ by $BV[a, b]$.

Theorem 4.1. *Let* $f_{a^+} \in I_{a+}^{\alpha}(L^p[a,b])$ *and* $g \in I_{b^-}^{1-\alpha}(L^q[a,b]) \cap BV[a,b]$, *for some* $p, q \geq 1, \frac{1}{p} + \frac{1}{q} \leq 1$, $0 < \alpha < 1$ *and*

$$\int_a^b I_{a+}^{\alpha}(|\mathcal{D}_{a+}^{\alpha} f|)(t) d|g|(t) < \infty,$$

$$\text{then} \qquad \int_a^b fdg = (LS) - \int_a^b fdg.$$

In 1936, Young [68] extended the Riemann - Stieltjes integrals to integrators of unbounded variation. More precisely, he proved the following theorem:

Theorem 4.2. *Let* $f \in \mathcal{W}_p$ *and* $g \in \mathcal{W}_q$ *for some* $1 \leq p, q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} > 1$. *Moreover assume that f and g do not have any common point of discontinuity. Then the Riemann - Stieltjes integral*

$$(RS) - \int_a^b fdg$$

exists.

Corollary 4.1. *Let* $f \in C^{\lambda}[a,b]$ *and* $g \in C^{\mu}[a,b]$ *with* $\lambda + \mu > 1$. *Then the Riemann - Stieltjes integral*

$$(RS) - \int_a^b fdg$$

exists.

Zähle showed that, as one would expect, the following result holds.

Theorem 4.3. *Let* $f \in C^{\lambda}[a,b]$ *and* $g \in C^{\mu}[a,b]$ *with* $\lambda + \mu > 1$. *Then the generalized Lebesgue - Stieltjes integral* $\int_a^b fdg$ *exists and*

$$\int_a^b fdg = (RS) - \int_a^b fdg.$$

4.2 Pathwise stochastic integration in fractional Besov spaces

This subsection is devoted to an approach to pathwise stochastic integration in fractional Besov-type spaces, which was introduced by Nualart and Răşcanu [51]. We start with the following definition.

Definition 4.4. *Let* $0 < \alpha < 1$.
(i) Denote by $W_0^{\alpha,\infty}[0,T]$, *the space of functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \Big(|f(t)| + \int_0^T \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} ds \Big) < \infty.$$

(ii) Denote by $W_0^{\alpha,1}[0,T]$, *the space of functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(t)|}{t^\alpha} dt + \int_0^T \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}} dsdt < \infty,$$

(iii) Denote by $W_T^{1-\alpha,\infty}[0,T]$, *the space of functions* $f : [0,T] \to \mathbb{R}$ *such that*

$$\|f\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|f(t)-f(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|f(y)-f(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Remark 4.3.1. *For any* $0 < \epsilon < \alpha \wedge (1-\alpha)$,

$$\begin{aligned} &C^{\alpha+\epsilon}[0,T] \subseteq W_{0\ (T)}^{\alpha,\infty}[0,T] \subseteq C^{\alpha-\epsilon}[0,T] \text{ and} \\ &C^{\alpha+\epsilon}[0,T] \subseteq W_0^{\alpha,1}[0,T]. \end{aligned} \tag{23}$$

Remark 4.3.2. *We remark that the trajectories of fractional Brownian motion* B^H, *for any* $0 < \alpha < H$, *belong to* $C^\alpha[0,T]$ *almost surely. Therefore, by (23), we obtain that the trajectories of* B^H *for any* $0 < \alpha < H$ *belong to* $W_T^{\alpha,\infty}[0,T]$ *almost surely.*

Let $f \in W_0^{\alpha,1}[0,T]$. Then the restriction of f to any interval $[0,t] \subset [0,T]$ belongs to $I_{0+}^{\alpha}(L^1[0,t])$. Similarly, if $g \in W_T^{1-\alpha,\infty}[0,T]$, then its restriction to the interval $[0,t]$ belongs to $I_{t^-}^{1-\alpha}(L^\infty[0,t])$, for all $t \in (0,T]$ and

$$\Lambda_{1-\alpha}(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t^-}^{1-\alpha} g_{t^-})(s)| \le \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T}.$$

Hence, if $f \in W_0^{\alpha,1}[0,T]$ and $g \in W_T^{1-\alpha,\infty}[0,T]$, then for any $t \in (0,T]$ the Lebesgue integral

$$\int_0^t (\mathcal{D}_{0+}^{\alpha} f_{0+})(s)(\mathcal{D}_{t^-}^{1-\alpha} g_{t^-})(s) ds$$

exists and we can define the generalized Lebesgue - Stieltjes integral $\int_0^t f dg$ which is equal to $\int_0^T f \mathbf{1}_{(0,t)} dg$, according to proposition 4.3. Moreover, we have the estimate

$$\Big| \int_0^t f dg \Big| \le \Lambda_{1-\alpha}(g) \|f\|_{\alpha,1} \quad \text{for } t \in [0,T]. \tag{24}$$

Fix $0 < \alpha < \frac{1}{2}$. We have the following result on the regularity of the integral function from [51].

Proposition 4.4. *Let $f \in W_0^{\alpha,1}[0,T]$ and $g \in W_T^{1-\alpha,\infty}[0,T]$. Let us to define the function*

$$G_t(f) := \int_0^t f dg \quad t \in [0,T].$$

(i) Then, for all $s < t$, we have

$$\big|G_t(f) - G_s(f)\big| \leq \Lambda_{1-\alpha}(g) \int_s^t \Big(\frac{|f(u)|}{(u-s)^\alpha} + \alpha \int_s^u \frac{|f(u) - f(v)|}{(u-v)^{1+\alpha}} dv \Big) du.$$

(ii) Let $f \in W_0^{\alpha,\infty}[0,T]$ and $g \in W_T^{1-\alpha,\infty}[0,T]$. Then $G_\cdot(f) \in C^{1-\alpha}[0,T]$. Moreover

$$\|G_\cdot(f)\|_{C^{1-\alpha}[0,T]} \leq C(\alpha,T)\Lambda_{1-\alpha}(g)\|f\|_{\alpha,\infty},$$

for some constant $C = C(\alpha,T)$ depending only on α and T.

Definition 4.5. *Let $0 < \alpha < 1$. We say that the stochastic process $f = \{f_t\}_{t\in[0,T]}$ belongs to the space $W_0^{\alpha,1}[0,T]$, if its trajectories belong to the space $W_0^{\alpha,1}[0,T]$ almost surely.*

We note that in the remark 4.3.2, the trajectories of fractional Brownian motion B^H, for any $0 < \alpha < H$ belong to $W_T^{\alpha,\infty}[0,T]$ almost surely. By Applying these results to fractional Brownian motion, we obtain the following proposition.

Proposition 4.5. *Assume $B^H = \{B_t^H\}_{t\in[0,T]}$ to be a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\alpha \in (1-H, \frac{1}{2})$.*
(i) Assume that stochastic process $f = \{f_t\}_{t\in[0,T]}$ belongs to the space $W_0^{\alpha,1}[0,T]$. Then the generalized Lebesgue - Stieltjes integral

$$\int_0^T f_t dB_t^H$$

exists almost surely.
(ii) Let f and $\{f^n\}$ belong to the space $W_0^{\alpha,1}[0,T]$. If $\|f^n - f\|_{\alpha,1} \to 0$ as n tends to infinity, then

$$\int_0^T f_t^n dB_t^H \to \int_0^T f_t dB_t^H \qquad \text{as } n \to \infty,$$

almost surely.

5 Summaries of the articles

I. On hedging European options in geometric fractional Brownian motion market model. In this article, we study a two-asset bond - stock frictionless market

$$B_t = 1 \qquad \text{and} \qquad S_t = S_0 e^{B_t^H} \quad t \in [0,T], \tag{25}$$

where the stock price S is a geometric fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then we pose two questions:

(i) Does the stochastic integral

$$\int_0^T f'_-(S_t) dS_t \tag{26}$$

exist? More precisely, in which sense does the integral exist?

(ii) Is it true that for a convex function f we have the following Itô formula:

$$f(S_T) = f(S_0) + \int_0^T f'_-(S_t) dS_t? \tag{27}$$

We answer these questions using the machinery described in subsection 4.2. We prove that the pathwise stochastic integral in (26) can be understood in the sense of the generalized Lebesgue - Stieltjes integral. We use the smoothness techniques with the help of convolution to show the Ito formula (27) and pass to the limit using proposition 4.5. It turns out that the pathwise stochastic integral in (26) is a Riemann - Stieltjes integral, i.e. for any sequence $\{\pi_n\}$ of the partitions of the interval $[0,T]$ such that $|\pi_n| \to 0$ as n tends to infinity, we have

$$\sum_{t_k^n \in \pi_n} f'_-(S_{t_{k-1}^n})(S_{t_k^n} - S_{t_{k-1}^n}) \xrightarrow{\text{a.s}} \int_0^T f'_-(S_t) dS_t. \tag{28}$$

The financial interpretation of these observations is that our frictionless and continuous trading pricing model based on geometric fractional Brownian motion behaves similarly to when the stock price is modeled by a continuous process of bounded variation. Although, in our pricing model one can hedge perfectly European options with convex payoff, but it allows to construct new arbitrage possibilities.

II. On the fractional Black-Scholes market with transaction costs. This article is a continuation of the previous one. A result of Guasoni [29] motivated us to add proportional transaction costs to our pricing model (25) which was based on geometric fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Let $T = 1$. For each n we divide the trading interval $[0, 1]$ into n subintervals $[t_{k-1}^n, t_k^n]$ where

$$t_k^n = \frac{k}{n} = k\Delta_n, \qquad k = 0, 1, \cdots, n.$$

We consider the *discretized version* of the hedging strategy, i.e.

$$\theta_t^n = \sum_{k=1}^{n} f_-^{'}(S_{t_{k-1}^n}) 1_{(t_{k-1}^n, t_k^n]}(t), \quad \text{for } t \in (0, 1].$$

In the presence of proportional transaction costs, the value of this portfolio at the terminal date is

$$V_1(\theta^n) = f(S_0) + \int_0^1 \theta_t^n dS_t - k \sum_{k=1}^{n} S_{t_{k-1}^n} |f_-^{'}(S_{t_k^n}) - f_-^{'}(S_{t_{k-1}^n})|$$

where we assume that the transaction costs coefficient k to be $k_n = k_0 n^{-(1-H)}$ for some $k_0 > 0$. Let μ be the positive Radon measure corresponding to the second derivative of the convex function f. Then the main result of the article states that

$$\mathbb{P}\text{-} \lim_{n \to \infty} V_1(\theta^n) = f(S_1) - \mathbf{J}, \tag{29}$$

where

$$\mathbf{J} = \mathbf{J}(k_0) := \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} \int_0^1 S_t l^H(\ln a, dt) \mu(da), \tag{30}$$

where l^H stands for the local time of fractional Brownian motion and the inner integral on the right hand side is understood as limit of Riemann-Stieltjes sums almost surely. The proof is rather technical. Put simply, first we prove (29) for the special case of the European call function $f(x) = (x - K)^+$ by approximating the local time of fractional Brownian motion by the number of level crossings, a result by Azaïs [2]. Then we pass to the general convex functions by the *convex linear approximation* technique for convex functions. The asymptotic hedging error

$$\mathbf{J} = \mathbf{J}(k_0) := \sqrt{\frac{2}{\pi}} k_0 \int_{\mathbb{R}} a l^H(\ln a, [0, 1]) \mu(da)$$

is strictly positive with positive probability. Therefore, with proportional transaction costs, the discretized replication strategy asymptotically subordinates rather than replicating the value of the convex European option $f(S_1)$ and the option is always subhedged in the limit. Another observation of our result is that there is a connection between transaction costs and quadratic variation.

III. When does fractional Brownian motion not behave as a continuous function with bounded variation? This work was motivated by the first article in which we showed that for fractional Brownian motion B^H with Hurst parameter $H > \frac{1}{2}$, the following statement (ii) holds. The statement (i) below is a classical result:

(i) For any function $f \in C^1(\mathbb{R})$, using the fact that $[B^H, B^H]_T = 0$, we have

$$f(B_T^H) = f(0) + \int_0^T f'(B_t^H) dB_t^H,$$

where the stochastic integral on the right hand side is understood in Riemann - Stieltjes sense.

(ii) For any convex function $f : \mathbb{R} \to \mathbb{R}$,

$$f(B_T^H) = f(0) + \int_0^T f'_-(B_t^H) dB_t^H,$$

where the stochastic integral on the right hand side is understood in the Riemann - Stieltjes sense.

Put simply, the change of variable formula holds for fractional Brownian motion trajectories. Hence, the above two cases suggest that fractional Brownian motion sometimes behaves as a continuous function with bounded variation. Therefore it is natural to ask how far this similarity goes?

Let $f : [0, T] \to \mathbb{R}$ be a continuous function with bounded variation. Assume that

$$f^*(t) := \max_{0 \le s \le t} f(s), \quad \text{for } t \in [0, T],$$

is its running maximum. Then we show the following representation for the running maximum:

$$f^*(T) = f(0) + \int_0^T \mathbf{1}_{\{f^*(t) = f(t)\}} df(t). \tag{31}$$

Let $E = \{t \in [0, T] : f^*(t) = f(t)\}$ be the record set and μ_f and μ_{f^*} stand for the signed measures induced by the bounded variation functions f and f^*. Obviously, the measure μ_{f^*} is supported on the record set E. Then the idea of the proof is to show that the measures μ_f and μ_{f^*} assign the same mass to the record set E, i.e. $\mu_f(E) = \mu_{f^*}(E)$. Then we proved that the same representation for the running maximum of fractional Brownian motion does not hold. More precisely, let

$$M_t := \max_{0 \le s \le t} B_s^H \quad \text{for } t \in [0, T].$$

Then the following representation cannot hold:

$$M_T = B_0 + \int_0^T \mathbf{1}_{\{B_t = M_t\}} dB_t,$$

neither in the case when the stochastic integral on the right hand side is a Riemann - Stieltjes integral nor when it is a generalized Lebesgue - Stieltjes integral. This was proved by the fact that

$$\mathbb{P}\{E_t\} = 0 \quad \forall t \in (0, T] \text{ and } \quad m(E_\omega) = 0 \quad \text{almost surely},$$

where m is Lebesgue measure and E_t and E_ω are the sections of the set

$$E = \{(t, \omega) \in [0, T] \times \Omega : M_t(\omega) = B_t(\omega)\}.$$

IV. Spectral characterization of the quadratic variation of mixed Brownian fractional Brownian motion. It is a classical result in stochastic processes theory that the quadratic variation of a semimartingale can be approximated by the limit in probability of the sums of the squared increments (*realized quadratic variation process in econometrics terminology*). In [26], Dzhaparidze and Spreij showed that one can approximate the bracket of semimartingales using *randomized periodogram*. We extend their result to the class of mixed Brownian fractional Brownian motions which contains both semimartingales and non semimartingales. The mixed Brownian fractional Brownian motion $X = \{X_t\}_{t\in[0,T]}$ is introduced by Cheridito in [18], and defined as

$$X_t = W_t + B_t^H \quad \text{for } t \in [0, T]$$

where W and B^H are independent Brownian motion and fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, respectively. For any $\lambda \in \mathbb{R}$, the *periodogram* of X evaluated at T is defined by

$$\begin{aligned} I_T(X; \lambda) &= \Big| \int_0^T e^{i\lambda t} dX_t \Big|^2 = \Big| e^{i\lambda T} X_T - i\lambda \int_0^T X_t e^{i\lambda t} dt \Big|^2 \\ &= 2\mathbf{Re} \int_0^T \int_0^t e^{i\lambda(t-s)} dX_s dX_t + [X, X]_T \quad \text{(by the Ito - Föllmer formula)}. \end{aligned}$$

Take $L > 0$ and let ξ be a symmetric random variable independent of the filtration $\mathbb{F}^X$ with a density g_ξ and hence real characteristic function φ_ξ. We define the *randomized periodogram* of X evaluated at T by

$$\mathbb{E}_\xi I_T(X; L\xi) = \int_{\mathbb{R}} I_T(X; Lx) g_\xi(x) dx. \tag{32}$$

First, under the assumption $\mathbb{E}\xi^2 < \infty$, we prove a parametrized stochastic Fubini type result in order to be able to change the place of the integrals in (32). Then the randomized periodogram takes the form

$$\mathbb{E}_\xi I_T(X; L\xi) = [X, X]_T + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dX_s dX_t.$$

Then we show that, the randomized periodogram converges to the quadratic variation of mixed Brownian fractional Brownian motion as L tends to infinity. Our proof involves breaking down the error term

$$2\int_0^T\int_0^t \varphi_\xi(L(t-s))dX_s dX_t$$

into four iterated integrals and showing that the second moment of these four integrals converges to 0 as L tends to infinity by using the independence between Brownian motion and fractional Brownian motion and the fact that

$$\varphi_\xi(L(t-s)) \to 0 \text{ as } L\to\infty \quad \text{for } s<t.$$

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