Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2006 A496

Dissection of the path-simplex in $\mathbf{R}^n$ into n path-subsimplices

Jan Brandts Sergey Korotov Michal Křížek

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2006 A496

Dissection of the path-simplex in $\mathbf{R}^n$ into n path-subsimplices

Jan Brandts Sergey Korotov Michal Křížek

Dedicated to Professor Miroslav Fiedler on the occasion of his 80-th birthday

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Jan Brandts, Sergey Korotov, Michal Křížek: *Dissection of the path-simplex in* $\mathbf{R}^n$ *into n path-subsimplices*; Helsinki University of Technology, Institute of Mathematics, Research Reports A496 (2006).

Abstract: *We review properties of acute and non-obtuse simplices, and of ortho-simplices and path-simplices. Dissection of path-simplices is considered, which leads to a new result: generalization of Coxeter's trisection of a path-tetrahedron into three path-subtetrahedra to arbitrary spatial dimension n. Moreover, following earlier results by Korotov and Křížek, we show that applying this procedure recursively in the proper way leads to a self-similar path-simplicial refinement towards a chosen vertex of the original path-simplex.*

AMS subject classifications: 51M20, 65N30

Keywords: n-simplex, path-simplex, Coxeter's trisection, self-similarity

Correspondence

brandts@science.uva.nl, sergey.korotov@hut.fi, krizek@math.cas.cz

ISBN 951-22-8099-X
ISSN 0784-3143

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Simplices and their facets

For given $n \in \mathbb{N}$, we define an n-simplex S as the convex hull of the origin $p_0 = 0$ and n linearly independent vectors $p_1, \dots, p_n \in \mathbb{R}^n$ called the vertices of S. If no confusion arises, we will write simplex instead of n-simplex. The $\frac{1}{2}n(n+1)$ convex hulls of arbitrary pairs of distinct vertices are called edges of S, whereas the $n+1$ convex hulls of n distinct vertices are called facets of S. For given $j \in \{0, \dots, n\}$, we write F_j for the facet of S that does not contain p_j, which is called the facet opposite to p_j. Let $P = (p_1|\dots|p_n)$ be the $n \times n$ matrix with the vertices of S as columns, and let $j \in \{1, \dots, n\}$. Write P_j for the $n \times (n-1)$ matrix that results from discarding p_j from P. Facet F_j is contained in the hyperplane $\mathcal{P}_j = \text{colspan}(P_j)$. The Euclidean distance of p_j to $\mathcal{P}_j$ is called the height h_j of S above F_j. Since P is non-singular, there exists $Q = (q_1|\dots|!q_n)$ such that Q^*P equals the $n \times n$ identity matrix I, and of course,

$$Q = P^{-*} = (P^{-1})^*. \tag{1}$$

In particular, $q_j^* P_j = 0$ shows that q_j is orthogonal to $\mathcal{P}_j$, and because $q_j^* p_j = 1$, both p_j and q_j lie in the same half-space defined by $\mathcal{P}_j$. For this reason, we will say that q_j is an inward normal to F_j. Since h_j is the component of p_j in the direction of q_j we find that

$$h_j = p_j^* \frac{q_j}{\|q_j\|} = \frac{1}{\|q_j\|}, \tag{2}$$

where $\|\cdot\|$ is the Euclidean norm. It remains to define an inward normal q_0 to F_0 such that its length is the inverse of the height h_0 of S above the facet F_0. This can be done by considering the simplex $\hat{S}$ with vertices $p_0 - p_1, \dots, p_n - p_1$, which is S translated along the vector $-p_1$. The facet of $\hat{S}$ that does not contain $-p_1$ corresponds to the facet F_0 of S. Now, write

$$e = e_1 + \dots + e_n, \tag{3}$$

for the sum of the canonical basis vectors of $\mathbb{R}^n$.

Proposition 1.1 *The inward normal q_0 to F_0 having the property that $\|q_0\| = h_0^{-1}$ equals*

$$q_0 = -Qe. \tag{4}$$

Proof. The facet $\hat{F}_0$ of $\hat{S}$ not containing $-p_1$ is spanned by the $n-1$ vectors $p_j - p_1$ for $j \in \{2, \dots, n\}$. Since

$$e^*Q^*(p_j - p_1) = e^*(e_j - e_1) = 1 - 1 = 0, \tag{5}$$

we see that q_0 defined by (4) is orthogonal to $\hat{F}_0$. Moreover,

$$-p_1^* q_0 = p_1^* Qe = e_1^* e = 1, \tag{6}$$

showing that the length of q_0 is the inverse of the height of $\hat{S}$ above $\hat{F}_0$. By back translation over p_1, the same is valid for S and F_0. $\square$

This completes the linear algebraic description of the simplex, its facets, and a set of inward normals to the facets with as lengths the inverses of the heights of S.

Now, let $n \geq 2$. To conclude this section, we will describe the facet F_1 of S seen as $(n-1)$-simplex in the hyperplane $\mathcal{P}_1$. For this, write $P = (p_1|P_1)$ and $Q = (q_1|Q_1)$, where both P_1 and Q_1 are $n \times (n-1)$ matrices, and let

$$(q_1|Q_1) = (u_1|U_1) \left[\begin{array}{c|c} \rho & r_1^* \\ \hline 0 & R_1 \end{array} \right] \quad \text{with} \quad (u_1|U_1)^*(u_1|U_1) = I \quad \text{and} \quad R_1 \text{ upper triangular} \tag{7}$$

be a QR-decomposition of Q, with $\rho = \|q_0\|$ and $r_1 \in \mathbb{R}^{n-1}$. Notice that the columns of U_1 form an orthonormal basis for $\mathcal{P}_1$.

Proposition 1.2 *The facet F_1 is represented by the matrix R_1^{-*}.*

Proof. Since $P = Q^{-*}$, we find from (7) that

$$(p_1|P_1) = (q_1|Q_1)^{-*} = (u_1|U_1) \left[\begin{array}{c|c} \rho^{-1} & 0 \\ \hline -\rho^{-1} R_1^{-*} r_1 & R_1^{-*} \end{array} \right]. \tag{8}$$

Comparing columns shows that $P_1 = U_1 R_1^{-*}$ and thus, R_1^{-*} is a matrix representation of the facet F_1 of P with respect to the columns of U_1. □

Consequently, the columns of R_1 are inward normals to the facets of F_1 with respect to the columns of U_1. These inward normals are the columns of $U_1 R_1$ in the standard basis of $\mathbb{R}^n$.

Proposition 1.3 *The columns of $U_1 R_1$ are the orthogonal projections onto the hyperplane $\mathcal{P}_1$ containing F_1 of the normals $q_2, \ldots, q_n$ to the facets $F_2, \ldots, F_n$ of S. Moreover, writing $\hat{e} = (1, \ldots, 1)^* \in \mathbb{R}^{n-1}$, the orthogonal projection onto $\mathcal{P}_1$ of the normal q_0 to F_0 equals $-R_1 \hat{e}$.*

Proof. Since (7) gives that $Q_1 = U_1 R_1 + u_1 r_1^*$, the statement is true for $q_2, \ldots, q_n$. From (7) we also find that

$$Qe = u_1(\rho + r_1^* \hat{e}) + U_1 R_1 \hat{e}, \tag{9}$$

showing that $-R_1 \hat{e}$ equals the projection on $\mathcal{P}_1$ of $q_0 = -Qe$. □

Notice that although the above explicitly describes the facet F_1 and its inward normals in $\mathcal{P}_1$, this is without loss of generality. By renumbering of the columns of P similar observations hold for the facets $F_2, \ldots, F_n$ of S, and by translation of S over $-p_1$ also for F_0.

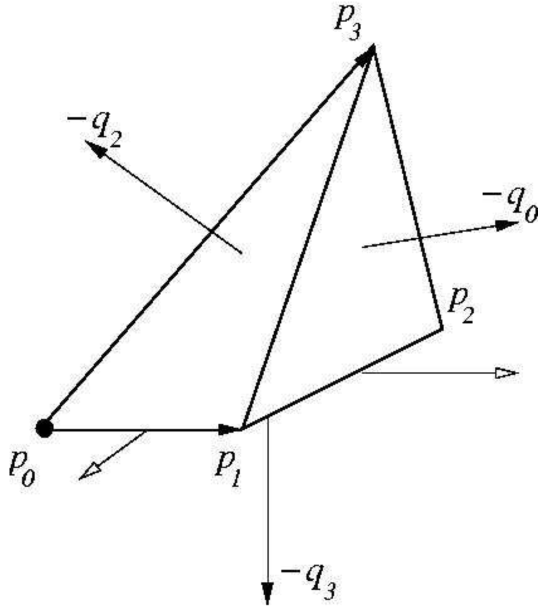

Figure 1. Illustration of notations and results of Section 1.

In Figure 1, three of the four normals to the facets of a tetrahedron are visible. For sake of clarity, outward normals are drawn. Also, the projections of two of them on the plane containing F_3 are depicted.

2 Acute and non-obtuse simplices

The inward normals $q_0, \ldots, q_n$ to the facets $F_0, \ldots, F_n$ of a simplex S can be employed to define the so-called dihedral angles between these facets.

Definition 2.1 For $i, j \in \{0, \ldots, n\}$ with $i \neq j$ and $n \geq 2$, let $\gamma_{ij} \in]0, \pi[$ be the angle between q_i and q_j. Then $\alpha_{ij} = \pi - \gamma_{ij}$ is called the *dihedral angle* between F_i and F_j, where $\alpha_{ij} \in]0, \frac{1}{2}\pi[$ is called *acute*, $\alpha_{ij} = \frac{1}{2}\pi$ *right*, and $\alpha_{ij} \in]\frac{1}{2}\pi, \pi[$ *obtuse*.

Since

$$q_i^* q_j = \|q_i\| \|q_j\| \cos \gamma_{ij} = -\|q_i\| \|q_j\| \cos \alpha_{ij}, \tag{10}$$

we conclude that that each negative off-diagonal entry of the (symmetric) matrix

$$(q_0|Q)^*(q_0|Q) = \left[\begin{array}{c|c} q_0^* q_0 & q_0^* Q \\ \hline Q^* q_0 & Q^* Q \end{array} \right] \tag{11}$$

corresponds to an acute dihedral angle, a zero entry to a right, and a positive off-diagonal entry to an obtuse dihedral angle. In fact, the type of angle between q_0 and the other inward normals can, using (4), be derived from the matrix Q^*Q since for $j \neq 0$,

$$q_j^* q_0 = -e_j^* Q^* Q e = -e_j^* Q^* (q_1 + \cdots + q_n) = -(q_j^* q_1 + \cdots + q_j^* q_n), \tag{12}$$

which is the negative j-th row sum of Q^*Q. The advantage of merely studying Q^*Q is, that any non-singular matrix Q represents a simplex, hence the study of the dihedral angles of a simplex reduces to the study of non-singular Gram matrices.

Definition 2.2 For given symmetric matrix M, let $\alpha_-(M), \alpha_0(M)$ and $\alpha_+(M)$ be half the numbers of off-diagonal entries of M that are negative, zero, and positive, and $\beta_-(M), \beta_0(M)$ and $\beta_+(M)$ the numbers of row sums of M that are negative, zero, and positive, respectively.

We will now prove that an n simplex has at least n acute dihedral angles. This result can be found at several places in the work by Fiedler [4, 6, 7, 8], but was rediscovered and published fifty years later as [13]. Here, we give a short proof based on a result for Gram matrices.

Lemma 2.3 *For $1 \leq k \leq n$, let V be a full rank real $n \times k$ matrix and set $M = V^*V$. Then,*

$$\beta_+(M) \geq 1 \quad \text{and} \quad \alpha_-(M) + \beta_+(M) \geq k. \tag{13}$$

Proof. Since $0 < \|Ve\|^2 = e^*Me$, the sum of all row sums of M is positive, hence $\beta_+(M) \geq 1$. Without loss of generality, assume that first row sum is positive. Write $V = (v_1|V_1)$ and $M_1 = V_1^*V_1$, then

$$M = \left[\begin{array}{c|c} v_1^*v_1 & v_1^*V_1 \\ \hline V_1^*v_1 & M_1 \end{array}\right]. \tag{14}$$

Let ℓ be the number of negative entries of $V_1^*v_1$. Then

$$\alpha_-(M) = \alpha_-(M_1) + \ell \quad \text{and} \quad \beta_+(M) \geq \max(\beta_+(M_1) - \ell, 0) + 1, \tag{15}$$

where the latter takes also the positive first row sum of M into account. Therefore,

$$\alpha_-(M)+\beta_+(M) \geq \alpha_-(M_1)+\ell+\max(\beta_+(M_1)-\ell, 0)+1 \geq \alpha_-(M_1)+\beta_+(M_1)+1. \tag{16}$$

The proof is now completed using an induction argument. □

Corollary 2.4 *Each simplex has at least n acute dihedral angles.*

Proof. The number of acute dihedral angles of S equals $\alpha_-(Q^*Q) + \beta_+(Q^*Q)$. Lemma 2.3 shows that this number is at least n. □

Remark 2.5 The fact that $\beta_+(Q^*Q) \geq 1$ reflects that each facet of S makes at least one acute dihedral angle with another facet. This is because the row sums of Q^*Q correspond to the dihedral angles between q_0 and $q_1, \ldots, q_n$, whereas the origin is an arbitrary vertex of S.

The simplex represented by the identity matrix I is an example of a simplex with precisely n acute dihedral angles. Simplices without any obtuse dihedral angles are of importance in many applications. This motivates the following nomenclature.

Definition 2.6 A simplex S is called *non-obtuse* if none of its $\frac{1}{2}n(n+1)$ dihedral angles are obtuse. A non-obtuse simplex without right dihedral angles is called *acute*.

The following characterizations are valid independent of the matrix P that is chosen to represent S, and therefore independent of Q:

- S is non-obtuse $\Leftrightarrow \alpha_+(Q^*Q) = 0$ and $\beta_-(Q^*Q) = 0$,
- S is acute $\Leftrightarrow \alpha_-(Q^*Q) = \frac{1}{2}(n-1)n$ and $\beta_+(Q^*Q) = n$.

The properties of non-obtuseness and acuteness of a simplex are inherited by its facets, and inductively by facets of facets and so on. A proof based on graph theory can be found in Fiedler's work [4, 7]. Here we present a linear algebraic proof.

Proposition 2.7 *For $n \geq 3$ the facets of an acute (non-obtuse) simplex S are acute (non-obtuse).*

Proof. Without loss of generality, we may assume that $j = 1$ and consider F_1 only. Using the QR-decomposition of $Q = (q_1|Q_1)$ from (7) we find that

$$Q^*Q = \left[\begin{array}{c|c} \rho^2 & \rho r_1^* \\ \hline \rho r_1 & R_1^*R_1 + r_1 r_1^* \end{array}\right]. \tag{17}$$

Assume that S is acute. Then the off-diagonal entries of Q^*Q are negative. This includes the entries of ρr_1, hence, $r_1 r_1^*$ has positive entries. Thus the off-diagonal entries of $R_1^*R_1$ are negative. Since Q^*Q has positive row sums,

$$\alpha = \rho^2 + \rho r_1^* \hat{e} > 0 \quad \text{and} \quad \rho r_1 + R_1^*R_1\hat{e} + r_1 r_1^* \hat{e} \quad \text{has positive entries.} \tag{18}$$

Because $\rho r_1 + r_1 r_1^* \hat{e} = \alpha\rho^{-1} r_1$ has negative entries, the vector of row sums $R_1^*R_1\hat{e}$ of $R_1^*R_1$ has positive entries. Since R_1 contains the inward normals to the facets of F_1, we conclude that F_1 is acute. For non-obtuse simplices the proof is similar. □

The converse of the above proposition is not valid: there exist obtuse tetrahedra of which all facets are acute triangles. An example is depicted in Figure 2.

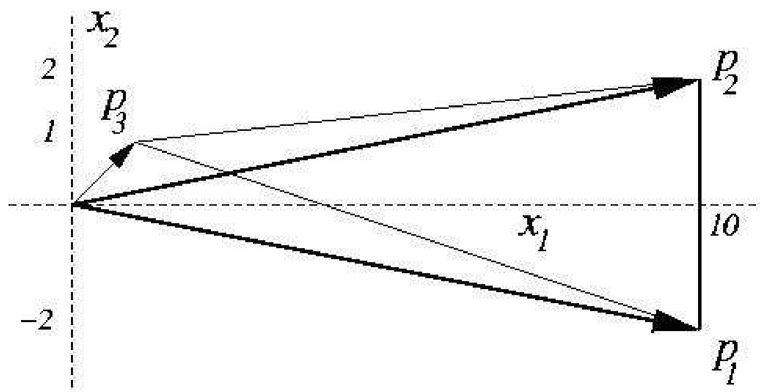

Figure 2. Example of an obtuse tetrahedron S with acute triangular facets.

The matrix representing the tetrahedron in Figure 2 is

$$P = \begin{bmatrix} 10 & 10 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 10 \end{bmatrix}. \tag{19}$$

The view is orthogonally from above, i.e., the x_3-direction is perpendicular to the bold triangular face which is in the (x_1, x_2)-plane. Clearly, F_3 and F_0 are acute triangles, and α_{13} is obtuse. Since p_3 is chosen high enough above the point $(1, 1)$, also F_1 and F_2 are acute.

Remark 2.8 Recalling that the volume of a simplex can be computed as

$$\text{Vol}(S) = \frac{h_j}{n}\text{Vol}(F_j), \tag{20}$$

we find by (2) and (10) a geometric interpretation of the inner product $q_i^* q_j$,

$$q_i^* q_j = -\frac{\text{Vol}(F_i)\text{Vol}(F_j)}{[n\text{Vol}(S)]^2} \cos \alpha_{ij}, \tag{21}$$

which was already derived for $n = 2$ in [9, 16] and for $n = 3$ in [12]. It proved relevant in the context of finite element methods for partial differential equations. □

3 Ortho-simplices and path-simplices

The simplex corresponding to the identity matrix I has exactly n acute dihedral angles. It has several additional interesting properties. For instance,

- its remaining $\frac{1}{2}(n-1)n$ dihedral angles are right,
- it has n mutually orthogonal edges,
- its facet F_0 makes acute dihedral angles with each of the other facets.

The latter property rephrases that $\beta_+(I) = n$. In fact, the facet F_0 itself, seen as an $(n-1)$-simplex, has only acute dihedral angles. The interest of properties like the above motivates the following terminology.

Definition 3.1 An *ortho-simplex* is a simplex having n mutually orthogonal edges. A *path-simplex* is an ortho-simplex whose n orthogonal edges form a path.

Ortho-simplices are, in fact, exactly the simplices with the maximal amount of $\frac{1}{2}(n-1)n$ right dihedral angles [5, 8]. This gives as an alternative characterization that:

- S is an ortho-simplex $\Leftrightarrow \alpha_0(Q^*Q) + \beta_0(Q^*Q) = \frac{1}{2}(n-1)n$.

Again, this is independent of the choice for the matrix P representing S, and thus independent of Q. The above equivalence immediately shows that ortho-simplices are non-obtuse. Consequently, also path-simplices are non-obtuse.

The canonical example of a path-simplex is the simplex S represented by the all-ones upper triangular $n \times n$ matrix T, i.e.,

$$T = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \quad \text{where} \quad T^{-*} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} \tag{22}$$

has inward normals to the facets $F_1, \dots, F_n$ as columns, whereas the remaining normal q_0 equals $-T^{-*}e = -e_1$. Clearly, S has a path of orthogonal edges from the origin to the point $e \in \mathbb{R}^n$, and those orthogonal edges are edges of a unit hypercube.

Unlike ortho-simplices in general, path-simplices have the additional property that each of their facets is again a path-simplex [7]. Also, an ortho-simplex is a path-simplex if and only if it contains the center of its circumscribed ball [1].

Now, let D be a non-singular diagonal matrix, then DT is also a path-simplex. The lengths of its consecutive edges belonging to the orthogonal path are the absolute values of the diagonal entries of D. If U is orthogonal, then UDT also represents a path-simplex. Since the columns of T are increasing in length from left to right, so are the columns of UDT. Therefore,

$$P = UDTE, \tag{23}$$

where E is a (column)permutation, is the general matrix representation of a path-simplex whose path of orthogonal edges starts at the origin.

Ortho-simplices and path-simplices in particular are very useful in spline approximation theory in general, and finite element methods in particular. They also have a central role in geometry. It was conjectured by Hadwiger [10] in 1957 that every simplex can be decomposed into a finite number of path-simplices. If this conjecture is correct, it would show that path-simplices are even more elementary geometric building blocks than simplices themselves. Because of the following theorem and its corollary, the difficulty of the conjecture is to decompose an arbitrary simplex into non-obtuse simplices.

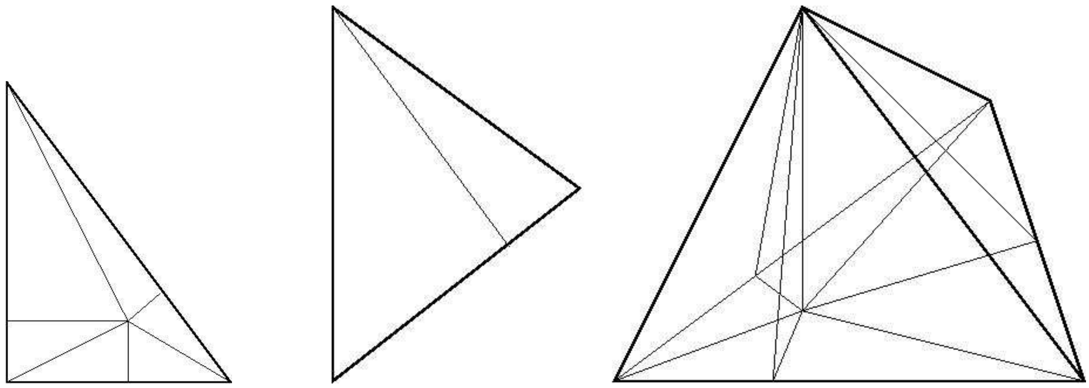

Figure 3. Dissection of non-obtuse and acute simplices into path-simplices.

Theorem 3.2 *Let x be a point in the interior of a non-obtuse n-simplex S. Then S can be dissected into $(n+1)!$ path-subsimplices whose orthogonal paths of n edges all end at x.*

Proof. By induction. The induction basis for $n = 2$ is illustrated left in Figure 3. Let S be a non-obtuse n-simplex with given interior point x. Define

$$S_j = \text{conv}(x, F_j), \quad j \in \{0, \dots, n\}. \tag{24}$$

Then $S_0, \dots, S_n$ is a dissection of S into $n+1$ subsimplices. The orthogonal projection x_j of x onto F_j is an interior point of F_j. By the induction hypothesis, we can dissect F_j into $n!$ path-subsimplices $\hat{S}_j^1, \dots, \hat{S}_j^{n!}$ with orthogonal paths ending at x_j. Then

$$S_j^i = \text{conv}(x, \hat{S}_j^i), \quad i \in \{1, \dots, n!\}, \tag{25}$$

is a dissection of S_j into $n!$ path-subsimplices, because $x - x_j$ is orthogonal to F_j, and thus it extends the orthogonal path of S_j to length n. Doing this for all j results in $(n+1)!$ path-subsimplices of S, proving the statement. □

The following corollary is immediate and is illustrated in the middle and right of Figure 3.

Corollary 3.3 *Each acute n-simplex S can be dissected into $n!$ path-subsimplices.*

Proof. The orthogonal projection x_j of p_j onto F_j is an interior point of F_j. Using Theorem 3.2, we can dissect F_j into $n!$ path-subsimplices $S_1, \dots, S_{n!}$ whose orthogonal paths all end at x_j. The $n!$ convex hulls of S_j and x form the required subdivision of S. □

If S is non-obtuse but not acute, degenerate results can be obtained due to the fact that some of the above path-subsimplices may dimensionally collapse. For example, see Corollary 4.3 as a degenerates case of Theorem 4.2 in the following section.

In 1960, Lenhardt [14] showed that each tetrahedron can be decomposed in at most 12 path-tetrahedra. Charsischwili [2] proved that each 4-simplex can be subdivided into a finite number of path-subsimplices. Tschirpke [17] solved the case $n = 5$. For $n \geq 6$ the conjecture remains open.

4 Dissection of path-simplices into n path-subsimplices

It is easy to dissect a right triangle into two right subtriangles. Much less trivial is that a path-tetrahedron can be dissected into three path-subtetrahedra. This was shown by Coxeter [3] in 1989. Here we will prove that path-simplices can be subdivided into $n+1$ path-subsimplices. As a degenerate case, a subdivision into n path-subsimplices follows. Coxeter's trisection then corresponds to our result for $n = 3$. First we prove a lemma.

Lemma 4.1 *Let $n \geq 2$, and let S be an n-simplex represented by $(p_1|\dots|p_n) = P = DT$. Then the orthogonal projection w of p_1 onto F_1 equals*

$$w = p_2 \frac{\|p_1\|^2}{\|p_2\|^2}, \tag{26}$$

which equals the orthogonal projection of p_1 onto p_2.

Proof. The explicit form of T and T^{-*} in (22) shows that the normal $q_1 = D^{-1}T^{-*}e_1$ to F_1 is a linear combination of p_1 and p_2, and that there exist non-zero α and β such that

$$p_1 = \alpha p_2 + \beta q_1, \quad \text{and } q_1 \perp p_2. \tag{27}$$

Thus, the orthogonal projection w of p_1 on F_1 equals the orthogonal projection of p_1 on p_2, which can be computed as

$$w = p_2 \frac{p_2^* p_1}{p_2^* p_2}. \tag{28}$$

Now, $p_2^* p_1 = (p_2 - p_1 + p_1)^* p_1 = p_1^* p_1$ because $p_2 - p_1 \perp p_1$, and the statement follows. □

We will now prove that a path-simplex S can be subdivided into $n+1$ path-subsimplices, such that their orthogonal paths all end at a point located at the first orthogonal edge $p_0 p_1$ of S. This is illustrated in Figure 4. For $n = 2$, two edges are drawn inside a right triangle p_0, p_1, p_2: one from $\alpha_1 p_1$ orthogonally onto the edge $p_0 p_2$, and one from $\alpha_1 p_1$ to the vertex p_2. This subdivides the right triangle into three right subtriangles, and the paths of orthogonal edges of the subtriangles end at $\alpha_1 p_1$.

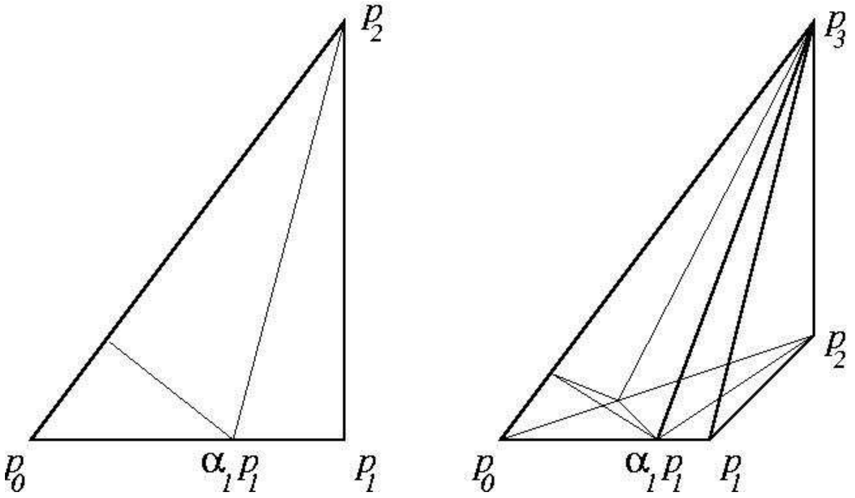

Figure 4. Subdivision of a path-simplex into $n+1$ path-subsimplices for $n=2$ and $n=3$.

For $n = 3$, consider the path-tetrahedron p_0, p_1, p_2, p_3 at the right of Figure 4. The point $\alpha_1 p_1$ trivially determines a path-tetrahedron with vertices $\alpha_1 p_1, p_1, p_2$ and p_3 and a second tetrahedron $p_0, \alpha_1 p_1, p_2, p_3$. The latter can be trisected using trisection of the right triangular face p_0, p_2, p_3 opposite $\alpha_1 p_1$ into three right triangles with paths ending at the orthogonal projection $\alpha_2 p_2$ of $\alpha_1 p_1$ onto p_2, and adding the edge between $\alpha_1 p_1$ and $\alpha_2 p_2$, which completes the orthogonal paths to length three. The fact that the trisection of the right triangle opposite $\alpha_1 p_1$ is used in the subdivision of the path-tetrahedron into four path-subtetrahedra suggests the following induction proof for arbitrary n.

Theorem 4.2 *Let $n \geq 2$, and let S be an n-simplex represented by $(p_1|\dots|p_n) = P = DT$. Then for each $\alpha_1 \in]0,1[$, S can be subdivided into $n+1$ path-subsimplices having the property that their $n+1$ orthogonal paths of n edges all end at $\alpha_1 p_1$.*

Proof. By induction. The induction basis for $n = 2$ and $n = 3$ is illustrated in Figure 4. Let S be an n-simplex represented by $P = DT$, and let $\alpha_1 \in]0,1[$. Then the point $\alpha_1 p_1$ uniquely determines a bisection of S into two simplices S_1 and S', where

$$S_1 = \text{conv}(\alpha_1 p_1, F_0) \quad \text{and} \quad S' = \text{conv}(\alpha_1 p_1, F_1). \tag{29}$$

Obviously, S_1 is a path-simplex with orthogonal path ending at $\alpha_1 p_1$. Consider S'. By Lemma 4.1, the orthogonal projection of $\alpha_1 p_1$ onto F_1 equals $\alpha_2 p_2$, where

$$0 < \alpha_2 = \alpha_1 \frac{\|p_1\|^2}{\|p_2\|^2} < 1. \tag{30}$$

By induction, since F_1 is an $(n-1)$ dimensional path-simplex with matrix representation $(p_2|\dots|p_n)$, it can be subdivided into n path-subsimplices $\hat{S}_1, \dots, \hat{S}_n$ whose orthogonal paths all end at $\alpha_2 p_2$. Defining

$$S_{j+1} = \text{conv}(\alpha_1 p_1, \hat{S}_j) \quad \text{for } j \in \{1, \dots, n\}, \tag{31}$$

then subdivides S' into n path-simplices $S_2, \ldots, S_{n+1}$, because the additional edge between $\alpha_2 p_2$ and $\alpha_1 p_1$ is orthogonal to F_1. Hence, S is subdivided into $n+1$ path-subsimplices $S_1, \ldots, S_{n+1}$, proving the theorem. □

For $\alpha_1 = 1$, a degenerate case results, which for $n = 3$ reduces to the trisection in [3] of the path-tetrahedron illustrated at the left in Figure 5.

Corollary 4.3 *Each path-simplex S represented by $P = DT$ can be subdivided into n path-subsimplices whose paths of orthogonal edges all end at p_1.*

Proof. Using Theorem 4.2, subdivide F_1 into n path-subsimplices $S_1, \ldots, S_n$ with paths ending at the orthogonal projection of p_1 on p_2. Then Lemma 4.1 shows that the convex hulls of p_1 with each of the S_j is the required subdivision of S. □

In Figure 5, the new vertices that resulted from the trisection of the path-tetrahedron are denoted by $y_1, \ldots, y_3$. From the construction in Theorem 4.2 for arbitrary n we see that y_j is the projection of y_{j-1} on the span $\mathcal{V}_j$ of $p_j, \ldots, p_n$. Since we have

$$\mathcal{V}_n \subset \cdots \subset \mathcal{V}_j \subset \cdots \subset \mathcal{V}_1, \tag{32}$$

we conclude that y_j is the projection of p_1 onto $\mathcal{V}_j$. Therefore, by Lemma 4.1, we even get that y_j is the projection of p_1 onto p_j, hence,

$$y_j = p_j \frac{\|p_1\|^2}{\|p_j\|^2}. \tag{33}$$

Korotov and Křížek observed in [11] that applying the trisection of Coxeter once more to the path-subtetrahedron y_0, y_3, y_2, y_1, the resulting path-tetrahedron z_0, z_1, z_2, z_3 indicated in the right of Figure 5, is similar to S in the sense that $z_j = \alpha p_j$ for a fixed $\alpha \in]0,1[$. This also holds for arbitrary n.

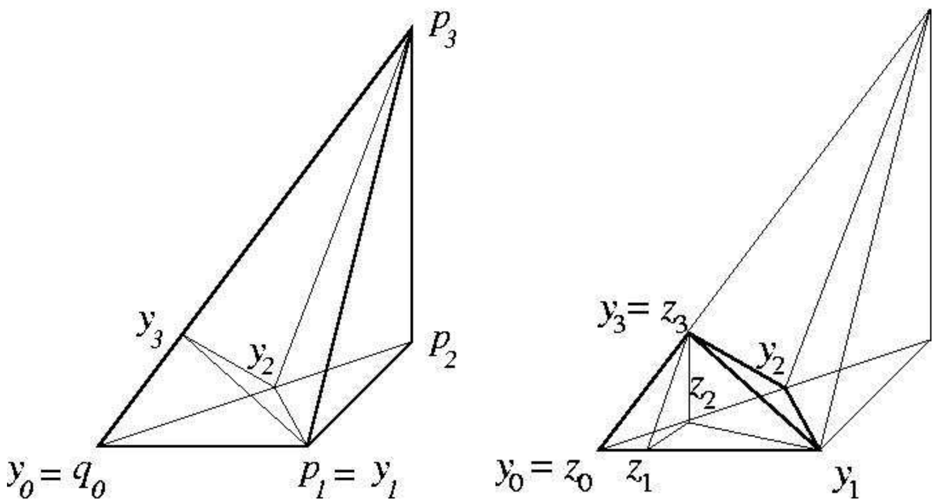

Figure 5. Left: Coxeter's trisection of the path-tetrahedron. Right: its double application.

Theorem 4.4 Given a path-simplex S with matrix representation $(p_1|\dots|p_n) = DT$ and its subdivision into n path-subsimplices $S_1,\dots,S_n$ according to Corollary 4.3. Apply the procedure again to the subsimplex S_1 having the origin as vertex. Then the resulting path-simplex $S_{1,1}$ having the origin as vertex is similar to S.

Proof. Using (33) twice, we find that $S_{1,1}$ has matrix representation $(z_1|\dots|z_n)$ where

$$z_j = y_j \frac{\|y_n\|^2}{\|y_j\|^2} = p_j \frac{\|p_1\|^2}{\|p_n\|^2}. \tag{34}$$

The scaling factor $\|p_1\|^2\|p_n\|^{-2}$ is independent of j and thus $S_{1,1}$ is similar with S. □

The above property may be used in local refinement towards a vertex on the longest diagonal of a given path-simplex, resulting in a self-similar non-obtuse face-to-face partition. As was proved in [15], a partition consisting of path-simplices is of Delaunay type.

Corollary 4.5 *Let $k \geq 3$ be an integer. Then each path-tetrahedron can be decomposed into k path-subtetrahedra.*

Proof. Follows directly from Corollary 4.3 and Theorems 4.1 and 4.4. See also Figures 4 and 5 for $k \in \{3,4,5\}$. □

Acknowledgments and contact details

Michal Křížek was supported by Grant No. A 1019201 of the Grant Agency of the Academy of Sciences of the Czech Republic. Sergey Korotov was supported by Grant No. 211 688 of the Academy of Finland.

Jan Brandts, Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands; e-mail: brandts@science.uva.nl.

Sergey Korotov, Institute of Mathematics, Helsinki University of Technology, P.O.Box 1100, FI-02015 TKK, Helsinki, Finland; e-mail: skorotov@cc.hut.fi.

Michal Křížek, Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67, Prague 1, Czech Republic; e-mail: krizek@math.cas.cz.

References

[1] M. Bern, P. Chew, D. Eppstein, and J. Ruppert (1995). *Dihedral bounds for mesh generation in high dimensions.* Proc. 6th Sympos. Discrete Algorithms, ACM and SIAM, 189–196.

[2] A.B. Charsischwili (1982). Orthogonale Simplexe im vierdimensionalen Raum, *Mitt. Akad. Wiss. der Georgischen SSR*, 88(1):33-36.

[3] H.S.M. Coxeter (1989). Trisecting an orthoscheme. *Computers Math. Applic.*, 17:59–71.

[4] M. Fiedler (1954). Geometrie simplexu I. *Čas. pěst. mat.*, 79:297–320.

[5] M. Fiedler (1957). Über qualitative Winkeleigenschaften der Simplexe. *Czechoslowak Math. J.* 7(82):463–478.

[6] M. Fiedler (1957). Einige Sätze aus der metrischen Geometrie der Simplexe in Euklidischen Räumen. In: Schriftenreihe d. Inst. f. Math. DAW, Heft 1, Berlin, 157.

[7] M. Fiedler (2001). *Matrices and graphs in Euclidean geometry* (in Czech). Publ. Dimatia MFF UK Praha, ISBN 80-238-7344-X.

[8] M. Fiedler (2005). *Matrices and graphs in Euclidean geometry. Electronic J. Linear Algebra*, 14:51–58

[9] H. Fujii (1973). Some remarks on finite element analysis of time-dependent field problems. In: Theory and Practice in Finite Element Structural Analysis, Univ. Tokyo Press, Tokyo, 91–106.

[10] H. Hadwiger (1957). *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer Verlag, Berlin-Göttingen-Heidelberg.

[11] S. Korotov and M. Křížek (2005). Global and local refinement techniques yielding nonobtuse tetrahedral partitions *Comput. Math. Appl.*, 50:1105–1113.

[12] M. Křížek and Q. Lin (1995). On the diagonal dominance of stiffness matrices in 3D. *East-West J. Numer. Math.*, 3:59–69.

[13] G. Leng (2003). The minimum number of acute dihedral angles of a simplex. *Proc. Amer. Math. Soc.*, 113(10):3039-3042.

[14] H.C. Lenhard (1960). Zerlegung von Tetraedern in Orthogonaltetraedern. *Elem. Math.*, 15:106–107.

[15] V.T. Rajan (1994). Optimality of the Delaunay triangulation in $\mathbb{R}^d$. *Discrete Comput. Geom.*, 12:189–202.

[16] V. Ruas Santos (1982). On the strong maximum principle for some piecewise linear finite element approximate problems of non-positive type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29:473–491.

[17] K. Tschirpke (1993). On the dissection of simplices into orthoschemes. *Geom. Dedicata*, 46:313–329.

(continued from the back cover)

A493 Giovanni Formica , Stefania Fortino , Mikko Lyly
A $vartheta$ method–based numerical simulation of crack growth in linear elastic fracture
February 2006

A492 Beirao da Veiga Lourenco , Jarkko Niiranen , Rolf Stenberg
A posteriori error estimates for the plate bending Morley element
February 2006

A491 Lasse Leskelä
Comparison and Scaling Methods for Performance Analysis of Stochastic Networks
December 2005

A490 Anders Björn , Niko Marola
Moser iteration for (quasi)minimizers on metric spaces
September 2005

A489 Sampsa Pursiainen
A coarse-to-fine strategy for maximum a posteriori estimation in limited-angle computerized tomography
September 2005

A487 Ville Turunen
Differentiability in locally compact metric spaces
May 2005

A486 Hanna Pikkarainen
A Mathematical Model for Electrical Impedance Process Tomography
April 2005

A485 Sampsa Pursiainen
Bayesian approach to detection of anomalies in electrical impedance tomography
April 2005

A484 Visa Latvala , Niko Marola , Mikko Pere
Harnack's inequality for a nonlinear eigenvalue problem on metric spaces
March 2005

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A498 Marcus Ruter , Sergey Korotov , Christian Steenbock
Goal-oriented Error Estimates based on Different FE-Spaces for the Primal and the Dual Problem with Applications to Fracture Mechanics
March 2006

A497 Outi Elina Maasalo
Gehring Lemma in Metric Spaces
March 2006

A496 Jan Brandts , Sergey Korotov , Michal Krizek
Dissection of the path-simplex in $\mathbf{R}^n$ into n path-subsimplices
March 2006

A495 Sergey Korotov
A posteriori error estimation for linear elliptic problems with mixed boundary conditions
March 2006

A494 Antti Hannukainen , Sergey Korotov
Computational Technologies for Reliable Control of Global and Local Errors for Linear Elliptic Type Boundary Value Problems
February 2006

ISBN 951-22-8099-X
ISSN 0784-3143