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Abstract: ***We show that it is possible to construct arbitrary order stable schemes for the homogeneous and heterogeneous wave equation in any dimension. The construction is elementary and uses formal power series techniques. We shall also calculate exact stability limits in various cases, and apparently this limit depends only on the dimension of the space.***

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1. Introduction

In recent years there has appeared some papers dealing with the construction of high order schemes for wave equation and other hyperbolic equations, see [1], [3], [4], [6], [8], [9], [10], [11], [14] and references therein.

In this report we show the existence of arbitrary order explicit schemes for wave equation using only very classical formal power series techniques which were used already by Boole, see [2]. This approach was introduced in [12] to treat the one-dimensional case and as will be seen it can also be applied to the many-dimensional case as well. As far as we know the many-dimensional case has not been treated previously in full generality. We treat first the constant propagation speed case. In this case the construction does not guarantee in itself the stability of the scheme, but we will prove that the schemes obtained are in fact stable.

To treat the heterogeneous case we construct an operator $\mathcal{T}$ such that $\mathcal{T}^t\mathcal{T}$ is an approximation to $-\Delta$; the stability then follows from the positive definiteness of $\mathcal{T}^t\mathcal{T}$, and this evidently yields a stable scheme also for the equation $u_{tt} - \nabla \cdot (\mu \nabla u) = 0$. This idea was already used in [11] and [14].

Finally we study the stability limits of the schemes. Various computations suggest that this limit depends only on the dimension of the space and not on the order of the scheme. In homogeneous one-dimensional case the relevant limit was proved in [8] using order stars. We prove this and the corresponding result for the heterogeneous scheme in an elementary way.

2. Preliminaries

We shall first introduce some notations which were used in [12]. Let us denote by ∂_i the partial derivative with respect to x_i and by ∂_t the partial derivative with respect to t. Let h denote the space discretization step and h_t the time step and define $\beta := h_t^2/h^2$. We define the following formal power series

$$\begin{aligned}
\delta_{i+} &:= (\exp(h\partial_i) - 1)/h \\
\delta t_+ &:= (\exp(h_t\partial_t) - 1)/h_t \\
\delta_{i-} &:= (1 - \exp(-h\partial_i))/h \\
\delta t_- &:= (1 - \exp(-h_t\partial_t))/h_t
\end{aligned}$$

Let us further define $\delta_+ := (\delta_{1+}, \ldots, \delta_{m+})$ and $\delta_- := (\delta_{1-}, \ldots, \delta_{m-})$. Now if $P = P(\delta_+, \delta_-, \delta t_+, \delta t_-)$ is some polynomial then Q is said to be *transpose* of P if $Q = P(-\delta_-, -\delta_+, -\delta t_-, -\delta t_+)$ and as usual we note this by $Q = P^t$. Moreover, we say that P is *symmetric* if $P = P^t$. In particular $\delta_{i+} = -\delta_{i-}^t$.

Let us finally recall the following lemma which is needed in stability questions.

Lemma 2.1. *Suppose that $\mathcal{A} : l^2(h\mathbb{Z}^m) \to l^2(h\mathbb{Z}^m)$ is symmetric and positive definite. Then the scheme*

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{h_t^2} + \mathcal{A}u^n = 0$$

is stable for $h_t^2 \|\mathcal{A}\| \leq 4$.

3. Construction of the schemes

Let us consider the following problem

$$\square u = u_{tt} - \Delta u = 0 \tag{3.1}$$

Theorem 3.1. *For all p there exists a symmetric difference approximation of order 2p for the equation* (3.1).

Proof. First using $\delta t_+ \delta t_-$ for the time discretization we get

$$\begin{aligned}\delta t_+ \delta t_- &= \partial_t^2 + \frac{h_t^2}{12}\partial_t^4 + \cdots + \frac{2h_t^{2n}}{(2n+2)!}\partial_t^{2n+2} + \cdots = \\ &\partial_t^2 + \frac{h_t^2}{12}\Delta^2 + \cdots + \frac{2h_t^{2n}}{(2n+2)!}\Delta^{n+1} + \cdots = \\ &\partial_t^2 + \frac{\beta h^2}{12}\Delta^2 + \cdots + \frac{2\beta^n h^{2n}}{(2n+2)!}\Delta^{n+1} + \ldots\end{aligned}$$

Let us put $\mathcal{T}_1 = -\sum_{i=1}^m \delta_{i+}\delta_{i-}$. Then clearly

$$\delta t_+ \delta t_- + \mathcal{T}_1 - \square = O(h^2)$$

and so the theorem holds for $p = 1$. Then let us suppose that it holds for some p; more precisely we suppose that there exist an operator $\mathcal{T}_p$ such that

$$\delta t_+ \delta t_- + \mathcal{T}_p - \square = O(h^{2p})$$

We can also take $\mathcal{T}_p$ to be symmetric because $\mathcal{T}_1$ is symmetric. To simplify the notations let us denote ∂_i^2 by d_i and let $\nu \in \mathbb{N}^m$ be a multi-index with $|\nu| = \nu_1 + \cdots + \nu_m$. Then the hypothesis above states that

$$\delta t_+ \delta t_- + \mathcal{T}_p - \square = h^{2p} \sum_{|\nu|=p} a_\nu d^\nu + O(h^{2p+2})$$

Note that by symmetry $a_\nu = a_\rho$ if ρ is some permutation of ν. Now introducing the notation $(\delta_+\delta_-)^\nu = \prod_{i=1}^m \delta_{i+}^{\nu_i}\delta_{i-}^{\nu_i}$ we obviously get

$$(\delta_+\delta_-)^\nu = d^\nu + O(h^2)$$

So if we put

$$\mathcal{T}_{p+1} = \mathcal{T}_p + h^{2p} \sum_{|\nu|=p} b_\nu (\delta_+\delta_-)^\nu \tag{3.2}$$

where $b_\nu = -a_\nu$ this clearly implies that

$$\delta t_+ \delta t_- + \mathcal{T}_{p+1} - \square = O(h^{2p+2})$$

Moreover every term is seen to be necessary so this construction is minimal. □

Let us then pass to the heterogeneous case.

$$u_{tt} - \nabla \cdot (\mu(x)\nabla u) = 0 \tag{3.3}$$

This problem is well-posed if $\mu(x) \geq \mu_* > 0$.

Theorem 3.2. *For all p there exist a stable symmetric difference scheme for the equation* (3.3) *which is of order 2p if μ is constant.*

Proof. To get stable schemes we explicitly construct discrete divergence $\mathcal{D}$ and discrete gradient $\mathcal{G}$ such that $\mathcal{D} = -\mathcal{G}^t$. Then $-\mathcal{D}\mu\mathcal{G}$ is positive definite if μ is positive and the stability follows by lemma 2.1. Hence to conclude we just need to show how to construct schemes of arbitrary order of the above form for the homogeneous case.

Let us start from the case $p = 1$ and put $\mathcal{D}_1 := (\delta_{1+}, \ldots, \delta_{m+})$ and $\mathcal{G}_1 := -\mathcal{D}_1^t = (\delta_{1-}, \ldots, \delta_{m-})^t$. Then clearly

$$\delta t_+ \delta t_- - \mathcal{D}_1\mathcal{G}_1 - \square = O(h^2)$$

Of course this is the same scheme that we started from in the previous theorem. Then let us suppose that we have a scheme of order $2p$, i.e.

$$\delta t_+ \delta t_- - \mathcal{D}_p\mathcal{G}_p - \square = h^{2p} \sum_{|\nu|=p} a_\nu d^\nu + O(h^{2p+2})$$

where $\mathcal{D}_p = -\mathcal{G}_p^t$. Then defining $q_j := -1 + \max_{i \neq j} \nu_i$ and denoting the j'th component of $\mathcal{D}_p$ by $\mathcal{D}_p^j$ we make the following ansatz

$$\mathcal{D}_{p+1}^j = \mathcal{D}_p^j + h^{2p}\delta_{j+} \sum_{\substack{|\nu|=p-1 \\ \nu_j \geq q_j}} c_\nu^j (\delta_+\delta_-)^\nu \tag{3.4}$$

We do the similar ansatz for other components. Now by symmetry $c_\nu^j = c_\rho^j$ if ν is some permutation of ρ and $\nu_j = \rho_i$. Then we evidently put $\mathcal{G}_{p+1}^j = -\left(\mathcal{D}_p^j\right)^t$ which leads to

$$\mathcal{D}_{p+1}^j \mathcal{G}_{p+1}^j = \mathcal{D}_p^j \mathcal{G}_p^j + 2h^{2p}\delta_{j+}\delta_{j-} \sum_{\substack{|\nu|=p-1 \\ \nu_j \geq q_j}} c_\nu^j (\delta_+\delta_-)^\nu + O(h^{2p+2})$$

Let us define $\nu + 1_j := (\nu_1, \ldots, \nu_j + 1, \ldots, \nu_m)$. Further let us define l_ν^j by

$$l_\nu^j := 1 + \sum_{\substack{\nu_i = 1+\nu_j \\ 1 \le i \le m}} 1$$

Then we get

$$\delta t_+ \delta t_- - \mathcal{D}_{p+1}\mathcal{G}_{p+1} - \Box = h^{2p} \sum_{\substack{|\nu|=p-1 \\ 1\le j\le m}} \left(a_{\nu+1_j} - 2 l_\nu^j c_\nu^j\right) d^{\nu+1_j} + O(h^{2p+2})$$

So we merely put $c_\nu = a_{\nu+1_j}/2l_\nu^j$ (the coefficient l_ν^j is needed because of some overlapping in the above sum). □

4. Some computations

We shall compute the coefficients for the schemes up to order 12 in three dimensional case. Note that the two dimensional coefficients are obtained simply by taking only the coefficients where the third index is zero and the one dimensional ones are those with only the first index nonzero. The computations were done using *axiom*, see [7]. By the construction of the theorem 3.1 we have

$$\mathcal{T}_p = \sum_{1 \le |\nu| \le p} b_\nu h^{2|\nu|-2} (\delta_+\delta_-)^\nu \tag{4.1}$$

Evidently $b_{(1,0,0)} = -1$ and in table 1 there are coefficients for $2 \le |\nu| \le 6$. Recall that by symmetry it is sufficient to give only the coefficients with $\nu_1 \ge \nu_2 \ge \nu_3$. Note that the table 1 yields also the coefficients for fourth and sixth order schemes in *any* dimension, since in these cases only two or three components of multi-indices can be nonzero.

For the heterogeneous scheme it is sufficient to consider D_p^1 which is given by

$$\mathcal{D}_p^1 = \delta_{1+} \sum_{\substack{0 \le |\nu| \le p-1 \\ \nu_1 \ge q_1}} h^{2|\nu|} c_\nu^1 (\delta_+\delta_-)^\nu \tag{4.2}$$

Now $c_{(0,0,0)} = 1$ and in table 2 we have the coefficients c_ν^1 for $1 \le |\nu| \le 5$ and $\nu_1 \ge q_1$. For same reasons as above these give also the relevant coefficients for fourth and sixth order schemes in any dimension.

In both tables there do not seem to be any immediate rule giving the polynomials explicitly, like in one-dimensional case, see below. However, define $\nu! := \nu_1! \ldots \nu_m!$. Then in table 1 we observe that the coefficient of the highest power of β is given by $-|\nu|!/\nu!$.

ν	$(2\|\nu\|)! b_\nu/2$	ν	$(2\|\nu\|)! b_\nu/2$
(2,0,0)	$1-\beta$	(1,1,0)	-2β
(3,0,0)	$-4+5\beta-\beta^2$	(2,1,0)	$5\beta-3\beta^2$
(1,1,1)	$-6\beta^2$	(4,0,0)	$36-49\beta+14\beta^2-\beta^3$
(3,1,0)	$-112\beta/3+28\beta^2-4\beta^3$	(2,2,0)	$-70\beta/3+28\beta^2-6\beta^3$
(2,1,1)	$28\beta^2-12\beta^3$	(5,0,0)	$-576+820\beta-273\beta^2+30\beta^3-\beta^4$
(4,1,0)	$540\beta-441\beta^2+90\beta^3-5\beta^4$	(3,2,0)	$280\beta-378\beta^2+120\beta^3-10\beta^4$
(3,1,1)	$-336\beta^2+180\beta^3-20\beta^4$	(2,2,1)	$-210\beta^2+180\beta^3-30\beta^4$
(6,0,0)	$14400-21076\beta+7645\beta^2-1023\beta^3+55\beta^4-\beta^5$		
(5,1,0)	$-12672\beta+10824\beta^2-2574\beta^3+220\beta^4-6\beta^5$		
(4,2,0)	$-5940\beta+8415\beta^2-3069\beta^3+385\beta^4-15\beta^5$		
(4,1,1)	$7128\beta^2-4158\beta^3+660\beta^4-30\beta^5$		
(3,3,0)	$-4928\beta+7392\beta^2-3036\beta^3+440\beta^4-20\beta^5$		
(3,2,1)	$3696\beta^2-3564\beta^3+880\beta^4-60\beta^5$		
(2,2,2)	$2310\beta^2-2970\beta^3+990\beta^4-90\beta^5$		

Table 1. Coefficients for the three-dimensional schemes of theorem 3.1 up to order 12.

ν	$4^{\lvert\nu\rvert}(2\lvert\nu\rvert+1)!c_\nu$	ν	$4^{\lvert\nu\rvert}(2\lvert\nu\rvert+1)!c_\nu$
(1,0,0)	$-1+\beta$	(0,1,0)	β
(2,0,0)	$9-10\beta+\beta^2$	(1,1,0)	$-10\beta+3\beta^2$
(0,1,1)	$2\beta^2$	(3,0,0)	$-225+259\beta-35\beta^2+\beta^3$
(2,1,0)	$497\beta/3-63\beta^2+4\beta^3$	(1,2,0)	$280\beta/3-42\beta^2+3\beta^3$
(1,1,1)	$-70\beta^2+12\beta^3$	(4,0,0)	$11025-12916\beta+1974\beta^2-84\beta^3+\beta^4$
(3,1,0)	$-6868\beta+11186\beta^2/5-72\beta^3-38\beta^4/5$		
(2,2,0)	$-6048\beta+18424\beta^2/5-516\beta^3+113\beta^4/5$		
(2,1,1)	$812\beta^2-96\beta^3-26\beta^4/5$	(1,2,1)	$1568\beta^2-456\beta^3+138\beta^4/5$
(5,0,0)	$-893025+1057221\beta-172810\beta^2+8778\beta^3-165\beta^4+\beta^5$		
(4,1,0)	$1519199\beta/3-157674\beta^2+35552\beta^3/3-4994\beta^4/3+171\beta^5$		
(3,2,0)	$340560\beta-217844\beta^2+33814\beta^3-1793\beta^4+15\beta^5$		
(3,1,1)	$-154748\beta^2/3+59576\beta^3/3-13882\beta^4/3+426\beta^5$		
(2,3,0)	$630784\beta/3-142912\beta^2+47344\beta^3/3+2948\beta^4/3-155\beta^5$		
(2,2,1)	$-342496\beta^2/3+87736\beta^3/3-2134\beta^4/3-105\beta^5$		
(1,2,2)	$-197120\beta^2/3+80960\beta^3/3-2200\beta^4-36\beta^5$		

TABLE 2. Coefficients for the three-dimensional schemes of theorem 3.2 up to order 12.

5. STABILITY

5.1. Arbitrary dimension, arbitrary order. As we have remarked the construction in theorem 3.1 does not in itself guarantee stability of the schemes. Let us prove that they are in fact stable for all p. To this end we use plane wave (or Fourier) analysis. This leads to replacing $\delta_{i+}\delta_{i-}$ by $-4\sin^2(k_ih)/h^2$ in equation (4.1) where k_i is the i'th component of the wave vector. Carrying out the substitutions, denoting $\sin^2(k_ih)$ by x_i where $0 \le x_i \le 1$ and canceling by the factor 4, the stability condition can be written in the form

$$0 \le \beta f_{(p,m)}(x,\beta) \le 1 \tag{5.1}$$

where m refers to the dimension of space. Note that $f_{(p,m)}$ does not explicitly depend on h. Referring to lemma 2.1 we note that the first inequality implies the positive definiteness of the operator and the second inequality gives the actual bound for the time-step. Now it is obvious that this second inequality is valid for β sufficiently small, so we just have to take care of the first one. This is done in the following

Theorem 5.1. *All the schemes obtained in theorem* 3.1 *are stable for sufficiently small* β.

Proof. We have to prove that $f_{(p,m)}(x,\beta) \ge 0$ for β small and for all $0 \le x_i \le 1$. Let $s_k(x) = x_1^k + \cdots + x_m^k$; then the construction shows that $f_{(p,m)}$ can be written in the form

$$f_{(p,m)}(x,\beta) = S_p(x) - \beta\tilde{S}_p(x,\beta) = \sum_{k=1}^{p} a_k s_k(x) - \beta\tilde{S}_p(x,\beta)$$

Now the coefficients of (4.1) in the one-dimensional case are given by, see [12]

$$b_k = (-1)^k \frac{2}{(2k)!}\prod_{l=1}^{k-1}\left(l^2-\beta\right) \tag{5.2}$$

Consequently we obtain

$$a_k = \frac{2^{2k-1}((k-1)!)^2}{(2k)!} = \frac{1}{k}\prod_{j=1}^{k-1}\frac{2j}{2j+1}$$

In particular all a_k are positive which implies that $S_p(0) = 0$ but $S_0(x) > 0$ for all $x \ne 0$. Further, it is seen that $\tilde{S}_p(x,\beta) = O(x^2)$ at the origin. But this means that if we are sufficiently near the origin then S_p is big with respect to $\tilde{S}_p$, so $f_{(p,m)}$ is positive at least in a small neighborhood of origin. But then outside this neighborhood S_p is bounded away from zero because of the positivity of the coefficients a_k and consequently $\min S_p/\max\tilde{S}_p > 0$ where the minimum and maximum are taken in the cube $0 \le x_i \le 1$ with some neighborhood of origin excluded. Hence if we choose some $\beta < \min S_p/\max\tilde{S}_p$, then $f_{(p,m)}$ is positive for all x. □

Note that the positivity of some coefficients coming from the *one*-dimensional case allowed us to prove the stability in *any* dimension. The above proof gives no indication of the actual stability limit. We shall in the following sections calculate the limits in various cases.

5.2. Arbitrary dimension, order up to six. In this section we shall prove some exact stability limits for low order schemes in any dimension. In case of schemes of theorem 3.2 we write the stability condition as

$$0 \le \beta g_{(p,m)}(x,\beta) \le 1$$

Let us start with the easy

Lemma 5.1. *The stability condition for the second order scheme is $\beta \le 1/m$.*

Proof. The claim follows because $f_{(1,m)} = g_{(1,m)} = x_1 + \cdots + x_m$. □

Proposition 5.1. *The stability condition for fourth order scheme of theorem* 3.1 *in m dimensions is $\beta \le 1/m$.*

Proof. In this case the stability condition is

$$0 \le \beta f_{(2,m)}(x,\beta) = \beta\Big(\sum_{i=1}^m x_i + \frac{1}{3}\sum_{i=1}^m x_i^2 - \frac{\beta}{3}\Big(\sum_{i=1}^m x_i\Big)^2\Big) \le 1$$

We regard this as an optimization problem: find the extrema of $f_{(2,m)}$ subject to constraints $0 \le x_i \le 1$. In general we should examine the boundaries and the zeros of gradient. In fact there is no need to check separately boundaries and the inside since it will turn out that *every* component of the gradient is positive. Hence the global minimum is attained at zero and the global maximum at $x = (1,\ldots,1)$. Evidently $f_{(2,m)}(0,\beta) = 0$, so this implies the first inequality above. Now using the notation $\gamma := m\beta$ and putting $x = (1,\ldots,1)$ we have

$$\gamma\big(4-\gamma\big) \le 3$$

which gives $\beta \le 1/m$. So to conclude we just compute the derivative

$$\frac{\partial f_{(2,m)}}{\partial x_k} = 1 + \frac{2}{3}x_k - \frac{2\beta}{3}\sum_{i=1}^m x_i \ge \frac{1+2x_k}{3} > 0$$

where we have used the estimate $\beta \sum x_i \le 1$. □

Proposition 5.2. *The stability condition for the sixth order scheme of theorem* 3.1 *in m dimensions is $\beta \le 1/m$.*

Proof. In this case we have

$$f_{(3,m)}(x,\beta) = \sum_{i=1}^m x_i + \frac{1}{3}\sum_{i=1}^m x_i^2 - \frac{\beta}{3}\Big(\sum_{i=1}^m x_i\Big)^2 + \frac{8}{45}\sum_{i=1}^m x_i^3 - \frac{2\beta}{9}\sum_{j=1}^m x_j^2 \sum_{j=1}^m x_j + \frac{2\beta^2}{45}\Big(\sum_{i=1}^m x_i\Big)^3$$

Evaluating at $(1,\ldots,1)$ gives

$$0 \le \gamma(2\gamma^2 - 25\gamma + 68) \le 45$$

This yields the condition $\beta \le 1/m$. Then calculating the derivative gives

$$\frac{\partial f_{(3,m)}}{\partial x_k} = 1 + \frac{2}{3}x_k - \frac{2\beta}{3}\sum_{i=1}^m x_i - \frac{2\beta}{9}\sum_{i=1}^m x_i^2 + \frac{2}{15}\Big(2x_k - \beta\sum_{i=1}^m x_i\Big)^2 + \frac{4\beta}{45}x_k\sum_{i=1}^m x_i \ge 1 + \frac{2}{3}x_k - \frac{2}{3} - \frac{2}{9} > 0$$

This shows that the global maximum is at the corner. □

Proposition 5.3. *The stability condition for fourth order scheme of theorem* 3.2 *in m dimensions is $\beta \le 1/m$.*

Proof. We have

$$g_{(2,m)}(x,\beta) = \sum_{i=1}^m x_i d_{(2,m,i)}^2$$

where

$$d_{(2,m,i)} = 1 + \frac{1}{6}x_i - \frac{\beta}{6}\sum_{j=1}^m x_j$$

Evaluating at the corner $x = (1,\ldots,1)$ leads to

$$\gamma(7-\gamma)^2 \le 36$$

which immediately yields the necessary condition $\beta \le 1/m$. To conclude we show that the global maximum of $g_{(2,m)}$ is attained at this corner for all $0 \le \beta \le 1/m$. As in the previous proposition we shall show that every component of the gradient is positive. Computing the derivative with respect to x_k gives

$$\frac{\partial g_{(2,m)}}{\partial x_k} = d_{(2,m,k)}^2 + 2\sum_{i=1}^m x_i d_{(2,m,i)} \frac{\partial d_{(2,m,i)}}{\partial x_k}$$

Straightforward estimates show that

$$\frac{5}{6} \le d_{(2,m,i)} \le \frac{7}{6}$$
$$\frac{\partial d_{(2,m,k)}}{\partial x_k} \ge \frac{m-1}{6m}$$
$$\frac{\partial d_{(2,m,i)}}{\partial x_k} \ge -\frac{1}{6m}$$

where in the last inequatlity $i \neq k$. From this we compute

$$\frac{\partial g_{(2,m)}}{\partial x_k} \ge \frac{25}{36} + \frac{5m-5}{18m}x_k - \frac{7}{18m}\sum_{i\neq k} x_i$$
$$\ge \frac{25}{36} - \frac{7m-7}{18m} > \frac{25}{36} - \frac{7}{18} = \frac{11}{36} > 0$$

Consequently the global maximum is attained at the corner. □

Proposition 5.4. *The stability condition for sixth order scheme of theorem 3.2 in m dimensions is $\beta \le 1/m$.*

Proof. We have

$$g_{(3,m)}(x,\beta) = \sum_{i=1}^{m} x_i d^2_{(3,m,i)}$$

where

$$d_{(3,m,i)} = d_{(2,m,i)} + \frac{3}{40}x_i^2 - \frac{\beta}{12}x_i\sum_{j=1}^{m} x_j + \frac{\beta^2}{120}\Big(x_i\sum_{j=1}^{m} x_j + 2\sum_{1\le j<l\le m} x_j x_l\Big)$$

Evaluating $g_{(3,m)}$ at the corner $x = (1, \dots, 1)$ leads to

$$\gamma(149 - 30\gamma + \gamma^2)^2 \le 14400$$

which gives the preliminary bound $\beta \le 1/m$. Then using the estimates of lemma 5.2 below we get for $m > 1$ (the case $m = 1$ can of course be checked directly)

$$\frac{\partial g_{(3,m)}}{\partial x_k} \ge \frac{25}{36} + \frac{5m-10}{36m}x_k - \frac{5}{8m}\sum_{i\neq k} x_i$$
$$\ge \frac{25}{36} - \frac{5m-5}{8m} > \frac{25}{36} - \frac{5}{8} = \frac{5}{72} > 0$$

Consequently every component of the gradient is positive, which implies that the golobal maximum is attained at the corner. □

Lemma 5.2.

$$\frac{5}{6} \le d_{(3,m,i)} \le \frac{5}{4}$$
$$\frac{\partial d_{(3,m,k)}}{\partial x_k} \ge \frac{m-2}{12m}$$
$$\frac{\partial d_{(3,m,i)}}{\partial x_k} \ge -\frac{1}{4m}$$

Proof. Using the inequality $\beta \sum x_j \le 1$ we obtain

$$d_{(3,m,i)} \ge 1 + \frac{1}{6}x_i - \frac{1}{6} - \frac{1}{12}x_i \ge \frac{5}{6}$$

On the other hand

$$d_{(3,m,i)} \le 1 + \frac{1}{6} + \frac{3}{40} + \frac{1}{120} = \frac{5}{4}$$

Then

$$\frac{\partial d_{(3,m,k)}}{\partial x_k} = \frac{1-\beta}{6} + \frac{18 - 10\beta - \beta^2}{120}x_k - \frac{\beta(10-3\beta)}{120}\sum_{j=1}^{m} x_j$$
$$\ge \frac{m-1}{6m} - \frac{1}{12} = \frac{m-2}{12m}$$

Finally for $i \neq k$ we get

$$\frac{\partial d_{(3,m,i)}}{\partial x_k} = -\frac{\beta}{6} - \frac{\beta}{12}x_i + \frac{\beta^2}{120}\Big(x_i + \sum_{j\neq k} x_j\Big) \ge -\frac{1}{4m}$$

□

5.3. One dimension, arbitrary order. We have showed above that all the schemes considered in this paper are stable for sufficiently small β in any dimension, and moreover we have determined the exact stability limit for fourth and sixth order schemes in any dimension. Let us now calculate the exact limit for the one-dimensional schemes of any order. For the schemes of theorem 3.1 this was proved in [8] using order stars. Our proof is elementary. We start with

Lemma 5.3.

$$\sum_{k=1}^{\infty} \frac{4^k(k-1)!(k+1)!}{(2k+2)!} = 1$$

Proof. Consulting [5, p. 45] we find that

$$\sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) = x - \sum_{k=1}^{\infty}(-1)^k \frac{4^k(k-1)!(k+1)!}{(2k+2)!}x^{2k+1}$$

Now the result follows by putting $x = i$ (where i is the imaginary unit), or by first transforming the formula to

$$\sqrt{1-x^2}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = x - \sum_{k=1}^{\infty}\frac{4^k(k-1)!(k+1)!}{(2k+2)!}x^{2k+1}$$

and then puttinng $x = 1$. □

Let us define the following polynomials.

$$Q_n(x) := x\tilde{Q}_n(x) := x\prod_{k=2}^{n}(k^2 - x)$$

Lemma 5.4. *$Q_n'(x) > 0$ for $0 \le x \le 1$ and for all n.*

Proof. Note that Q_n is of degree n and its zeros are $x = k^2$, $k = 0, 2, \dots, n$. Because all zeros are real, the derivative of Q_n has exactly one zero between the zeros of Q_n. It is easily seen that $Q_n'(0) > 0$ and $Q_n'(4) < 0$. Since there can be only one zero between zero and four it is sufficient to show that $Q_n'(1) > 0$. Now explicit derivation shows that

$$Q_n'(x) = \tilde{Q}_n(x)\Big(1 - x\sum_{k=2}^{n}\frac{1}{k^2 - x}\Big)$$

In [5, p. 8] we find that

$$\sum_{k=2}^{\infty}\frac{1}{k^2-1} = 3/4$$

The result follows by observing that $\tilde{Q}_n(1) > 0$. □

Then we are ready for

Proposition 5.5. *The one dimensional schemes of theorem 3.1 are stable for $\beta \le 1$.*

Proof. Using the coefficients in (5.2), the stability condition for the scheme of order $2p$ can be written as

$$0 \le \beta\Big(x + \sum_{k=2}^{p}\frac{2^{2k-1}x^k}{(2k)!}\prod_{l=1}^{k-1}(l^2 - \beta)\Big) \le 1$$

It is evident that the first inequality is true for $0 \le x, \beta \le 1$ and moreover for the second it is sufficient to consider the case $x = 1$. Let us define

$$\tilde{N}_p(\beta) := -1 + \beta\Big(1 + \sum_{k=2}^{p}\frac{2^{2k-1}}{(2k)!}\prod_{l=1}^{k-1}(l^2 - \beta)\Big)$$

Obviously $\tilde{N}_p(1) = 0$. Factoring $\tilde{N}_p$ gives

$$\tilde{N}_p(\beta) = (1-\beta)\Big(-1 + \beta\sum_{k=1}^{p-1}\frac{2^{2k+1}}{(2k+2)!}\prod_{l=2}^{k}(l^2 - \beta)\Big)$$

Denote the second factor by $N_p(\beta)$. To prove the proposition we need to show that $N_p(\beta) < 0$ for all $0 \le \beta \le 1$. Obviously $N_p(0) < 0$. Putting $\beta = 1$ we get after some manipulation

$$N_p(1) = -1 + \sum_{k=1}^{p-1}\frac{4^k(k-1)!(k+1)!}{(2k+2)!}$$

Now lemma 5.3 says that $N_p(1) < 0$. To conclude we show that $N_p'(\beta) > 0$ for $0 \le \beta \le 1$. This is true for $N_2(\beta) = -1 + \beta/3$. The general case follows since

$$N_{p+1}(\beta) = N_p(\beta) + \frac{2^{2p+1}}{(2p+2)!}Q_p(\beta)$$

and lemma 5.4 tells us that $Q_p'(\beta) > 0$ for $0 \le \beta \le 1$. □

The proof for the schemes of theorem 3.2 is quite similar. It is interesting to note that also in this case the lemma 5.3 is needed.

Proposition 5.6. *The one dimensional schemes of theorem 3.2 are stable for $\beta \le 1$.*

Proof. The coefficients in (4.2) in one dimensional case are given by ([12])

$$c_k = \frac{(-1)^k}{4^k(2k+1)!}\prod_{l=1}^{k}\left((2l-1)^2 - \beta\right)$$

Hence the stability condition for the scheme of order $2p+2$ can be written as

$$0 \le x\beta\Big(1 + \sum_{k=1}^{p}\frac{x^k}{(2k+1)!}\prod_{l=1}^{k}\left((2l-1)^2 - \beta\right)\Big)^2 \le 1$$

Again it is sufficient to consider the case $x = 1$. Define

$$\tilde{R}_p(\beta) := -1 + \beta\Big(1 + \sum_{k=1}^{p}\frac{1}{(2k+1)!}\prod_{l=1}^{k}\left((2l-1)^2 - \beta\right)\Big)^2$$

Moreover let

$$R_p(\beta) := \sum_{k=1}^{p}\frac{1}{(2k+1)!}\prod_{l=2}^{k}\left((2l-1)^2 - \beta\right)$$

Then

$$\tilde{R}_p(\beta) = (1-\beta)\Big(-1 + 2\beta R_p(\beta) + \beta(1-\beta)R_p^2(\beta)\Big)$$

Denote the second factor by $M_p(\beta)$. We need to show that $M_p(\beta) < 0$ for $0 \le \beta \le 1$. Proceeding as before we first note that $M_p(0) < 0$ and

$$M_p(1) = -1 + 2\sum_{k=1}^{p} \frac{1}{(2k+1)!}\prod_{l=2}^{k}\big((2l-1)^2 - 1\big) = -1 + \sum_{k=1}^{p}\frac{4^k(k-1)!(k+1)!}{(2k+2)!}$$

Again lemma 5.3 says that $M_p(1) < 0$ and in particular $R_p(1) < 1/2$. It remains to prove that $M_p'(\beta) > 0$ for $0 \le \beta \le 1$. Taking the derivative gives

$$M_p' = R_p(R_p + 2\beta R_p') + 2(R_p + \beta R_p')(1 - \beta R_p)$$

Putting $L = 1 - \beta R_p$ we see that $L(0) = 1$ and $L(1) = 1 - R_p(1) > 1/2$. Hence L remains positive if $L' = -R_p - \beta R_p' < 0$ for $0 \le \beta \le 1$. Hence to prove that $M_p' > 0$ we just need to show that $R_p + 2\beta R_p' > 0$, because $R_p > 0$ and $R_p' < 0$. However, this follows directly from lemma 5.5 below. □

Lemma 5.5. *Let $A_p(\beta) = cR_p + \beta R_p'$; then $A_p > 0$ for $0 \le \beta \le 1$ if $c \ge 1/4$.*

Proof. Let us first observe that $A_1 > 0$ for all $c > 0$ and an easy computation shows that $A_2 > 0$ if $c > 1/48$. To prove the general case we define

$$B_p(\beta) := \prod_{l=2}^{p}\big((2l-1)^2 - \beta\big)$$
$$\tilde{B}_p(\beta) := cB_p + \beta B_p'$$

Then

$$R_p = R_{p-1} + \frac{1}{(2p+1)!}B_p$$
$$A_p = A_{p-1} + \frac{1}{(2p+1)!}\tilde{B}_p = cR_{p-1} + \beta R_{p-1}' + \frac{1}{(2p+1)!}\Big(cB_p + \beta B_p'\Big)$$

Let us suppose that $A_{p-1} > 0$ for all $c > c_* > 0$ where c_* is some constant. Hence it is sufficient to show that $\tilde{B}_p > 0$ for some c. Calculating B_p' gives

$$\tilde{B}_p(\beta) = B_p(\beta)\Big(c - \beta\sum_{l=2}^{p}\frac{1}{(2l-1)^2 - \beta}\Big)$$

Now evidently $B_p > 0$ and moreover the second factor of $\tilde{B}_p$ attains its minimum at $\beta = 1$. Computing

$$\sum_{l=2}^{\infty}\frac{1}{(2l-1)^2 - 1} = \frac{1}{4}$$

we see that one can take $c \ge 1/4$. □

5.4. Dimensions 2 and 3, order up to 12. Let us then try to calculate the stability limits of the schemes given in tables 1 and 2. We start with the homogeneous case. For convenience we use here the notation $x_1 = x$, $x_2 = y$ and $x_3 = z$.

Proposition 5.7. *The stability condition for the eighth order scheme of theorem 3.1 is $\beta \le 1/m$ for $m \le 3$.*

Proof. Now $f_{(4,3)}$ is given by

$$f_{(4,3)} = f_{(3,3)} + \frac{36}{315}\big(x^4 + y^4 + z^4\big) - \frac{\beta}{27}\big(x^2 + y^2 + z^2\big)^2 - \frac{\beta^3}{315}\big(x + y + z\big)^4 -$$
$$\frac{16\beta}{135}\big((x^3 + y^3 + z^3)(x + y + z)\big) + \frac{2\beta^2}{45}\big(x^2 + y^2 + z^2\big)\big(x + y + z\big)^2$$

Evidently $f_{(4,2)}(x,y) = f_{(4,3)}(x,y,0)$ and $f_{(4,1)}(x) = f_{(4,3)}(x,0,0)$. Evaluating at the corners give for $m = 1,2,3$

$$0 \le \gamma(-\gamma^3 + 28\gamma^2 - 224\gamma + 512) \le 315$$

This implies that $\beta \le 1/m$.

Then we will prove that the components of gradient are positive. Now we have already calculated in proposition 5.2 that

$$\begin{aligned}
\frac{\partial f_{(3,3)}}{\partial x} =& 1 + \frac{2}{3}x - \frac{2\beta}{3}(x + y + z) - \frac{2\beta}{9}(x^2 + y^2 + z^2) + \\
& \frac{2}{15}\big(2x - \beta(x + y + z)\big)^2 + \frac{4\beta}{45}x(x + y + z) \qquad (5.3)\\
\ge& 1 + \frac{4}{9}x - \frac{2\beta}{3}(y + z) - \frac{2}{15}x^2 - \frac{2\beta}{9}(y^2 + z^2) + \frac{4\beta}{45}x(y + z)
\end{aligned}$$

so it is sufficient to consider $w_4 := f_{(4,3)} - f_{(3,3)}$. Calculating and rearranging we obtain

$$\begin{aligned}
\frac{\partial w_4}{\partial x} =& \frac{16}{35}x^3 - \frac{4\beta}{27}x(x^2 + y^2 + z^2) - \frac{16\beta}{135}\big(x^3 + y^3 + z^3 + 3x^2(x + y + z)\big) + \\
& \frac{4\beta^2}{45}\big((x + y + z)(x^2 + y^2 + z^2) + x(x + y + z)^2\big) - \frac{4\beta^3}{315}(x + y + z)^3 \\
\ge& \frac{32}{315}x^3 + \frac{4}{45}x\big(2x - \beta(x + y + z)\big)^2 - \frac{4\beta}{27}x(x^2 + y^2 + z^2) - \\
& \frac{16\beta}{135}\big(x^3 + y^3 + z^3\big) + \frac{8\beta^2}{105}\big((x + y + z)(x^2 + y^2 + z^2)\big) \\
\ge& \frac{4}{315}x^3 - \frac{4\beta}{27}x(y^2 + z^2) - \frac{16\beta}{135}\big(y^3 + z^3\big)
\end{aligned}$$

Combining this with (5.3) we get

$$\begin{aligned}\frac{\partial f_{(4,3)}}{\partial x} &\geq 1 + \frac{4}{9}x - \frac{2}{15}x^2 - \frac{8\beta}{9}(y+z) + \frac{32}{315}x^3 - \frac{8\beta}{135}x(y+z) - \frac{16\beta}{135}\left(y^3+z^3\right)\\ &\geq 1 + \frac{164}{405}x - \frac{2}{15}x^2 + \frac{32}{315}x^3 - \frac{272}{405} > 0\end{aligned}$$

Similar calculation proves also the case $m = 2$. □

Proposition 5.8. *The stability condition for the tenth order scheme is $\beta \leq 1/m$ for $m \leq 3$.*

Proof. Now $f_{(5,3)}$ is given by

$$\begin{aligned}f_{(5,3)} = f_{(4,3)} &+ \frac{128}{1635}\left(x^5+y^5+z^5\right) - \frac{8\beta}{105}\left((x^4+y^4+z^4)(x+y+z)\right) -\\ &\frac{16\beta}{405}\left(x^2+y^2+z^2\right)\left(x^3+y^3+z^3\right) + \frac{2\beta^2}{135}\left(x^2+y^2+z^2\right)^2\left(x+y+z\right) +\\ &\frac{16\beta^2}{675}\left(x^3+y^3+z^3\right)\left(x+y+z\right)^2 - \frac{4\beta^3}{945}\left(x^2+y^2+z^2\right)\left(x+y+z\right)^3 +\\ &\frac{2\beta^4}{14175}\left(x+y+z\right)^5\end{aligned}$$

Evaluating at the corners gives for $m \leq 3$

$$0 \leq \gamma(2\gamma^4 - 105\gamma^3 + 1806\gamma^2 - 11720\gamma + 24192) \leq 14175$$

which yields $\beta \leq 1/m$. Put $w_5 := f_{(5,3)} - f_{(4,3)}$; then we compute

$$\begin{aligned}\frac{\partial w_5}{\partial x} =& \frac{128}{327}x^4 - \frac{8\beta}{105}\left(4x^3(x+y+z) + x^4+y^4+z^4\right) -\\ &\frac{16\beta}{405}\left(2x(x^3+y^3+z^3) + 3x^2(x^2+y^2+z^2)\right) +\\ &\frac{2\beta^2}{135}\left(4x(x^2+y^2+z^2)(x+y+z) + (x^2+y^2+z^2)^2\right) +\\ &\frac{16\beta^2}{675}\left(3x^2(x+y+z)^2 + 2(x^3+y^3+z^3)(x+y+z)\right) -\\ &\frac{4\beta^3}{945}\left(2x(x+y+z)^3 + 3(x^2+y^2+z^2)(x+y+z)^2\right) + \frac{2\beta^4}{2835}\left(x+y+z\right)^4\\ =& -\frac{124112}{721035}x^4 + \frac{2}{135}\left(4x^2 - \beta(x^2+y^2+z^2)\right)^2 - \frac{8\beta}{105}\left(x^4+y^4+z^4\right) +\\ &\frac{16\beta^2}{315}\left(x(x^2+y^2+z^2)(x+y+z)\right) + \frac{16}{11025}x^2\left(15x - 7\beta(x+y+z)\right)^2 -\\ &\frac{32\beta}{405}\left(x(x^3+y^3+z^3)\right) + \frac{32\beta^2}{675}\left((x^3+y^3+z^3)(x+y+z)\right) -\\ &\frac{4\beta^3}{315}\left((x^2+y^2+z^2)(x+y+z)^2\right) + \frac{2\beta^4}{2835}\left(x+y+z\right)^4\\ \geq& -\frac{1452728}{6489315}x^4 - \frac{16}{315} - \frac{64}{1215}x\end{aligned}$$

We have used above the following simple consequence of Hölder's inequality

$$(x^2+y^2+z^2)(x+y+z)^2 \leq 9(x^3+y^3+z^3)(x+y+z)$$

Combined to the estimates in previous proposition we obtain

$$\begin{aligned}\frac{\partial f_{(5,3)}}{\partial x} &\geq 1 + \frac{428}{1215}x - \frac{2}{15}x^2 - \frac{793496}{6489315}x^3 - \frac{2048}{2835}\\ &\geq \frac{787}{2835} + \frac{4646}{48069}x > 0\end{aligned} \tag{5.4}$$

The case $m = 2$ is similar. □

Proposition 5.9. *The stability condition for the twelfth order scheme is $\beta \leq 1/m$ for $m \leq 3$.*

Proof. Now $f_{(6,3)}$ is given by

$$\begin{aligned} f_{(6,3)} = f_{(5,3)} &+ \frac{128}{2079}\left(x^6+y^6+z^6\right) - \frac{256\beta}{4725}\left((x^5+y^5+z^5)(x+y+z)\right) - \\ &\frac{8\beta}{315}\left(x^2+y^2+z^2\right)\left(x^4+y^4+z^4\right) - \frac{64\beta}{6075}\left(x^3+y^3+z^3\right)^2 + \\ &\frac{2\beta^2}{1215}\left(x^2+y^2+z^2\right)^3 + \frac{8\beta^2}{525}\left(x^4+y^4+z^4\right)\left(x+y+z\right)^2 + \\ &\frac{32\beta^2}{2025}\left(x^3+y^3+z^3\right)\left(x^2+y^2+z^2\right)\left(x+y+z\right) - \\ &\frac{2\beta^3}{945}\left(x^2+y^2+z^2\right)^2\left(x+y+z\right)^2 - \frac{32\beta^3}{14175}\left(x^3+y^3+z^3\right)\left(x+y+z\right)^3 + \\ &\frac{2\beta^4}{8505}\left(x^2+y^2+z^2\right)\left(x+y+z\right)^4 - \frac{2\beta^5}{467775}\left(x+y+z\right)^6 \end{aligned}$$

Evaluating at the corners gives for $m \leq 3$

$$0 \leq \gamma(-2\gamma^5 + 176\gamma^4 - 5511\gamma^3 + 74888\gamma^2 - 428912\gamma + 827136) \leq 467775$$

which yields $\beta \leq 1/m$. Put $w_6 := f_{(6,3)} - f_{(5,3)}$; then we compute

$$\begin{aligned} \frac{\partial w_6}{\partial x} =& \frac{256}{693}x^5 - \frac{256\beta}{4725}\left(5x^4(x+y+z) + x^5+y^5+z^5\right) - \\ &\frac{16\beta}{315}x\left(2x^2(x^2+y^2+z^2) + x^4+y^4+z^4\right) - \\ &\frac{128\beta}{2025}x^2\left(x^3+y^3+z^3\right) + \frac{4\beta^2}{405}x\left(x^2+y^2+z^2\right)^2 + \\ &\frac{16\beta^2}{525}\left((x^4+y^4+z^4)(x+y+z) + 2x^3(x+y+z)^2\right) + \\ &\frac{32\beta^2}{2025}\left(3x^2(x^2+y^2+z^2)(x+y+z) + \right. \\ &\qquad \left. 2x(x^3+y^3+z^3)(x+y+z) + (x^3+y^3+z^3)(x^2+y^2+z^2)\right) - \\ &\frac{4\beta^3}{945}\left(2x(x^2+y^2+z^2)(x+y+z)^2 + (x^2+y^2+z^2)^2(x+y+z)\right) - \\ &\frac{32\beta^3}{4725}\left(x^2(x+y+z)^3 + (x^3+y^3+z^3)(x+y+z)^2\right) + \\ &\frac{4\beta^4}{8505}\left(x(x+y+z)^4 + 2(x^2+y^2+z^2)(x+y+z)^3\right) - \frac{4\beta^5}{155925}\left(x+y+z\right)^5 \end{aligned}$$

Then simple estimates give

$$\begin{aligned} \frac{\partial w_6}{\partial x} \geq& -\frac{126272}{654885}x^5 - \frac{16\beta}{315}x\left(x^4+y^4+z^4\right) \\ &- \frac{256\beta}{4725}\left(x^5+y^5+z^5\right) - \frac{128\beta}{2025}x^2\left(x^3+y^3+z^3\right) \\ &+ \frac{32}{42525}x^3\left(20x - 9\beta(x+y+z)\right)^2 + \frac{4}{19845}x\left(36x - 7\beta(x^2+y^2+z^2)\right)^2 \\ &+ \frac{124\beta^2}{14175}\left((x^2+y^2+z^2)^2(x+y+z)\right) + \frac{4736\beta^2}{581175}\left((x^3+y^3+z^3)(x+y+z)^2\right) \\ &+ \frac{16832\beta^2}{581175}x^2(x+y+z)^3 + \frac{88\beta^2}{127575}x(x+y+z)^4 \\ &+ \frac{428\beta^4}{467775}\left((x^2+y^2+z^2)(x+y+z)^3\right) \\ \geq& -\frac{512}{14175} - \frac{32}{945}x - \frac{256}{6075}x^2 - \frac{33184048}{140800275}x^5 \geq -\frac{512}{14175} - \frac{585136}{1877337}x \end{aligned}$$

Combining this with the estimate (5.4) we obtain

$$\frac{\partial f_{(6,3)}}{\partial x} \geq \frac{1076671}{4459455} - \frac{44001802}{204629733}x \geq \frac{143081867}{5418898485} > 0$$

The case $m = 2$ is similar. □

Let us then pass to the schemes of theorem 3.2. We first derive a little sharper form of lemma 5.2 in case $m = 3$. For notational simplicity we set $d_{(p,x)} := d_{(p,3,1)}$, $d_{(p,y)} := d_{(p,3,2)}$ and $d_{(p,z)} := d_{(p,3,3)}$.

Lemma 5.6.

$$\begin{aligned} d_{(3,x)} &\geq \frac{8}{9} + \frac{1}{18}x \\ d_{(3,y)} &\leq \frac{56}{45} + \frac{1}{216}x \\ \frac{\partial d_{(3,x)}}{\partial x} &\geq \frac{1}{18} + \frac{13}{135}x \\ \frac{\partial d_{(3,y)}}{\partial x} &\geq -\frac{1}{12} \end{aligned}$$

Proof. We have

$$\begin{aligned} d_{(3,x)} =& 1 + \frac{1-\beta}{6}x - \frac{\beta}{6}(y+z) + \frac{9-10\beta+\beta^2}{120}x^2 - \frac{10\beta-3\beta^2}{120}x(y+z) + \frac{\beta^2}{60}yz \\ \geq& 1 + \frac{1-\beta}{6}x - \frac{\beta}{6}(y+z) - \frac{\beta}{12}x(y+z) \geq \frac{8}{9} + \frac{1}{18}x \end{aligned}$$

Then we get

$$\begin{aligned}d_{(3,y)} =& 1 + \frac{1-\beta}{6}y - \frac{\beta}{6}(x+z) + \frac{9-10\beta+\beta^2}{120}y^2 - \frac{10\beta-3\beta^2}{120}y(x+z) + \frac{\beta^2}{60}xz \\ \le& 1 + \frac{1}{6} + \frac{3}{40} + \frac{1}{360} + \frac{1}{360}x + \frac{1}{540}x \le \frac{56}{45} + \frac{1}{216}x\end{aligned}$$

Next taking the derivative gives

$$\frac{\partial d_{(3,x)}}{\partial x} = \frac{1-\beta}{6} + \frac{9-10\beta+\beta^2}{60}x - \frac{10\beta-3\beta^2}{120}(y+z) \ge \frac{1}{18} + \frac{13}{135}x$$

Finally we compute

$$\begin{aligned}\frac{\partial d_{(3,y)}}{\partial x} &= -\frac{\beta}{6} - \frac{10\beta-3\beta^2}{120}y + \frac{\beta^2}{60}z \\ &\ge -\frac{\beta}{6} - \frac{\beta}{12}y \ge -\frac{1}{12}\end{aligned}$$

By symmetry there is no need to to compute the x-derivative with respect to $d_{(3,z)}$. □

Proposition 5.10. *The stability condition for the eighth order scheme of theorem 3.2 is $\beta \le 1/m$ for $m \le 3$.*

Proof. We have

$$g_{(4,3)} = xd^2_{(4,x)} + yd^2_{(4,y)} + zd^2_{(4,z)}$$

where

$$\begin{aligned}d_{(4,x)} =& d_{(3,x)} + \frac{5}{112}x^3 - \frac{\beta}{2160}\left(111x^3 + 71x^2(y+z) + 40x(y^2+z^2)\right) + \\ & \frac{\beta^2}{720}\left(5x^3 + 9x^2(y+z) + 6x(y^2+z^2) + 10xyz\right) - \\ & \frac{\beta^3}{5040}\left(x^3 + 4x^2(y+z) + 3x(y^2+z^2) + 12xyz\right)\end{aligned}$$

At the corner we have

$$\gamma(\gamma^3 - 77\gamma^2 + 1519\gamma - 6483)^2 \le 25401600$$

which gives $\beta \le 1/m$. Then using the estimates of lemma 5.7 below we get

$$\begin{aligned}\frac{\partial g_{(4,3)}}{\partial x} =& d^2_{(4,x)} + 2\left(xd_{(4,x)}\frac{\partial d_{(4,x)}}{\partial x} + yd_{(4,y)}\frac{\partial d_{(4,y)}}{\partial x} + zd_{(4,z)}\frac{\partial d_{(4,z)}}{\partial x}\right) \\ \ge& \left(\frac{8}{9} + \frac{7}{162}x\right)^2 + 2x\frac{8}{9}\cdot\frac{7}{162} - 4\left(\frac{58613}{45360} + \frac{11}{1296}x\right)\left(\frac{611}{6480} + \frac{1}{81}x\right) \\ \ge& \frac{22248257}{73483200} + \frac{84871}{979776}x + \frac{19}{13122}x^2 > 0\end{aligned}$$

By symmetry all the other derivatives are also positive. □

Lemma 5.7.

$$\begin{aligned}d_{(4,x)} &\ge \frac{8}{9} + \frac{7}{162}x \\ d_{(4,y)} &\le \frac{58613}{45360} + \frac{11}{1296}x \\ \frac{\partial d_{(4,x)}}{\partial x} &\ge \frac{7}{162} + \frac{20687}{1496880}x \\ \frac{\partial d_{(4,y)}}{\partial x} &\ge -\frac{611}{6480} - \frac{1}{81}x\end{aligned}$$

Proof. Put $\tilde{w}_4 := d_{(4,x)} - d_{(3,x)}$ and $\bar{w}_4 := d_{(4,y)} - d_{(3,y)}$; then

$$\begin{aligned}\tilde{w}_4 \ge& \frac{5}{112}x^3 - \frac{\beta}{2160}\left(111x^3 + 71x^2(y+z) + 40x(y^2+z^2)\right) + \\ & \frac{\beta^2}{15120}\left(104x^3 + 185x^2(y+z) + 123x(y^2+z^2) + 198xyz\right) \\ \ge& \frac{20291}{5987520}x^3 + \frac{11}{1680}x\left(\frac{497}{198}x - \beta(y+z)\right)^2 - \frac{\beta}{54}x(y^2+z^2) \ge -\frac{1}{81}x\end{aligned}$$

Combining with the results of lemma 5.6 we obtain

$$d_{(4,x)} \ge \frac{8}{9} + \frac{7}{162}x$$

Then

$$\begin{aligned}\bar{w}_4 \le& \frac{5}{112}y^3 + \frac{\beta^2}{720}\left(5y^3 + 9y^2(x+z) + 6y(x^2+z^2) + 10xyz\right) \\ \le& \frac{5}{112} + \frac{\beta^2}{720}\left(20 + 25x\right) \le \frac{433}{9072} + \frac{5}{1296}x\end{aligned}$$

which implies that

$$d_{(4,y)} \le \frac{58613}{45360} + \frac{11}{1296}x$$

Taking the x-derivative yields

$$\begin{aligned}
\frac{\tilde{w}_4}{\partial x} =& \frac{15}{112}x^2 - \frac{\beta}{2160}\Big(333x^2 + 142x(y+z) + 40(y^2+z^2)\Big) + \\
& \frac{\beta^2}{720}\Big(15x^2 + 18x(y+z) + 6(y^2+z^2) + 10yz\Big) - \\
& \frac{\beta^3}{5040}\Big(3x^2 + 8x(y+z) + 3(y^2+z^2) + 12yz\Big) \\
\geq & \frac{15}{112}x^2 - \frac{\beta}{2160}\Big(333x^2 + 142x(y+z) + 40(y^2+z^2)\Big) + \\
& \frac{\beta^2}{15120}\Big(312x^2 + 370x(y+z) + 123(y^2+z^2) + 198yz\Big) \\
\geq & -\frac{123457}{1496880}x^2 + \frac{11}{1680}x\Big(\frac{497}{99}x - \beta(y+z)\Big)^2 - \frac{\beta}{54}(y^2+z^2) \geq -\frac{123457}{1496880}x - \frac{1}{81}
\end{aligned}$$

Consequently

$$\frac{\partial d_{(4,x)}}{\partial x} \geq \frac{7}{162} + \frac{20687}{1496880}x$$

Finally we compute

$$\begin{aligned}
\frac{\bar{w}_4}{\partial x} =& -\frac{\beta}{2160}\Big(71y^2 + 80yx\Big) + \\
& \frac{\beta^2}{720}\Big(9y^2 + 12yx + 10yz\Big) - \frac{\beta^3}{5040}\Big(4y^2 + 6yx + 12yz\Big) \\
\geq & -\frac{\beta}{2160}\Big(71y^2 + 80yx\Big) \geq -\frac{71}{6480} - \frac{1}{81}x
\end{aligned}$$

which gives the last inequality. □

Proposition 5.11. *The stability condition for the tenth order scheme of theorem* 3.2 *is $\beta \leq 1/m$ for $m \leq 3$.*

Proof. We have

$$g_{(5,3)} = xd_{(5,x)}^2 + yd_{(5,y)}^2 + zd_{(5,z)}^2$$

where

$$\begin{aligned}
d_{(5,x)} =& d_{(4,x)} + \frac{35}{1152}x^4 - \frac{\beta}{90720}\Big(3229x^4 + 1717x^3(y+z) + 1512x^2(y^2+z^2)\Big) + \\
& \frac{\beta^2}{129600}\Big(705x^4 + 799x^3(y+z) + 1316x^2(y^2+z^2) + 290x^2yz + 560xyz(y+z)\Big) - \\
& \frac{\beta^3}{30240}\Big(7x^4 + 6x^3(y+z) + 43x^2(y^2+z^2) + 8x^2yz + 38xyz(y+z)\Big) + \\
& \frac{\beta^4}{1814400}\Big(5x^4 - 38x^3(y+z) + 113x^2(y^2+z^2) - 26x^2yz + 138xyz(y+z)\Big)
\end{aligned}$$

At the corner we have

$$\gamma(\gamma^4 - 156\gamma^3 + 7518\gamma^2 - 122284\gamma + 477801)^2 \leq 131681894400$$

which gives $\beta \leq 1/m$. Then using the lemma 5.8 below we get

$$\begin{aligned}
\frac{\partial g_{(5,3)}}{\partial x} =& d_{(5,x)}^2 + 2\Big(xd_{(5,x)}\frac{\partial d_{(5,x)}}{\partial x} + yd_{(5,y)}\frac{\partial d_{(5,y)}}{\partial x} + zd_{(5,z)}\frac{\partial d_{(5,z)}}{\partial x}\Big) \\
\geq & \Big(\frac{8}{9} + \frac{13}{405}x\Big)^2 + 2x\frac{8}{9}\cdot\frac{1}{29} - 4\Big(\frac{7}{5} + \frac{11}{95}x\Big)\Big(\frac{14}{139} + \frac{19}{810}x\Big) \\
\geq & \frac{12728}{56295} - \frac{83247386}{1395834525}x - \frac{1613}{164025}x^2 \geq \frac{1967552309}{12562510725}
\end{aligned}$$

□

Lemma 5.8.

$$\begin{aligned}
d_{(5,x)} &\geq \frac{8}{9} + \frac{13}{405}x \\
d_{(5,y)} &\leq \frac{7}{5} + \frac{11}{950}x \\
\frac{\partial d_{(5,x)}}{\partial x} &\geq \frac{1}{29} \\
\frac{\partial d_{(5,y)}}{\partial x} &\geq -\frac{14}{139} - \frac{19}{810}x
\end{aligned}$$

Proof. Put $\tilde{w}_5 := d_{(5,x)} - d_{(4,x)}$ and $\bar{w}_5 := d_{(5,y)} - d_{(4,y)}$; then

$$\begin{aligned}
\tilde{w}_5 \geq & \frac{20159}{1088640}x^4 - \frac{\beta}{90720}\Big(1717x^3(y+z) + 1512x^2(y^2+z^2)\Big) + \\
& \frac{\beta^2}{907200}\Big(5533x^3(y+z) + 4875x^2(y+z)^2 + 3540xyz(y+z)\Big) \\
\geq & \frac{72833}{39312000}x^4 + \frac{65}{12096}x^2\Big(\frac{1717}{975}x - \beta(y+z)\Big)^2 - \frac{\beta}{60}x^2(y^2+z^2) \geq -\frac{1}{90}x
\end{aligned}$$

Combined with the results of lemma 5.7 we get the first inequality. Then

$$\begin{aligned}
\bar{w}_5 \leq & \frac{35}{1152}y^4 + \frac{\beta^4}{1814400}\Big(5y^4 + 113y^2(x^2+z^2) + 138xyz(x+z)\Big) + \\
& \frac{\beta^2}{129600}\Big(705y^4 + 799y^3(x+z) + 1316y^2(x^2+z^2) + 290y^2xz + 560xyz(x+z)\Big) \\
\leq & \frac{35}{1152} + \frac{\beta^2}{25920}\Big(676 + 593x\Big) + \frac{\beta^4}{1814400}\Big(256 + 251x\Big) \\
\leq & \frac{1747}{52488} + \frac{187}{73483}x
\end{aligned}$$

which implies the second inequality. Taking the x-derivative yields

$$\begin{aligned}\frac{\tilde{w}_5}{\partial x} =& \frac{35}{288}x^3 - \frac{\beta}{90720}\big(12916x^3 + 5151x^2(y+z) + 3024x(y^2+z^2)\big) + \\ & \frac{\beta^2}{129600}\big(2820x^3 + 2397x^2(y+z) + 2632x(y^2+z^2) + 580xyz + 560yz(y+z)\big) - \\ & \frac{\beta^3}{15120}\big(14x^3 + 9x^2(y+z) + 43x(y^2+z^2) + 8xyz + 19yz(y+z)\big) + \\ & \frac{\beta^4}{1814400}\big(20x^3 - 114x^2(y+z) + 226x(y^2+z^2) - 52xyz + 138yz(y+z)\big) \\ \geq & \frac{1123612751}{14696640000}x^3 - \frac{\beta}{90720}\big(5151x^2(y+z) + 3024x(y^2+z^2)\big) + \\ & \frac{\beta^2}{907200}\big(16599x^2(y+z) + 9750x(y+z)^2 + 3540yz(y+z)\big) \\ \geq & \frac{279253223}{191056320000}x^3 + \frac{65}{6048}x\Big(\frac{1717}{650}x - \beta(y+z)\Big)^2 - \frac{\beta}{30}x(y^2+z^2) \geq -\frac{1}{45}x\end{aligned}$$

Finally we compute

$$\begin{aligned}\frac{\tilde{w}_5}{\partial x} =& -\frac{\beta}{90720}\big(1717y^3 + 3024y^2x\big) + \\ & \frac{\beta^2}{129600}\big(799y^3 + 2632y^2x + 290y^2z + 560yz(2x+z)\big) - \\ & \frac{\beta^3}{30240}\big(6y^3 + 86y^2x + 8y^2z + 38yz(2x+z)\big) + \\ & \frac{\beta^4}{1814400}\big(-38y^3 + 226y^2x - 26y^2z + 138yz(2x+z)\big) \\ \geq & -\frac{\beta}{90720}\big(1717y^3 + 3024y^2x\big) \geq -\frac{1717}{272160} - \frac{1}{90}x\end{aligned}$$

which gives the last inequality. □

Proposition 5.12. *The stability condition for the twelfth order scheme of theorem 3.2 is $\beta \leq 1/m$ for $m \leq 3$.*

Proof. We have

$$g_{(6,3)} = xd^2_{(6,x)} + yd^2_{(6,y)} + zd^2_{(6,z)}$$

where

$$\begin{aligned}d_{(6,x)} =& d_{(5,x)} + \frac{63}{2816}x^5 - \frac{\beta}{10886400}\big(288333x^5 + 138109x^4(y+z) + \\ & 92880x^3(y^2+z^2) + 57344x^2(y^3+z^3)\big) + \\ & \frac{\beta^2}{5443200}\big(23565x^5 + 21501x^4(y+z) + 29706x^3(y^2+z^2) + \\ & 7034x^3yz + 19488x^2(y^3+z^3) + 15568x^2yz(y+z) + 8960xy^2z^2\big) - \\ & \frac{\beta^3}{5443200}\big(1197x^5 + 1616x^4(y+z) + 4611x^3(y^2+z^2) + \\ & 2708x^3yz + 2152x^2(y^3+z^3) + 3988x^2yz(y+z) + 3680xy^2z^2\big) + \\ & \frac{\beta^4}{10886400}\big(45x^5 + 454x^4(y+z) + 489x^3(y^2+z^2) + \\ & 1262x^3yz - 268x^2(y^3+z^3) + 194x^2yz(y+z) + 600xy^2z^2\big) - \\ & \frac{\beta^5}{39916800}\big(x^5 + 171x^4(y+z) + 15x^3(y^2+z^2) + \\ & 426x^3yz - 155x^2(y^3+z^3) - 105x^2yz(y+z) - 36xy^2z^2\big)\end{aligned}$$

At the corner we have

$$\gamma(\gamma^5 - 275\gamma^4 + 25938\gamma^3 - 999790\gamma^2 + 14508461\gamma - 53451135)^2 \leq 1593350922240000$$

which gives $\beta \leq 1/m$. Then using the lemma 5.9 below we get

$$\begin{aligned}\frac{\partial g_{(6,3)}}{\partial x} =& d^2_{(6,x)} + 2\Big(xd_{(6,x)}\frac{\partial d_{(6,x)}}{\partial x} + yd_{(6,y)}\frac{\partial d_{(6,y)}}{\partial x} + zd_{(6,z)}\frac{\partial d_{(6,z)}}{\partial x}\Big) \\ \geq & \Big(\frac{8}{9} + \frac{179}{8100}x\Big)^2 + 2x\Big(\frac{8}{9} + \frac{179}{8100}x\Big)\frac{3}{928} - 4\Big(\frac{57}{40} + \frac{11}{760}x\Big)\Big(\frac{2}{19} + \frac{1}{29}x\Big) \\ \geq & \frac{77}{405} - \frac{30071996}{190797525}x - \frac{197404561}{144604440000}x^2 \geq \frac{85573794941}{2747484360000}\end{aligned}$$

□

Lemma 5.9.

$$\begin{aligned} d_{(6,x)} &\ge \frac{8}{9} + \frac{179}{8100}x \\ d_{(6,y)} &\le \frac{57}{40} + \frac{11}{760}x \\ \frac{\partial d_{(6,x)}}{\partial x} &\ge \frac{3}{928} \\ \frac{\partial d_{(6,y)}}{\partial x} &\ge -\frac{2}{19} - \frac{1}{29}x \end{aligned}$$

Proof. Put $\tilde{w}_6 := d_{(6,x)} - d_{(5,x)}$ and $\bar{w}_6 := d_{(6,y)} - d_{(5,y)}$; then

$$\begin{aligned} \tilde{w}_6 \ge& \frac{90103}{6652800}x^5 - \frac{\beta}{10886400}\big(138109x^4(y+z) + 92880x^3(y^2+z^2) + 57344x^2(y^3+z^3)\big) + \\ & \frac{\beta^2}{48988800}\big(188661x^4(y+z) + 253521x^3(y^2+z^2) + 55182x^3yz + \\ & \quad 1668802x^2(y^3+z^3) + 128148x^2yz(y+z) + 69600xy^2z^2\big) \\ \ge& -\frac{7182778403}{14961455308800}x^5 + \frac{11713}{4082400}x^3\Big(\frac{414327}{187408}x - \beta(y+z)\Big)^2 - \\ & \frac{\beta}{10886400}\big(92880x^3(y^2+z^2) + 57344x^2(y^3+z^3)\big) \\ \ge& -\frac{52135395955}{5386123911168}x \ge -\frac{1}{100}x \end{aligned}$$

Combined with the results of lemma 5.8 we get the first inequality. Then

$$\begin{aligned} \bar{w}_6 \le& \frac{63}{2816}y^5 + \frac{\beta^2}{5443200}\big(23565y^5 + 21501y^4(x+z) + 29706y^3(x^2+z^2) + \\ & \quad 7034y^3xz + 19488y^2(x^3+z^3) + 15568y^2xz(x+z) + 8960yx^2z^2\big) + \\ & \frac{\beta^4}{10886400}\big(45y^5 + 454y^4(x+z) + 489y^3(x^2+z^2) + \\ & \quad 1262y^3xz + 194y^2xz(x+z) + 600yx^2z^2\big) + \\ & \frac{\beta^5}{39916800}\big(155y^2(x^3+z^3) + 105y^2xz(x+z) + 36yx^2z^2\big) \\ \le& \frac{63}{2816} + \frac{\beta^2}{362880}\big(6284 + 7855x\big) + \frac{\beta^4}{10886400}\big(988 + 3193x\big) + \frac{\beta^5}{39916800}\big(155 + 401x\big) \\ \le& \frac{117839789}{4849891200} + \frac{11682437}{4849891200}x \le \frac{1}{40} + \frac{1}{400}x \end{aligned}$$

which implies the second inequality. Taking the x-derivative yields

$$\begin{aligned} \frac{\tilde{w}_6}{\partial x} =& \frac{315}{2816}x^4 - \frac{\beta}{10886400}\big(1441665x^4 + 552436x^3(y+z) + \\ & \quad 278640x^2(y^2+z^2) + 114688x(y^3+z^3)\big) + \\ & \frac{\beta^2}{5443200}\big(117825x^4 + 86004x^3(y+z) + 89118x^2(y^2+z^2) + \\ & \quad 21102x^2yz + 38976x(y^3+z^3) + 31136xyz(y+z) + 8960y^2z^2\big) - \\ & \frac{\beta^3}{5443200}\big(5985x^4 + 6464x^3(y+z) + 13833x^2(y^2+z^2) + \\ & \quad 8124x^2yz + 4304x(y^3+z^3) + 7976xyz(y+z) + 3680y^2z^2\big) + \\ & \frac{\beta^4}{10886400}\big(225x^4 + 1816x^3(y+z) + 1467x^2(y^2+z^2) + \\ & \quad 3786x^2yz - 572x(y^3+z^3) + 388xyz(y+z) + 600y^2z^2\big) - \\ & \frac{\beta^5}{39916800}\big(5x^4 + 684x^3(y+z) + 45x^2(y^2+z^2) + \\ & \quad 1278x^2yz - 310x(y^3+z^3) - 210xyz(y+z) - 36y^2z^2\big) \\ \ge& -\frac{19855404941}{2805153120000}x^4 + \frac{937}{108864}x^2\Big(\frac{138109}{46850}x - \beta(y+z)\Big)^2 - \\ & \frac{\beta}{10886400}\big(278640x^2(y^2+z^2) + 114688x(y^3+z^3)\big) \\ \ge& -\frac{262267799623}{8415459360000}x \ge -\frac{1}{32}x \end{aligned}$$

Finally we compute

$$\begin{aligned} \frac{\bar{w}_6}{\partial x} =& -\frac{\beta}{10886400}\big(138109y^4 + 185760y^3x + 172032y^2x^2\big) + \\ & \frac{\beta^2}{5443200}\big(21501y^4 + 59412y^3x + 7034y^3z + 58464y^2x^2 + 15568y^2z(2x+z) + 17920yxz^2\big) - \\ & \frac{\beta^3}{5443200}\big(1616y^4 + 9222y^3x + 2708y^3z + 6456y^2x^2 + 3988y^2z(2x+z) + 7360yxz^2\big) + \\ & \frac{\beta^4}{10886400}\big(454y^4 + 978y^3x + 1262y^3z - 804y^2x^2 + 194y^2z(2x+z) + 1200yxz^2\big) - \\ & \frac{\beta^5}{39916800}\big(171y^4 + 30y^3x + 426y^3z - 465y^2x^2 - 105y^2z(2x+z) - 72yxz^2\big) \\ \ge& -\frac{\beta}{10886400}\big(138109y^4 + 185760y^3x + 172032y^2x^2\big) \ge -\frac{138109}{32659200} - \frac{3727}{340200}x \end{aligned}$$

which gives the last inequality. □

Let us finish with a conjecture. All the stability results obtained above indicate that it might be true.

Conjecture 5.1. *The stability condition for the schemes constructed in theorems* 3.1 *and* 3.2 *is* $\beta \leq 1/m$ *where* m *is the dimension of the space.*

6. CONCLUSION

We have used elementary formal power series techniques to prove some results which seem to be new. The proofs are rather easy and straightforward because the relevant constructions have nice hierarchical properties with respect to order and dimension: the coefficients of low order schemes are 'contained' in high order ones, and high dimensional schemes can immediately be 'specialized' to low dimensional ones. We have used the same idea also in another context, namely in constructing absorbing boundary conditions for wave equation, see [13]. However, in that context it is more useful to regard δ_{i+} etc. as indeterminates in some polynomial ring rather than formal power series.

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