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PIECEWISE POLYNOMIAL COLLOCATION METHODS FOR LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS

Hermann Brunner Arvet Pedas Gennadi Vainikko

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Abstract: *The smoothness of the solution of a linear weakly singular Volterra integro-differential equation is studied. Two piecewise polynomial collocation methods are discussed to solve such equations. Global convergence estimates have been derived and the superconvergence effect at collocation points has been analyzed.*

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Hermann Brunner Department of Mathematics & Statistics, Memorial University of Newfoundland, St. John's, Nfld., Canada A1C 5S7, hbrunner@morgan.ucs.mun.ca

Arvet Pedas Department of Applied Mathematics, University of Tartu, Liivi 2–206, Tartu 50409, Estonia, Arvet.Pedas@ut.ee

Gennadi Vainikko Institute of Mathematics, Helsinki University of Technology, P.O.Box 1100, FIN–02015 HUT, Finland, Gennadi.Vainikko@hut.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:*math@hut.fi* downloadables:*http://www.math.hut.fi/*

1. INTRODUCTION

We study approximate solution of initial-value problem for a linear integro-differential equation,

$$y'(t) = a(t)y(t) + b(t) + \int_0^t K(t,s)y(s)ds, \quad t \in [0,T], \ T > 0, \tag{1.1}$$

$$y(0) = y_0, \quad y_0 \in \mathbb{C}, \tag{1.2}$$

by piecewise polynomial collocation method. Our aim is to construct approximations which possess a maximal convergence order on the interval $[0,T]$. We assume that $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. Here $C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$ is defined as collection of all m times continuously differentiable functions $x: \ (0,T] \to \mathbb{C}$ such that the estimation

$$|x^{(k)}(t)| \le c_k \begin{cases} 1 & \text{if} \quad k < 1-\nu, \\ 1 + |\log t| & \text{if} \quad k = 1-\nu, \\ t^{1-\nu-k} & \text{if} \quad k > 1-\nu \end{cases} \tag{1.3}$$

holds with a constant $c_k = c_k(x)$ for all $t \in (0,T]$ and $k = 0, 1, \dots, m$. The set $\mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, with

$$\Delta_T = \{(t,s) \in \mathbb{R}^2 : \ 0 \le t \le T, \ 0 \le s < t\},$$

consists of m times continuously differentiable functions $K\colon \ \Delta_T \to \mathbb{C}$ satisfying

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s) \right| \le c \begin{cases} 1 & \text{if} \quad \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if} \quad \nu + i > 0, \end{cases} \tag{1.4}$$

with a constant $c = c(K)$ for all $(t,s) \in \Delta_T$ and all non-negative integers i and j such that $i + j \le m$.

We remark that the asymmetry of (1.4) with respect t and s is only apparent. Actually, using the equality $\partial/\partial s = (\partial/\partial t + \partial/\partial s) - \partial/\partial t$ we can deduce from (1.4) the estimations

$$\left| \left(\frac{\partial}{\partial s}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s) \right| \le c \begin{cases} 1 & \text{if} \quad \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if} \quad \nu + i > 0, \end{cases} \tag{1.4'}$$

with $(t,s) \in \Delta_T$ and $i + j \le m$.

It follows from (1.4) (with $i = j = 0$, $0 \le \nu < 1$) that the kernel $K(t,s)$ of equation (1.1) may possess a weak singularity as $s \to t$. In the case $\nu < 0$, the kernel $K(t,s)$ is bounded on Δ_T, but its derivatives may be singular as $s \to t$. Often the kernel K of equation (1.1) has the form

$$K_\nu(t,s) = \kappa(t,s)(t-s)^{-\nu}, \quad 0 < \nu < 1, \tag{1.5}$$

or

$$K_0(t,s) = \kappa(t,s)\log(t-s), \tag{1.6}$$

where $\kappa \in C^m(\overline{\Delta}_T)$ with $\overline{\Delta}_T = \{(t,s) \in \mathbb{R}^2 : \ 0 \le s \le t \le T\}$. Clearly, $K_\nu \in \mathcal{W}^{m,\nu}(\Delta_T)$ and $K_0 \in \mathcal{W}^{m,0}(\Delta_T)$. This remains true even if the derivatives of $\kappa(t,s)$ have certain singularities at $t = s$; we do not go into details here.

In sequel we consider two equivalent reformulations of problem $\{(1.1),(1.2)\}$. The first one is based on the change of unknown function,

$$y' = z. \tag{1.7}$$

Using (1.2), equation (1.1) may be rewritten as a linear Volterra integral equation of the second kind with respect to z:

$$\begin{aligned} z(t) &= b(t) + y_0 a(t) + a(t)\int_0^t z(s)ds + \\ &+ \int_0^t K(t,s)\big[y_0 + \int_0^s z(\tau)d\tau\big]ds, \quad t \in [0,T], \end{aligned} \tag{1.8}$$

or, due to Dirichlet formula

$$\int_0^t \int_0^\tau \Phi(\tau,s)dsd\tau = \int_0^t \int_s^t \Phi(\tau,s)d\tau ds, \ 0 \le s \le \tau \le t, \tag{1.9}$$

in the form

$$z(t) = f_1(t) + \int_0^t K_1(t,s)z(s)ds, \quad t \in [0,T], \tag{1.10}$$

where

$$f_1(t) = b(t) + y_0 a(t) + y_0 \int_0^t K(t,s)ds, \quad t \in [0,T], \tag{1.11}$$

$$K_1(t,s) = a(t) + \int_s^t K(t,\tau)d\tau, \quad 0 \le s < t \le T. \tag{1.12}$$

The second reformulation is based on the integration both sides of (1.1) over $(0,t)$. Using (1.2) and the Dirichlet formula (1.9), equation (1.1) may be rewritten as a linear Volterra integral equation of the second kind with respect to y:

$$y(t) = f_2(t) + \int_0^t K_2(t,s)y(s)ds, \quad t \in [0,T], \tag{1.13}$$

where

$$f_2(t) = y_0 + \int_0^t b(s)ds, \quad t \in [0,T], \tag{1.14}$$

$$K_2(t,s) = a(s) + \int_s^t K(\tau,s)d\tau, \quad 0 \le s < t \le T. \tag{1.15}$$

Piecewise polynomial collocation methods for Fredholm and Volterra integral equations have been extensively examined by many authors. We refer here to monographs [1, 2, 8, 11, 13, 26, 28] and the literature given there; see also [9, 19, 23]. In this paper, we apply results of [9] to integral equations (1.8) and (1.13) obtained from the Cauchy problem $\{(1.1), (1.2)\}$. Using special graded grids, we derive global convergence estimates and analyze a superconvergence effect at the collocation points. The main results of the paper extend known ones (see [3–8, 14, 15, 24, 29] and references in these works) and are formulated in Theorems 2.1 and 4.1–4.6. Of course, our analysis needs a smoothness result for the solution of the Cauchy problem $\{(1.1), (1.2)\}$. This is given in Theorem 2.1 which is based on [25] and the integral equation reformulation of $\{(1.1), (1.2)\}$ proving the compactness of the underlying integral operator in $C^{m,\nu}(0,T]$. Similar results for integral equations see also in [25–28, 8–12, 16, 18, 20–22].

2. SMOOTHNESS OF THE SOLUTION

In order to study regularity properties of the solution of problem $\{(1,1), (1,2)\}$ we first establish some auxiliary results.

For $\lambda \in \mathbb{R}$ introduce the weight functions $w_\lambda(t)$, $t \in (0,T]$, by

$$w_\lambda(t) = \begin{cases} 1 & \text{if} \quad \lambda < 0, \\ (1 + |\log t|)^{-1} & \text{if} \quad \lambda = 0, \\ t^\lambda & \text{if} \quad \lambda > 0. \end{cases} \tag{2.1}$$

We redefine the space $C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$ as the collection of all m times continuously differentiable functions $x\colon (0,T] \to \mathbb{C}$ such that

$$||x||_{m,\nu} := \sum_{k=0}^{m} \sup_{0<t\le T} \left(w_{k-(1-\nu)}(t) \left| x^{(k)}(t) \right| \right) < \infty. \tag{2.2}$$

In other words, an m times continuously differentiable function x on $(0,T]$ belongs to $C^{m,\nu}(0,T]$ if the growth of its derivatives can be estimated by (1.3). Actually, for $x \in C^{m,\nu}(0,T]$, we have

$$||x||_{m,\nu} = \sum_{k=0}^{m} c_k^0, \tag{2.3}$$

where c_k^0 $(k = 0, \dots, m)$ is the smallest value of c_k for which (1.3) yet holds. Equipped with the norm $||\cdot||_{m,\nu}$, $C^{m,\nu}(0,T]$ is complete (is a Banach space).

Notice also that $C^m[0,T] \subset C^{m,\nu}(0,T]$. On the other hand, a function $x \in C^{m,\nu}(0,T]$ can be extended up to a continuous function on $[0,T]$.

Lemma 2.1. *If $x_1, x_2 \in C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $\nu < 1$, then $x_1 x_2 \in C^{m,\nu}(0,T]$, and*

$$||x_1 x_2||_{m,\nu} \le c||x_1||_{m,\nu}||x_2||_{m,\nu} \tag{2.4}$$

with a constant c which is independent of x_1 and x_2.

Proof. Let k be a non-negative integer not exceeding m. Since $x_1, x_2 \in C^{m,\nu}(0,T]$, we have for all $t \in (0,T]$ and $j = 0, 1, \dots, k$ that

$$\left|x_1^{(j)}(t)\right| \le ||x_1||_{m,\nu} \begin{cases} 1 & \text{if} \quad j < 1-\nu, \\ 1+|\log t| & \text{if} \quad j = 1-\nu, \\ t^{1-\nu-j} & \text{if} \quad j > 1-\nu, \end{cases}$$

$$\left|x_2^{(k-j)}(t)\right| \le ||x_2||_{m,\nu} \begin{cases} 1 & \text{if} \quad k-j < 1-\nu, \\ 1+|\log t| & \text{if} \quad k-j = 1-\nu, \\ t^{1-\nu-(k-j)} & \text{if} \quad k-j > 1-\nu. \end{cases}$$

Combining these estimates we obtain

$$\left|x_1^{(j)}(t)x_2^{(k-j)}(t)\right| \le c_{jk}||x_1||_{m,\nu}||x_2||_{m,\nu} \begin{cases} 1 & \text{if} \quad k < 1-\nu, \\ 1+|\log t| & \text{if} \quad k = 1-\nu, \\ t^{1-\nu-k} & \text{if} \quad k > 1-\nu, \end{cases} \tag{2.5}$$

with some positive constants c_{jk}, independent of $t \in (0,T]$. Using (2.5) and the Leibnitz rule

$$(x_1 x_2)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} x_1^{(j)} x_2^{(k-j)},$$

we obtain that the estimate

$$\left|(x_1 x_2)^{(k)}(t)\right| \le c_k||x_1||_{m,\nu}||x_2||_{m,\nu} \begin{cases} 1 & \text{if} \quad k < 1-\nu, \\ 1+|\log t| & \text{if} \quad k = 1-\nu, \\ t^{1-\nu-k} & \text{if} \quad k > 1-\nu, \end{cases}$$

holds for $k = 0, 1, \dots, m$ with some constant c_k for all $t \in (0,T]$. Thus, $x_1 x_2 \in C^{m,\nu}(0,T]$ and the estimation (2.4) holds. □

Lemma 2.2. *Let $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. Then the operator S defined by*

$$(Sx)(t) = \int_0^t K(t,s)x(s)ds, \quad t \in [0,T], \tag{2.6}$$

is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$. Moreover, S is compact as an operator from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$.

Proof. Since $K \in \mathcal{W}^{m,\nu}(\Delta_T)$ is at most weakly singular (see (1.4) with $i = j = 0$), the first statement is well known. The second statement follows from a similar assertion for Fredholm integral operators. A function $x \in C^{m,\nu}(0,T]$ can be extended up to a function $\tilde{x} \in C^{m,\nu}(0,2T]$, cf. formula (4.1) in [9]. Analogously, the kernel $K(t,s)$ can be extended up to a kernel $\tilde{K}(t,s)$ (cf. formula (4.4) in [9]) which is m times continuously differentiable with respect to $t, s \in [0,2T]$, $t \neq s$, and satisfies

$$(2.7) \qquad \Big|\Big(\frac{\partial}{\partial t}\Big)^i \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big)^j \tilde{K}(t,s)\Big| \le c \begin{cases} 1 & \text{if } \ \nu + i < 0, \\ 1 + |\log|t-s|| & \text{if } \ \nu + i = 0, \\ |t-s|^{-\nu-i} & \text{if } \ \nu + i > 0, \end{cases}$$

with a constant $c = c(\tilde{K})$ for all $t, s \in [0,2T]$, $t \neq s$, and all non-negative integers i and j such that $i + j \le m$. Then (see [25]) operator $\tilde{S}$, defined by

$$(2.8) \qquad (\tilde{S}\tilde{x})(t) = \int_0^{2T} \tilde{K}(t,s)\tilde{x}(s)ds, \quad t \in (0,2T),$$

is compact as an operator from $C^{m,\nu}(0,2T)$ to $C^{m,\nu}(0,2T)$. Here (cf. formula (3.2) in [9]) $C^{m,\nu}(0,2T)$ is defined as collection of all m times continuously differentiable functions $\tilde{x}\colon\ (0,2T) \to \mathbb{C}$ such that the estimation

$$|\tilde{x}^{(k)}(t)| \le c_k \begin{cases} 1 & \text{if } \ k < 1-\nu, \\ 1 + |\log \rho(t)| & \text{if } \ k = 1-\nu, \\ \rho(t)^{1-\nu-k} & \text{if } \ k > 1-\nu, \end{cases}$$

holds with a constant $c_k = c_k(\tilde{x})$ for all $t \in (0,2T)$ and $k = 0,1,\dots,m$, where $\rho(t) = \min\{t, 2T-t\}$. Restricting $\tilde{x}$, $\tilde{K}$ and $\tilde{S}$ respectively to x, K and S, we obtain the statement of Lemma 2.2. □

Lemma 2.3. *Let $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$. Then for $0 \le s < t \le T$ and $j = 0,1,\dots,m$,*

$$(2.9) \qquad \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big)^j \int_s^t K(\tau,s)d\tau = \int_s^t \Big(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial s}\Big)^j K(\tau,s)d\tau,$$

$$(2.10) \qquad \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\Big)^j \int_s^t K(t,\tau)d\tau = \int_s^t \Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\Big)^j K(t,\tau)d\tau.$$

Proof. For $0 \le s < t$, let $d > 0$ be such that $s < \tau < s + d < t$. Then (cf. [26, p. 28])

$$(2.11) \qquad \frac{d}{ds} \int_s^{s+d} K(\tau,s)d\tau = \int_s^{s+d} \Big(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial s}\Big) K(\tau,s)ds,$$

$$(2.12)\qquad \frac{d}{ds}\int\limits_{s+d}^{t} K(\tau,s)d\tau = \int\limits_{s+d}^{t} \frac{\partial K(\tau,s)}{\partial s}d\tau - K(s+d,s).$$

Indeed, (2.11) follows from the behaviour of the difference quotient which corresponds to the derivative $(d/ds)\int_s^{s+d} K(\tau,s)d\tau$:

$$\Big[\int\limits_{s+h}^{s+h+d} K(\tau,s+h)d\tau - \int\limits_{s}^{s+d} K(\tau,s)d\tau\Big]/h =$$
$$= \int\limits_{s}^{s+d} \{K(\tau+h,s+h) - K(\tau,s)]/h\}d\tau \to \int\limits_{s}^{s+d} \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau,s)d\tau \quad \text{as } h\to 0.$$

The last convergence can be argued using Lebesgue theorem about the limiting process under the integral sign. First, the integrands converge almost everywhere for $\tau\in(s,s+d)$,

$$\frac{1}{h}[K(\tau+h,s+h) - K(\tau,s)] \to \big(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\big)K(\tau,s) \quad \text{as} \quad h\to 0 \quad \text{for} \quad s<\tau<s+d.$$

Secondly, due to (1.4),

$$\big|\tfrac{1}{h}\big[K(\tau+h,s+h) - K(\tau,s)\big]\big| = \big|\tfrac{1}{h}\int\limits_0^1 \tfrac{d}{d\sigma}K(\tau+h\sigma,s+h\sigma)d\sigma\big| =$$
$$= \big|\int\limits_0^1 \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau+h\sigma,s+h\sigma)d\sigma\big| \le c\ \Psi_\nu(\tau-s),$$

where c is a constant, independent of s and τ, $0\le s<\tau<s+d$, and

$$\Psi_\nu(\tau) = \begin{cases} 1 & \text{if} \quad \nu<0, \\ 1+|\log\tau| & \text{if} \quad \nu=0, \\ \tau^{-\nu} & \text{if} \quad \nu>0. \end{cases}$$

In other words, the integrands $[K(\tau+h,s+h) - K(\tau,s)]/h$, $h\to 0$, are bounded by a function $c\Psi_\nu(\tau-s)$ which is integrable with respect to τ on $(s,s+d)$.

The equality (2.12) holds trivially since there is no singularity in the integrands of (2.12).

Further, using (2.11) and (2.12) with $s<s+d<t$ we obtain

$$\big(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial s}\big)\int\limits_s^t K(\tau,s)d\tau = K(t,s) + \tfrac{\partial}{\partial s}\big(\int\limits_s^{s+d} K(\tau,s)d\tau + \int\limits_{s+d}^{t} K(\tau,s)d\tau\big) =$$
$$= K(t,s) + \int\limits_s^{s+d} \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau,s)d\tau + \int\limits_{s+d}^{t} \tfrac{\partial K(\tau,s)}{\partial s}d\tau - K(s+d,s) =$$
$$= K(t,s) \quad + \int\limits_s^{s+d} \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau,s)d\tau +$$
$$+ \int\limits_{s+d}^{t} \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau,s)d\tau - \int\limits_{s+d}^{t} \tfrac{\partial K(\tau,s)}{\partial\tau}d\tau - K(s+d,s) =$$
$$= \int\limits_s^t \big(\tfrac{\partial}{\partial\tau} + \tfrac{\partial}{\partial s}\big)K(\tau,s)d\tau.$$

Now (2.9) follows by induction. The statement (2.10) is a consequence of (2.9). □

Lemma 2.4. *For $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, let K_3 and K_4 be defined by*

$$K_3(t,s) = \int_s^t K(t,\tau)d\tau, \qquad 0 \le s < t \le T; \tag{2.13}$$

$$K_4(t,s) = \int_s^t K(\tau,s)d\tau, \qquad 0 \le s < t \le T; \tag{2.14}$$

Then $K_3, K_4 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$.

Proof. Due to Lemma 2.3 and inequality (1.4), we have

$$\left|\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K_3(t,s)\right| \le \text{const}, \quad (t,s) \in \Delta_T, \ j = 0,1,\dots,m;$$

$$\left|\left(\frac{\partial}{\partial s}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K_3(t,s)\right| = \left|\left(\frac{\partial}{\partial s}\right)^{i-1} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s)\right| \le$$

$$\le \text{const} \begin{cases} 1 & \text{if} \quad \nu + i - 1 < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i - 1 = 0, \\ (t-s)^{-\nu-(i-1)} & \text{if} \quad \nu + i - 1 > 0 \end{cases}$$

for $(t,s) \in \Delta_T$ and all integers $i \ge 1$, $j \ge 0$ such that $i + j \le m$. Therefore (cf. (1.4) and (1.4')) the same estimate holds for $\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K_3(t,s)$. Thus, $K_3 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$. The proof of the second statement is similar. □

The smoothness of the solution of problem $\{(1.1),(1.2)\}$ is characterized by the following statement.

Theorem 2.1. *Let $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $y_0 \in \mathbb{C}$. Then the Cauchy problem $\{(1.1),(1.2)\}$ has a unique solution $y \in C^{m+1,\nu-1}(0,T]$.*

Proof. As we know from Section 1, problem $\{(1.1),(1.2)\}$ is equivalent to the integral equation (1.10) where $z = y'$ and the forcing function f_1 and the kernel K_1 are given by (1.11) and (1.12), respectively. We rewrite (1.10) in the form

$$z = S_1 z + f_1 \tag{2.15}$$

where

$$(S_1 z)(t) = \int_0^t K_1(t,s)z(s)ds. \quad t \in [0,T]. \tag{2.16}$$

Using (1.12) and (2.13), operator S_1 can be presented as follows:

$$S_1 = AJ + S_3, \tag{2.17}$$

where

$$(S_3 z)(t) = \int_0^t K_3(t,s)z(s)ds, \quad t \in [0,T], \tag{2.18}$$

$$(Jz)(t) = \int_0^t z(s)ds, \quad t \in [0,T], \tag{2.19}$$

$$(Az)(t) = a(t)z(t), \quad t \in [0,T]. \tag{2.20}$$

By Lemma 2.4 we obtain that $K_3 \in \mathcal{W}^{m,\nu-1}(\Delta_T) \subset \mathcal{W}^{m,\nu}(\Delta_T)$. Due to Lemma 2.2, S_3 is compact as an operator from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$. Since $1 \in \mathcal{W}^{m,\nu}(\Delta_T)$, it follows from (2.19) and Lemma 2.2 that J is compact as an operator from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$. Using (2.20) and Lemma 2.1 we obtain that A is bounded as an operator from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$. Thus, AJ and S_1 are linear and compact as operators from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$.

Further, it follows from $a \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$ that $f_1 \in C^{m,\nu}(0,T]$. Indeed, $f_1 = g_1 + g_2$, where (see (1.11)) $g_1(t) = b(t) + y_0 a(t)$, $t \in [0,T]$, and

$$g_2(t) = y_0 \int_0^t K(t,s)ds, \quad t \in [0,T].$$

Clearly $g_1 \in C^{m,\nu}(0,T]$ and $g_2 = y_0 S1$, where the operator S is defined by (2.6). Since $1 \in C^{m,\nu}(0,T]$ and S is bounded as an operator from $C^{m,\nu}(0,T]$ to $C^{m,\nu}(0,T]$ (see Lemma 2.2), $g_2 \in C^{m,\nu}(0,T]$.

By Fredholm alternative theorem, $1 - S_1$ has a bounded inverse $(1 - S_1)^{-1} : C^{m,\nu}(0,T] \to C^{m,\nu}(0,T]$ and equation (2.15) has a unique solution $z = (1 - S_1)^{-1} f_1 \in C^{m,\nu}(0,T]$. In other words, $y' \in C^{m,\nu}(0,T]$ implying $y \in C^{m+1,\nu-1}(0,T]$. □

Remark 2.1. Notice that $C^{m+1,\nu-1}(0,T] \subset C^1[0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$..

Remark 2.2. For $a \in C^m[0,T]$, $b(t) = b_1(t) + b_2(t)t^{-\nu}$, $b_1, b_2 \in C^m[0,T]$, $K_\nu(t,s) = \kappa(t,s)(t-s)^{-\nu}$, $\kappa \in C^m(\overline{\Delta_T})$, $t \in [0,T]$, $s \in [0,T)$, $m \in \mathbb{N}$, $0 < \nu < 1$, the statement of Theorem 2.1 can be derived also from Theorem 1.3.16 in [8]; see also [5, 14].

Remark 2.3. Let $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$. Then the integral equation (1.10) has a unique solution $z \in C^{m,\nu}(0,T]$.

This was established in the proof of Theorem 2.1.

3. PIECEWISE POLYNOMIAL INTERPOLATION

For $N \in \mathbb{N}$, let

$$\Pi_N = \{t_0, \dots, t_N\colon\ 0 = t_0 < t_1 < \dots < t_N = T\} \tag{3.1}$$

be a partition of the interval $[0, T]$ by grid points

$$t_j = t_j^{N,r} = T(j/N)^r, \quad j = 0, \dots, N. \tag{3.2}$$

Here the real number $r \in [1, \infty)$ characterizes the non-uniformity of the grid Π_N. If $r = 1$ then the grid points (3.2) are distributed uniformly; for $r > 1$, the grid points (3.2) are more densely clustered near the left endpoint of the interval $[0, T]$. Let

$$\sigma_j = [t_{j-1}, t_j], \quad h_j = t_j - t_{j-1}, \quad j = 1, \dots, N;$$

$$h = h^{(N)} = \max\{h_j\colon\ j = 1, \dots, N\}. \tag{3.3}$$

It is easily to seen that

$$h_j \le h \le rTN^{-1}, \quad j = 1, \dots, N. \tag{3.4}$$

For given integers $m \ge 0$ and $-1 \le d \le m-1$, let $S_m^{(d)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid $\{(3.1), (3.2)\}$:

$$\begin{aligned} S_m^{(d)}(\Pi_N) \ &= \{u\colon\ u|_{\sigma_j} =: u_j \in \pi_m, \quad j = 1, \dots, N; \\ &\quad u_j^{(k)}(t_j) = u_{j+1}^{(k)}(t_j), \quad k = 0, \dots, d;\ j = 1, \dots, N-1\}, \end{aligned} \tag{3.5}$$

where π_m denotes the set of polynomials of degree not exceeding m. Note that elements of $S_m^{(-1)}(\Pi_N) = \{u\colon\ u|_{\sigma_j} \in \pi_m,\ j = 1, \dots, N\}$ may have jump discontinuities at interior grid points $t_1, \dots, t_{N-1}$. The dimension of $S_m^{(d)}(\Pi_N)$ is given by

$$\dim S_m^{(d)}(\Pi_N) = N(m-d) + d + 1, \quad -1 \le d \le m-1. \tag{3.6}$$

In every subinterval $\sigma_j = [t_{j-1}, t_j]$, $j = 1, \dots, N$ we define $m \in \mathbb{N}$ interpolation points $t_{j1} < \dots < t_{jm}$,

$$t_{jk} = t_{j-1} + \eta_k h_j, \quad k = 1, \dots, m;\ j = 1, \dots, N, \tag{3.7}$$

where $\eta_1, \dots, \eta_m$ do not depend on j and N and satisfy

$$0 \le \eta_1 < \dots < \eta_m \le 1. \tag{3.8}$$

To a continuous function $x :\ [0, T] \to \mathbb{C}$ we assign a piecewise polynomial interpolation function $P_N x = P_N^{(m)} x \in S_{m-1}^{(-1)}(\Pi_N)$ which interpolates x at the points (3.7): $(P_N x)(t_{jk}) = x(t_{jk})$, $k = 1, \dots, m$; $j = 1, \dots, N$. Thus, $(P_N x)(t)$ is independently defined in every subinterval σ_j, $j = 1, \dots, N$ and may be discontinuous at the interior grid points $t = t_j$, $j = 1, \dots, N-1$; we may treat $P_N x$ as a two-valued function at these points. Note that in the case $\eta_1 = 0$, $\eta_m = 1$ (see (3.8)), $P_N x$ is a continuous function on $[0,T]$.

We introduce also an interpolation operator $P_N = P_N^{(m)}$ which assigns to every continuous function $x: [0,T] \to \mathbb{C}$ its piecewise polynomial interpolation function $P_N x$.

Lemma 3.1. *Let $x \in C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, and let the interpolation nodes (collocation points) (3.7) with grid points (3.2) and parameters (3.8) be used. Then the following error estimates hold:*

1) if $m < 1-\nu$ then

$$||x - P_N x||_\infty \le ch^m \qquad \text{for} \quad r \ge 1; \tag{3.9}$$

2) if $m = 1-\nu$ then

$$||x - P_N x||_\infty \le c \begin{cases} h^m(1 + |\log h|) & \text{for} \quad r = 1, \\ h^m & \text{for} \quad r > 1, \end{cases} \tag{3.10}$$

and

$$||x - P_N x||_{L^p(0,T)} \le ch^m \qquad \text{for} \quad r \ge 1,\ 1 \le p < \infty; \tag{3.11}$$

3) if $m > 1-\nu$ then

$$||x - P_N x||_\infty \le c \begin{cases} h^{r(1-\nu)} & \text{for} \quad 1 \le r \le m/(1-\nu), \\ h^m & \text{for} \quad r > m/(1-\nu), \end{cases} \tag{3.12}$$

and for $1 \le p < \infty$,

(3.13)

$$||x - P_N x||_{L^p(0,T)} \le c \begin{cases} h^{r(1-\nu+p^{-1})} & \text{if} \quad 1 \le r < \frac{m}{1-\nu+p^{-1}}, \\ & \quad\ m > 1-\nu+p^{-1}, \\ h^m(1+|\log h|)^{1/p} & \text{if} \quad r = \frac{m}{1-\nu+p^{-1}}, \\ & \quad\ m \ge 1-\nu+p^{-1}, \\ h^m & \text{if} \quad r > \frac{m}{1-\nu+p^{-1}},\ r \ge 1. \end{cases}$$

Here h is defined by (3.3), c is a positive constant which is independent of N (of h) and

$$||x - P_N x||_\infty = \max_{j=1,\dots,N} \max_{t_{j-1} \le t \le t_j} |x(t) - (P_N x)(t)|.$$

Proof. Applying to $x \in C^{m,\nu}(0,T]$ the estimate (7.15) from [26, p.116] we obtain that

$$\max_{t_{j-1} \le t \le t_j} |x(t) - (P_N x)(t)| \le ch_j^m \begin{cases} 1 & \text{if} \quad m < 1-\nu \\ 1 + |\log h_j| & \text{if} \quad m = 1-\nu \\ t_j^{1-\nu-m} & \text{if} \quad m > 1-\nu \end{cases} \tag{3.14}$$

for $j = 1, \dots, N$, with a constant c which is independent of j and N. Estimates (3.9)–(3.13) are easy consequences of (3.14); see [26] for details. □

In sequel, for Banach spaces E and F, by $\mathcal{L}(E, F)$ is denoted the Banach space of linear bounded operators $A\colon\ E \to F$ with the norm

$$||A|| = \sup_{x \in E, ||x|| \leq 1} ||Ax||.$$

Lemma 3.2. *Let* $S\colon\ L^\infty(0, T) \to C[0, T]$ *be a linear compact operator. Then*

$$||S - P_N S||_{\mathcal{L}(L^\infty(0,T), L^\infty(0,T))} \to 0 \quad \text{as} \quad N \to \infty.$$

Proof. We have

$$(3.15) \qquad ||x - P_N x||_{L^\infty(0,T)} \to 0 \quad \text{as } N \to \infty \text{ for every } x \in C[0, T].$$

Indeed, this convergence takes place for $x \in C^{m,\nu}(0, T]$ (see Lemma 3.1), and an easy observation shows that

$$(3.16) \qquad ||P_N||_{\mathcal{L}(C[0,T], L^\infty(0,T))} = ||P||_{\mathcal{L}(C[0,1], C[0,1])}, \quad N \in \mathbb{N},$$

with the interpolation operator $P \in \mathcal{L}(C[0, 1], C[0, 1])$ corresponding to the nodes (3.8):

$$(Px)(t) = \sum_{k=1}^{m} x(\eta_k) \prod_{\substack{l=1 \\ l \neq k}}^{m} (t - \eta_l)/(\eta_k - \eta_l), \quad x \in C[0, 1], \ t \in [0, 1].$$

Now (3.15) follows by Banach-Steinhaus theorem, and together with the compactness of $S\colon\ L^\infty(0, T) \to C[0, T]$ we obtain the assertion of the Lemma. □

4. COLLOCATION METHODS

In order to solve the Cauchy problem $\{(1.1),(1.2)\}$ we construct two collocation methods which solve equivalent problems (1.8) and (1.13), respectively.

Method 1. We look for an approximate solution v to equation (1.8) in $S_{m-1}^{(-1)}(\Pi_N)$, $m, N \in \mathbb{N}$. The approximation $v = v^{(N)} \in S_{m-1}^{(-1)}(\Pi_N)$ will be determined from the following collocation conditions:

$$
\begin{aligned}
v_j(t_{jk}) &= f_1(t_{jk}) + a(t_{jk}) \int_0^{t_{jk}} v(s)ds \\
&+ \int_0^{t_{jk}} K(t_{jk}, s)\Big(\int_0^s v(\tau)d\tau\Big)ds, \quad k = 1, \dots, m;\ j = 1, \dots, N,
\end{aligned} \tag{4.1}
$$

where the function f_1 and the set of collocation points $\{t_{jk}\}$ are given by (1.11) and (3.7), respectively,

$$
f_1(t) = b(t) + y_0 a(t) + y_0 \int_0^t K(t,s)ds, \quad t \in [0,T],
$$

$$
t_{jk} = t_{j-1} + \eta_k h_j, \quad k = 1, \dots, m;\ j = 1, \dots, N.
$$

Recall that $v_j = v|_{\sigma_j}$ is the restriction of v to $\sigma_j = [t_{j-1}, t_j]$, $j = 1, \dots, N$. The function $v \in S_{m-1}^{(-1)}(\Pi_N)$ is defined on each interval σ_j, $j = 1, \dots, N$ separately as a polynomial of degree $\le m-1$. Therefore, if $\eta_1 > 0$ or $\eta_m < 1$ (see (3.8)), then $v(t)$ has generally two different values $v_j(t_j)$ and $v_{j+1}(t_j)$ at $t = t_j$, $j = 1, \dots, N-1$. If $\eta_1 = 0$, $\eta_m = 1$, then $t_{jm} = t_{j+1,1}$, $j = 1, \dots, N-1$, and for every $j = 1, \dots, N-1$, we actually have the same equation in (4.1) for $v_j(t_{jm})$ and $v_{j+1}(t_{j+1,1})$, i.e. $v_j(t_{jm}) = v_{j+1}(t_{j+1,1})$ whenever

$$
g(t) = f_1(t) + a(t)\int_0^t v(s)ds + \int_0^t K(t,s)\Big(\int_0^s v(\tau)d\tau\Big)ds, \quad t \in [0,T],
$$

is a continuous function on $[0,T]$ (and this holds under conditions of Theorem 4.1 below). In other words, the choice of parameters (3.8) with $\eta_1 = 0$, $\eta_m = 1$ in Method 1 actually implies that the resulting collocation approximation v belongs to the smoother polynomial spline space $S_{m-1}^{(0)}(\Pi_N)$; of course, the repeated collocation conditions at $t_{jm} = t_{j+1,1}$ will be taken only once in this case.

Having determined the approximation v for $z = y'$, we can determine also the approximation u for y, the solution of the Cauchy problem $\{(1.1),(1.2)\}$, setting

$$
u(t) = y_0 + \int_0^t v(s)ds, \quad t \in [0,T].
$$

Clearly,

$$
\begin{aligned}
&u|_{\sigma_j} = u_j, \quad j = 1, \dots, N; \\
(4.2) \qquad &u_j(t) = u_j(t_{j-1}) + \int_{t_{j-1}}^{t} v(s)ds, \quad t \in \sigma_j,\ j = 1, \dots, N; \\
&u_1(0) = y_0.
\end{aligned}
$$

Note also that $v \in S_{m-1}^{(-1)}(\Pi_N)$ implies $u \in S_m^{(0)}(\Pi_N)$ and $v \in S_{m-1}^{(0)}(\Pi_N)$ implies $u \in S_m^{(1)}(\Pi_N)$.

Conditions (4.1) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$ (or in $S_{m-1}^{(0)}(\Pi_N)$ if $\eta_1 = 0$, $\eta_m = 1$). For instance, the approximation v can be generated recursively by successive computation of its restrictions v_j to the subintervals σ_j, $j = 1, \dots, N$. In order to render the collocation conditions (4.1) into a form which more clearly exhibits the recursive nature, rewrite (4.1) as follows:

$$
\begin{aligned}
&v_j(t_{jk}) = f_1(t_{jk}) \\
&+a(t_{jk})\Big[\sum_{l=1}^{j-1} h_l \int_0^1 v_l(t_{l-1} + \tau h_l)d\tau + h_j \int_0^{\eta_k} v_j(t_{j-1} + \tau h_j)d\tau\Big] \\
&+\sum_{l=1}^{j-1} h_l \Big[\int_0^1 K(t_{jk}, t_{l-1} + sh_l)ds \sum_{p=1}^{l-1} h_p \int_0^1 v_p(t_{p-1} + \tau h_p)d\tau\Big] \\
(4.3) \qquad &+\sum_{l=1}^{j-1} h_l^2 \int_0^1 K(t_{jk}, t_{l-1} + sh_l)\Big(\int_0^s v_l(t_{l-1} + \tau h_l)d\tau\Big)ds \\
&+h_j \int_0^{\eta_k} K(t_{jk}, t_{j-1} + sh_j)ds \sum_{p=1}^{j-1} h_p \int_0^1 v_p(t_{p-1} + \tau h_p)d\tau \\
&+h_j^2 \int_0^{\eta_k} K(t_{jk}, t_{j-1} + sh_j)\Big(\int_0^s v_j(t_{j-1} + \tau h_j)d\tau\Big)ds, \\
&\qquad\qquad k = 1, \dots, m;\ j = 1, \dots, N.
\end{aligned}
$$

For v_j we may use the representation

$$
(4.4) \quad v_j(t_{j-1} + \tau h_j) = \sum_{q=1}^{m} c_{jq} L_q(\tau),\ t_{j-1} + \tau h_j \in [t_{j-1}, t_j],\ j = 1, \dots, N,
$$

with $L_q(\tau)$ denoting qth Lagrange fundamental polynomial associated with the collocation parameters (3.8),

$$
(4.5) \qquad L_q(\tau) = \prod_{i=1, i\neq q}^{m} (\tau - \eta_i)/(\eta_q - \eta_i), \quad \tau \in [0, 1],
$$

and with

$$
(4.6) \qquad c_{jq} = c_{jq}^{(N)} = v_j(t_{jq}), \quad q = 1, \dots, m.
$$

Let

$$
(4.7) \qquad \Lambda_q(\tau) = \int_0^{\tau} L_q(s)ds, \quad \tau \in [0, 1],\ q = 1, \dots, m.
$$

With these observations, (4.3) may be written in the following form:

$$
\begin{aligned}
(4.8)\qquad c_{jk} &= f_1(t_{jk}) + a(t_{jk})\big[\sum_{l=1}^{j-1} h_l \sum_{q=1}^{m} c_{lq}\Lambda_q(1) + h_j \sum_{q=1}^{m} c_{jq}\Lambda_q(\eta_k)\big] \\
&+ \sum_{l=1}^{j-1} h_l \int_0^1 K(t_{jk}, t_{l-1}+sh_l)\big[\sum_{p=1}^{l-1} h_p \sum_{q=1}^{m} c_{pq}\Lambda_q(1) + h_l \sum_{q=1}^{m} c_{lq}\Lambda_q(s)\big]ds \\
&+ h_j \int_0^{\eta_k} K(t_{jk}, t_{j-1}+sh_j)\big[\sum_{p=1}^{j-1} h_p \sum_{q=1}^{m} c_{pq}\Lambda_q(1) + h_j \sum_{q=1}^{m} c_{jq}\Lambda_q(s)\big]ds, \\
& k = 1,\dots,m;\ j = 1,\dots,N.
\end{aligned}
$$

Thus, for each $j = 1,\dots,N$ the collocation conditions (4.1) can be treated as a linear system in $\mathbb{C}^m$ for the vector $(c_{j1},\dots,c_{jm})$; once its components have been found, the approximation v can be composed by formula (4.4).

Theorem 4.1. *Let $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $y_0 \in \mathbb{C}$, and let the collocation points (3.7) with grid points (3.2) and parameters (3.8) be used.*

Then for all sufficiently large $N \in \mathbb{N}$, say $N \geq N_0$, and for every choice of collocation parameters (3.8) with $\eta_1 > 0$ or $\eta_m < 1$, the equations (4.2) and (4.1) determine unique approximations $u \in S_m^{(0)}(\Pi_N)$ and $v \in S_{m-1}^{(-1)}(\Pi_N)$ (with $v|_{\sigma_j} = (u|_{\sigma_j})'$, $j = 1,\dots,N$) to the solution y of the Cauchy problem $\{(1.1),(1.2)\}$ and its derivative y', respectively. If $\eta_1 = 0$, $\eta_m = 1$, then $u \in S_m^{(1)}(\Pi_N)$ and $v = u' \in S_{m-1}^{(0)}(\Pi_N)$. The following error estimates hold for $k = 0$ and $k = 1$:

1) if $m < 1-\nu$ then

$$
(4.9)\qquad ||u^{(k)} - y^{(k)}||_\infty \leq ch^m \qquad \text{for} \quad r \geq 1;
$$

2) if $m = 1-\nu$ then

$$
(4.10)\qquad ||u^{(k)} - y^{(k)}||_\infty \leq c \begin{cases} h^m(1+|\log h|) & \text{for} \quad r = 1, \\ h^m & \text{for} \quad r > 1; \end{cases}
$$

3) if $m > 1-\nu$ then

$$
(4.11)\qquad ||u^{(k)} - y^{(k)}||_\infty \leq c \begin{cases} h^{r(1-\nu)} & \text{for} \quad 1 \leq r \leq m/(1-\nu), \\ h^m & \text{for} \quad r > m/(1-\nu). \end{cases}
$$

Here h is defined by (3.3), c is a positive constant which is independent of N (of h) and

$$
||u^{(k)} - y^{(k)}||_\infty = \max_{j=1,\dots,N} \max_{t_{j-1}\leq t\leq t_j} \left|u_j^{(k)}(t) - y^{(k)}(t)\right|, \quad u_j = u|_{\sigma_j}.
$$

Proof. The Cauchy problem $\{(1.1),(1.2)\}$ is equivalent to the integral equation (1.10) with $z = y'$. We write (1.10) in the form $z = S_1 z + f_1$, with f_1 defined by (1.11) and operator S_1 defined by (2.16),

$$
(S_1 z)(t) = \int_0^t K_1(t,s)z(s)ds, \quad t \in [0,T],
$$

where

$$K_1(t,s) = a(t) + \int_s^t K(t,\tau)d\tau, \quad 0 \le s < t \le T.$$

It follows from $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$ that $f_1 \in C^{m,\nu}(0,T] \subset L^\infty(0,T)$ and S_1 is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$ and to $L^\infty(0,T)$, also. As the homogeneous equation $z = S_1 z$ has only the trivial solution $z = 0$, equation $z = S_1 z + f_1$ has a unique solution $z \in L^\infty(0,T)$. Moreover, $z \in C^{m,\nu}(0,T]$ by Remark 2.3.

On the other hand, the conditions (4.1) are equivalent to the conditions

(4.12)

$$v_j(t_{jk}) = f_1(t_{jk}) + \int_0^{t_{jk}} K_1(t_{jk},s)v(s)ds, \quad k = 1,\dots,m;\ j = 1,\dots,N.$$

This follows from the application of Dirichlet formula (1.9) to the second integral on the right-hand side of (4.1):

$$\int_0^{t_{jk}} K(t_{jk},s)\Big(\int_0^s v(\tau)d\tau\Big)ds = \int_0^{t_{jk}}\Big(\int_\tau^{t_{jk}} K(t_{jk},s)ds\Big)v(\tau)d\tau$$
$$= \int_0^{t_{jk}}\Big(\int_s^{t_{jk}} K(t_{jk},\tau)d\tau\Big)v(s)ds, \quad k = 1,\dots,m;\ j = 1,\dots,N.$$

In its turn, collocation conditions (4.12) have the operator equation representation

(4.13) $$v = P_N S_1 v + P_N f_1$$

with P_N defined in Section 3. From Lemma 3.2 and from the boundedness of $(1-S_1)^{-1}$ in $L^\infty(0,T)$ we obtain that $1 - P_N S_1$ is invertible in $L^\infty(0,T)$ for all sufficiently large N, say $N \ge N_0$, and

(4.14) $$||(1-P_N S_1)^{-1}||_{\mathcal{L}(L^\infty(0,T),L^\infty(0,T))} \le c', \quad c' = \text{const},\ N \ge N_0.$$

Thus, for $N \ge N_0$ equation (4.13) (Method 1) provides a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N)$. We have for it and z, the solution of equation $z = S_1 z + f_1$, that $(1 - P_N S_1)(v - z) = P_N z - z$. Therefore,

$$v - z = (1 - P_N S_1)^{-1}(P_N z - z),$$

and, due to (4.14),

$$||v - z||_{L^\infty(0,T)} \le c'||P_N z - z||_{L^\infty(0,T)}, \quad N \ge N_0.$$

Applying Lemma 3.1 we obtain the estimations (4.9)–(4.11) with $k = 1$, $z = y'$, $v = u'$ for $u' - y'$. Further, due to (4.2),

$$\begin{aligned} u_j(t) - y(t) &= u_j(t_{j-1}) + \int_{t_{j-1}}^{t} v(s)ds - \Big(\int_{t_{j-1}}^{t} y'(s)ds + y(t_{j-1}) \Big) \\ &= u_j(t_{j-1}) - y(t_{j-1}) + \int_{t_{j-1}}^{t} [u_j'(s) - y'(s)]ds, \quad t \in [t_{j-1}, t_j], j = 1, \dots, N, \end{aligned}$$

with $u_1(0) - y(0) = 0$. Applying (4.9)–(4.11) with $k = 1$ we obtain the estimations (4.9)–(4.11) with $k = 0$ for $u - y$. □

According to (4.11), in the case $m > 1 - \nu$, the convergence order $||y^{(k)} - u^{(k)}||_\infty \le ch^m$, $k = 0, 1$, is guarantied for $r \ge m/(1-\nu)$. For a great ν ($\nu < 1$), this condition on r may be too restrictive. To obtain the convergence order $||y - u||_\infty \le ch^m$, the condition on r can be essentially relaxed.

Theorem 4.2. *Let the conditions of Theorem 4.1 be fulfilled. Then in the case $m > 1 - \nu$, with designations of Theorem 4.1,*

$$||u - y||_\infty \le ch^m \quad \textit{for} \quad r > m/(2 - \nu), \ r \ge 1,$$

where c is a constant which is independent of N (of h).

Proof. Using the equality

$$(1 - P_N S_1)^{-1} = 1 + (1 - P_N S_1)^{-1} P_N S_1, \quad N \ge N_0,$$

we rewrite the formula for the error $v - z = (1 - P_N S_1)^{-1}(P_N z - z)$ in the form

$$v - z = P_N z - z + (1 - P_N S_1)^{-1} P_N S_1 (P_N z - z).$$

Together with (4.14), (3.16) and the boundedness of the kernel $K_1(t, s)$ (see (1.12) and Lemma 2.4) we obtain for $0 \le t \le T$ that

$$|u(t) - y(t)| = \Big| \int_0^t \big[v(s) - z(s)\big] ds \Big| \le \int_0^t \big|v(s) - z(s)\big| ds \le c \int_0^t \big|(P_N z)(s) - z(s)\big| ds,$$

where c is a constant which is independent of N. Let $t \in [t_{j-1}, t_j]$, $1 \le j \le N$. Then, by (3.14),

$$|u(t) - y(t)| \le c \sum_{l=1}^{j} \int_{t_{l-1}}^{t_l} |(P_N z)(s) - z(s)| ds \le c' \sum_{l=1}^{j} (t_l - t_{l-1})^{m+1} t_l^{1-\nu-l},$$

with a constant c' which is independent of j and N. Further, we have

$$t_l^{1-\nu-m} = T^{1-\nu-m} N^{-r(1-\nu-m)} l^{r(1-\nu-m)},$$

$$(t_l - t_{l-1})^{m+1} \le T^{m+1} r^{m+1} N^{-r(m+1)} l^{(r-1)(m+1)},$$

$$(t_l - t_{l-1}) t_l^{1-\nu-m} \le T^{m+1} r^{m+1} N^{-r(2-\nu)} l^{r(2-\nu)-m-1}.$$

Therefore, since $r > m/(2-\nu)$,

$$|u(t) - y(t)| \le c'' N^{-r(2-\nu)} \sum_{l=1}^{j} l^{r(2-\nu)-m-1} \le c''' N^{-r(2-\nu)} j^{r(2-\nu)-m} \le c''' N^{-m},$$

with some constants c'' and c''' which do not depend on N. In other words, the statement of Theorem holds. □

Remark 4.1. In more special case ($a, b \in C^m[0,T]$, $K(t,s) = \kappa(t,s)(t-s)^{-\nu}$, $\kappa \in C^m(\overline{\Delta_T})$, $m \in \mathbb{N}$, $0 < \nu < 1$, b and K do not vanish identically), the condition $r > m/(2-\nu)$ for r was proposed and justified in [24].

Method 2. We look for an approximate solution u of equation (1.13) in $S_m^{(-1)}(\Pi_N)$, $m, N \in \mathbb{N}$. The approximation $u = u^{(N)} \in S_m^{(-1)}(\Pi_N)$ will be determined from the following collocation conditions:

$$(4.15) \qquad u_j(t_{jk}) = f_2(t_{jk}) + \int_0^{t_{jk}} K_2(t_{jk}, s) u(s) ds, \ j = 1, \dots, N.$$

Here $u_j = u|_{\sigma_j}$ ($j = 1, \dots, N$) is the restriction of $u \in S_m^{(-1)}(\Pi_N)$ to $\sigma_j = [t_{j-1}, t_j]$, f_2 and K_2 are defined by (1.14) and (1.15), respectively,

$$f_2(t) = y_0 + \int_0^t b(s) ds, \quad t \in [0,T],$$

$$K_2(t,s) = a(s) + \int_s^t K(\tau, s) d\tau, \quad 0 \le s < t \le T.$$

The collocation points $\{t_{jk}\}$ are given by

$$(4.16) \qquad t_{jk} = t_{j-1} + \eta_k h_j, \quad k = 1, \dots, m+1; \ j = 1, \dots, N,$$

where

$$(4.17) \qquad 0 \le \eta_1 < \ldots < \eta_{m+1} \le 1$$

is some fixed system of collocation parameters, the same for every j and N.

It is easily seen (cf. Method 1) that for collocation parameters (4.17) with $\eta_1 = 0$, $\eta_{m+1} = 1$, the resulting collocation approximation u is an element of the smoother polynomial spline space $S_m^{(0)}(\Pi_N)$.

Conditions (4.15) form a system of equations whose exact form is determined by the choice of a basis in $S_m^{(-1)}(\Pi_N)$. For instance, in each subinterval σ_j ($j = 1, \dots, N$) we may use the representation

$$(4.18) \qquad u_j(s) = \sum_{q=1}^{m+1} c_{jq} \varphi_{jq}(s), \quad s \in \sigma_j,$$

where $\varphi_{jq}(s) = L_q((s - t_{j-1})/h_j)$ with $L_q(\tau)$ denoting qth Lagrange fundamental polynomial associated with $m+1$ collocation parameters $\eta_1, \dots, \eta_{m+1}$ in (4.17) (cf. (4.4), (4.5)). The collocation conditions (4.15) then lead to the following system of algebraic equations for coefficients $c_{jq} = c_{jq}^{(N)} = u_j(t_{jq})$:

(4.19)

$$\begin{aligned} c_{jk} &= f_2(t_{jk}) + \sum_{l=1}^{j-1} \sum_{q=1}^{m+1} \Big(\int_{t_{l-1}}^{t_l} K_2(t_{jk}, s)\varphi_{lq}(s)ds\Big) c_{lq} + \\ &+ \sum_{q=1}^{m+1} \Big(\int_{t_{j-1}}^{t_{jk}} K_2(t_{jk}, s)\varphi_{jq}(s)ds\Big) c_{jq}, \ k = 1, \dots, m+1; \ j = 1, \dots, N. \end{aligned}$$

This system can be solved by a recursive process. First, the coefficients $c_{11}, \dots, c_{1,m+1}$ can be found by solving the system

$$c_{1k} = f_2(t_{1k}) + \sum_{q=1}^{m+1} \left(\int_0^{t_{1k}} K_2(t_{1k}, s)\varphi_{1q}(s)ds \right) c_{1q}, \ q = 1, \dots, m+1. \tag{4.20}$$

Having determined $c_{11}, \dots, c_{1,m+1}$, one can find $c_{21}, \dots, c_{2,m+1}$ from the $m+1$ equations (4.19) with $j = 2$. Generally, having determined $c_{11}, \dots, c_{1,m+1}, \dots, c_{j-1,1}, \dots, c_{j-1,m+1}$, the coefficients $c_{j1}, \dots, c_{j,m+1}$ can be found from the $m+1$ equations (4.19) with corresponding j. Once all $\{c_{jq}\}$ have been found, the collocation approximation u can be composed by formula (4.18).

We now analyze the convergence order of Method 2. First we formulate some auxiliary results (Lemmas 4.1–4.3) which we need for this analysis.

Lemma 4.1. [17, p.174] *Let $P(t)$ be a polynomial of order not exceeding n, and let*

$$|P(t)| \le M, \quad t \in [\alpha, \beta], \quad -\infty < \alpha < \beta < \infty. \tag{4.21}$$

Then

$$|P'(t)| \le \frac{2Mn^2}{\beta - \alpha}, \quad t \in [\alpha, \beta]. \tag{4.22}$$

Next consider an integral equation

$$y(t) = f(t) + \int_0^t K(t, s)y(s)ds, \quad t \in [0, T], \tag{4.23}$$

and collocation method to solve it:

(4.24)

$$u_j(t_{jk}) = f(t_{jk}) + \int_0^{t_{jk}} K(t_{jk}, s)u(s)ds, \quad k = 1, \dots, m+1; \quad j = 1, \dots, N,$$

with $\{t_{jk}\}$ defined by (4.16) and $u \in S_m^{(-1)}(\Pi_N)$, $u_j = u|_{\sigma_j}$, $j = 1, \dots, N$. Let ε_N be the maximal error of the approximation u at the collocation points (4.16):

$$\varepsilon_N = \max_{k=1,\dots,m+1;j=1,\dots,N} |u(t_{jk}) - y(t_{jk})|. \tag{4.25}$$

Lemma 4.2. [9] *Let the following conditions be fulfilled:*

1) $f \in C^{m+1,\nu}(0,T]$, $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$.

2) The collocation points (4.16) with grid points (3.2) and collocation parameters (4.17) are used.

3) The scaling parameter $r = r(m,\nu) \geq 1$ *is restricted by conditions*

$$\begin{array}{lll} r > \frac{m+1}{2(1-\nu)} & \textit{for} & 0 \leq \nu < 1, \\ r > \frac{m+1}{2-\nu} & \textit{for} & 1-m \leq \nu < 0, \\ r \geq 1 & \textit{for} & \nu < 1-m. \end{array} \tag{4.26}$$

Then there exists an $N_0 \in \mathbb{N}$ *such that for* $N \geq N_0$ *the collocation conditions (4.24) define a unique approximation* $u \in S_m^{(-1)}(\Pi_N)$ *(if* $\eta_1 = 0$, $\eta_{m+1} = 1$ *then* $u \in S_m^{(0)}(\Pi_N)$*) to the solution* y *of equation (4.23) and*

$$\varepsilon_N \leq c h^{m+1}, \tag{4.27}$$

with h *defined by (3.3) and a constant* c*, independent of* N *(of* h*).*

Lemma 4.3. [9] *Let the following conditions be fulfilled:*

1) $f \in C^{m+\mu+2,\nu}(0,T)$, $K \in \mathcal{W}^{m+\mu+2,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\mu \in \mathbb{Z}$, $0 \leq \mu \leq m$, $-\infty < \nu < 1$.

2) The collocation points (4.16) are generated by grid points (3.2) and parameters (4.17) which are choosen so that the quadrature approximation

$$\int_0^1 \varphi(s)ds \approx \sum_{q=1}^{m+1} A_q \varphi(\eta_q), \quad 0 \leq \eta_1 < \ldots < \eta_{m+1} \leq 1, \tag{4.28}$$

with appropriate weights $\{A_q\}$*, is exact for all polynomials of degree* $m+1+\mu$.

3) The scaling parameter $r = r(m,\nu,\mu) \geq 1$ *is subject to the restrictions*

$$\begin{array}{lll} r > \frac{m+1}{1-\nu},\ r \geq \frac{m+2-\nu}{2-\nu} & \textit{if} & 1-\nu < \mu+1, \\ r > \frac{m+1}{1-\nu},\ r > \frac{m+\mu+2}{2-\nu} & \textit{if} & \mu+1 \leq 1-\nu \leq m+1, \\ r > \frac{m+\mu+2}{2-\nu} & \textit{if} & 1-\nu > m+1. \end{array} \tag{4.29}$$

Then

$$\varepsilon_N \leq c h^{m+1} \begin{cases} h & \textit{if} \quad \nu < 0, \\ h(1+|\log h|) & \textit{if} \quad \nu = 0, \\ h^{1-\nu} & \textit{if} \quad \nu > 0, \end{cases} \tag{4.30}$$

where h *and* ε_N *are defined by (3.3) and (4.25), respectively, and* c *is a positive constant which is independent of* N *(of* h*).*

Theorem 4.3. *Let $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $y_0 \in \mathbb{C}$, and let the collocation points (4.16) with grid points (3.2) and collocation parameters (4.17) be used.*

Then there exists an $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the collocation conditions (4.15) define a unique approximation $u \in S_m^{(-1)}(\Pi_N)$ (if $\eta_1 = 0$, $\eta_{m+1} = 1$ then $u \in S_m^{(0)}(\Pi_N)$) to y, the solution of the Cauchy problem $\{(1.1), (1.2)\}$, and the following error estimates hold:

1) if $m < 1 - \nu$ then

$$||u - y||_\infty \leq ch^{m+1} \quad \text{for} \quad r \geq 1; \tag{4.31}$$

$$||u' - y'||_\infty \leq ch^{m} \quad \text{for} \quad r = 1; \tag{4.32}$$

$$||u' - y'||_{\varepsilon,\infty} \leq c_\varepsilon h^{m} \quad \text{for} \quad r > 1; \tag{4.33}$$

2) if $m = 1 - \nu$ then

$$||u - y||_\infty \leq c \begin{cases} h^{m+1}(1 + |\log h|) & \text{for} \quad r = 1, \\ h^{m+1} & \text{for} \quad r > 1; \end{cases} \tag{4.34}$$

$$||u' - y'||_\infty \leq ch^{m}(1 + |\log h|) \quad \text{for} \quad r = 1; \tag{4.35}$$

$$||u' - y'||_{\varepsilon,\infty} \leq c_\varepsilon h^{m} \quad \text{for} \quad r > 1; \tag{4.36}$$

3) if $m > 1 - \nu$ then

$$||u - y||_\infty \leq c \begin{cases} h^{r(2-\nu)} & \text{for} \quad 1 \leq r \leq (m+1)/(2-\nu), \\ h^{m+1} & \text{for} \quad r > (m+1)/(2-\nu); \end{cases} \tag{4.37}$$

$$||u' - y'||_\infty \leq ch^{1-\nu} \quad \text{for} \quad r = 1; \tag{4.38}$$

$$||u' - y'||_{\varepsilon,\infty} \leq c_\varepsilon \begin{cases} h^{r(1-\nu)} & \text{for} \quad 1 < r < m/(1-\nu), \\ h^{m} & \text{for} \quad r > m/(1-\nu). \end{cases} \tag{4.39}$$

Here h is defined by (3.3), the constants c and c_ε in (4.31)–(4.39) are independent of N (of h), and

$$\begin{aligned} ||u^{(k)} - y^{(k)}||_\infty &= \sup_{t\in[0,T])} |u^{(k)}(t) - y^{(k)}(t)| \\ &= \max_{j=1,\dots,N} \max_{t_{j-1}\leq t\leq t_j} |u_j^{(k)}(t) - y^{(k)}(t)|, \quad u_j^{(k)} = (u|_{\sigma_j})^{(k)},\ k = 0, 1; \end{aligned} \tag{4.40}$$

$$\begin{aligned} ||u' - y'||_{\varepsilon,\infty} &= \sup_{t\in[\varepsilon,T]} |u'(t) - y'(t)| \\ &= \max_{j=1,\dots,N} \max_{t\in[t_{j-1},t_j]\cap[\varepsilon,T]} |u_j'(t) - y'(t)|, \quad u_j' = (u|_{\sigma_j})',\ 0 < \varepsilon < T. \end{aligned} \tag{4.41}$$

Proof. The Cauchy problem $\{(1.1), (1.2)\}$ is equivalent to the integral equation (1.13) with the forcing function f_2 and the kernel K_2 defined by (1.14) and (1.15), respectively. We write (1.13) in the form

$$y = S_2 y + f_2 \tag{4.42}$$

where

$$(S_2 y)(t) = \int_0^t \big[a(s) + K_4(t,s)\big] y(s) ds, \quad t \in [0,T], \tag{4.43}$$

with K_4 defined by (2.14). We may rewrite

$$S_2 = JA + S_4, \tag{4.44}$$

where J and A are defined by (2.19) and (2.20), respectively, and

$$(S_4 y)(t) = \int_0^t K_4(t,s) y(s) ds, \quad t \in [0,T]. \tag{4.45}$$

Clearly $A \in \mathcal{L}(L^\infty(0,T), L^\infty(0,T))$ and J is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$. Due to Lemma 2.4, $K_4 \in \mathcal{W}^{m,\nu-1}(\Delta_T) \subset \mathcal{W}^{m,\nu}(\Delta_T)$. Together with Lemma 2.2 this implies that S_4 is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$. Thus, S_2 is compact as an operator from $L^\infty(0,T)$ to $C[0,T]$ and to $L^\infty(0,T)$, also.

Further, it follows from $b \in C^{m,\nu}(0,T]$ that $f_2 \in C[0,b]$. As the homogeneous equation $y = S_2 y$ has only the trivial solution $y = 0$, equation (4.42) has a unique solution $y \in L^\infty(0,T)$. Due to Theorem 2.1, $y \in C^{m+1,\nu-1}(0,T]$.

On the other hand, it is easy to see that the collocation conditions (4.15) have the operator equation representation

$$u = P_N S_2 u + P_N f_2, \tag{4.46}$$

with an interpolation operator $P_N = P_N^{(m+1)}$ which assigns to every continuous function $x \colon [0,T] \to \mathbb{C}$ its piecewise polynomial interpolation function $P_N x \in S_m^{(-1)}(\Pi_N)$ interpolating x at the points (4.16) (see Section 3). From Lemma 3.2 and from the boundedness of $(1 - S_2)^{-1}$ in $L^\infty(0,T)$ we obtain that $1 - P_N S_2$ is invertible in $L^\infty(0,T)$ for all sufficiently large N, say $N \geq N_0$, and

$$||(1 - P_N S_2)^{-1}||_{\mathcal{L}(L^\infty(0,T), L^\infty(0,T))} \leq c, \quad c = \text{const}, \ N \geq N_0. \tag{4.47}$$

Thus, for $N \geq N_0$, equation (4.46) (equations (4.15)) provides a unique solution $u \in S_m^{(-1)}(\Pi_N)$. We have for it and y, the solution of (4.42) that

$(1 - P_N S_2)(y - u) = y - P_N y$. Therefore, $y - u = (1 - P_N S_2)^{-1}(y - P_N y)$, and, due to (4.47),

$$(4.48) \qquad ||y - u||_{L^\infty(0,T)} \le c\, ||y - P_N y||_{L^\infty(0,T)} \quad \text{for} \quad N \ge N_0.$$

Applying Lemma 3.1 we obtain the estimations (4.31),(4.34) and (4.37) of Theorem 4.3.

Let $y \in C^{m+1,\nu-1}(0,T]$ be the solution of the Cauchy problem $\{(1.1),(1.2)\}$ and let $u \in S_m^{(-1)}(\Pi_N)$ denote the collocation approximation determined by (4.15) for $N \ge N_0$. We shall estimate the error

$$|u_j'(t) - y'(t)|, \quad u_j'(t) = (u|_{\sigma_j})'(t), \quad t \in \sigma_j, \quad j = 1, \dots, N.$$

Let $P_N y = P_N^{(m+1)} y \in S_m^{(-1)}(\Pi_N)$ (see Section 3) interpolate y at the collocation points (4.16): $(P_N y)(t_{jk}) = y(t_{jk})$, $k = 1, \dots, m+1$; $j = 1, \dots, N$. Let

$$(P_N y)_j = (P_N y)|_{\sigma_j}, \quad (P_N y)_j'(t) = ((P_N y)|_{\sigma_j})'(t), \quad t \in \sigma_j, \quad j = 1, \dots, N.$$

Thus, the derivative $(P_N y)'(t)$ of $(P_N y)(t)$ at $t = t_j$ ($j = 1, \dots, N-1$) is understood as two valued function with values $(P_N y)_j'(t_j - 0)$ and $(P_N y)_{j+1}'(t_j + 0)$. We have

$$(4.49) \qquad \begin{array}{c} |u_j'(t) - y'(t)| \le |u_j'(t) - (P_N y)_j'(t)| + |(P_N y)_j'(t) - y'(t)|, \\ t \in \sigma_j, \ j = 1, \dots, N. \end{array}$$

Clearly $(P_N y)' \in S_{m-1}^{(-1)}(\Pi_N)$, and by Rolle's theorem, in every σ_j ($j = 1, \dots, N$) there exist m points $t_{jk}' \in (t_{jk}, t_{j,k+1}) \subset \sigma_j$, $k = 1, \dots, m$ so that $(P_N y)_j'$ interpolates y' at these points:

$$(P_N y)_j'(t_{jk}') = y'(t_{jk}), \quad t_{jk} < t_{jk}' < t_{j,k+1}, \quad k = 1, \dots, m; \ j = 1, \dots, N.$$

This allows to use Lemma 3.1 to derive upper bounds for $|(P_N y)_j'(t) - y'(t)|$, $t \in \sigma_j$, $j = 1, \dots, N$. By (3.9), (3.10) and (3.12) we find the following estimations for $||(P_N y)' - y'||_\infty$:

1) if $m > 1 - \nu$ then

$$(4.50) \qquad ||(P_N y)' - y'||_\infty \le c h^m \quad \text{for} \quad r \ge 1;$$

2) if $m = 1 - \nu$ then

$$(4.51) \qquad ||(P_N y)' - y'||_\infty \le c \begin{cases} h^m (1 + |\log h|) & \text{for} \quad r = 1, \\ h^m & \text{for} \quad r > 1; \end{cases}$$

3) if $m > 1 - \nu$ then

$$(4.52) \qquad ||(P_N y)' - y'||_\infty \le c \begin{cases} h^{r(1-\nu)} & \text{for} \quad 1 \le r \le m/(1-\nu), \\ h^m & \text{for} \quad r > m/(1-\nu). \end{cases}$$

We now return to $|u'_j(t) - (P_N y)'_j(t)|$, $t \in \sigma_j$, $j = 1, \dots, N$, the first summand on the right-hand side of the inequality (4.49). We have

$$||u - P_N y||_\infty \leq ||u - y||_\infty + ||y - P_N y||_\infty. \tag{4.53}$$

We have already found the estimations for $||u - y||_\infty$ and we may use Lemma 3.1 to estimate $||y - P_N y||_\infty$. By (3.9), (3.10), (3.12), (4.31), (4.34), (4.37) and (4.53) we find the following estimations for $||u - P_N y||_\infty$:

1) if $m > 1 - \nu$ then

$$||u - P_N y||_\infty \leq c h^{m+1} \quad \text{for} \quad r \geq 1; \tag{4.54}$$

2) if $m = 1 - \nu$ then

$$||u - P_N y||_\infty \leq c \begin{cases} h^{m+1}(1 + |\log h|) & \text{for} \quad r = 1, \\ h^{m+1} & \text{for} \quad r > 1; \end{cases} \tag{4.55}$$

3) if $m > 1 - \nu$ then

$$||u - P_N y||_\infty \leq c \begin{cases} h^{r(2-\nu)} & \text{for} \quad 1 \leq r \leq (m+1)/(2-\nu), \\ h^{m+1} & \text{for} \quad r > (m+1)/(2-\nu). \end{cases} \tag{4.56}$$

We now use Lemma 4.1 to derive upper bounds for the derivative of $u - P_N y$ since $u_j(t) - (P_N y)_j(t)$, $t \in \sigma_j$ $(j = 1, \dots, N)$ is a polynomial of degree not exceeding m. For $r = 1$ we obtain the following estimates:

1) if $m > 1 - \nu$ then

$$||u' - (P_N y)'||_\infty \leq c h^m; \tag{4.57}$$

2) if $m = 1 - \nu$ then

$$||u' - (P_N y)'||_\infty \leq c h^m (1 + |\log h|); \tag{4.58}$$

3) if $m > 1 - \nu$ then

$$||u' - (P_N y)'||_\infty \leq c h^{1-\nu}. \tag{4.59}$$

If $r > 1$ then we choose a small $\varepsilon > 0$ $(\varepsilon < T)$ and estimate $|u'(t) - (P_N y)'(t)|$ for $t \in [\varepsilon, T]$. If $t \in [\varepsilon, T]$ when $t \in [t_{j_0-1}, t_{j_0}]$ with some $j_0 \in \{1, \dots, N\}$. For $j_0 = 1$, $t \in [\varepsilon, t_1]$ we have

$$h^{m+1}/t_1 \leq h^{m+1}/\varepsilon \leq (T/\varepsilon) h^m.$$

Let $2 \leq j_0 \leq N$. Then

$$t \geq t_{j_0-1} = T(j_0 - 1)^r N^{-r} \geq \varepsilon$$

implies

$$\begin{aligned} t_{j_0} - t_{j_0-1} &= T N^{-r}(j_0^r - (j_0 - 1)^r) \geq T N^{-r} r (j_0 - 1)^{r-1} \\ &\geq (j_0 - 1)^{-1} r \varepsilon \geq N^{-1} r \varepsilon. \end{aligned}$$

Therefore, by (3.14),

$$h^{m+1}/(t_{j_0} - t_{j_0-1}) \le (T/\varepsilon)h^m.$$

Since $t_{j+1} - t_j \ge t_j - t_{j-1}$, $j = 1, \dots, N$, we now obtain that

$$h^{m+1}/(t_j - t_{j-1}) \le (T/\varepsilon)h^m, \quad j = 1, \dots, N.$$

Using Lemma 4.1 we obtain from this and (4.54)–(4.56) that, for $N \ge N_0$, the following error estimates hold:

1) if $m \le 1 - \nu$ then

$$||u' - (P_N y)'||_{\varepsilon,\infty} \le c_\varepsilon h^m \quad \text{for} \quad r > 1; \tag{4.60}$$

2) if $m > 1 - \nu$ then

$$||u' - (P_N y)'||_{\varepsilon,\infty} \le c_\varepsilon \begin{cases} h^{r(2-\nu)-1} & \text{for} \quad 1 < r \le \frac{m+1}{2-\nu}, \\ h^m & \text{for} \quad r > \frac{m+1}{2-\nu}. \end{cases} \tag{4.61}$$

Here c_ε is a positive constant which is independent of N (of h), possibly $c_\varepsilon \to \infty$ as $\varepsilon \to 0$, and (see (4.41)

$$||u'(t) - (P_N y)'||_{\varepsilon,\infty} = \sup_{\varepsilon \le t \le T} |u'(t) - (P_N y)'(t)|, \quad 0 < \varepsilon \le T.$$

Combining (4.49)–(4.52) and (4.57)–(4.61) we obtain the statements (4.32), (4.33), (4.35), (4.36), (4.38) and (4.39) of Theorem 4.3. □

It follows from Theorem 4.3 that, for Method 2, the maximal possible order $O(h^{m+1})$ for the error $||u - y||_\infty$ can be achieved using sufficiently large values of r, the scaling parameter of the grid $\{(3.1), (3.2)\}$. There are possibilities to reduce $r \ge 1$ restricting ourselves to $L^P(0,T)$ estimates of the error $u - y$ (Theorem 4.4) or to uniform estimates at the collocations points (4.16) only (Theorem 4.5). Moreover, the convergence rate at the collocation points will be higher than $O(h^{m+1})$ for special choice of collocation parameters (4.17) (Theorem 4.6).

Lemma 4.4. *Let $a \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, and let S_2 be defined by (4.43). Then S_2 is compact as an operator from $L^p(0,T)$, $p > 1$, to $C[0,T]$.*

Proof. We have $S_2 = A + S_4$ where S_4 is defined by (4.45) and

$$(Ay)(t) = \int_0^t a(s)y(s)ds, \quad t \in [0,T].$$

Due to Lemma 2.4, $K_4 \in \mathcal{W}^{m,\nu-1}(\Delta_T)$. Let $y \in L^p(0,T)$, $||y||_{L^p(0,T)} \le 1$, $p > 1$, $q = p/(p-1)$. Then $(1/p) + (1/q) = 1$ and, by Hölder inequality,

$$\max_{0 \le t \le T} \big|(Ay)(t)\big| \le c_1, \quad \max_{0 \le t \le T} \big|(S_4 y)(t)\big| \le c_2,$$

where

$$c_1 = \Big(\int\limits_0^T |a(s)|^q ds\Big)^{1/q}, \quad c_2 = \max_{0\le t\le T}\Big(\int\limits_0^t |K_4(t,s)|^q ds\Big)^{1/q}.$$

Further, for any $\varepsilon > 0$ there exists a $\delta > 0$ such the, for $0 \le t_1 < t_2 \le T$, $t_2 - t_1 \le \delta$, and $y \in L^p(0,T)$, $||y||_{lp(0,T)} \le 1$, $p > 1$, we obtain

$$\big|(Ay)(t_1) - (Ay)(t_2)\big| \le \big(\int\limits_{t_1}^{t_2} |a(s)|^q ds)^{1/\varepsilon} \le \varepsilon,$$

$$\big|(S_4y)(t_1) - (S_4y)(t_2)\big| \le \int\limits_0^{t_1} |K_4(t_1,s) - K_4(t_2,s)||y(s)|ds + \int\limits_{t_1}^{t_2} |K_4(t_2,s)||y(s)|ds$$
$$\le [(\int\limits_0^{t_1} |K_4(t_1,s) - K_4(t_2,s)|^q ds)^{1/q} + (\int\limits_{t_1}^{t_2} |K_4(t_2,s)|^q ds)^{1/q}] \le \varepsilon.$$

By Arzelá-Ascoli theorem, A and S_4 and thus also S_2 are compact as operators from $L^p(0,T)$ to $C[0,T]$, $p > 1$. □

Theorem 4.4. *Let the conditions of Theorem 4.3 be fulfilled. Then, with designations of Theorem 4.3, the following error estimates hold:*

1) if $m \le 1 - \nu$ *then*

$$(4.62) \qquad ||u - y||_{L^p(0,T)} \le ch^{m+1} \quad \textit{for} \quad r \ge 1, \ 1 \le p < \infty;$$

2) if $m > 1 - \nu$ *then for* $1 < p < \infty$,

(4.63)

$$||u-y||_{L^p(0,T)} \le c \begin{cases} h^{r(2-\nu+\frac{1}{p})} & \textit{for} \quad 1 \le r < (m+1)/(2-\nu+\frac{1}{p}), \\ & \qquad m+1 > 2-\nu+\frac{1}{p}, \\ h^{m+1}(1+|\log h|)^{\frac{1}{p}} & \textit{for} \quad r = (m+1)/(2-\nu+\frac{1}{p}), \\ & \qquad m+1 \le 2-\nu+\frac{1}{p}, \\ h^{m+1} & \textit{for} \quad r > (m+1)/(2-\nu+\frac{1}{p}), \ r \ge 1. \end{cases}$$

Proof. We repeat the argument of the proof of the estimates (4.31), (4.34) and (4.37) in Theorem 4.3 with $L^p(0,T)$ instead of $L^\infty(0,T)$.

We use the operator equation representations (4.42) and (4.47) for the Cauchy problem $\{(1.1), (1.2)\}$ and collocation conditions (4.15), respectively. By Lemma 4.4, S_2 is compact as an operator from $L^p(0,T)$ to $C[0,T]$ for $p > 1$.

As $f_2 \in C[0,T]$ and the homogeneous equation $y = S_2y$ has only the trivial solution $y = 0$, equation (4.42) has a unique solution $y \in L^p(0,T)$, $p > 1$. Due to Theorem 2.1, $y \in C^{m+1,\nu-1}(0,T]$.

Further, we have

$$||x - P_Nx||_{L^p(0,T)} \to 0 \quad \text{as} \quad N \to \infty \quad \text{for every} \quad x \in C[0,T],$$

and together with the compactness of $S_2\colon\ L^p(0,T) \to C[0,T]$ we derive that

$$||S_2 - P_N S_2||_{\mathcal{L}(L^p(0,T),L^p(0,T)} \to 0 \quad \text{as} \quad N \to \infty.$$

>From this and from the boundedness of $(1-S_2)^{-1}$ in $L^p(0,T)$ we obtain that $I - P_N S_2$ is invertible in $L^p(0,T)$ for all sufficiently large N, say $N \geq N_0$, and

$$||(1-P_N S_2)^{-1}||_{\mathcal{L}(L^p(0,T),L^p(0,T))} \leq \text{const}, \quad N \geq N_0,\ p > 1.$$

Thus, for $N \geq N_0$, equation (4.47) provides a unique solution $u \in S_m^{(-1)}(\Pi_N)$ and we establish that

$$||u - y||_{L^p(0,T)} \leq \text{const}||y - P_N y||_{L^p(0,T)}, \quad N \geq N_0,\ p > 1,$$

with y, the solution of the Cauchy problem $\{(1.1),(1.2)\}$. Applying Lemma 3.1 we obtain the estimations (4.62) and 4.63) for $p > 1$. Since the right-hand side of the inequality (4.62) does not depend on p, (4.62) holds also by smaller p, i.e. for $1 \leq p < \infty$. □

Theorem 4.5. *Let $a \in C^{m+1}[0,T]$, $b \in C^m[0,T]$, $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, $y_0 \in \mathbb{C}$, and let the collocation points (4.16) with grid points (3.2) and parameters (4.17) be used. Let the scaling parameter $r = r(m,\nu \geq 1$ be restricted by the conditions (4.26).*

Then there exists $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the collocation condition (4.15) define a unique approximation $u \in S_m^{(-1)}(\Pi_N)$ (if $\eta_1 = 0$, $\eta_{m+1} = 1$ then $u \in S_m^{(0)}(\Pi_N)$) to y, the solution of the Cauchy problem $\{(1.1),(1.2)\}$, and the error estimate (4.27) holds.

Proof. Due to Section 1, the Cauchy problem $\{(1.1),(1.2)\}$ is equivalent to the integral equation (1.13) with the forcing function f_2 and the kernel K_2 defined by (1.14) and (1.15), respectively. It follows from $b \in C^m[0,T]$ that $f_2 \in C^{m+1}[0,T] \subset C^{m+1,\nu}(0,T]$. By Lemma 2.4, $K_2 \in \mathcal{W}^{m+1,\nu}(\Delta_T)$. Applying Lemma 4.2 we obtain the estimation (4.27). □

Theorem 4.6. *Let the following conditions be fulfilled:*

1) $a \in C^{m+\mu+2}[0,T]$, $b \in C^{m+\mu+1}[0,T]$ $K \in \mathcal{W}^{m+\mu+2,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\mu \in \mathbb{Z}$, $0 \leq \mu \leq m$, $-\infty < \nu < 1$.

2) The collocation points (4.16) are generated by the grid points (3.2) and by the knots (4.17) of a quadrature approximation (4.28) which is exact for all polynomials of degree $m+1+\mu$, where μ is an integer satisfying $0 \leq \mu \leq m$.

3) The scaling parameter $r = r(m,\nu,\mu) \geq 1$ is subject to restrictions (4.29).

Then there exists $N_0 \in \mathbb{N}$ such that, for $N \geq N_0$, the collocation conditions (4.15) determine a unique approximation $u \in S_m^{(-1)}(\Pi_N)$ (if $\eta_1 = 0$, $\eta_{m+1} = 1$ then $u \in S_m^{(0}_)(\Pi_N)$) to the solution y of the Cauchy problem $\{(1.1),(1.2)\}$ and the error estimate (4.30) holds.

Proof. Using arguments analogous to those in the proof of Theorem 4.5 we find that the kernel (1.15) of the equation (1.13) belongs to the set $\mathcal{W}^{m+\mu+2,\nu}(\Delta_T)$ and the forcing function (1.14) belongs to the set $C^{m+\mu+2,\nu}(0,T]$. Applying Lemma 4.3 we obtain the estimate (4.30). □

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