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Abstract: *Let T be a bounded linear operator in a Banach space, with $\sigma(T) = \{1\}$. In 1990, M. Berkani presented a conjecture on the decay of differences $(I - T)\, T^n$ as $n \to \infty$. More precisely, either*

$$\liminf_{n\to\infty} (n+1) \| (I - T)\, T^n \| \geq 1/e$$

or $T = I$. We prove this claim and discuss some of its consequences.

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1 Introduction

Let $T \in \mathcal{L}(X)$; a bounded linear operator in a (complex) Banach space X. The following result by J. Esterle holds, see [1, Corollary 9.5]:

Proposition 1. *Let $T \in \mathcal{L}(X)$ satisfy $\sigma(T) = \{1\}$. If $T \neq I$ then*

$$\liminf_{n\to\infty} (n+1)\|(I-T)T^n\| \geq \frac{1}{96}.$$

M. Berkani improved the lower bound to 1/12, and he conjectured that the best lower bound is $1/e$, see [6]. That $1/e$ has a special role in related estimates can also be seen in the following remark by O. Nevanlinna, see [7, Theorem 4.5.1]:

Proposition 2. *Assume that there exists $\{\lambda_j\} \subset \sigma(T)$ such that $|\lambda_j| < 1$ and $|\lambda_j| \to 1$ as $j \to \infty$. Then*

$$\limsup_{n\to\infty} (n+1)\|(I-T)T^n\| \geq \frac{1}{e}.$$

The constant $1/e$ appears also in the well-known "continuous time" case [2, Theorem 10.3.6].

In this paper, we show that M. Berkani's and J. Esterle's conjecture is right in the sense that Proposition 1 holds with 1/96 replaced by $1/e$. We use a related but more careful analysis that has already been used in [1], involving the univalent functions $g_n(z) = z(1-z)^n$. We give also another variant of Proposition 2 without restrictions on $\sigma(T)$.

All of these results were first presented in [9] (Z. Yuan, 2002) with somewhat longer proofs. That $1/e$ in Proposition 1 is a valid lower bound, is also proved in [3] (N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, and Y. Tomilov, 2002) by quite different means. Both of the existing approaches can be generalized to a larger class of results, but these respective classes are different (and we shall not discuss these generalizations here). An example is given in [3], indicating that the constant $1/e$ is the best possible. The construction is a modification of an example given in [4] (Lyubich, 2001); see also [5] (O. E. Maasalo, 2003).

2 Estimating $\liminf_{n\to\infty} (n+1)\|(I-T)T^n\|$

Denote $\mathbb{D}(R) := \{z \in \mathbb{C} : |z| < R\}$, and let $g : \mathbb{D}(R) \to \mathbb{C}$ be an analytic function satisfying $g(0) = 0$ and $g'(0) \neq 0$. Then there exists a maximal radius R_u, $0 < R_u \leq R$, such that g is an *univalent* (i.e. an injective analytic) function on the disk $\mathbb{D}(R_u)$. It is then easy to see that the image of $g(\mathbb{D}(R_u))$ contains an open disc, centered at origin. Let $0 < c < \infty$ be the largest radius such that $\mathbb{D}(c) \subset g(\mathbb{D}(R_u))$. Then there exists an analytic function $f : \mathbb{D}(c) \to \mathbb{D}(R_u)$ such that

$$(g \circ f)(z) := g(f(z)) = z \quad \text{for all} \quad z \in \mathbb{D}(c). \tag{1}$$

We denote the spectral radius of $L \in \mathcal{L}(X)$ by $\rho(L)$. If $\rho(L) = 0$, then L is called *quasi-nilpotent*. With these notations, we can prove the following proposition:

Proposition 3. *Let* $g : \mathbb{D}(R) \to \mathbb{C}$ *be an analytic function such that* $g(0) = 0$ *and* $g'(0) \neq 0$*. Let the constants* c *and* R_u *be as above. Then for all* $0 < \eta < 1$

$$\inf \{\|g(L)\| : \ L \in \mathcal{L}(X), \ \rho(L) = 0, \ \|L\| \geq R_u \eta (1-\eta)^{-1}\} \geq \eta c.$$

Proof. The proof is carried out by showing that the set

$$\{L \in \mathcal{L}(X) : \ \rho(L) = 0, \ \|g(L)\| < \eta c, \ \|L\| \geq R_u \eta (1-\eta)^{-1}\}$$

is empty for all $0 < \eta < 1$. This is achieved by using the Cauchy estimates for the function f defined in (1). Denote the power series representations

$$f(z) = \sum_{j \geq 1} a_j z^j \quad \text{and} \quad g(z) = \sum_{j \geq 1} b_j z^j.$$

Clearly $f : \mathbb{D}(c) \to \mathbb{D}(R_u)$ means that $\sup_{|z|<c} |f(z)| \leq R_u$, and then the Cauchy estimates give $|a_j| r^j \leq R_u$ for each $r < c$ and $j \geq 1$. Letting $r \to c-$, we get that $|a_j| c^j \leq R_u$ for all $j \geq 1$.

Let $L \in \mathcal{L}(X)$ be an arbitrary quasi-nilpotent operator. Then $g(L)$ is quasi-nilpotent by the spectral mapping theorem, as $g(0) = 0$. Similarly $Y := f(g(L))$ is also quasi-nilpotent. Let now $0 < \eta < 1$, and assume that $\|g(L)\| < \eta c$. It now follows from the above Cauchy estimates that

$$\|Y\| \leq \sum_{j \geq 1} |a_j| \cdot \|g(L)\|^j < \sum_{j \geq 1} |a_j| c^j \cdot \eta^j \leq R_u \eta (1-\eta)^{-1};$$

hence $\|Y\| < R_u \eta (1-\eta)^{-1}$.

We proceed to show that $Y = L$. Since Y is quasi-nilpotent, $g(Y)$ is well-defined. By the associativity

$$g(Y) = g[f(g(L))] = g(f[g(L)]) = (g \circ f)(g(L)) = g(L)$$

because $(g \circ f)(z) = z$ for any $z \in \mathbb{D}(c)$. As $g(0) = 0$, it follows that $\sigma(g(L)) = \{0\} \subset \mathbb{D}(c)$. Using the power series of g, we get

$$\begin{aligned} 0 = g(Y) - g(L) &= \sum_{j \geq 1} b_j Y^j - \sum_{j \geq 1} b_j L^j \qquad (2) \\ &= (Y - L) \left(b_1 I + \sum_{j \geq 2} b_j \left[Y^{j-1} + Y^{j-2} L + \dots + L^{j-1} \right] \right) \\ &= (Y - L)(b_1 I + U), \end{aligned}$$

where $b_1 = g'(0) \neq 0$ and $U := \sum_{j \geq 2} b_j \left[Y^{j-1} + Y^{j-2} L + \dots + L^{j-1} \right]$.

We know that $Y = f(g(L))$ is quasi-nilpotent, and it is actually a function of L. We now consider function h defined in $\mathbb{D}(R_u)$ as follows

$$h(z) := \sum_{j \geq 2} b_j \left[f(g(z))^{j-1} + f(g(z))^{j-2} z + \dots + z^{j-1} \right].$$

Then $h(z)$ is analytic in $\mathbb{D}(R_u)$ and $h(0) = 0$. So $h(L)$ is well-defined and $U = h(L)$. Since both L and Y are quasi-nilpotent, we see that U is quasi-nilpotent. Therefore $b_1 I + U$ is boundedly invertible. This together with

(2) implies that $Y = L$. Hence for any $0 < \eta < 1$ and any quasi-nilpotent $L \in \mathcal{L}(X)$

$$\|g(L)\| < \eta c \quad \Rightarrow \quad \|L\| = \|Y\| < R_u \eta (1-\eta)^{-1}.$$

This proves the claim. □

A somewhat analogous result to the previous proposition is [3, Theorem 4.5]. We proceed to study the functions

$$g_n(z) := (1-z)^n z \quad \text{for } n \geq 1 \tag{3}$$

that made their appearance also in J. Esterle's original argument. We shall make use of the constants $R_u^{(n)}$ and $c^{(n)}$ defined as follows:

(i) $R_u^{(n)} > 0$ is the largest radius of an open disc $\mathbb{D}(R_u^{(n)})$ such that $g_n(z)$ is univalent in $\mathbb{D}(R_u^{(n)})$.

(ii) $c^{(n)} > 0$ is the largest radius of an open disc $\mathbb{D}(c^{(n)})$ such that

$$\mathbb{D}(c^{(n)}) \subset g_n(\mathbb{D}\left(0, R_u^{(n)}\right)).$$

Because $g_n'(z) = (1-z)^{n-1}(1-(n+1)z)$ and hence $g_n'(1/(n+1)) = 0$, it follows by elementary function theory that $R_u^{(n)} \leq 1/(n+1)$. In essence, the proof of Theorem 1 reduces to showing that equality holds here. For this, we shall provide two different proofs. In the first proof, we shall use (with some modifications) the *positive real univalence criterion*, see e.g. [8, p. 16]:

Lemma 1. *Suppose Ω is a convex region, $f \in H(\Omega)$, and $\Re f'(z) > 0$ for all $z \in \Omega$. Then f is univalent in Ω.*

It clearly follows for any angle $\gamma \in [0, 2\pi)$ that if $\Re[e^{i\gamma} f'(z)] \neq 0$ for all $z \in \Omega$, then f is univalent in Ω. In particular it follows that $\Im f'(z) \neq 0$ for all $z \in \Omega$ implies that f is univalent in Ω.

Let the function $g_n(z)$ be defined by (3). Define $\mathbb{D}_+(0, 1/(n+1)) := \{z \in \mathbb{D}(1/(n+1)) : \Im z > 0\}$ and $\mathbb{D}_-(0, 1/(n+1)) := \{z \in \mathbb{D}(1/(n+1)) : \Im z < 0\}$. We take an arbitrary $z \in \mathbb{D}(1/(n+1))$ and write it as $z = re^{i\theta} = a + bi$, where $\theta \in (-\pi, \pi]$. Then

$$1 - z = 1 - r\cos\theta - ir\sin\theta = |1-z|e^{i\alpha}$$

for some $\alpha \in (-\pi, \pi]$. Since $z \in \mathbb{D}(1/(n+1))$, we have the estimate

$$|\alpha| < |\tan\alpha| = \frac{|b|}{|1-a|} < \frac{1/(n+1)}{1-1/(n+1)} = \frac{1}{n}. \tag{4}$$

On the other hand, we get by a direct computation

$$\begin{aligned}\Im g_n'(z) &= \Im\big(|1-z|^{n-1} e^{i(n-1)\alpha}(1-(n+1)a-(n+1)bi)\big) \\ &= |1-z|^{n-1}\left(-(n+1)b\cos(n-1)\alpha + (1-(n+1)a)\sin(n-1)\alpha\right).\end{aligned}$$

Assume now that $z \in \mathbb{D}_+(0, 1/(n+1))$; i.e. $b > 0$ and $0 < \theta < \pi$. Then we have $-1 < n\alpha < 0$ by (4) and some geometric reasoning, and moreover

$$-\pi/2 < -1 - \alpha < (n-1)\alpha < -\alpha < 0.$$

It now follows immediately that $-(n+1)b\cos{(n-1)\alpha} < 0$ and $(1-(n+1)a)\sin{(n-1)\alpha} < 0$, and hence $\Im g_n'(z) < 0$. By a similar argument, $\Im g_n'(z) > 0$ for all $z \in \mathbb{D}_-(0, 1/(n+1))$. By Lemma 1, the function g_n is univalent in both $\mathbb{D}_+(0, 1/(n+1))$ and $\mathbb{D}_-(0, 1/(n+1))$. A more refined analysis is required to prove the following proposition:

Proposition 4. *The functions $g_n(z) = (1-z)^n z$ are univalent in the disc $\mathbb{D}(1/(n+1))$ for all $n \geq 1$.*

Proof. Fix $n \geq 1$. If the claim did not hold for this n, then there would exist $z_1, z_2 \in \mathbb{D}(1/(n+1))$, such that $g_n(z_1) = g_n(z_2)$ but $z_1 \neq z_2$. From the preceding discussion, both z_1 and z_2 cannot be in the same half disc $\mathbb{D}_+(0, 1/(n+1))$ or $\mathbb{D}_-(0, 1/(n+1))$. Then z_1 and z_2 would have to satisfy (without loss of generality) one of the following conditions:

(i) $z_1 \in \mathbb{D}_+(0, 1/(n+1))$ and $z_2 \in \mathbb{D}_-(0, 1/(n+1))$, or

(ii) z_1 is a pure real number satisfying $-1/(n+1) < z_1 < 1/(n+1)$.

To show that (i) cannot hold, we write $z_1 = r_1 e^{i\theta_1} = a_1 + b_1 i$, $z_2 = r_2 e^{i\theta_2} = a_2 + b_2 i$, where θ_1 and θ_2 are angles satisfying $0 < \theta_1 < \pi$ and $-\pi < \theta_2 < 0$. Then we have by (4) and geometric reasoning

$$\begin{aligned} 1 - z_1 &= |1 - z_1| e^{i\alpha_1} \text{ where } -1 < n\alpha_1 < 0, \text{ and} \\ 1 - z_2 &= |1 - z_2| e^{i\alpha_2} \text{ where } 1 > n\alpha_2 > 0. \end{aligned}$$

Since $z_1 \in \mathbb{D}_+(0, 1/(n+1))$, we have $b_1 > 0$ and

$$\begin{aligned} \sin|\alpha_1| &= \frac{b_1}{|1-z_1|} = \frac{b_1}{r_1}\frac{r_1}{|1-z_1|} = \sin\theta_1 \frac{r_1}{|1-z_1|} \\ &< \sin\theta_1 \frac{1/(n+1)}{1 - 1/(n+1)} = \frac{1}{n}\sin\theta_1. \end{aligned}$$

Therefore, $n \sin|\alpha_1| < \sin\theta_1$. Because the function $h(x) := n\sin x - \sin nx$ for $0 < x < 1/n$ has a positive derivative, we get $\sin n|\alpha_1| < n\sin|\alpha_1| < \sin\theta_1$. As $0 < -n\alpha_1 < 1$ and sin is increasing in $[0, \pi/2]$, it follows that $0 < -n\alpha_1 < \theta_1$ if $0 < \theta_1 \leq \pi/2$. On the other hand, if $\pi/2 < \theta_1 < \pi$ then trivially $0 < -n\alpha_1 < 1 < \pi/2 < \theta_1$. We conclude that the estimate $0 < -n\alpha_1 < \theta_1$ holds always, and hence

$$0 < n\alpha_1 + \theta_1 < \theta_1 < \pi. \tag{5}$$

Similarly, for $z_2 \in \mathbb{D}_-(0, 1/(n+1))$, we get

$$0 < -n\alpha_2 - \theta_2 < \pi, \tag{6}$$

and adding up (5) and (6) gives

$$0 < (n\alpha_1 + \theta_1) - (n\alpha_2 + \theta_2) < 2\pi. \tag{7}$$

For contradiction, let us assume that $g_n(z_1) = g_n(z_2)$. Then

$$|1-z_1|^n e^{in\alpha_1} r_1 e^{i\theta_1} = |1-z_2|^n e^{in\alpha_2} r_2 e^{i\theta_2},$$

and the angles would satisfy for some integer k

$$n\alpha_1 + \theta_1 = n\alpha_2 + \theta_2 + 2k\pi. \tag{8}$$

This contradicts with inequality (7), and case (i) has therefore been excluded.

Suppose now that case (ii) holds. Then $n\alpha_1 = 0$, and $\theta_1 = 0$, or π. For contradiction, assume again that $g_n(z_1) = g_n(z_2)$ which leads to equality (8). But (8) implies now $n\alpha_2 + \theta_2 = k\pi$ for some $k \in \mathbb{Z}$. But by inequalities (5) and (6), we get $0 < |n\alpha_2 + \theta_2| < \pi$ if $z_2 \in \mathbb{D}_+(0, 1/(n+1)) \cup \mathbb{D}_-(0, 1/(n+1))$. Thus z_2 is also a real number.

Since $g_n(z_1) = g_n(z_2)$ for real $-1/(n+1) < z_1, z_2 < 1/(n+1)$ implies trivially $z_1 = z_2$, the proof is now complete. □

Now comes the other, shorter proof for Proposition 4:

Proof. Let $z = re^{i\phi} \in \mathbb{C}$, where $0 \le r < 1/(n+1)$ and $\phi \in \mathbb{R}$. Now

$$g_n(z) \quad = \quad R(r,\phi)\; e^{i\Phi(r,\phi)},$$

where $r_\phi = \sqrt{1 - 2r\cos(\phi) + r^2}$, $\Phi(r,\phi) = \phi - n \arcsin(r\sin(\phi)/r_\phi)$ and $R(r,\phi) = r \cdot r_\phi^n$; note that $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ is the inverse function of $\sin : [-\pi/2, \pi/2] \to [-1,1]$. Mapping $\phi \mapsto \Phi(r,\phi)$ is injective on $\mathbb{R}$, because by writing $t = \cos(\phi)$,

$$\begin{aligned}
\frac{\partial \Phi(r,\phi)}{\partial \phi} &= \left(1 - (n+2)rt + (n+1)r^2\right)\left(1 - 2rt + r^2\right)^{-1} \\
&\ge \left(1 - (n+2)r + (n+1)r^2\right)\left(1 - 2rt + r^2\right)^{-1} \\
&= (1-r)\left(1 - (n+1)r\right)\left(1 - 2rt + r^2\right)^{-1} > 0,
\end{aligned}$$

where the last estimate follows as $r < 1/(n+1)$. Notice furthermore that $\Phi(r, 2\pi k) = 2\pi k$ for every $k \in \mathbb{Z}$. Moreover, if ϕ is fixed then

$$\frac{\partial R(r,\phi)}{\partial r} = \frac{\partial \Phi(r,\phi)}{\partial \phi}\left(1 - 2rt + r^2\right)^{n/2} > 0.$$

Hence $r \mapsto R(r,\phi)$ is injective on $[0, 1/(n+1))$, and the claim follows. □

In other words, we have now proved that $R_u^{(n)} = 1/(n+1)$ for all $n \ge 1$. The other sequence of constants can be determined easily:

Proposition 5. *The constants $c^{(n)}$ (as introduced earlier) satisfy*

$$c^{(n)} = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n \quad \textit{for all } n \ge 1.$$

Proof. Clearly for any fixed n,

$$c^{(n)} = \inf_{z \in \partial \mathbb{D}(R_u^{(n)})} |g_n(z)|.$$

Since $|(1-z)^n z| \geq (1-R_u^{(n)})^n R_u^{(n)}$ for all z satisfying $|z| = R_u^{(n)}$, we get $c^{(n)} \geq \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^n$ as $R_u^{(n)} = 1/(n+1)$ by Proposition 4. By choosing $z = R_u^{(n)}$, we see that even the equality holds. □

Now we are prepared to prove our main result. The required improvement of Proposition 1 follows by taking $L = I - T$ in the following theorem.

Theorem 1. *Let $L \in \mathcal{L}(X)$, $L \neq 0$, be quasi-nilpotent. Then*

$$\liminf_{n\to\infty} (n+1)\|(I-L)^n L\| \geq \frac{1}{e}.$$

Proof. Define the functions g_n and the constants $R_u^{(n)}$, $c^{(n)}$ as earlier. Let $0 < \eta < 1$ be arbitrary. Since by Proposition 4

$$R_u^{(n)}\eta(1-\eta)^{-1} = \frac{1}{n+1}\cdot \eta(1-\eta)^{-1} \to 0 \text{ as } n \to \infty,$$

there exists $N(\eta) < \infty$, such that for all $n \geq N(\eta)$ we have

$$\|L\| \geq R_u^{(n)}\eta(1-\eta)^{-1}.$$

By Proposition 3 (with $g = g_n$) and Proposition 5, we have for all $n \geq N(\eta)$,

$$\|(I-L)^n L\| \geq \eta c^{(n)} = \eta \frac{1}{n+1}\left(1-\frac{1}{n+1}\right)^{n+1}.$$

Since $\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n+1} = 1/e$, we get by letting $n \to \infty$

$$\liminf_{n\to\infty} (n+1)\|(I-L)^n L\| \geq \eta/e.$$

Because $0 < \eta < 1$ is arbitrary, the claim follows by letting $\eta \to 1$. □

3 Estimating $\limsup_{n\to\infty} (n+1)\|(I-T)T^n\|$

Theorem 2. *For any $T \in \mathcal{L}(X)$ either*

(i) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| \geq 1/e$ *or*

(ii) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| = 0$ *holds.*

Proof. If $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| = \infty$ or $T = I$, then the claim holds. It remains to consider the case when $\sup_{n\geq 0}(n+1)\|(I-T)T^n\| < \infty$ and $T \neq I$. By [7, Theorem 4.2.2], $\sigma(T) \subset \mathbb{D}(1) \cup \{1\}$.

If $1 \notin \sigma(T)$, then $\|T^n\| \leq Mr^n$ for some $0 \leq r < 1$ and (ii) follows. If 1 is an accumulation point of $\sigma(T)$, then (i) holds by Proposition 2. If 1 is an isolated point, then either $\sigma(T) = \{1\}$ or there is a positive distance between 1 and $\sigma(T) \setminus \{1\}$. If $\sigma(T) = \{1\}$, then (i) holds by Theorem 1.

To complete the proof, we can assume $\operatorname{dist}(1, \sigma(T) \setminus \{1\}) > 0$. There exist closed, nonintersecting curves Γ_1 and Γ_2 with following properties: Γ_1 lies strictly inside the open unit disc $\mathbb{D}(1)$ and it surrounds the set $\sigma(T) \setminus \{1\}$; Γ_2 surrounds point 1. Define the bounded spectral projections P_1 and P_2, together with the corresponding closed subspaces

$$P_1 := \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda - T)^{-1} d\lambda, \quad P_2 := \frac{1}{2\pi i} \int_{\Gamma_2} (\lambda - T)^{-1} d\lambda,$$
$$X_1 := P_1 X \quad \text{and} \quad X_2 := P_2 X.$$

Both X_1 and X_2 are invariant for T, $X_1 \cap X_2 = \{0\}$ and $X = X_1 + X_2$. They inherit their norms from X, and X itself is isometrically isomorphic to the exterior direct sum $\begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix}$, equipped with the norm

$$\| \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \|_{X_1 \times X_2} := \|x_1 + x_2\| \quad \text{for all} \quad x_1 \in X_1, \quad x_2 \in X_2.$$

Define the bounded operators L and M by $L := T|X_1 \in \mathcal{L}(X_1)$ and $M := T|X_2 \in \mathcal{L}(X_2)$. Then T is isometrically equivalent to the block matrix $\left[\begin{smallmatrix} L & 0 \\ 0 & M \end{smallmatrix}\right] : \begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix} \to \begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix}$, and $(I - T)T^n$ is represented (apart from an isometric isomorphism) by $\left[\begin{smallmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & (I_{X_2} - M)M^n \end{smallmatrix}\right]$. By the triangle inequality

$$\begin{aligned} \|(I - T)T^n\| &= \left\| \left[\begin{smallmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & (I_{X_2} - M)M^n \end{smallmatrix}\right] \right\|_{\mathcal{L}(X_1 \times X_2)} \\ &\geq \left\| \left[\begin{smallmatrix} 0 & 0 \\ 0 & (I_{X_2} - M)M^n \end{smallmatrix}\right] \right\|_{\mathcal{L}(X_1 \times X_2)} - \left\| \left[\begin{smallmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & 0 \end{smallmatrix}\right] \right\|_{\mathcal{L}(X_1 \times X_2)} \\ &= \|(I_{X_2} - M)M^n\|_{\mathcal{L}(X_2)} - \|(I_{X_1} - L)L^n\|_{\mathcal{L}(X_1)}. \end{aligned} \tag{9}$$

The spectra of L and M satisfy $\sigma(L) = \sigma(T) \setminus \{1\} \subset \mathbb{D}(1)$ and $\sigma(M) = \{1\}$. It follows again immediately that $\lim_{n \to \infty} (n+1) \|(I_{X_1} - L)L^n\|_{\mathcal{L}(X_1)} = 0$. By Theorem 1

$$\limsup_{n \to \infty} (n + 1) \|(I_{X_2} - M)M^n\|_{\mathcal{L}(X_2)} \geq 1/e.$$

Therefore (9) implies

$$\limsup_{n \to \infty} (n + 1) \|T^n(T - 1)\| \geq \limsup_{n \to \infty} (n + 1) \|(I_{X_2} - M)M^n\|_{\mathcal{L}(X_2)} \geq 1/e,$$

and the proof is completed. □

The lower bound $1/e$ in Theorem 2 can be reached, see [7, Example 4.5.2].

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