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TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Helsinki University of Technology
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Abstract: *This paper presents a method for the automatic simulation of quasi–static crack growth in 2D linear elastic bodies with existing cracks. A finite element algorithm, based on the so-called ϑ method, provides the load vs. crack extension curves in the case of stable rectilinear crack propagation. Since the approach is both theoretically general and simple to be performed from a computational point of view, it appears very suitable for the extension to curvilinear crack propagation in nonlinear materials.*

AMS subject classifications: 47A10, 65F10

Keywords: Linear Elastic Fracture Mechanics (LEFM), ϑ method, Stable crack propagation, Finite Element Method (FEM)

Correspondence

Giovanni Formica
Department of Structural Engineering, University of "Roma 3",
Via Vito Volterra 62, 00146 Rome, Italy

Stefania Fortino
Laboratory of Structural Mechanics, Helsinki University of Technology,
P.O. Box 2100, FIN-02015 TKK, Espoo, Finland

Mikko Lyly
CSC-Scientific Computing Ltd,
P.O. Box 405, FIN-02101, Espoo, Finland

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

The evaluation of crack growth in quasi–brittle bodies with existing cracks is a fundamental topic when the structural reliability related to a certain crack length has to be studied (see [1]). In particular, the crack length extension under a given load increment is an important variable to be analyzed during a loading process [2]. In this context, the Griffith criterion [3, 4] still represents a strong theoretical tool for establishing the onset of crack growth in LEFM. Furthermore, several criteria for determining the direction of crack propagation under various mode loading can be found in the fracture mechanics literature (see references in [16]). Instead, as already observed in [16, 17], a general theoretical model for automatically evaluating the increments of crack growth during a loading process does not exist yet. Nowadays this limit still characterizes the computer codes for simulation of crack growth in elastic and elastic–plastic materials (e.g. FRANC2D, FRANC3D, FRANC2D/L, ZENCRACK and, recently, ABAQUS). In the presence of fatigue crack growth, the cracks are driven by the empirical Paris law [5] while, in general cases of monotonic loading, the crack increments are usually assigned after the determination of the crack growth direction. This approach is also used in the recent computational methods for LEFM like the extended finite element method (X-FEM) [8, 9] and the meshless techniques based on the element-free Galerkin method (EFGM) [10, 11, 12].

An attempt to define a general analysis which allows the increments of crack growth during a given loading process to be automatically determined was done by Fortino and Bilotta in [16]. In particular, an incremental displacement-crack growth approach for 2D problems of stable elastic crack propagation was proposed. The theoretical analysis, originally introduced by Nguyen *et al.* ([14, 15]) and written in rate terms, used the classical equations of linear elasticity, the Griffith criterion of crack growth in the form proposed by Irwin [6] and some discontinuity conditions on the stresses and displacements. The presence of the discontinuities was due to a description of crack growth based on the definition of a subdomain moving with the crack tip during crack extension. The computational method based on this theory gave promising results for the determination of curves load vs. crack growth in linear elastic fracture. The method can be also extended to incremental elastoplasticity as suggested in [17] but the mentioned discontinuity conditions could become difficult to handle from a computational point of view because of the necessity to define both a plastic fracture zone and a moving subdomain around the crack tip.

In the present work a new crack growth formulation based on the so–called ϑ method of Destuynder and Djaoua [18] is introduced. From a theoretical point of view, the new approach is more general than that proposed in [16] and, since it doesn't contain discontinuity conditions, it is also simpler to be implemented into a computer program. The ϑ method was originally introduced for giving a mathematical interpretation of the Griffith criterion of crack growth in elasticity. The basic idea is to work in a fixed configuration by using a method of domain variation which transforms the variables of the current equilibrium problem into the variables of the perturbed problem in a one–to–one manner. All physical quantities of the perturbed configuration are then rewritten in the current. This operation provides the convergence of the solution of the perturbed equilibrium problem to the solution of the problem in the current domain. As a direct result, the method introduces an energetic domain parameter known in literature as the $G\theta$ and characterized by a smooth value vector function $\boldsymbol{\vartheta}$ defined in a subdomain of the current configuration around the crack tip ([18], [19]). The parameter coincides with the Griffith energy release rate during quasi-static crack growth.

Note that, although the use of the $G\vartheta$ parameter in contexts where the Rice's J integral [7] is usually required would avoid the eventual transformation from path integrals to domain integrals as done by standard computational approaches [3], this parameter is not widely used in literature. After Destuynder and Djaoua, some French researchers used the $G\vartheta$ for problems of crack growth in viscoelastic materials (see [21] and relative references). Anyway, also in these works the increments of crack extensions are assigned. Furthermore, an extension of the $G\theta$ parameter to incremental elastoplasticity was introduced by Debruyne in [19] for studying problems of ductile tearing [20]. Debruyne presented a fracture criterion but did not calculate the crack increments during crack propagation. Note that a method very similar to the Destuynder–Djaoua approach was introduced and justified from a mathematical point of view by some Russian researchers (see [22] and relative references) in order to differentiate energy functionals with respect to the crack length for both cases of rectilinear and curvilinear cracks. In those works the crack growth during a loading process was not investigated.

In the present paper, successive $G\vartheta$–based crack growth problems written in the current configuration furnish the increments of crack growth relative to given load increments. Furthermore, at each load step the equilibrium is imposed in the current configuration with updated crack length. For sake of simplicity, the method is defined for cases of stable rectilinear crack growth but it can be extended to curvilinear crack propagation as suggested in Section 5. The extension to elastoplasticity is also possible starting from the results obtained in [19]. The proposed algorithm is implemented by using the computational tools of the computer code ELMER (Scientific Computing LTD., Espoo, Finland). Remeshing and standard FEM discretization are employed.

The paper is organized as follows. In Section 2 the Destuynder and Djaoua ϑ method in linear elastic fracture is briefly recalled. A $G\vartheta$–based crack growth formulation providing the increment of crack length is described in Section 3. Some implementation details for the case of stable rectilinear crack growth are illustrated in Section 4. In the same section, some numerical results show the effectiveness of the proposed ϑ method–based simulation of crack growth in LEFM. Finally, some remarks about further developments of the method can be found in Section 5.

2 The ϑ method in 2D LEFM

Let us analyze the quasi–static evolution of a two-dimensional elastic cracked body of current domain $\Omega \in R^2$ and unit thickness characterized by a rectilinear crack of length a_0. No traction is applied along the surface of the crack and the body forces are neglected. As shown in Figure 1, the body in the current configuration is subjected to a load $\mathbf{f}(\lambda) = \lambda\hat{\mathbf{f}}$ applied on the boundary S_{f}, where $\lambda > 0$ represents a control parameter and $\hat{\mathbf{f}}$ is a fixed load. During a crack propagation along a given direction and after a load increment $\delta\lambda\,\hat{\mathbf{f}}$, the cracked body evolves into the updated configuration $\Omega_{\delta a}$ with crack length $a = a_0 + \delta a$ and crack length increment $\delta a > 0$.

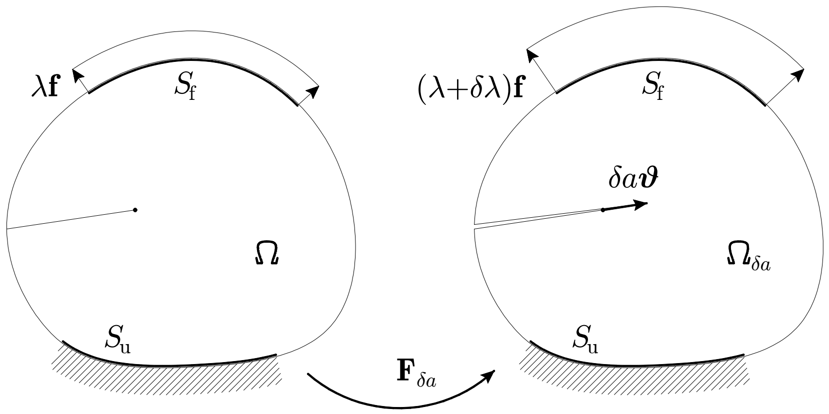

Figure 1: Current (Ω) and perturbed ($\Omega_{\delta a}$) cracked domains; S_{f} = boundary with applied forces; S_{u} = boundary with applied displacements; δa = crack length increment; $\boldsymbol{\vartheta}$ = vector field.

Let us introduce the displacement and stress spaces

$$\mathcal{V} = \{\mathbf{v} \in (H^1(\Omega))^2,\ \mathbf{v} = 0 \text{ on } S_u\} \tag{1}$$

$$\Sigma = \{\boldsymbol{\tau} \in (L^2(\Omega))^4,\ \boldsymbol{\tau}^T = \boldsymbol{\tau}\} \tag{2}$$

The elastic solution $(\mathbf{u}_0, \boldsymbol{\sigma}_0) \in \mathcal{V} \times \Sigma$ for the problem of the current cracked body subjected to the load $\lambda\hat{\mathbf{f}}$ is obtained from the Hellinger-Reissner integral equations

$$\begin{cases} \int_\Omega \boldsymbol{\sigma}_0 : \nabla \mathbf{v} = \lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} & \forall \mathbf{v} \in \mathcal{V} \\ \int_\Omega \mathbf{C}\,\boldsymbol{\sigma}_0 : \boldsymbol{\tau} - \int_\Omega \nabla \mathbf{u}_0 : \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \Sigma \end{cases} \tag{3}$$

where $\mathbf{C}$ represents the compliance tensor and ∇ is the gradient operator. Let us further introduce the spaces

$$\mathcal{V}_{\delta a} = \{\mathbf{v} \in (H^1(\Omega_{\delta a}))^2,\ \mathbf{v} = 0 \text{ on } S_u\} \tag{4}$$

$$\Sigma_{\delta a} = \{\boldsymbol{\tau} \in (L^2(\Omega_{\delta a}))^4,\ \boldsymbol{\tau}^T = \boldsymbol{\tau}\} \tag{5}$$

The elastic solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{V}_{\delta a} \times \Sigma_{\delta a}$ in the updated configuration $\Omega_{\delta a}$ after a load increment $\delta\lambda\,\hat{\mathbf{f}}$, is obtained from the integral equations

$$\begin{cases} \int_{\Omega_{\delta a}} \boldsymbol{\sigma} : \nabla \mathbf{v} = (\lambda + \delta\lambda) \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \mathcal{V}_{\delta a} \\ \\ \int_{\Omega_{\delta a}} \mathbf{C}\,\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_{\Omega_{\delta a}} \nabla \mathbf{u} : \boldsymbol{\tau} = 0 \qquad \forall \boldsymbol{\tau} \in \Sigma_{\delta a} \end{cases} \tag{6}$$

The Destuynder–Djaoua ϑ method consists of a linear perturbation from the current domain Ω into the updated configuration $\Omega_{\delta a}$ (see Figure 1):

$$\mathbf{F}_{\delta a}\mathbf{x} = \mathbf{x} + \delta a\, \boldsymbol{\vartheta}(\mathbf{x})\,, \quad \forall \mathbf{x} \in \Omega \tag{7}$$

where $\boldsymbol{\vartheta}$ represents a smooth vector valued function defined in Ω such that $|\boldsymbol{\vartheta}| = 1$ at the crack tip and $\boldsymbol{\vartheta} = 0$ on $\partial\Omega \backslash S_f$. $\mathbf{F}_{\delta a}$ permits to rewrite each function $\mathbf{q}$ of the updated configuration $\Omega_{\delta a}$ as an associated function $\mathbf{q}_{\delta a}$ of the current configuration Ω:

$$\mathbf{q}_{\delta a} = \mathbf{q} \circ \mathbf{F}_{\delta a} \tag{8}$$

Starting from (8), the following equalities hold:

$$\int_{\Omega_{\delta a}} \mathbf{q} = \int_{\Omega} \mathbf{q}_{\delta a} \mid \mathbf{J}_{\delta a} \mid \tag{9}$$

$$\mathbf{J}_{\delta a} = \nabla \mathbf{F}_{\delta a} = \mathbf{I} + \delta a \nabla \boldsymbol{\vartheta} \tag{10}$$

$$\mid \mathbf{J}_{\delta a} \mid = 1 + \delta a\,(div \boldsymbol{\vartheta}) + \delta a^2 \;\mid \nabla \boldsymbol{\vartheta} \mid \tag{11}$$

$$\nabla \mathbf{q} \circ \mathbf{F}_{\delta a} = \nabla \mathbf{q}_{\delta a}\, \mathbf{J}_{\delta a}^{-1} \tag{12}$$

$$\mathbf{J}_{\delta a}^{-1} = \mathbf{I} - \delta a \nabla \boldsymbol{\vartheta} + \delta a^2 (\nabla \boldsymbol{\vartheta})^2 + ... + (-1)^n \delta a^n (\nabla \boldsymbol{\vartheta})^n + ... \tag{13}$$

where $\mathbf{J}_{\delta a}$ is the Jacobian matrix of transformation (7).

By using formulae (9 - 13), the associated solution $(\boldsymbol{\sigma}_{\delta a}, \mathbf{u}_{\delta a})$ in the configuration Ω is the unique solution in $\mathcal{V} \times \Sigma$ of the problem

$$\begin{cases} \int_{\Omega} \boldsymbol{\sigma}_{\delta a} : (\nabla \mathbf{v}\, \mathbf{J}_{\delta a}^{-1}) \mid \mathbf{J}_{\delta a} \mid = (\lambda + \delta\lambda) \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \mathcal{V} \\ \\ \int_{\Omega} \mathbf{C}\,\boldsymbol{\sigma}_{\delta a} : \boldsymbol{\tau} \mid \mathbf{J}_{\delta a} \mid - \int_{\Omega} (\nabla \mathbf{u}_{\delta a}\, \mathbf{J}_{\delta a}^{-1}) : \boldsymbol{\tau} \mid \mathbf{J}_{\delta a} \mid = 0 \quad \forall \boldsymbol{\tau} \in \Sigma \end{cases} \tag{14}$$

As pointed out in [18], the couple $(\boldsymbol{\sigma}_{\delta a}, \mathbf{u}_{\delta a})$ represents a calculation tool which permits to transfer problem (6) into the current configuration.

Let we approximate the determinant of the Jacobian $\mathbf{J}_{\delta a}$ as

$$\mid \mathbf{J}_{\delta a} \mid \approx 1 + \delta a\,(div \boldsymbol{\vartheta}) \tag{15}$$

such that

$$\mathbf{J}_{\delta a}^{-T} \approx \mathbf{I} - \delta a\,(\nabla \boldsymbol{\vartheta})^T \tag{16}$$

By substituting these expressions into (14), the associated solution $(\mathbf{u}_{\delta a}, \boldsymbol{\sigma}_{\delta a})$ is read as

$$\mathbf{u}_{\delta a} \;=\; \mathbf{u}_0 + \delta \mathbf{u} + R_u \tag{17}$$

$$\boldsymbol{\sigma}_{\delta a} \;=\; \boldsymbol{\sigma}_0 + \delta \boldsymbol{\sigma} + R_\sigma \tag{18}$$

with $(\mathbf{u}_0, \boldsymbol{\sigma}_0)$ solution of (3) and

$$\delta a^{-1}(||R_{\mathbf{u}}||_{\mathcal{V}} + ||R_{\boldsymbol{\sigma}}||_{\Sigma}) \rightarrow 0 \text{ as } \delta a \rightarrow 0 \tag{19}$$

and where the increments $(\delta\mathbf{u}, \delta\boldsymbol{\sigma})$ are the unique solution in $\mathcal{V} \times \Sigma$ of the following problem (see proof in [18] for the case of null load increments):

$$\begin{cases} \int_\Omega \delta\boldsymbol{\sigma} : \nabla\mathbf{v} - \delta a \int_\Omega \mathbf{s}_0 : \nabla\mathbf{v} = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} & \forall \mathbf{v} \in \mathcal{V} \\ \int_\Omega \mathbf{C}\,\delta\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_\Omega \nabla\delta\mathbf{u} : \boldsymbol{\tau} - \delta a \int_\Omega \mathbf{r}_0 : \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \Sigma \end{cases} \tag{20}$$

In (20) we have posed

$$\mathbf{s}_0 = \boldsymbol{\sigma}_0 \nabla\boldsymbol{\vartheta}^T - (div\boldsymbol{\vartheta})\boldsymbol{\sigma}_0 \tag{21}$$

and

$$\mathbf{r}_0 = -\frac{1}{2}(\nabla\mathbf{u}_0\nabla\boldsymbol{\vartheta} + (\nabla\mathbf{u}_0\nabla\boldsymbol{\vartheta})^T) \tag{22}$$

2.1 The Gϑ parameter

Let the strain energy of the system in the updated configuration be:

$$W(\Omega_{\delta a}) = -\frac{1}{2}\int_{\Omega_{\delta a}} \boldsymbol{\sigma} : \nabla\mathbf{u} \tag{23}$$

where $(\mathbf{u}, \boldsymbol{\sigma})$ is the solution of problem (6). As done in [18], let us introduce the derivative of (23) along $\boldsymbol{\vartheta}$ with respect to the current configuration Ω:

$$\frac{\partial W}{\partial\Omega}(\Omega)\boldsymbol{\vartheta} = \lim_{\delta a \rightarrow 0} \frac{W(\Omega_{\delta a}) - W(\Omega)}{\delta a} \tag{24}$$

Destuynder and Djaoua proved that (24) is equivalent to the following expression:

$$\begin{aligned} \frac{\partial W}{\partial\Omega}(\Omega)\boldsymbol{\vartheta} &= -\frac{1}{2}\int_\Omega (div\boldsymbol{\vartheta})\,\boldsymbol{\sigma}_0 : \nabla\mathbf{u}_0 + \int_\Omega \boldsymbol{\sigma}_0 : \nabla\mathbf{u}_0\nabla\boldsymbol{\vartheta} = \\ &= -\frac{1}{2}\int_\Omega \mathbf{s}_0 : \nabla\mathbf{u}_0 + \frac{1}{2}\int_\Omega \mathbf{r}_0 : \boldsymbol{\sigma}_0 \end{aligned} \tag{25}$$

which is the opposite of the so-called $G\vartheta$ parameter (see [19], [21]):

$$G\vartheta = G\vartheta(\mathbf{u}_0, \boldsymbol{\sigma}_0) = \frac{1}{2}\int_\Omega \mathbf{s}_0 : \nabla\mathbf{u}_0 - \frac{1}{2}\int_\Omega \mathbf{r}_0 : \boldsymbol{\sigma}_0 \tag{26}$$

The $G\vartheta$ has the same meaning as the Griffith energy release rate at the onset of crack growth in LEFM and coincides with the Rice J integral for all subdomains $\Omega_\vartheta \in \Omega$ (see [18] and [19]). The same result was found in [22].

In the present work we refer to materials that locally exhibit flat resistance curves (or R-curves), that is, curves R versus crack size, where R represents the crack growth resistance ([3],[2]). The constant value of R is denoted by G_{f} which represents the fracture energy. The Griffith criterion for crack growth in the form proposed by Irwin [6] can be written in terms of the $G\vartheta$:

$$\begin{cases} G\vartheta < G_{\text{f}} \Rightarrow \delta a = 0 & \text{(no propagation)} \\ G\vartheta = G_{\text{f}} \Rightarrow \delta a \geq 0 & \text{(propagation could start)} \end{cases} \tag{27}$$

where G_{f} is also considered as the critical energy release rate.

3 A $G\vartheta$ based incremental crack growth formulation

From a theoretical point of view, the new idea of this work is to use domain transformation (7) for solving a problem of stable crack propagation during a loading process. In particular, the objective of the present study is to determine the curves load multiplier vs. crack length during crack growth. Under a given loading process, if the perturbed configuration at each load step is in equilibrium, as it holds in the case of quasi-static crack growth, it is possible to write the energy release rate during crack propagation. In fact, the $G\vartheta(\mathbf{u}, \boldsymbol{\sigma})$ of the perturbed domain $\Omega_{\delta a}$, taking into account (17) and (18), can be written in the current configuration Ω as

$$\begin{aligned} G\vartheta(\mathbf{u}_0 + \delta\mathbf{u}, \boldsymbol{\sigma}_0 + \delta\boldsymbol{\sigma}) &\approx G\vartheta(\mathbf{u}_0, \boldsymbol{\sigma}_0) + G\vartheta(\delta\mathbf{u}, \boldsymbol{\sigma}_0) + \\ &+ G\vartheta(\mathbf{u}_0, \delta\boldsymbol{\sigma}) + G\vartheta(\delta\mathbf{u}, \delta\boldsymbol{\sigma}) \approx \\ &\approx G\vartheta(\mathbf{u}_0, \boldsymbol{\sigma}_0) + \delta G\vartheta \end{aligned} \tag{28}$$

which, after some manipulations, furnishes the increment $\delta G\vartheta$ in the form:

$$\delta G\vartheta \approx \int_\Omega \mathbf{s}_0 : \nabla\delta\mathbf{u} - \int_\Omega \mathbf{r}_0 : \delta\boldsymbol{\sigma} \tag{29}$$

In the case of stable crack growth (see [2] and [3]), the following conditions must be locally satisfied for each load increment:

$$\left\{ \begin{array}{lcllc} G\vartheta & = & G_f & \Rightarrow & \int_\Omega \mathbf{s}_0 : \nabla\mathbf{u}_0 - \int_\Omega \mathbf{r}_0 : \boldsymbol{\sigma}_0 = 2G_f \\ \\ \delta G\vartheta & = & 0 & \Rightarrow & \int_\Omega \mathbf{s}_0 : \nabla\delta\mathbf{u} - \int_\Omega \mathbf{r}_0 : \delta\boldsymbol{\sigma} = 0 \end{array} \right. \tag{30}$$

By coupling equations (20) and the second condition of (30) we obtain a system in the unknowns $(\delta\mathbf{u}, \delta\boldsymbol{\sigma}, \delta a)$:

$$\left\{ \begin{array}{ll} \int_\Omega \delta\boldsymbol{\sigma} : \nabla\mathbf{v} - \delta a \int_\Omega \mathbf{s}_0 : \nabla\mathbf{v} = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} & \forall \mathbf{v} \in V \\ \\ \int_\Omega \mathbf{C}\,\delta\boldsymbol{\sigma} : \boldsymbol{\tau} - \int_\Omega \boldsymbol{\tau} : \nabla\delta\mathbf{u} - \delta a \int_\Omega \mathbf{r}_0 : \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \Sigma \\ \\ \int_\Omega \mathbf{s}_0 : \nabla\delta\mathbf{u} - \int_\Omega \mathbf{r}_0 : \delta\boldsymbol{\sigma} = 0 & \end{array} \right. \tag{31}$$

It is worth to note that the increments $\delta\mathbf{u}$ and $\delta\boldsymbol{\sigma}$, as well as the associated functions $(\mathbf{u}_{\delta a}, \boldsymbol{\sigma}_{\delta a})$ have to be considered as computational tools for the definition of the crack growth problem in the current configuration. It means that $(\delta\mathbf{u}, \delta\boldsymbol{\sigma})$ don't correspond to physical quantities of the cracked domain $\Omega_{\delta a}$.

For some practical problems, it can be convenient to reduce the unknowns of system (31) to $(\delta\mathbf{u}, \delta a)$ by using the constitutive equations of elasticity. We obtain:

$$\delta\boldsymbol{\sigma} = \mathbf{E}[\boldsymbol{\varepsilon} + \delta a\, \mathbf{r}_0] \tag{32}$$

where $\mathbf{E}$ is the elasticity tensor and $\boldsymbol{\varepsilon}$ represents the strain tensor. After some manipulations, we can rewrite (31) as

$$\begin{cases} \int_\Omega \mathbf{E}\,\boldsymbol{\varepsilon}(\delta\mathbf{u}) \, : \, \boldsymbol{\varepsilon}(\mathbf{v}) - \delta a \int_\Omega \mathbf{t}_0 : \nabla\mathbf{v} = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v} \\ - \int_\Omega \mathbf{t}_0 : \nabla\,\delta\mathbf{u} + \delta a \int_\Omega \mathbf{E}\mathbf{r}_0 : \mathbf{r}_0 = 0 \end{cases} \tag{33}$$

which holds $\forall\mathbf{v} \in V$, $\delta a > 0$, and where we have posed

$$\mathbf{t}_0 = \mathbf{s}_0 - \mathbf{E}\mathbf{r}_0. \tag{34}$$

Therefore, system (33) provides the crack length increment δa relative to a given load increment $\delta\lambda$.

4 Numerical testing

4.1 Computational framework

The proposed approach is implemented using the computational tools of the FEM computer code ELMER (CSC–Scientific Computing Ltd., Espoo, Finland). Successive linear elastic analyses are performed for calculating the quantities required to recover the equilibrium path of the cracked body during crack growth. Quadratic triangular FE are used. During crack propagation, remeshing is performed. The used mesh generator is the program Easy Mesh, available on the web site *www-dinma.univ.trieste.it/nirftc/research/easymesh/*.

Let us describe the framework exploited to perform the crack growth analysis. In particular, we refer to the operator form of system (33):

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & C \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta a \end{bmatrix} = d\lambda \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{35}$$

where $\mathbf{A}$ represents the stiffness matrix of the problem, while $\mathbf{B}$ and C are respectively the vector and the scalar arising from the discretization of the corresponding terms in (33).

Starting from a state defined by the index $i = 0$ where the initial crack length of the body is a_0, the proposed algorithm for crack growth is characterized by successive analyses consisting of the following steps:

1. Linear elastic analysis for the cracked body of domain Ω_i, i.e. solution of equation (3) for the load factor λ_i. In operator form we have:

$$\mathbf{A}_i\mathbf{u}_i = \lambda_i\mathbf{F} \tag{36}$$

2. Evaluation of the crack increment in the updated configuration $\Omega_{\delta a_i}$ through the equation obtained by solving system (35):

$$\delta a_i = \frac{1}{\mathbf{B}_i^T\mathbf{A}_i^{-1}\mathbf{B}_i - C_i}\frac{d\lambda_i}{\lambda_i}\mathbf{B}_i^T\mathbf{u}_i \tag{37}$$

3. Updating of the new current configuration Ω_{i+1} and remeshing.

4. Updating of the load factor:

$$\lambda_{i+1} := \lambda_i + d\lambda_i \tag{38}$$

Note that equation (37) can be used either for evaluating δa_i in the case of assigned $d\lambda_i$ or, conversely, for calculating the load increment relative to a given increment of crack growth.

The solution of equation (36) allows to compute the parameter $G\vartheta(\mathbf{u}_i, \boldsymbol{\sigma}_i)$ in the equilibrium configuration Ω_i. The vector field $\boldsymbol{\vartheta}$, necessary for evaluating $G\vartheta$, represents a smooth vector valued function defined in Ω_i, with values $\boldsymbol{\vartheta} = \mathbf{0}$ on $\partial\Omega_i \setminus S_f$ and $|\boldsymbol{\vartheta}| = 1$ at the crack tip. If the direction of the crack propagation is known, the crack tip moves to the new crack tip in Ω_{i+1}, so that the two points will be spaced of δa along the same direction.

The choice of the ϑ field has the same general meaning as that of the path along which the J integral function is integrated. As suggested in [18], in the implemented code the vector $\boldsymbol{\vartheta}$ is treated as well as a displacement field. In particular, it corresponds to the solution of the same (but homogenous) linear elastic problem with Dirichelet boundary conditions of zero value displacement on S_f and linear displacement distribution along the direction of the crack length (with unitary value at the crack tip). Once the crack increment δa is provided by equation (37), the direction of crack growth is assumed to be the same as that defined by the initial orientation of the crack length. Actually this approach is correct when the crack growth is rectilinear. However, the method can be extended to the curvilinear case either by describing the path as a linear piecewise curve or taking into account the real shape of the crack growth curve as done in [22] (see Section 5).

4.2 Some results

In this section the rectilinear crack propagation in a single edge plate in plane strain subject to a traction at the top (see Figure 2) is analyzed. The example was also tested in [11, 12]. In those papers the authors referred to the analytical solution given by Tada et al. in [13] which provides the mode I Stress Intensity Factor in function of the crack length in the form

$$K_I^a = \frac{\sqrt{2\,w\,\tan(\beta)}}{\cos(\beta)}\left(0.752 + 2.02\,\alpha + 0.37\,(1-\sin(\beta))^3\right) \tag{39}$$

where $\alpha = a/w$ and $\beta = \alpha\pi/2$, a and w being the crack length and the plate width respectively.

The algorithm implemented in the present work performs the crack growth analysis starting from a initial crack length $a = 3.5$. The Stress Intensity Factor K_I is numerically carried out directly from $G\vartheta$ in the case of plane strain state:

$$K_I = \frac{E}{2(1-\nu^2)}G\vartheta \tag{40}$$

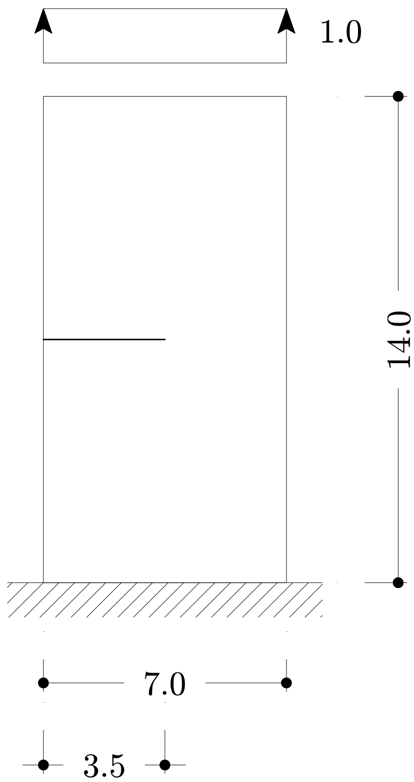

Figure 2: Geometry of the test. Initial crack length: a= 3.5; Elasticity modulus $E = 30 \times 10^6$; Poisson ratio $\nu = 0.25$. SI units.

As shown in Figure 3, the curve stress intensity factor vs. crack length computed by equation (40) is very accurate, in comparison with the analytical provided by equation (39). The algorithm furnishes the curves load parameter vs. crack length (Figure 4) and load parameter vs. displacement norm (Figure 5). In particular, the displacement norm corresponds to an Euclidean measure of the nodal displacements. Finally, in Figure 6 the final configuration corresponding to the last evaluated crack length ($a = 6.0$) is shown. For the same crack configuration, the stress distribution in the plate is reported in Figure 7.

5 Concluding remarks

In this paper a general approach for the evaluation of the increments of quasi–static crack growth during a given loading process in 2D linear elastic bodies with existing cracks is proposed. The theoretical formulation is based on successive one-to-one domain transformations from the current into the updated configuration and can be considered as an extension of the so-called ϑ method. For sake of simplicity, an algorithm for the case of stable rectilinear crack growth is performed but the formulation is very suitable to be extended to general cases of curvilinear crack growth in nonlinear materials.

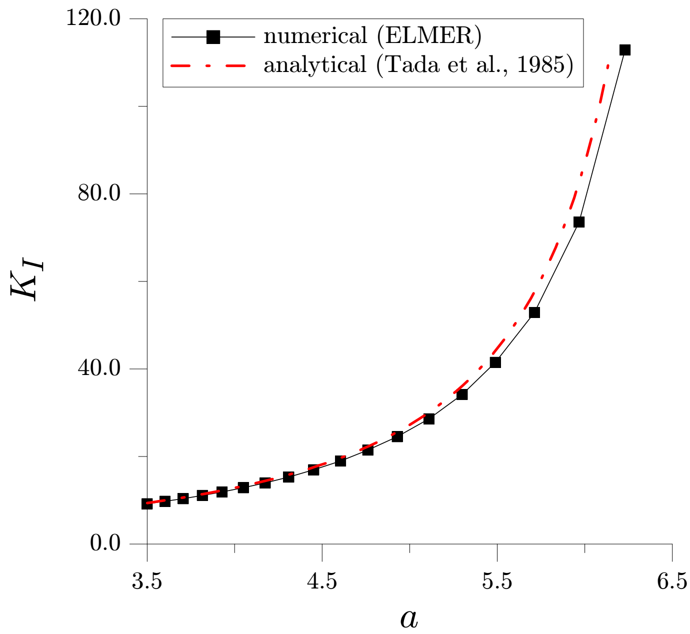

Figure 3: Test of Figure2: comparison between stress intensity factors.

In order to directly extend the method to curvilinear crack growth, the simplest way is to represent the curvilinear path by means of a linear piecewise curve. Then, the described algorithm can be used for determining the crack growth increments along each line of the curve after calculating the direction of crack extension by using one of the several methods existing in literature (e.g. the maximum energy release rate criterion, the maximum circumferential stress criterion, the minimum strain energy density criterion, etc., see references in [16]). A more rigorous approach should take into account the real shape of the path and define the direction of crack growth like a variable of the problem. As mentioned in the introduction, a theoretical formulation for defining an energy release parameter in the cases of curvilinear cracks was presented in [22]. Anyway, also in that work the shape of the path has to be known before calculating the derivative of the functional with respect to the crack length.

The proposed approach is also very suitable to be extended to elastic-plastic materials. In fact, since the idea of the method is to impose the equilibrium in the current configuration at each load step, a complete incremental elastic-plastic analysis can be performed in the current configuration without taking into account the previous plastic deformation history. To obtain the equivalent elastic-plastic of problem (33), systems (14) and (20) have to be modified using the classical equations of elastoplasticity. Furthermore, an elastic-plastic condition of crack growth based on the θ method has to be used. Note that a fracture criterion with an elastic-plastic version of the $G\vartheta$ was already introduced in [19]. In the equivalent elastic-plastic of system (35), matrix $\mathbf{A}$ will represent the tangent matrix.

Finally, with respect to the incremental displacement-crack growth ap-

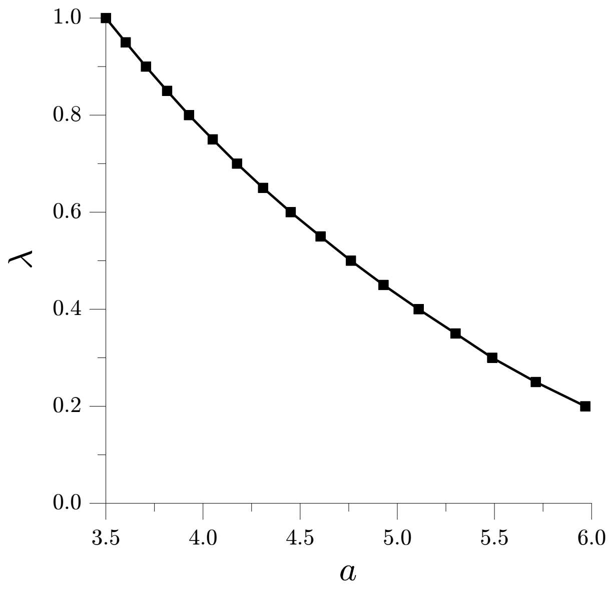

Figure 4: Test of Figure 2: curve load parameter vs. crack length.

proach introduced in [16], the present method is much simpler from a computational point of view and can furnish more accurate curves load vs. crack extension for the general case of curvilinear crack growth because of the strong convergence of the equilibrium problem in the perturbed domain to the equilibrium problem in the current configuration.

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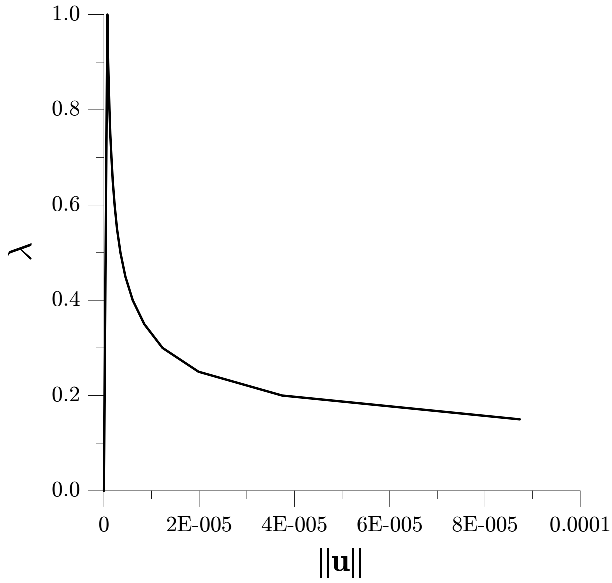

Figure 5: Test of Figure 2: curve load parameter vs. displacement norm (Euclidean norm of the vector $\mathbf{u}$).

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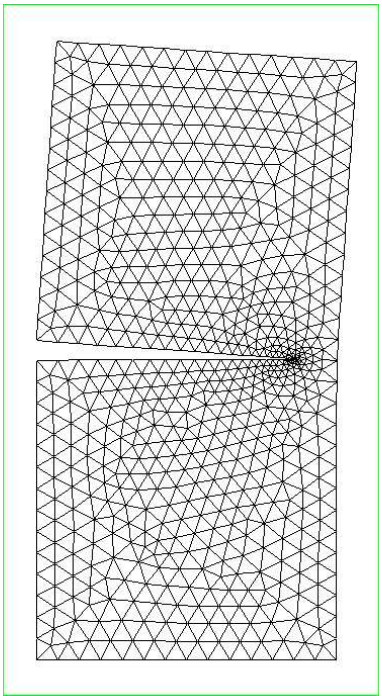

Figure 6: Test of Figure 2: final configuration corresponding to the last evaluated crack length ($a = 6.0$).

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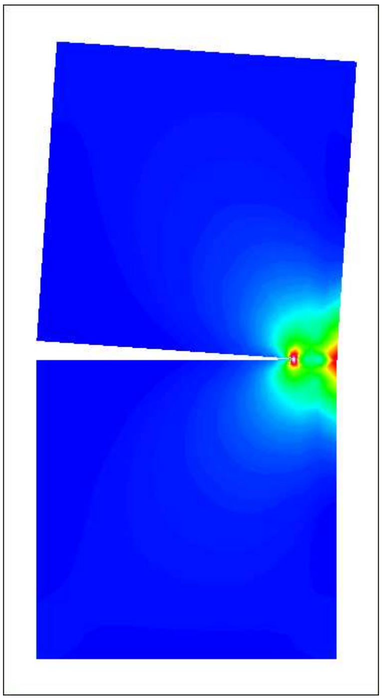

Figure 7: Test of Figure 2: stress distribution, vertical component. Final crack configuration ($a = 6.0$).

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