Esim. $\int \ln x \, dx = ?$ 27.11.2007

osit. integr. $\begin{cases} U = \ln x & dV = dx \\ dU = \frac{dx}{x} & V = x \end{cases}$

$$\int \underbrace{\ln x}_{U} \cdot \underbrace{dx}_{dV} = \underset{V}{x} \underset{U}{\ln x} - \int \underset{V}{x} \underbrace{\frac{dx}{x}}_{dU}$$

$$= x \ln x - \int dx = x \ln x - x + C$$

Esim: $\int x^2 \sin x \, dx = ?$

osit. integr. $\begin{cases} U = x^2 & dV = \sin x \, dx \\ dU = 2x \, dx & V = -\cos x \end{cases}$

$$\int \underset{U}{x^2} \underbrace{\sin x \, dx}_{dV} = -x^2 \cos x - \int (-\cos x) \cdot 2x \, dx$$

$$= -x^2 \cos x + 2 \int x \underline{\cos x \, dx}$$

osit. integr. $\begin{cases} U = x & dV = \cos x \, dx \\ dU = dx & V = \sin x \end{cases}$

$$\int x \cos x \, dx = x \sin x - \int \underset{V}{\underline{\sin x}} \, dx$$

$$= x \sin x + \cos x .$$

Siis

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x .$$

Ehim:

$$\int \underbrace{\arcsin x}_{U}\, \underbrace{dx}_{dV} = ?$$

Ositi integr.

$$\begin{cases} U = \arcsin x & dV = dx \\ dU = \dfrac{dx}{\sqrt{1-x^2}} & V = x \end{cases}$$

$$\int \underbrace{\arcsin x}_{U}\, \underbrace{dx}_{dV} = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} .$$

Muutl. vahetus.

$$\begin{cases} u = 1 - x^2 \\ du = -2x\,dx \\ \Rightarrow x\,dx = -\frac{1}{2}\,du \end{cases}$$

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{2} \cdot 2 u^{1/2} + C = -u^{1/2} + C = -\sqrt{1-x^2} + C$$

Siis

$$\int \arcsin x\,dx = x \arcsin x - \sqrt{1-x^2} + C .$$

Esim:

$$I = \int \underbrace{e^{ax}}_{u} \underbrace{\cos bx \, dx}_{dv}$$

Ositt. Integr.

$$\begin{cases} u = e^{ax} & dv = \cos bx \, dx \\ du = a e^{ax} & v = \frac{1}{b} \sin bx \end{cases}$$

$$(*) \quad \int \underbrace{e^{ax}}_{u} \underbrace{\cos bx \, dx}_{dv} = \frac{1}{b} e^{ax} \sin bx$$

$$- \frac{a}{b} \int \underbrace{e^{ax}}_{u} \underbrace{\sin bx \, dx}_{dv}.$$

Ositt. Integr.

$$\begin{cases} u = e^{ax} & dv = \sin bx \, dx \\ du = a e^{ax} & v = -\frac{1}{b} \cos bx \end{cases}$$

$$\int \overset{u}{e^{ax}} \overbrace{\sin bx \, dx}^{dv}$$

$$(**) \quad = -\frac{1}{b} e^{ax} \cos bx - \frac{a}{b} \int e^{ax} \cos bx \, dx$$

$$= -\frac{1}{b} e^{ax} \cos bx - \frac{a}{b} I$$

Yhdistämällä (*) ja (**) saadaan

$$I = \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \left(-\frac{1}{b} e^{ax} \cos bx - \frac{a}{b} I \right)$$

Ratkaisemalla I saadaan

$$I = \frac{b e^{ax} \sin bx + a e^{ax} \cos bx}{a^2 + b^2} + C.$$

Örnek: $\int x^3 (\ln x)^2 \, dx = ?$

Kısm. İnteg.
$$\begin{cases} U = (\ln x)^2 & dV = x^3 \, dx \\ dU = \dfrac{2 \ln x}{x} \, dx & V = \dfrac{1}{4} x^4 \end{cases}$$

$$\int \underbrace{(\ln x)^2}_{U} \cdot \underbrace{x^3 dx}_{dV} = \frac{1}{4} x^4 (\ln x)^2$$
$$- \frac{1}{2} \int x^4 \cdot \frac{\ln x}{x} \, dx$$
$$= \frac{1}{4} x^4 \ln^2 x - \frac{1}{2} \int x^3 \ln x \, dx$$

Kısm. İnteg.
$$\begin{cases} U = \ln x & dV = x^3 dx \\ dU = \dfrac{dx}{x} & V = \dfrac{1}{4} x^4 \end{cases}$$

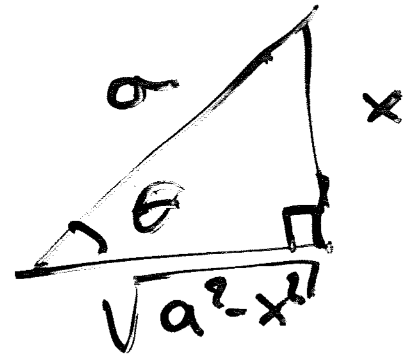
$$\int x^3 \ln x \, dx = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int \underbrace{x^4 \cdot \frac{dx}{x}}_{x^3 dx}$$
$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \cdot \frac{1}{4} x^4 + C$$

Bu'ş koko kaydığı integrali
$$= \frac{1}{4} x^4 \ln^2 x - \frac{1}{8} x^4 \ln x + \frac{1}{32} x^4 + C.$$

Erittäin hyviä sijoitus-vinkkejä!

1 Integraalit, jotka sisältävät $\sqrt{a^2-x^2}$ tyyppiä olevia lausekkeita.

Tarkastellaan kolmiota



sijoitus
$$\begin{cases} \sin\theta = \dfrac{x}{a} & \cos\theta = \dfrac{1}{a}\sqrt{a^2-x^2} \\ dx = a\cos\theta\, d\theta. & \tan\theta = \dfrac{x}{\sqrt{a^2-x^2}} \end{cases}$$

Esim:
$$\int \frac{dx}{(5-x^2)^{3/2}}$$

sijoitus
$$\begin{cases} x = \sqrt{5}\sin\theta \\ 5-x^2 = 5-5\sin^2\theta = 5\cos^2\theta \\ dx = \sqrt{5}\cos\theta\, d\theta \end{cases}$$

$$\int \frac{dx}{(5-x^2)^{3/2}} = \int \frac{\sqrt{5}\cdot\cos\theta\, d\theta}{(\sqrt{5})^3\cdot\cos^3\theta}$$

$$= \frac{1}{5}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{5}\tan\theta + C.$$

koska

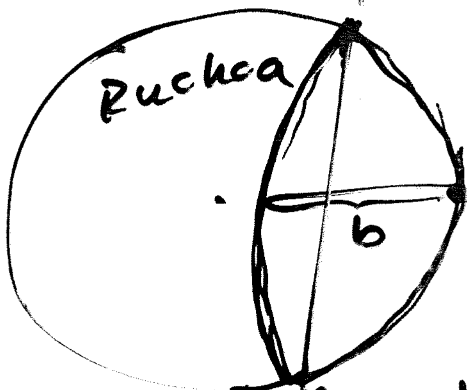
$$\frac{d \tan\theta}{d\theta} = 1 + \tan^2\theta$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}.$$

Siis

$$\int \frac{dx}{(5-x^2)^{3/2}} = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C.$$

Esim: Lammas liikkuu ympyrässä, $x^2+y^2=1$



Kuinka pitkä lieka $b<2$ jotta lammas söisi puolet pinta-alasta. Siis tulee osata laskea ympyräsegmentin pinta-alaa: $b<a$

$$\int_b^a \sqrt{a^2-x^2}\,dx = ?$$

sijoitus

$$\begin{cases} x = a\sin\theta \\ a^2-x^2 = a^2\cos^2\theta \\ dx = a\cos\theta\, d\theta \end{cases}$$

$$\int \sqrt{a^2 - x^2}\,dx = \int a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta \qquad \text{(parillinen potenssi: siinä [illegible] kannattaa!)}$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)$$

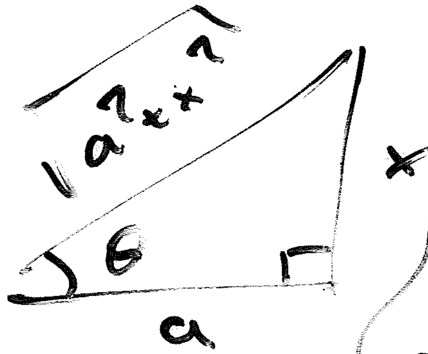
Nyt $\frac{x}{a} = \underbrace{\sin\theta}_{x/a} \Rightarrow \theta = \arcsin\frac{x}{a}$

$$\sin 2\theta = 2\overbrace{\sin\theta}^{x/a}\cos\theta = 2\cdot\frac{x}{a}\cdot\sqrt{1 - \frac{x^2}{a^2}}.$$

Siis saadaan

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{2x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right). \qquad ∎$$

2 Integraalit jotka sisältävät lausekkeen muotoa $a^2 + x^2$.



$$\begin{cases} \sin\theta = \dfrac{x}{\sqrt{a^2+x^2}} \\ \cos\theta = \dfrac{a}{\sqrt{a^2+x^2}} \\ \tan\theta = \dfrac{x}{a} \\ dx = a(1+\tan^2\theta)\,d\theta \end{cases}$$

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Esim: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

$4+x^2 = 4(1+\tan^2\theta)$

Sij: $\begin{cases} x = 2\tan\theta \\ dx = 2(1+\tan^2\theta)\,d\theta \end{cases}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2(1+\tan^2\theta)\,d\theta}{2(1+\tan^2\theta)^{1/2}}$$

$$= \int (1+\tan^2\theta)^{1/2}\,d\theta$$

Merkitsevät!

$$= \int \frac{d\theta}{\cos\theta}$$

$$= \ln\left|\frac{1}{\cos\theta} + \tan\theta\right| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

$$= \ln\left(\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right) + C$$

Tämä integraali lasketaan usein eri

(i) sijoituksella, joka käyttää identiteettiä

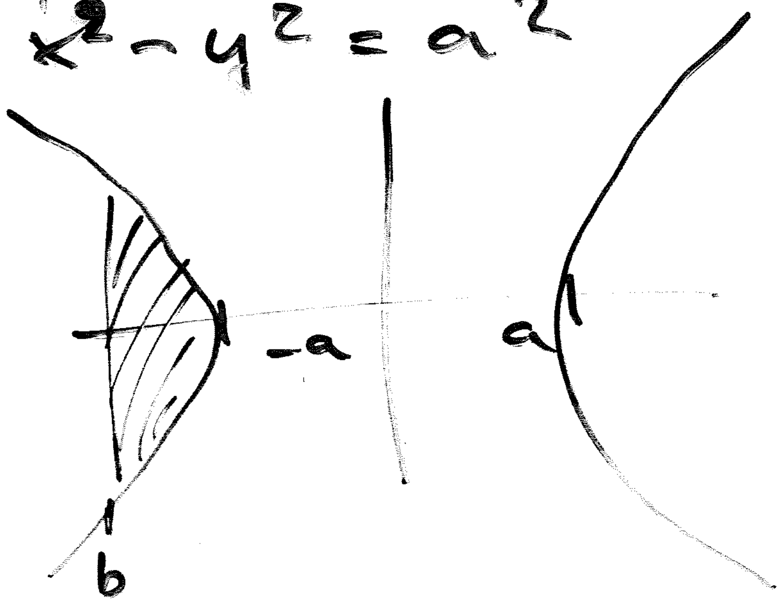
$$\cosh^2\theta - \sinh^2\theta = 1$$

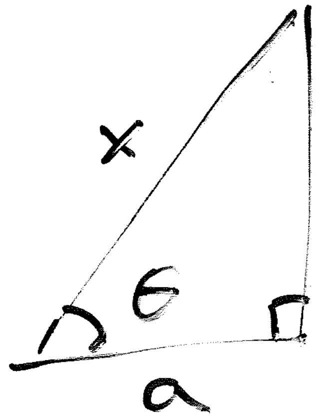
(ii) $-\frac{x}{2} + t = \sqrt{4+x^2}$

uusi muuttuja t $\boxed{\text{ns. Eulerin sijoitus.}}$

3. Integraalit, jotka sisältävät lausekkeen muotoa x^2-a^2 $(x>a)$

Liittyvät hyperbeleihin $x^2-y^2=a^2$





$\sqrt{x^2-a^2}$, $x>a$

$\tan\theta = \frac{1}{a}\sqrt{x^2-a^2}$

$\cos\theta = \frac{a}{x}$ $\quad \sin\theta = \frac{\sqrt{x^2-a^2}}{x}$

$x = \frac{a}{\cos\theta} \Rightarrow dx = -a\frac{1}{\cos^2\theta}\cdot(-\sin\theta)\,d\theta$

$= \frac{a\sin\theta\, d\theta}{\cos^2\theta}$

Esim: $\int \frac{dx}{\sqrt{x^2-a^2}} = ?$ $\quad |x|>|a|$

$$x^2 - a^2 = \left(\frac{a}{\cos\theta}\right)^2 - a^2$$

$$= a^2\left(\frac{1}{\cos^2\theta} - 1\right) = a^2\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)$$

$$= a^2 \tan^2\theta .$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \underbrace{\frac{a \sin\theta \, d\theta}{\cos^2\theta}}_{dx} \cdot \underbrace{\frac{1}{a\tan\theta}}_{= \frac{\cos\theta}{a\sin\theta}}$$

$$= \int \frac{d\theta}{\cos\theta} = \ln\left|\frac{1}{\cos\theta} + \tan\theta\right| + C'$$

$$= \ln\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2-a^2}\right| + C'$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| \underbrace{- \ln|a| + C'}_{=:C}$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C. \quad \blacksquare$$

<u>Neliöksi täydentäminen integraalimerkin alla</u>

<u>Esim:</u>

$$\int \frac{dx}{\sqrt{2x-x^2}} = ?$$

$$2x - x^2 = 1 - (x-1)^2$$

$$-x^2 + 2x - 1$$

Muuttujanvaihto: $u = x-1$, $dx = du$. .

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{du}{\sqrt{1-u^2}} \qquad |u|<1$$

$$= \arcsin u + C$$

$$= \arcsin (x-1) + C. \quad (\text{kun } |x-1| < 1)$$

Esim:
$$\int \frac{x\,dx}{4x^2+12x+13}$$

$$= \int \frac{x\,dx}{4\left(x^2+3x+\frac{13}{4}\right)}$$

$$= \frac{1}{4}\int \frac{x\,dx}{\left(x+\frac{3}{2}\right)^2 \underbrace{-\frac{9}{4}+\frac{13}{4}}_{4/4=1}}$$

$$= \frac{1}{4}\int \frac{x\,dx}{1+\left(x+\frac{3}{2}\right)^2}$$

Muuttujanvaihto: $\begin{cases} u = x+\frac{3}{2} \\ du = dx \end{cases}$

$$= \frac{1}{4}\int \frac{\left(u-\frac{3}{2}\right)du}{1+u^2}$$

muista $\int \frac{f'(u)}{f(u)}\,du = \ln|f(u)| + C$.

$$\int \frac{u-\frac{3}{2}}{1+u^2}\,du = \frac{1}{2}\int \frac{2u}{1+u^2}\,du - \frac{3}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\ln(1+u^2) - \frac{3}{2}\arctan u + C$$

$$= \frac{1}{2}\ln(4x^2+12x+13) - \frac{3}{2}\arctan\left(x+\frac{3}{2}\right) + C. \blacksquare$$

<u>Esim:</u> Juurilausekkeen poistaminen integraalista.

$$\int \frac{dx}{1+\sqrt{2x}} \qquad x \geq 0$$

<u>Sijoitus:</u>

$$\begin{cases} u^2 = 2x \\ 2u\,du = 2\,dx \\ (u\,du = dx) \end{cases}$$

$$\int \frac{dx}{1+\sqrt{2x}} = \int \frac{u\,du}{1+u}$$

$$= \int \frac{1+u-1}{1+u}\,du = \int\left(1 - \frac{1}{1+u}\right)du$$

$$= u - \int \frac{du}{1+u} = u - \ln|1+u| + C$$

$$= \sqrt{2x} - \ln(1+\sqrt{2x}) + C. \qquad \blacksquare$$

Esim:
$$\int \frac{x\,dx}{\sqrt[3]{3x+2}} = ?$$

Sijoitus:
$$\begin{cases} u^3 = 3x+2 \\ 3u^2\,du = 3\,dx \\ (u^2\,du = dx) \end{cases}$$

$$= \frac{1}{3}\int \frac{\overbrace{(u^3-2)}^{x}\cdot\overbrace{u^2\,du}^{dx}}{u}$$

$$= \frac{1}{3}\int (u^4 - 2u)\,du$$

$$= \frac{1}{3}\cdot\left(\frac{1}{5}u^5 - u^2\right) + C$$

(Laske itse.) ■

$$= \cdots\cdots$$