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Analysis of $H(\mathrm{div})$-conforming finite elements for the Brinkman problem

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December 18, 2009

Abstract

The Brinkman equations describe the flow of a viscous fluid in a porous matrix. Mathematically the Brinkman model is a parameter-dependent combination of the Darcy and Stokes models. We introduce a dual mixed framework for the problem, and use $H(\mathrm{div})$-conforming finite elements together with Nitsche's method to obtain a stable formulation. We show the formulation to be stable in a mesh-dependent norm for all values of the parameter. We also introduce a postprocessing scheme for the pressure along with a residual-based a posteriori estimator, which is shown to be efficient and reliable for all parameter values.

1 Introduction

In soil mechanics, the Brinkman equation describes the flow of a viscous fluid in a very porous medium. For a derivation of and details on the equations we refer to [17, 1, 2, 3, 21]. As opposed to the Darcy model widely used in soil mechanics, the Brinkman model adds an effective viscosity to the equations. Typical applications of the equations lie in oil exploration, groundwater modelling, and some special applications, such as heat pipes [15]. The Brinkman model is also often used as a coupling layer between a free surface flow and a porous Darcy flow [10]. Mathematically, the Brinkman equations are a parameter-dependent combination of the Darcy and Stokes equations.

We study the application of $H(\mathrm{div})$-conforming finite elements designed for the Darcy problem to the more complicated Brinkman problem. $H(\mathrm{div})$-elements have been considered for the closely related Stokes problem in [9, 26, 14]. Our model constitutes an approximation with non-conforming basis functions, since in the discretizations of the $H(\mathrm{div})$-space only the normal component of the velocity is continuous on interelement boundaries. To enforce the tangential continuity, we use the so-called Nitsche's method introduced in [20]. This in turn requires the use of a mesh-dependent bilinear form. The method has a strong resemblance to totally discontinuous Galerkin methods for the Stokes

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equation, cf. [12]. The motivation for using this non-conforming approximation is the fact that $H(\text{div})$-conforming elements are widely used in industry for solving the Darcy equations, and we want to derive a way of easily incorporating viscosity to the existing solvers, along with a rigorous mathematical analysis. In Part II of this paper we will report on numerical benchmark studies with our methods.

Methods based on Stokes elements are studied in e.g. [13, 5].

The structure of the article is as follows. First, we formulate the Brinkman problem mathematically and introduce the corresponding variational formulation. Then we introduce the finite element spaces used for the discretization along with the mesh-dependent energy norms in which the error is measured. We derive a priori convergence results for this discretization, which implies the possibility of improving the accuracy of the pressure approximation via postprocessing. This is addressed in a separate section, where we provide optimal a priori convergence results for the postprocessed solution. Finally, we end the paper with the a posteriori error analysis. A residual-based a posteriori error estimator is introduced. It is then shown that the estimator is both reliable and efficient for all values of the viscosity parameter.

We use standard notation throughout the paper. We denote by C, C_1, C_2 etc. generic constants, that are not always identical in value, but are always independent of the parameter t and the mesh size h.

2 The Brinkman model

Let $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$, be a domain with a polygonal or polyhedral boundary. We denote by $\boldsymbol{u}$ the velocity field of the fluid and by p the pore pressure. The equations are scaled as presented in [11], with the single parameter t representing the effective viscosity of the fluid, which is assumed constant for simplicity. With this notation, the Brinkman equations are

$$-t^2\Delta\boldsymbol{u} + \boldsymbol{u} - \nabla p = \boldsymbol{f}, \quad \text{in } \Omega \tag{2.1}$$

$$\text{div } \boldsymbol{u} = g, \quad \text{in } \Omega \tag{2.2}$$

For $t > 0$, the equations are formally a Stokes problem. The solution $(\boldsymbol{u}, p)$ is sought in $\boldsymbol{V} \times Q = [H_0^1(\Omega)]^n \times L_0^2(\Omega)$. For the case $t = 0$ we get the Darcy problem, and accordingly the solution space can be chosen as $\boldsymbol{V} \times Q = H(\text{div}, \Omega) \times L_0^2(\Omega)$. For simplicity of the mathematical analysis, we consider for the case $t > 0$ homogenous Dirichlet boundary conditions for the velocity field. Thus the boundary conditions are

$$\boldsymbol{u} = \boldsymbol{0}. \tag{2.3}$$

For the limiting Darcy case $t = 0$ we assume Neumann conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0. \tag{2.4}$$

In the following, we denote by $(\cdot\,, \cdot)_D$ the standard L^2 inner product over a set $D \subset \mathbb{R}^n$. If $D = \Omega$, the subscript is dropped for convenience. Similarly,

$\langle \cdot , \cdot \rangle_B$ is the L^2 inner product over a set $B \subset \mathbb{R}^{n-1}$. We define the following bilinear forms

$$a(\boldsymbol{u}, \boldsymbol{v}) = t^2(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}), \tag{2.5}$$

$$b(\boldsymbol{v}, p) = (\text{div } \boldsymbol{v}, p), \tag{2.6}$$

and

$$\mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q). \tag{2.7}$$

The weak formulation of the Brinkman problem then reads: Find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ such that

$$\mathcal{B}(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + (g, q), \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q. \tag{2.8}$$

3 Solution by mixed finite elements

Let $\mathcal{K}_h$ be a shape-regular partition of Ω into simplices. As usual, the diameter of an element K is denoted by h_K, and the global mesh size h is defined as $h = \max_{K \in \mathcal{K}_h} h_K$. We denote by $\mathcal{E}_h$ the set of all faces of $\mathcal{K}_h$. We write h_E for the diameter of a face E.

We introduce the jump and average of a piecewise smooth scalar function f as follows. Let $E = \partial K \cap \partial K'$ be an interior face shared by two elements K and K'. Then the jump of f over E is defined by

$$[\![f]\!] = f|_K - f|_{K'}. \tag{3.1}$$

and the average as

$$\{f\} = \frac{1}{2}(f|_K + f|_{K'}). \tag{3.2}$$

For vector valued functions, we define the jumps and averages analogously.

3.1 The mixed method and the norms

Mixed finite element discretization of the problem is based on finite element spaces $\boldsymbol{V}_h \times Q_h \subset H(\text{div}, \Omega) \times L^2_0(\Omega)$ of piecewise polynomial functions with respect to $\mathcal{K}_h$. We will focus here on the Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) families of elements [8]. In three dimensions the counterparts are the Nédélec elements [19] and the BDDF elements [7].That is, for an approximation of order $k \geq 1$, the flux space $\boldsymbol{V}_h$ is taken as one of the following two spaces

$$\boldsymbol{V}_h^{RT} = \{\boldsymbol{v} \in H(\text{div}, \Omega) \mid \boldsymbol{v}|_K \in [P_{k-1}(K)]^n \oplus \boldsymbol{x}\tilde{P}_{k-1}(K) \ \forall K \in \mathcal{K}_h\}, \tag{3.3}$$

$$\boldsymbol{V}_h^{BDM} = \{\boldsymbol{v} \in H(\text{div}, \Omega) \mid \boldsymbol{v}|_K \in [P_k(K)]^n \ \forall K \in \mathcal{K}_h\}, \tag{3.4}$$

where $\tilde{P}_{k-1}(K)$ denotes the homogeneous polynomials of degree $k-1$. The pressure is approximated in the space

$$Q_h = \{q \in L^2_0(\Omega) \mid q|_K \in P_{k-1}(K) \ \forall K \in \mathcal{K}_h\}. \tag{3.5}$$

Notice that $\boldsymbol{V}_h^{RT} \subset \boldsymbol{V}_h^{BDM}$ and $Q_h^{BDM} = Q_h^{RT}$. The combination of spaces satisfies the following equilibrium property:

$$\text{div } \boldsymbol{V}_h \subset Q_h. \tag{3.6}$$

To assure the stability of the non-conforming approximation, we use Nitsche's method [13, 20] with a suitably chosen stabilization parameter α. We define the following mesh-dependent bilinear form

$$\mathcal{B}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = a_h(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q), \tag{3.7}$$

in which

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) &+ t^2 \sum_{K \in \mathcal{K}_h} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_K \\ &+ t^2 \sum_{E \in \mathcal{E}_h} \{\frac{\alpha}{h_E} \langle [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] \rangle_E - \langle \{\frac{\partial \boldsymbol{u}}{\partial n}\}, [\![\boldsymbol{v}]\!] \rangle_E - \langle \{\frac{\partial \boldsymbol{v}}{\partial n}\}, [\![\boldsymbol{u}]\!] \rangle_E \}. \end{aligned} \tag{3.8}$$

Then the discrete problem is to find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $p_h \in Q_h$ such that

$$\mathcal{B}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + (g, q), \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h. \tag{3.9}$$

We introduce the following mesh-dependent norms for the problem. For the velocity we use

$$\|\boldsymbol{u}\|_{t,h}^2 = \|\boldsymbol{u}\|^2 + t^2 \left(\sum_{K \in \mathcal{K}_h} \|\nabla \boldsymbol{u}\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[\![\boldsymbol{u} \cdot \boldsymbol{\tau}]\!]\|_{0,E}^2 \right), \tag{3.10}$$

and for the pressure

$$|\!|\!| p |\!|\!|_{t,h}^2 = \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla p\|_{0,K}^2 + \sum_{E \in \mathcal{E}_h} \frac{h_E}{h_E^2 + t^2} \|[\![p]\!]\|_{0,E}^2. \tag{3.11}$$

Note that both of the norms are also parameter dependent.

3.2 A priori analysis

First we prove the consistency of the modified method. For the exact solution $(\boldsymbol{u}, p)$ it holds $[\![\boldsymbol{u}]\!]|_E = \mathbf{0}$. Inserting $\boldsymbol{u}$ into the modified part of the bilinear form, we have using element-by-element partial integration

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) &= (\boldsymbol{u}, \boldsymbol{v}) + t^2 \left(\sum_{K \in \mathcal{K}_h} (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_K - \sum_{E \in \mathcal{E}_h} \langle \{\frac{\partial \boldsymbol{u}}{\partial n}\}, [\![\boldsymbol{v}]\!] \rangle_E \right) \\ &= (\boldsymbol{u}, \boldsymbol{v}) + t^2 \sum_{K \in \mathcal{K}_h} \{(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_K - \langle \frac{\partial \boldsymbol{u}}{\partial n}, \boldsymbol{v} \rangle_{\partial K}\} \\ &= (\boldsymbol{u}, \boldsymbol{v}) + t^2 \sum_{K \in \mathcal{K}_h} (-\Delta \boldsymbol{u}, \boldsymbol{v})_K \\ &= (-t^2 \Delta \boldsymbol{u} + \boldsymbol{u}, \boldsymbol{v}). \end{aligned}$$

This gives us the following result.

Theorem 3.1. *The exact solution* $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ *satisfies*

$$\mathcal{B}_h(\boldsymbol{u}, p; \boldsymbol{v}, q) = (\boldsymbol{f}, \boldsymbol{v}) + (g, q), \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h. \tag{3.12}$$

Next we prove the stability of $a_h(\cdot\,,\cdot\,)$ in the mesh-dependent norm (3.10). The stability only holds in the discrete space $\boldsymbol{V}_h$. First we recall, as shown in [24], that the normal derivative can be estimated as

$$h_E\|\frac{\partial \boldsymbol{v}}{\partial n}\|_{0,E}^2 \le C_I\|\nabla \boldsymbol{v}\|_{0,K}^2, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{3.13}$$

Lemma 3.2. *The bilinear form $a_h(\cdot\,,\cdot\,)$ is coercive in the discrete space $\boldsymbol{V}_h$; there exists a positive constant C such that*

$$a_h(\boldsymbol{v},\boldsymbol{v}) \ge C\|\boldsymbol{v}\|_{t,h}^2, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{3.14}$$

Proof. First we note that by Young's inequality

$$\begin{aligned}
-2\sum_{E\in\mathcal{E}_h}\langle\{\frac{\partial \boldsymbol{v}}{\partial n}\},[\![\boldsymbol{v}]\!]\rangle_E &= -\sum_{K\in\mathcal{K}_h}\langle\frac{\partial \boldsymbol{v}}{\partial n},[\![\boldsymbol{v}]\!]\rangle_{\partial K}\\
&\ge -\sum_{K\in\mathcal{K}_h}\|\frac{\partial \boldsymbol{v}}{\partial n}\|_{0,\partial K}\|[\![\boldsymbol{v}]\!]\|_{0,\partial K}\\
&\ge -(\sum_{K\in\mathcal{K}_h}h_K\|\frac{\partial \boldsymbol{v}}{\partial n}\|_{0,\partial K}^2)^{1/2}(\sum_{K\in\mathcal{K}_h}h_K^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,\partial K}^2)^{1/2}\\
&\ge -\frac{1}{2\epsilon}\sum_{K\in\mathcal{K}_h}h_K\|\frac{\partial \boldsymbol{v}}{\partial n}\|_{0,\partial K}^2-\frac{\epsilon}{2}\sum_{K\in\mathcal{K}_h}h_K^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,\partial K}^2\\
&\ge -\frac{C_I}{2\epsilon}\sum_{K\in\mathcal{K}_h}\|\nabla \boldsymbol{v}\|_{0,K}^2-\frac{\epsilon}{2}\sum_{K\in\mathcal{K}_h}h_K^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,\partial K}^2,
\end{aligned}$$

for an arbitrary $\varepsilon > 0$. This immediately gives

$$\begin{aligned}
a_h(\boldsymbol{v},\boldsymbol{v}) &= \|\boldsymbol{v}\|_0^2 + t^2\sum_{K\in\mathcal{K}_h}\|\nabla \boldsymbol{v}\|_{0,E}^2 + t^2\sum_{E\in\mathcal{E}_h}(\frac{\alpha}{h_K}\|[\![\boldsymbol{v}]\!]\|_{0,E}^2 - 2\langle\frac{\partial \boldsymbol{v}}{\partial n},[\![\boldsymbol{v}]\!]\rangle_E)\\
&\ge \min\{1-\frac{C_I}{2\epsilon},\alpha-\frac{\epsilon}{2}\}\|\boldsymbol{v}\|_{t,h}^2.
\end{aligned}\tag{3.15}$$

Here C_I is the constant from the discrete trace inequality (3.13). Since ϵ and α are free parameters, choosing $\epsilon > C_I/2$ and $\alpha > \epsilon/2$, we have the desired result. □

Next, we prove the discrete Brezzi-Babuska stability condition. Recall that we only have to prove the condition in the Raviart-Thomas case since $\boldsymbol{V}_h^{RT} \subset \boldsymbol{V}_h^{BDM}$.

Lemma 3.3. *There exists a positive constant C such that*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{t,h}} \ge C|\!|\!|q|\!|\!|_{t,h}, \quad \forall q \in Q_h. \tag{3.16}$$

Proof. We recall that the local degrees of freedom for the RT family are

$$\langle \boldsymbol{v}\cdot \boldsymbol{n}, z\rangle_E, \quad \forall z \in P_{k-1}(E), \tag{3.17}$$

$$(\boldsymbol{v},\boldsymbol{z})_K, \quad \forall \boldsymbol{z} \in [P_{k-2}(K)]^n. \tag{3.18}$$

Thus, for a given $q \in Q_h$ we define $\boldsymbol{v}$ by

$$\langle \boldsymbol{v}\cdot \boldsymbol{n}, z\rangle_E = \frac{h_K}{h_K^2+t^2}\langle [\![q]\!], z\rangle_E, \quad \forall z \in P_{k-1}(E), \tag{3.19}$$
$$(\boldsymbol{v}, \boldsymbol{z})_K = -\frac{h_K^2}{h_K^2+t^2}(\nabla q, \boldsymbol{z})_K \quad \forall \boldsymbol{z} \in [P_{k-2}(K)]^n.$$

Choosing $z = [\![q]\!] \in P_{k-1}(E)$ and $\boldsymbol{z} = \nabla q \in [P_{k-2}(K)]^n$ gives

$$\langle \boldsymbol{v}\cdot \boldsymbol{n}, [\![q]\!]\rangle_E = \frac{h_K}{h_K^2+t^2}\|[\![q]\!]\|_{0,E}^2, \tag{3.20}$$

$$(\boldsymbol{v}, \nabla q)_K = -\frac{h_K^2}{h_K^2+t^2}\|\nabla q\|_{0,K}^2. \tag{3.21}$$

An explicit inspection of the degrees of freedom yields the relation

$$\frac{h_K^2+t^2}{h_K^2}\|\boldsymbol{v}\|_{0,K}^2 \le \frac{h_K}{h_K^2+t^2}\|[\![q]\!]\|_{0,E}^2 + \frac{h_K^2}{h_K^2+t^2}\|\nabla q\|_{0,K}^2. \tag{3.22}$$

Thus, using scaling arguments, we have

$$\|\boldsymbol{v}\|_{t,h}^2 \le C \sum_{K\in\mathcal{K}_h} \frac{h_K^2+t^2}{h_K^2}\|\boldsymbol{v}\|_{0,K}^2 \le C|\!|\!| q|\!|\!|_{t,h}^2. \tag{3.23}$$

Next we use element-by-element partial integration on $b(\boldsymbol{v}, q)$, and apply the definitions (3.19) to get

$$b(\boldsymbol{v}, q) = \sum_{K\in\mathcal{K}_h} -(\boldsymbol{v}, \nabla q)_K + \sum_{E\in\mathcal{E}_h} \langle \boldsymbol{v}\cdot \boldsymbol{n}, q\rangle_E \tag{3.24}$$
$$= \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2+t^2}(\nabla q, \nabla q)_K + \sum_{E\in\mathcal{E}_h} \frac{h_K}{h_K^2+t^2}\langle [\![q]\!], [\![q]\!]\rangle_E \tag{3.25}$$
$$= |\!|\!| q|\!|\!|_{t,h}^2. \tag{3.26}$$

Combining (3.23) and (3.24) gives (3.16). □

By the above stability results for $a_h(\cdot\,,\cdot)$ and $b(\cdot\,,\cdot)$ the following stability result holds, see e.g. [8].

Lemma 3.4. *For some positive constant C it holds*

$$\sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_h\times Q_h} \frac{\mathcal{B}_h(\boldsymbol{r}, s; \boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{t,h} + |\!|\!| q|\!|\!|_{t,h}} \ge C(\|\boldsymbol{r}\|_{t,h} + |\!|\!| s|\!|\!|_{t,h}), \quad \forall(\boldsymbol{r}, s) \in \boldsymbol{V}_h \times Q_h. \tag{3.27}$$

For interpolation in $H(\mathrm{div})$, a special interpolation operator is required. We use the interpolation operator $\boldsymbol{R}_h : H(\mathrm{div}, \Omega) \to \boldsymbol{V}_h$ introduced by Schöberl in [22] satisfying

$$(\mathrm{div}\ (\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}), q) = 0, \quad \forall q \in Q_h. \tag{3.28}$$

The interpolant satisfies the following properties. We denote by $P_h : L^2(\Omega) \to Q_h$ the L^2-projection. The equilibrium property (3.6) implies

$$(\mathrm{div}\ \boldsymbol{v}, q - P_h q) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{3.29}$$

Furthermore, we have the commuting diagram property:

$$\text{div } \boldsymbol{R}_h = P_h \text{div}. \tag{3.30}$$

Traditionally the interpolation operator has been defined based on the moments of the normal component of the velocity on the boundaries. By using the alternative interpolation operator of [22] we need not the extra regularity assumption of the edge-based interpolation operators. In the following, the standard Sobolev norm of order k is denoted $\|\cdot\|_k$. The main result of the chapter is the following quasioptimal a priori result:

Theorem 3.5. *There is a positive constant C such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,h} + |||P_h p - p_h|||_{t,h} \leq C\|\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}\|_{t,h}. \tag{3.31}$$

Proof. By Lemma 3.4 there exists functions $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$ such that $\|\boldsymbol{v}\|_{t,h} + |||q|||_{t,h} \leq C$, and

$$\begin{aligned} \|\boldsymbol{u}_h - \boldsymbol{R}_h\boldsymbol{u}\|_{t,h} + |||p_h - P_h p|||_{t,h} &\leq \mathcal{B}_h(\boldsymbol{u}_h - \boldsymbol{R}_h\boldsymbol{u}, p_h - P_h p; \boldsymbol{v}, q) \\ &= a_h(\boldsymbol{u}_h - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}) + (\text{div } \boldsymbol{v}, p_h - P_h p) + (\text{div } (\boldsymbol{u}_h - \boldsymbol{R}_h\boldsymbol{u}), q) \\ &= a_h(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}) + (\text{div } \boldsymbol{v}, p - P_h p) + (\text{div } (\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}), q), \end{aligned} \tag{3.32}$$

where the last line follows from the consistency of the method given by Theorem 3.1. By using the interpolation properties (3.28) and (3.29), we arrive at

$$\|\boldsymbol{u}_h - \boldsymbol{R}_h\boldsymbol{u}\|_{t,h} + |||p_h - P_h p|||_{t,h} \leq a_h(\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}, \boldsymbol{v}) \leq C\|\boldsymbol{u} - \boldsymbol{R}_h\boldsymbol{u}\|_{t,h}. \tag{3.33}$$

Using the triangle inequality yields the result of the theorem. □

Analogously to the dual mixed formulation of the Poisson problem discussed e.g. in [4, 23, 18], we have a superconvergence result for $|||p_h - P_h p|||_{t,h}$. This implies that the pressure solution can be improved by local postprocessing. Assuming full regularity, we conclude the chapter with the following a priori result:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,h} + |||P_h p - p_h|||_{t,h} \leq \begin{cases} C(h^k + th^{k-1})\|\boldsymbol{u}\|_k, & \text{for RT,} \\ C(h^{k+1} + th^k)\|\boldsymbol{u}\|_{k+1}, & \text{for BDM.} \end{cases} \tag{3.34}$$

4 Postprocessing method

In this section we present a postprocessing method for the pressure in the spirit of [18]. We seek the postprocessed pressure in an augmented space $Q_h^* \supset Q_h$, defined as

$$Q_h^* = \begin{cases} \{q \in L_0^2(\Omega) \mid q|_K \in P_k(K) \ \forall K \in \mathcal{K}_h\}, & \text{for RT,} \\ \{q \in L_0^2(\Omega) \mid q|_K \in P_{k+1}(K) \ \forall K \in \mathcal{K}_h\}, & \text{for BDM.} \end{cases} \tag{4.1}$$

The postprosessing method is: find $p_h^* \in Q_h^*$ such that

$$P_h p_h^* = p_h \tag{4.2}$$

$$(\nabla p_h^*, \nabla q)_K = (-t^2\Delta\boldsymbol{u}_h + \boldsymbol{u}_h - \boldsymbol{f}, \nabla q)_K, \quad \forall q \in (I - P_h)Q_h^*|_K. \tag{4.3}$$

The method can be compactly treated as an integral part of the problem by embedding it into the bilinear form. We introduce the modified bilinear form

$$\mathcal{B}_h^*(\boldsymbol{u}, p^*; \boldsymbol{v}, q^*) = \mathcal{B}_h(\boldsymbol{u}, p^*; \boldsymbol{v}, q^*) + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} (-\nabla p^* + \boldsymbol{u} - t^2 \Delta \boldsymbol{u}, \nabla (I - P_h) q^*)_K. \tag{4.4}$$

The postprocessed problem is then: find $(\boldsymbol{u}_h, p_h^*) \in \boldsymbol{V}_h \times Q_h^*$ such that for every pair $(\boldsymbol{v}, q^*) \in \boldsymbol{V}_h \times Q_h^*$ it holds

$$\mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{v}, q^*) = \mathcal{L}_h(\boldsymbol{f}, P_h g; \boldsymbol{v}, q^*), \tag{4.5}$$

in which

$$\mathcal{L}_h(\boldsymbol{f}, g; \boldsymbol{v}, q^*) = (\boldsymbol{f}, \boldsymbol{v}) + (g, q^*) + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} (\boldsymbol{f}, \nabla (I - P_h) q^*)_K. \tag{4.6}$$

We have the following theorem relating the solution of the postprocessed problem to the original problem.

Theorem 4.1. *Let* $(\boldsymbol{u}_h, p_h^*) \in \boldsymbol{V}_h \times Q_h^*$ *be the solution of the problem* (4.5) *and set* $p_h = P_h p_h^*$. *Then* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *is the solution of the original problem* (3.9). *Conversely, if* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ *is the solution of the original problem* (3.9) *and* p_h^* *is defined as above, then* $(\boldsymbol{u}_h, p_h^*) \in \boldsymbol{V}_h \times Q_h^*$ *is the solution to* (4.5).

Proof. Testing with $(\boldsymbol{v}, 0) \in \boldsymbol{V}_h \times Q_h^*$ and using the equilibrium property (3.6) yields

$$\begin{aligned} \mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{v}, 0) &= a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p_h^*) \\ &= a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, P_h p_h^*) \\ &= a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\operatorname{div} \boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}). \end{aligned}$$

On the other hand, testing with $(\boldsymbol{0}, P_h q^*) \in \boldsymbol{V}_h \times Q_h \subset \boldsymbol{V}_h \times Q_h^*$ gives

$$\mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{0}, P_h q^*) = (\operatorname{div} \boldsymbol{u}_h, P_h q^*) = (g, P_h q^*).$$

Combining the above two equations yields the original problem (3.9) and first part of the assertion is proved. Next take $(\boldsymbol{u}_h, p_h)$ to be the solution of (3.9), and p_h^* the postprocessed pressure defined above. Using the definition of the postprocessed pressure and the equilibrium property, we have

$$
\begin{aligned}
\mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{v}, q^*) &= \mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{v}, P_h q^*) + \mathcal{B}_h^*(\boldsymbol{u}_h, p_h^*; \boldsymbol{0}, (I-P_h)q^*) \\
&= a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\mathrm{div}\ \boldsymbol{v}, p_h^*) + (\mathrm{div}\ \boldsymbol{u}_h, P_h q^*) \\
&\quad + \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2+t^2}(-t^2\Delta \boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p_h^*, \nabla(I-P_h)P_h q^*)_K \\
&\quad + (\mathrm{div}\ \boldsymbol{u}_h, (I-P_h)q^*) + \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2+t^2}(\boldsymbol{f}, \nabla(I-P_h)^2 q^*)_K \\
&\quad + \frac{h_K^2}{h_K^2+t^2}(-t^2\Delta \boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p_h^* - \boldsymbol{f}, \nabla(I-P_h)^2 q^*)_K \\
&= a_h(\boldsymbol{u}_h, \boldsymbol{v}) + (\mathrm{div}\ \boldsymbol{v}, P_h p_h^*) + (\mathrm{div}\ \boldsymbol{u}_h, P_h q^*) \\
&\quad + (\mathrm{div}\ \boldsymbol{u}_h, P_h(I-P_h)q^*) + \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2+t^2}(\boldsymbol{f}, \nabla(I-P_h)^2 q^*)_K \\
&= \mathcal{L}_h(\boldsymbol{f}, P_h g; \boldsymbol{v}, q^*)
\end{aligned}
$$

for arbitrary $(\boldsymbol{v}, q^*) \in \boldsymbol{V}_h \times Q_h^*$. Thus the second part of the assertion is valid. □

Next we show that the postprocessed method is stable in the discrete spaces. For this we need the following lemma, essentially proved in [18].

Lemma 4.2. *There exists positive constants C_1, C_2 such that for every $q^* \in Q_h^*$ it holds*

$$\|\!|q^*|\!\|_{t,h} \le \|\!|P_h q^*|\!\|_{t,h} + \|\!|(I-P_h)q^*|\!\|_{t,h} \le C_2 \|\!|q^*|\!\|_{t,h}, \tag{4.7}$$

$$C_1 \|\!|q^*|\!\|_{t,h} \le \|\!|P_h q^*|\!\|_{t,h} + \left(\sum_{K\in\mathcal{K}_h} \|\nabla(I-P_h)q^*\|_{0,K}^2 \right)^{1/2} \le C_2 \|\!|q^*|\!\|_{t,h}. \tag{4.8}$$

Since $(I-P_h)q^*$ is L^2-orthogonal to piecewise constant functions, we furthermore have the following estimate, with $C_3 > 0$,

$$\|\!|(I-P_h)q^*|\!\|_{t,h} \le C_3 \left(\sum_{K\in\mathcal{K}_h} \|\nabla(I-P_h)q^*\|_{0,K}^2 \right)^{1/2}. \tag{4.9}$$

We are now ready to prove the main stability result.

Theorem 4.3. *There exists $C > 0$ such that for every $(\boldsymbol{u}, p^*) \in \boldsymbol{V}_h \times Q_h^*$ it holds*

$$\sup_{(\boldsymbol{v},q^*)\in \boldsymbol{V}_h\times Q_h^*} \frac{\mathcal{B}_h^*(\boldsymbol{u}, p^*; \boldsymbol{v}, q^*)}{\|\boldsymbol{v}\|_{t,h} + \|\!|q^*|\!\|_{t,h}} \ge C(\|\boldsymbol{u}\|_{t,h} + \|\!|p^*|\!\|_{t,h}). \tag{4.10}$$

Proof. Let $(\boldsymbol{u}, p^*) \in \boldsymbol{V}_h \times Q_h^*$ be arbitrary. Choosing $q^* = q \in Q_h$ we have

$$
\begin{aligned}
\mathcal{B}_h^*(\boldsymbol{u}, p^*; \boldsymbol{v}, q) &= a_h(\boldsymbol{u}, \boldsymbol{v}) + (\mathrm{div}\ \boldsymbol{v}, p^*) + (\mathrm{div}\ \boldsymbol{u}, q) \\
&= a_h(\boldsymbol{u}, \boldsymbol{v}) + (\mathrm{div}\ \boldsymbol{v}, P_h p^*) + (\mathrm{div}\ \boldsymbol{u}, q) \\
&= \mathcal{B}_h(\boldsymbol{u}, P_h p^*; \boldsymbol{v}, q).
\end{aligned}
$$

Thus Lemma 3.4 guarantees that there exists $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times Q_h$ such that $\|\boldsymbol{v}\|_{t,h} + |\!|\!| q |\!|\!|_{t,h} \le C(\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h})$ and

$$\mathcal{B}_h(\boldsymbol{u}, p^*; \boldsymbol{v}, q) \ge C(\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h}). \tag{4.11}$$

Next, we choose $(\boldsymbol{v}, q^*) = (\mathbf{0}, (I - P_h)p^*) \in \boldsymbol{V}_h \times Q_h^*$.

$$\begin{aligned}
&\mathcal{B}_h^*(\boldsymbol{u}, p^*; \mathbf{0}, (I - P_h)p^*) \\
&= (\text{div } \boldsymbol{u}, (I - P_h)p^*) + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} (-\nabla p^* + \boldsymbol{u} - t^2 \Delta \boldsymbol{u}, \nabla (I - P_h)p^*)_K \\
&\ge -C(\|\boldsymbol{u}\|_{t,h} |\!|\!| (I - P_h)p^* |\!|\!|_{t,h}) + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla (I - P_h)p^*\|_{0,K}^2 \\
&\quad + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} (-\nabla P_h p^* + \boldsymbol{u} - t^2 \Delta \boldsymbol{u}, \nabla (I - P_h)p^*)_K \\
&\ge -C(\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h}) |\!|\!| (I - P_h)p^* |\!|\!|_{t,h} + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla (I - P_h)p^*\|_{0,K}^2 \\
&\quad - \left(\sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|t^2 \Delta \boldsymbol{u}\|_{0,K}^2 \right)^{1/2} |\!|\!| (I - P_h)p^* |\!|\!|_{t,h}.
\end{aligned}$$

We can estimate the term containing the Laplacian using the inverse inequality as follows:

$$\sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|t^2 \Delta \boldsymbol{u}\|_{0,K}^2 \le \sum_{K \in \mathcal{K}_h} \frac{t^2 h_K^2}{h_K^2 + t^2} \frac{1}{h_K^2} \|t \nabla \boldsymbol{u}\|_{0,K}^2 \le t^2 \sum_{K \in \mathcal{K}_h} \|\nabla \boldsymbol{u}\|_{0,K}^2.$$

Using Young's inequality and the norm equivalence (4.9), we have for any $\epsilon > 0$

$$\begin{aligned}
\mathcal{B}_h^*(\boldsymbol{u}, p^*; \mathbf{0}, (I - P_h)p^*) &\ge -\frac{C_1}{2\epsilon} (\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h})^2 - \frac{\epsilon}{2} |\!|\!| (I - P_h)p^* |\!|\!|_{t,h}^2 \\
&\quad + \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla (I - P_h)p^*\|_{0,K}^2 \\
&\ge -\frac{C_1}{2\epsilon} (\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h})^2 \\
&\quad + \left(1 - \frac{\tilde{C}_2 \epsilon}{2}\right) \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla (I - P_h)p^*\|_{0,K}^2.
\end{aligned}$$

Choosing $\epsilon = 1/C_1$ yields, with $C_3 = C_1 C_2 / 2$,

$$\begin{aligned}
\mathcal{B}_h^*(\boldsymbol{u}, p^*; \mathbf{0}, (I - P_h)p^*) \ge &- C_3 (\|\boldsymbol{u}\|_{t,h} + |\!|\!| P_h p^* |\!|\!|_{t,h})^2 \\
&+ \frac{1}{2} \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2} \|\nabla (I - P_h)p^*\|_{0,K}^2.
\end{aligned} \tag{4.12}$$

Combining estimates (4.11) and (4.12), the norm equivalence (4.8), and choosing

δ sufficiently small, we have

$$\begin{aligned}\mathcal{B}_h^*(\boldsymbol{u},p^*;\boldsymbol{v},q+\delta(I-P_h)p^*) &\geq (1-\delta C_3)(\|\boldsymbol{u}\|_{t,h}+|||P_hp^*|||_{t,h})^2\\ &\quad \delta C_3\sum_{K\in\mathcal{K}_h}\|\nabla(I-P_h)p^*\|_{0,K}^2\\ &\geq C(\|\boldsymbol{u}\|_{t,h}+|||p^*|||_{t,h})^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|\boldsymbol{v}\|_{t,h}+|||q+\delta(I-P_h)p^*|||_{t,h} &\leq \|\boldsymbol{v}\|_{t,h}+|||q|||_{t,h}+\delta|||(I-P_h)p^*|||_{t,h}\\ &\leq C(\|\boldsymbol{u}\|_{t,h}+|||P_hp^*|||_{t,h})+\delta|||(I-P_h)p^*|||_{t,h}\\ &\leq C(\|\boldsymbol{u}\|_{t,h}+|||p^*|||_{t,h}),\end{aligned}$$

yielding the desired result. □

We have the following a priori result, which shows that given sufficient regularity, the postprocessed displacement converges with an optimal rate.

Theorem 4.4. *For the postprocessed solution* $(\boldsymbol{u}_h,p_h^*)$ *it holds*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}+|||p-p_h^*|||_{t,h} \leq C\inf_{q^*\in Q_h^*}&\Big\{\|\boldsymbol{u}-R_h\boldsymbol{u}\|_{t,h}+|||p-q^*|||_{t,h}\\ &+\Big(\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}\|-\nabla q^*+R_h\boldsymbol{u}-t^2\Delta R_h\boldsymbol{u}-\boldsymbol{f}\|_{0,K}^2\Big)^{1/2}\Big\}.\end{aligned} \tag{4.13}$$

Proof. Let $q^*\in Q_h^*$. From Theorem 4.3 it follows that we have a pair $(\boldsymbol{v},r^*)\in\boldsymbol{V}_h\times Q_h^*$ such that $\|\boldsymbol{v}\|_{t,h}+|||r^*|||_{t,h}\leq C$ and

$$\|\boldsymbol{u}_h-R_h\boldsymbol{u}\|_{t,h}+|||p_h^*-q^*|||_{t,h}\leq C\mathcal{B}_h^*(\boldsymbol{u}_h-R_h\boldsymbol{u},p_h^*-q^*;\boldsymbol{v},r^*).$$

Combining the definition of the postprocessed problem and the consistency result 3.1 gives

$$\begin{aligned}&\|\boldsymbol{u}_h-R_h\boldsymbol{u}\|_{t,h}+|||p_h^*-q^*|||_{t,h}\leq C\mathcal{B}_h^*(\boldsymbol{u}-R_h\boldsymbol{u},p-q^*;\boldsymbol{v},r^*)-(g-P_hg,r^*)\\ &=a_h(\boldsymbol{u}-R_h\boldsymbol{u},\boldsymbol{v})+(\operatorname{div}\boldsymbol{v},p-q^*)+(\operatorname{div}(\boldsymbol{u}-R_h\boldsymbol{u}),r^*)-(g-P_hg,r^*)\\ &\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}(-\nabla(p-q^*)+(\boldsymbol{u}-R_h\boldsymbol{u})-t^2\Delta(\boldsymbol{u}-R_h\boldsymbol{u}),\nabla(I-P_h)r^*)_K.\end{aligned}$$

The last two terms on the second line cancel by the commuting diagram property (3.30). Inserting $\boldsymbol{f}$ into the last equation we have

$$\begin{aligned}\|\boldsymbol{u}_h-R_h\boldsymbol{u}\|_{t,h}+|||p_h^*-q^*|||_{t,h}\leq C\{&\|\boldsymbol{u}-R_h\boldsymbol{u}\|_{t,h}\|\boldsymbol{v}\|_{t,h}+|||p-q^*|||_{t,h}\|\boldsymbol{v}\|_{t,h}\\ &+\Big(\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}\|\nabla q^*-R_h\boldsymbol{u}+t^2\Delta R_h\boldsymbol{u}+\boldsymbol{f}\|_{0,K}^2\Big)^{1/2}|||r^*|||_{t,h}.\end{aligned}$$

Thus the assertion is proved using the triangle equality and the above result. □

Assuming full regularity, we have the following optimal a priori result for the postprocessed problem.

Theorem 4.5. *For the solution* $(\boldsymbol{u}_h,p_h^*)$ *of the postprocessed problem* (4.5), *assuming sufficient regularity of the solution, it holds*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}+|||p-p_h^*|||_{t,h}\leq\begin{cases}C(h^k+th^{k-1})\|\boldsymbol{u}\|_k, & \text{for RT},\\ C(h^{k+1}+th^k)\|\boldsymbol{u}\|_{k+1}, & \text{for BDM}.\end{cases} \tag{4.14}$$

5 A posteriori estimates

In this section we derive a residual-based a posteriori estimator for the postprocessed solution. It should be noted that the postprocessing is vital for a properly functioning estimator. Our derivation of the a posteriori estimator is based on the following saturation assumption. Let $\mathcal{K}_{h/2}$ be a uniformly refined subtriangulation of $\mathcal{K}_h$, then the saturation assumption is [6, 18]

$$\|\boldsymbol{u} - \boldsymbol{u}_{h/2}\|_{t,h/2} + \|\!|p - p^*_{h/2}|\!\|_{t,h/2} \leq \beta(\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,h} + \|\!|p - p^*_h|\!\|_{t,h}), \tag{5.1}$$

where the constant $\beta < 1$. The triangle inequality gives

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,h} + \|\!|p - p^*_h|\!\|_{t,h} \leq \frac{1}{1-\beta}(\|\boldsymbol{u}_{h/2} - \boldsymbol{u}_h\|_{t,h/2} + \|\!|p^*_{h/2} - p^*_h|\!\|_{t,h/2}). \tag{5.2}$$

We divide the estimator into two distinct parts, one defined over the elements and one over the edges of the mesh. The elementwise and edgewise estimators are defined as

$$\eta_K^2 = \frac{h_K^2}{h_K^2 + t^2}\| - t^2\Delta \boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p^*_h - \boldsymbol{f}\|^2_{0,K} + (t^2 + h_K^2)\|g - P_h g\|^2_{0,K}, \tag{5.3}$$

$$\eta_E^2 = \frac{t^2}{h_E}\|[\![\boldsymbol{u}_h]\!]\|^2_{0,E} + \frac{h_E}{h_E^2 + t^2}\|[\![p^*_h]\!]\|^2_{0,E} + \frac{h_E}{h_E^2 + t^2}\|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|^2_{0,E}. \tag{5.4}$$

The global estimator is

$$\eta = \left(\sum_{K\in\mathcal{K}_h} \eta_K^2 + \sum_{E\in\mathcal{E}_h} \eta_E^2\right)^{1/2}. \tag{5.5}$$

Note that setting $t = 0$ gives the standard estimator for the Darcy problem, see e.g. [16, 18]. In the following, we address the reliability and efficiency of the estimator and show the terms of the estimator to be properly matched to one another.

5.1 Reliability

First we focus on the reliability and prove the following theorem. Note that the upper bound holds uniformly for all values of the parameter t with the constant C independent of t.

Theorem 5.1. *Suppose that the saturation assumption holds. Then there exists a constant $C > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{t,h} + \|\!|p - p^*_h|\!\|_{t,h} \leq C\eta. \tag{5.6}$$

Proof. Due to (5.2) we only have to prove the result

$$\|\boldsymbol{u}_{h/2} - \boldsymbol{u}_h\|_{t,h/2} + \|\!|p^*_{h/2} - p^*_h|\!\|_{t,h/2} \leq C\eta. \tag{5.7}$$

By the stability result of Theorem 4.3 we can find $(\boldsymbol{v}, q^*) \in \boldsymbol{V}_{h/2} \times Q^*_{h/2}$ such that $\|\boldsymbol{v}\|_{t,h/2} + \|\!|q^*|\!\|_{t,h/2} \leq C$ for which it holds

$$\|\boldsymbol{u}_{h/2} - \boldsymbol{u}_h\|_{t,h/2} + \|\!|p^*_{h/2} - p^*_h|\!\|_{t,h/2} \leq \mathcal{B}^*_{h/2}(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p^*_{h/2} - p^*_h; \boldsymbol{v}, q^*). \tag{5.8}$$

Next, we add and subtract $\boldsymbol{R}_h\boldsymbol{v} \in \boldsymbol{V}_h$ and $P_h q^* \in Q_h^*$ from the test functions $\boldsymbol{v}$ and q^*. For the projections $P_{h/2}$ and P_h it holds $P_{h/2}P_h = P_h$, which implies $(I - P_{h/2})(I - P_h) = (I - P_{h/2})$. This property will be important in the analysis to follow. We introduce the following notation:

$$\mathcal{B}_{h/2}^*(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p_{h/2}^* - p_h^*; \boldsymbol{v}, q^*) = I + II, \tag{5.9}$$

in which

$$I = \mathcal{B}_{h/2}^*(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p_{h/2}^* - p_h^*; \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}, q^* - P_h q^*), \tag{5.10}$$

$$II = \mathcal{B}_{h/2}^*(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p_{h/2}^* - p_h^*; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*). \tag{5.11}$$

Keeping in mind that $(\boldsymbol{u}_{h/2}, p_{h/2}^*)$ is the solution on the refined mesh, we have

$$\begin{aligned} I &= \mathcal{L}_{h/2}(\boldsymbol{f}, P_{h/2}g; \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}, q^* - P_h q^*) - \mathcal{B}_{h/2}(\boldsymbol{u}_h, p_h^*; \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}, q^* - P_h q^*) \\ &= I_a + I_b + I_c, \end{aligned}$$

in which

$$I_a = (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}) - a_{h/2}(\boldsymbol{u}_h, \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}) - (\operatorname{div}(\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}), p_h^*), \tag{5.12}$$

$$I_b = (P_{h/2}g, q^* - P_h q^*) - (\operatorname{div}\, \boldsymbol{u}_h, q^* - P_h q^*), \tag{5.13}$$

$$I_c = \sum_{K \in \mathcal{K}_h} \frac{h_K^2}{h_K^2 + 4t^2} (\nabla p_h^* - \boldsymbol{u}_h + t^2 \Delta \boldsymbol{u}_h + \boldsymbol{f}, \nabla (I - P_{h/2}) q^*). \tag{5.14}$$

We have the following interpolation estimate for $\boldsymbol{R}_h\boldsymbol{v} \in \boldsymbol{V}_h$:

$$\frac{h_K^2 + t^2}{h_K^2} \|\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}\|_{0,K}^2 \le C(\|\boldsymbol{v}\|_{0,K}^2 + t^2 \|\nabla \boldsymbol{v}\|_{0,K}^2) \le \|\boldsymbol{v}\|_{t,h}^2. \tag{5.15}$$

The following shorthand notation for the residual is used

$$R_K = (-t^2 \Delta \boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p_h^* - \boldsymbol{f})|_K. \tag{5.16}$$

To estimate the term I_a we first integrate by parts in the first and second term. This gives

$$\begin{aligned} I_a = &\sum_{K \in \mathcal{K}_h} \{(t^2 \Delta \boldsymbol{u}_h - \boldsymbol{u}_h + \nabla p_h^* + \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v})_K - \frac{t^2}{2} \langle [\![\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!], \boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v} \rangle_{\partial K} \} \\ &+ \sum_{E \in \mathcal{E}_h} \{ t^2 \langle \frac{\partial (\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v})}{\partial \boldsymbol{n}}, [\![\boldsymbol{u}_h]\!] \rangle_E - \frac{\alpha t^2}{h_K} \langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}]\!] \rangle_E \\ &- \langle (\boldsymbol{v} - \boldsymbol{R}_h\boldsymbol{v}) \cdot \boldsymbol{n}, [\![p_h^*]\!] \rangle_E \}. \end{aligned}$$

Using the inequality (3.13), scaling arguments, and the inequality (5.15) the term I_a can be estimated as

$$
\begin{aligned}
I_a \leq & \sum_{K\in\mathcal{K}_h} \{ \left(\frac{h_K^2}{h_K^2+t^2}\right)^{1/2} \|R_K\|_{0,K} \left(\frac{h_K^2+t^2}{h_K^2}\right)^{1/2} \|\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v}\|_{0,K} \\
& + \left(\frac{h_K}{h_K^2+t^2}\right)^{1/2} \|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,\partial K} \left(\frac{h_K^2+t^2}{h_K}\right)^{1/2} h_K^{-1/2}\|\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v}\|_{0,K} \\
& + t\|\nabla(\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v})\|_{0,K}\frac{t}{h^{1/2}}\|[\![\boldsymbol{u}_h]\!]\|_{0,\partial K}\} \\
& + \sum_{E\in\mathcal{E}_h}\{\frac{t}{h_K^{1/2}}\|[\![\boldsymbol{u}_h]\!]\|_{0,E}\frac{t}{h_K^{1/2}}\|[\![\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v}]\!]\|_{0,E} \\
& + \left(\frac{h_K}{h_K^2+t^2}\right)^{1/2} \|[\![p_h^*]\!]\|_{0,E} \left(\frac{h_K^2+t^2}{h_K}\right)^{1/2} h_K^{-1/2}\|\boldsymbol{v}-\boldsymbol{R}_h\boldsymbol{v}\|_{0,K}\} \\
\leq & C\left(\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}\|R_K\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h}\{\frac{t^2}{h_K}\|[\![\boldsymbol{u}_h]\!]\|_{0,E}^2\right. \\
& \left. + \frac{h_K}{h_K^2+t^2}\|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}^2 + \frac{h_K}{h_K^2+t^2}\|[\![p_h^*]\!]\|_{0,E}^2\}\right)^{1/2} \|\boldsymbol{v}\|_{t,h}.
\end{aligned}
$$

Turning to the term I_b, we have by the equilibrium property (3.6) the result

$$\text{div } \boldsymbol{u}_h = P_h g.$$

Adding and subtracting the loading g gives

$$
\begin{aligned}
I_b &= (P_{h/2}g - \text{div } \boldsymbol{u}_h, q^* - P_h q^*) = (P_{h/2}g - g + g - P_h g, (I-P_h)q^*) \\
&\leq \sum_{K\in\mathcal{K}_h}\{\|P_{h/2}g - g\|_{0,K} + \|P_h g - g\|_{0,K}\}\|(I-P_h)q^*\|_{0,K} \\
&\leq C\sum_{K\in\mathcal{K}_h}(t^2+h_K^2)^{1/2}\|P_h g - g\|_{0,K}\left(\frac{h_K^2}{h_K^2+t^2}\right)^{1/2}\|\nabla q^*\|_{0,K} \\
&\leq C\left(\sum_{K\in\mathcal{K}_h}(t^2+h_K^2)\|P_h g - g\|_{0,K}^2\right)^{1/2} |\!|\!| q^* |\!|\!|_{t,h}.
\end{aligned}
$$

Finally, for the term I_c we have by straightforward estimation and the inequality (4.7)

$$
\begin{aligned}
I_c &\leq \sum_{K\in\mathcal{K}_h}\left(\frac{h_K^2}{h_K^2+t^2}\right)^{1/2}\|R_K\|_{0,K}\left(\frac{h_K^2}{h_K^2+t^2}\right)^{1/2}\|\nabla(I-P_{h/2})q^*\|_{0,K} \\
&\leq \left(\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}\|R_K\|_{0,K}^2\right)^{1/2} |\!|\!|(I-P_{h/2})q^*|\!|\!|_{t,h/2} \\
&\leq C\left(\sum_{K\in\mathcal{K}_h}\frac{h_K^2}{h_K^2+t^2}\|R_K\|_{0,K}^2\right)^{1/2} |\!|\!| q^* |\!|\!|_{t,h/2}.
\end{aligned}
$$

Combining the above results we have

$$I = I_a + I_b + I_c \leq C\eta. \tag{5.17}$$

Employing the fact that $(I - P_{h/2})P_h = 0$, we have for the second term

$$
\begin{aligned}
II &= \mathcal{B}^*_{h/2}(\boldsymbol{u}_{h/2}, p^*_{h/2}; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) - \mathcal{B}^*_{h/2}(\boldsymbol{u}_h, p^*_h; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) \\
&= \mathcal{B}^*_h(\boldsymbol{u}_h, p^*_h; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) - \mathcal{B}^*_{h/2}(\boldsymbol{u}_h, p^*_h; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) \\
&\quad - \mathcal{L}_h(P_h g, \boldsymbol{f}; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) + \mathcal{L}_{h/2}(P_{h/2} g, \boldsymbol{f}; \boldsymbol{R}_h\boldsymbol{v}, P_h q^*) \\
&= t^2 \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_K}\langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{R}_h\boldsymbol{v}]\!]\rangle_E - t^2 \sum_{E\in\mathcal{E}_h} \frac{2\alpha}{h_K}\langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{R}_h\boldsymbol{v}]\!]\rangle_E \\
&\quad + \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2 + t^2}(R_K, \nabla(I - P_h)P_h q^*)_K \\
&\quad - \sum_{K\in\mathcal{K}_h} \frac{h_K^2}{h_K^2 + 4t^2}(R_K, \nabla(I - P_{h/2})P_h q^*)_K \\
&\quad + (P_h g, P_h q^*) - (P_{h/2} g, P_h q^*)
\end{aligned}
$$

$$
\begin{aligned}
&= -t^2 \sum_{E\in\mathcal{E}_h} \frac{\alpha}{h_K}\langle [\![\boldsymbol{u}_h]\!], [\![\boldsymbol{R}_h\boldsymbol{v}]\!]\rangle_E + (g, P_h^2 q^* - P_{h/2}P_h q^*) \\
&\le \sum_{E\in\mathcal{E}_h} \frac{\alpha t}{h_K^{1/2}} \|[\![\boldsymbol{u}_h]\!]\|_{0,E} \frac{t}{h_K^{1/2}} \|[\![\boldsymbol{R}_h\boldsymbol{v}]\!]\|_{0,E} \\
&\le C\left(\sum_{E\in\mathcal{E}_h} \frac{t^2}{h_K}\|[\![\boldsymbol{u}_h]\!]\|^2_{0,E}\right)^{1/2} \|\boldsymbol{v}\|_{t,h}.
\end{aligned}
$$

Combining the estimates for parts I and II gives

$$
\mathcal{B}^*_{h/2}(\boldsymbol{u}_{h/2} - \boldsymbol{u}_h, p^*_{h/2} - p^*_h; \boldsymbol{v}, q^*) \le C\eta, \tag{5.18}
$$

and thus the theorem holds. □

5.2 Efficiency

Showing the estimator to be efficient proves to be more tedious than for the case of the pure Darcy flow treated in [18]. Indeed, we have to resort to the standard bubble function techniques to obtain the desired result. In the following we denote by ω_E the union of elements sharing an edge or face E. In addition, two cut-off functions Ψ_K and Ψ_E are introduced. The function Ψ_K has its support in K and $0 \le \Psi_K \le 1$, whilst Ψ_E is supported in ω_E and $0 \le \Psi_E \le 1$. Finally, we need an extension operator $\chi : L^2(E) \to L^2(\omega_E)$ such that on the edge E it coincides with the identity operator. We have the following lemma [25].

Lemma 5.2. *For an element K with an edge E it holds, for any polynomials p and σ,*

$$
\begin{aligned}
\|\Psi_K p\|_{0,K} \le \|p\|_{0,K} &\le C\|\Psi_K^{1/2} p\|_{0,K}, \\
\|\nabla(\Psi_K p)\|_{0,K} &\le C h_K^{-1}\|\Psi_K p\|_{0,K}, \\
\|\sigma\|_{0,E} &\le C\|\Psi_E^{1/2} p\|_{0,E}, \\
C h_E^{1/2}\|\sigma\|_{0,E} \le \|\Psi_E \chi\sigma\|_{0,K} &\le C h_E^{1/2}\|\sigma\|_{0,E}, \\
\|\nabla(\Psi_E \chi\sigma)\|_{0,K} &\le C h_K^{-1}\|\Psi_E \chi\sigma\|_{0,K}.
\end{aligned}
$$

Using this cut-off function allows us to prove the following theorem.

Theorem 5.3. *There exists a constant $C > 0$ such that*

$$\eta^2 \leq C\Big\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}^2 + |||p-p_h^*|||_{t,h}^2 \\ + \sum_{K\in\mathcal{K}_h}\Big(\frac{h_K^2}{h_K^2+t^2}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2 + (t^2+h_K^2)\|g-P_hg\|_{0,K}^2\Big)\Big\}. \tag{5.19}$$

Proof. We treat the terms separately. As before, we denote the first part of the residual by R_K, and further introduce the notation

$$R_{K,\mathrm{red}} = (-t^2\Delta\boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p_h^* - \boldsymbol{f}_h)|_K,$$
$$\boldsymbol{w} = \Psi_K R_{K,\mathrm{red}}.$$

We proceed by integrating by parts. Note that this gives no boundary terms due to the cut-off function. Inserting the exact solution and using the results of Lemma 5.2, we have

$$\begin{aligned}\|R_{K,\mathrm{red}}\|_{0,K}^2 &\leq C\|\Psi_K^{1/2}R_{K,\mathrm{red}}\|_{0,K}^2 = C(R_{K,\mathrm{red}},\boldsymbol{w})_K = C(R_K+\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{w})_K \\ &= C\{t^2(\nabla(\boldsymbol{u}_h-\boldsymbol{u}),\nabla\boldsymbol{w})_K + (\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{w})_K \\ &\quad - (\nabla(p_h^*-p),\boldsymbol{w})_K + (\boldsymbol{f}-\boldsymbol{f}_h,\boldsymbol{w})_K\} \\ &\leq C\|R_{K,\mathrm{red}}\|_{0,K}\{t^2h_K^{-1}\|\nabla(\boldsymbol{u}_h-\boldsymbol{u})\|_{0,K} + \|\boldsymbol{u}_h-\boldsymbol{u}\|_{0,K} \\ &\quad + \|\nabla(p-p_h^*)\|_{0,K} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}\}.\end{aligned}$$

Keeping in mind that $\|R_K\|_{0,K} \leq \|R_{K,\mathrm{red}}\|_{0,K} + \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}$, we have

$$\begin{aligned}&\frac{h_K^2}{h_K^2+t^2}\|-t^2\Delta\boldsymbol{u}_h + \boldsymbol{u}_h - \nabla p_h^* - \boldsymbol{f}\|_{0,K}^2 \\ &\qquad \leq C\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}^2 + |||p-p_h^*|||_{t,h}^2 + \frac{h_K^2}{h_K^2+t^2}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2\}.\end{aligned}$$

For the second part of the elementwise estimator the result holds trivially, thus we proceed to the edgewise estimators. Since for the exact solution $[\![p]\!] = 0$, and $[\![\boldsymbol{u}]\!] = 0$ on the element edges, we have

$$\frac{h_K}{h_K^2+t^2}\|[\![p_h^*]\!]\|_{0,E}^2 = \frac{h_K}{h_K^2+t^2}\|[\![p-p_h^*]\!]\|_{0,E}^2 \leq |||p-p_h^*|||_{t,h}^2,$$
$$\frac{t^2}{h_K}\|[\![\boldsymbol{u}_h]\!]\|_{0,E}^2 = \frac{t^2}{h_K}\|[\![\boldsymbol{u}-\boldsymbol{u}_h]\!]\|_{0,E}^2 \leq \|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}^2.$$

Finally, we must bound the normal jumps of the flux in the estimator. We use the cut-off function Ψ_E and the extension χ to define $\boldsymbol{w} = \Psi_E\chi[\![t^2\frac{\partial\boldsymbol{u}_h}{\partial\boldsymbol{n}}]\!]$.

Lemma 5.2 and integration by parts yield

$$\begin{aligned}
\|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}^2 &\le C\|\Psi_E^{1/2}[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}^2 = C\langle[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!],\boldsymbol{w}\rangle_E \\
&= C\{t^2(\Delta\boldsymbol{u}_h,\boldsymbol{w})_{\omega_E} + t^2(\nabla\boldsymbol{u}_h,\nabla\ \boldsymbol{w})_{\omega_E}\} \\
&= C\{t^2(\Delta\boldsymbol{u}_h,\boldsymbol{w})_{\omega_E} + t^2(\nabla\boldsymbol{u},\nabla\ \boldsymbol{w})_{\omega_E} + t^2(\nabla(\boldsymbol{u}_h-\boldsymbol{u}),\nabla\ \boldsymbol{w})_{\omega_E}\} \\
&= C\{(R_K,\boldsymbol{w})_{\omega_E} + (\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{w})_{\omega_E} - (\nabla(p-p_h^*),\boldsymbol{w})_{\omega_E} \\
&\quad + t^2(\nabla(\boldsymbol{u}_h-\boldsymbol{u}),\nabla\ \boldsymbol{w})_{\omega_E}\} \\
&\le C\|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}\{t^2h_K^{-1/2}\|\nabla(\boldsymbol{u}_h-\boldsymbol{u})\|_{0,\omega_E} + h_K^{1/2}\|\boldsymbol{u}_h-\boldsymbol{u}\|_{0,\omega_E} \\
&\quad + h_K^{1/2}\|\nabla(p-p_h^*)\|_{0,\omega_E} + h_K^{1/2}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,\omega_E}\}.
\end{aligned}$$

This gives

$$\begin{aligned}
&\frac{h_K}{h_K^2+t^2}\|[\![t^2\frac{\partial \boldsymbol{u}_h}{\partial \boldsymbol{n}}]\!]\|_{0,E}^2 \\
&\qquad \le C\{\|\boldsymbol{u}-\boldsymbol{u}_h\|_{t,h}^2 + |\!|\!|p-p_h^*|\!|\!|_{t,h}^2 + \frac{h_K^2}{h_K^2+t^2}\|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,K}^2\}.
\end{aligned}$$

Combining all of the above estimates proves the claim. □

Thus for the displacement $\boldsymbol{u}_h$ and the postprocessed pressure p_h^* we have by Theorems 5.1 and 5.3 a reliable and efficient indicator for all values of the parameter t.

6 Conclusions

We have shown that Nitsche's method can be successfully applied to $H(\mathrm{div})$-conforming elements as a non-conforming approximation for the Brinkman problem. The method is stable for all values of the viscosity parameter t. It was also shown that via postprocessing one achieves optimal convergence rate for both of the variables. Furthermore, we introduced a residual-based a posteriori error indicator with parameter-independent optimal convergence properties. The numerical performance of the method along with the usefulness of the estimator in adaptive procedures will be studied in a separate paper.

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