Esim: $I = \int e^{ax} \cos bx \, dx$. 29.11.2006

Osittais integroimalla

$$U = e^{ax} \qquad dV = \cos bx \, dx$$

$$dU = a e^{ax} dx \qquad V = \frac{1}{b} \sin bx$$

$$I = \int \overbrace{e^{ax}}^{U} \cdot \overbrace{\cos bx \, dx}^{dV}$$

$$= \frac{1}{b} e^{ax} \sin bx - \frac{a}{b} \int \sin bx \cdot e^{ax} \, dx$$

Viimeiselle integraalille sama temppu:

$$U = e^{ax} \qquad dV = \sin bx \, dx$$

$$dU = a e^{ax} dx$$

$$V = -\frac{1}{b} \cos bx$$

$$\int e^{ax} \sin bx \, dx$$

$$= -\frac{1}{b} e^{ax} \cos bx$$

$$-\frac{a}{b}\int e^{ax} (-\cos bx) \, dx$$

$$= \frac{1}{b} e^{ax} \cos bx$$

$$+ \frac{a}{b} \int e^{ax} \cos bx \, dx \quad (+C_2)$$

Siis yhdistämällä nämä osittaisintegrointitemput saadaan

$$\underline{I} = \frac{1}{b} e^{ax} \sin bx$$

$$- \frac{a}{b}\left(\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} I \right)$$

$$= \frac{1}{b} e^{ax} \sin bx$$

$$- \frac{a}{b^2} e^{ax} \cos bx$$

$$- \frac{a^2}{b^2} I$$

Saadaan ratkaisuna

$$I = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{b^2+a^2}$$

$+ C.$

Esim: $\int_1^e x^3 (\ln x)^2 dx$

Osittaisintegrointi

$u = (\ln x)^2, \quad dv = x^3 dx$

$du = 2\ln x \cdot \frac{1}{x} \cdot dx$

$v = \frac{1}{4} x^4$

$\int x^3 (\ln x)^2 dx$

$= \frac{1}{4} x^4 (\ln x)^2$

$- \frac{1}{2} \int x^3 \ln x \, dx.$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{dx}{x} \qquad v = \frac{1}{4} x^4$

$\int x^3 \ln x \, dx$

$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$

$= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4.$

Kasaamalla yhteen

$\int x^3 (\ln x)^2 dx$

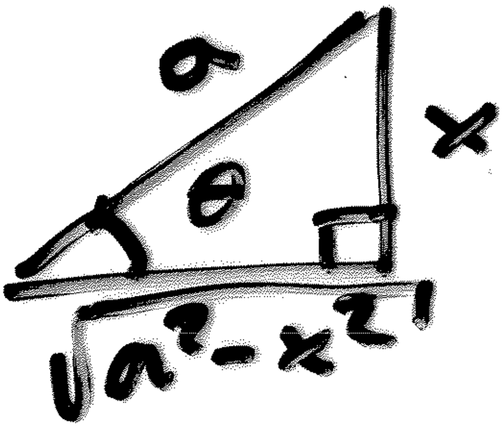
$= \frac{1}{4} x^4 (\ln x)^2$

$- \frac{1}{8} x^4 \ln x + \frac{1}{32} x^4.$

■

Erittäin hyviä sijoitusvinkkejä

1. Integraalit jotka sisältävät lausekkeita tyyppiä $\sqrt{a^2-x^2}$



$$\sin\theta = \frac{x}{a}$$

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$$

Esim:

$$\int \frac{dx}{(5-x^2)^{3/2}}$$

$$x = \sqrt{5}\sin\theta$$

$$5-x^2 = 5-5\sin^2\theta$$

$$= 5\cos^2\theta .$$

Dus $\dfrac{1}{(5-x^2)^{3/2}} = \dfrac{1}{(5\cos^2\theta)^{3/2}}$

$= \dfrac{1}{5\sqrt{5}} \cdot \dfrac{1}{\cos^3\theta}$

Tovsaalks

$x = \sqrt{5}\sin\theta$

$dx = \sqrt{5}\cos\theta\, d\theta$

Dus

$$\frac{dx}{(5-x^2)^{3/2}} = \frac{\sqrt{5}}{5\sqrt{5}} \cdot \frac{\cos\theta}{\cos^3\theta}\, d\theta$$

$$= \frac{1}{5}\,\frac{d\theta}{\cos^2\theta}$$

$$\frac{1}{\cos^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}$$

$$= 1 + \tan^2\theta = \frac{d}{d\theta}\tan\theta$$

$$\int \frac{dx}{(5-x^2)^{3/2}} = \frac{1}{5}\int \frac{d\theta}{\cos^2\theta}$$

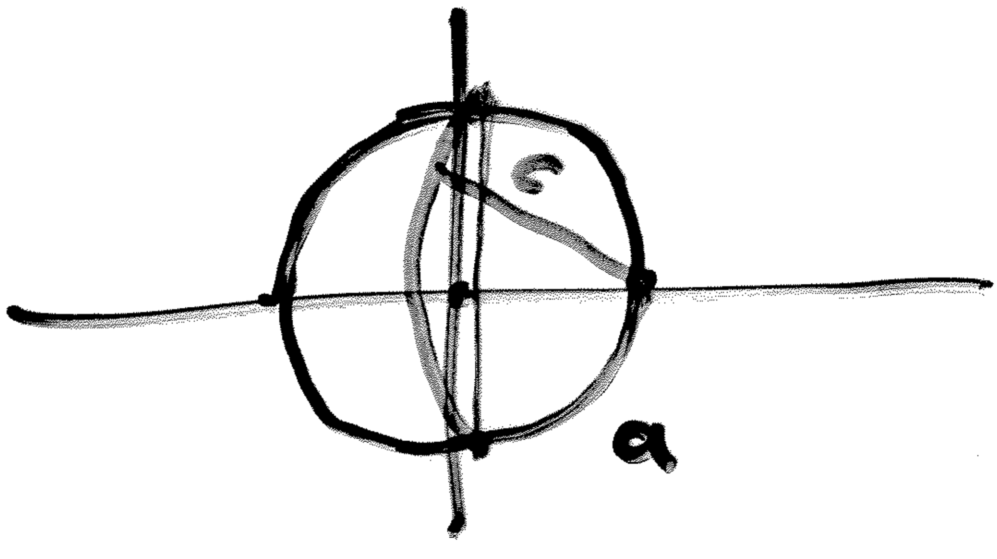
$$= \frac{1}{5}\tan\theta + C$$

$$= \frac{1}{5}\,\frac{\sin\theta}{\cos\theta} + C$$

$$= \frac{1}{5}\,\frac{\frac{x}{\sqrt{5}}}{\sqrt{1-\frac{x^2}{5}}} + C$$

$$= \frac{1}{5}\,\frac{x}{\sqrt{5-x^2}} + C.$$

Esim: Lammas liikenneympyrässä:

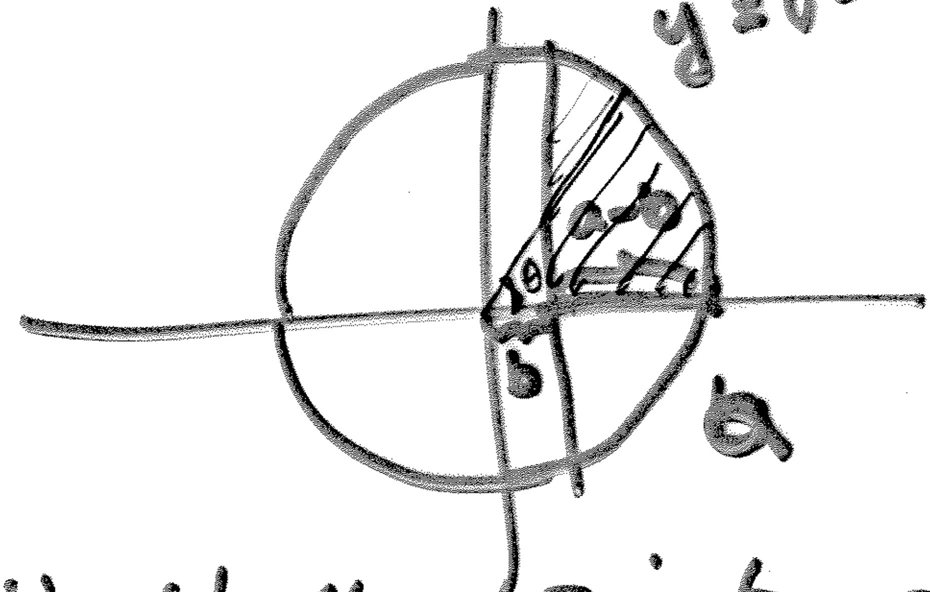


Pisteeseen $(a,0)$ laitetaan narun päässä lammas, jonka narun pituus on $c>0$

Kuinka pitkä tulee b olla, jotta lammas söisi tasan puolet ruohosta.

Huom: lampaan syömä pinta-ala on kahden ympyräsegmentin unioni ⇒ pitää osata laskea ympyräsegmentin pinta-ala.

$y = \sqrt{a^2 - x^2}$



Viivoitettu pinta-ala

$$\int_b^a \sqrt{a^2 - x^2}\, dx = ?$$

$$x = a \sin \theta$$

$$dx = a \cos \theta \, d\theta$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta}$$

$$= a \cos \theta$$

(kuan: $\theta \in (0, \frac{\pi}{2})$!)

Fis

$$\int \sqrt{a^2 - x^2} \, dx$$

$$= \int a \cos \theta \cdot a \cos \theta \, d\theta$$

$$= a^2 \int \cos^2 \theta \, d\theta$$

$$= a^2 \int \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2} \cdot \frac{1}{2} \sin 2\theta$$

$$= a^2 \left(\frac{\theta}{2} + \frac{1}{4} \cdot 2 \sin \theta \cos \theta \right)$$

$$= \frac{a^2}{2} (\theta + \sin \theta \cos \theta)$$

$$\theta = \arcsin \frac{x}{a}$$

$$\sin \theta = \frac{x}{a}$$

$$\cos \theta = \sqrt{1 - \frac{x^2}{a^2}}\,.$$

Siis integraali on

$$\int \sqrt{a^2 - x^2}\, dx$$

$$= \frac{a^2}{2}\left(\arcsin \frac{x}{a} + \frac{x}{a} \cdot \sqrt{1 - \frac{x^2}{a^2}} \right)$$

$$= \frac{a^2}{2}\left(\arcsin \frac{x}{a} + x\sqrt{a^2 - x^2} \right)$$

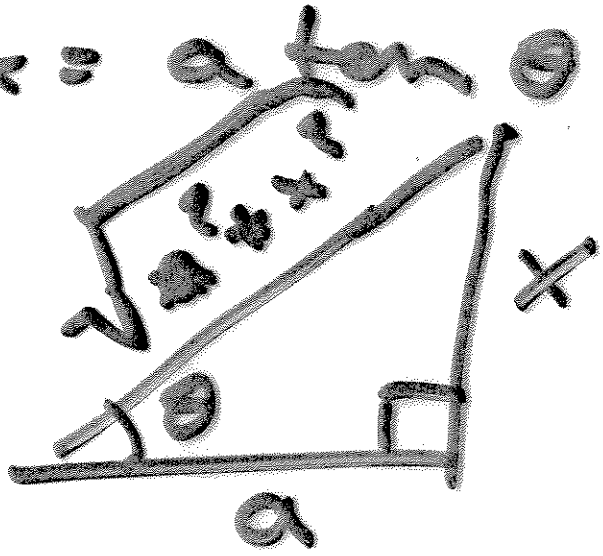
2. Integraalit jotka sisältävät lausekkeen

$$a^2 + x^2$$

Esim: $\sqrt{a^2+x^2}$, $\frac{1}{a^2+x^2}$.

Sijoitus

$$x = a\tan\theta$$



Esim: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}} = ?$

$$x = 2\tan\theta$$
$$dx = 2(1+\tan^2\theta)\,d\theta$$
$$4+x^2 = 4+4\tan^2\theta$$
$$= 4(1+\tan^2\theta)$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2(1+\tan^2\theta)\,d\theta}{2(1+\tan^2\theta)^{1/2}}$$
$$= \int \sqrt{1+\tan^2\theta}\,d\theta$$
$$= \int \frac{d\theta}{\cos\theta}$$

Koska $\sqrt{\dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}$

$$= \frac{1}{\cos\theta}.$$

Muistetaan että

$$\int \frac{d\theta}{\cos\theta} = \ln\left|\frac{1}{\cos\theta} + \tan\theta\right| + C$$

$$\cos\theta = \frac{a}{\sqrt{a^2+x^2}}$$

$$\tan\theta = \frac{x}{a},$$ joten

$$\int \frac{dx}{\sqrt{4+x^2}} = \ln\left|\frac{1}{2}\sqrt{4+x^2} + \frac{x}{2}\right| + C$$

$$= \ln\left|\sqrt{4+x^2} + x\right| \underbrace{- \ln 2 + C}_{=: C'}.$$

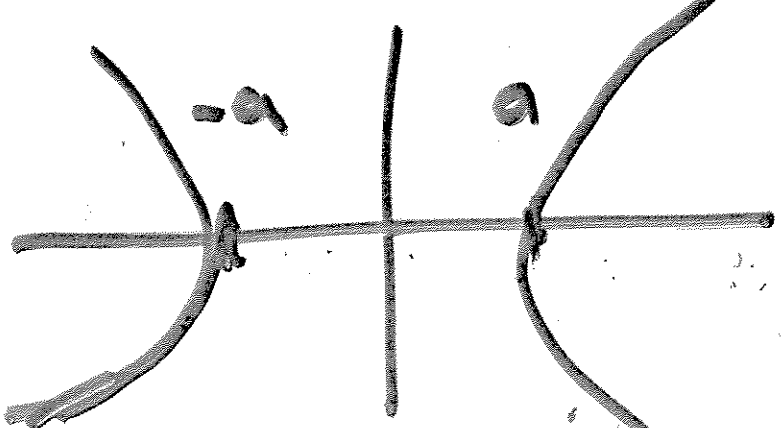
Tämä integraali voidaan laskea eri tavoin.

(i) $\cosh^2\theta - \sinh^2\theta = 1$

(ii) $-x + t = \sqrt{4+x^2}$
t on uusi muuttuja.

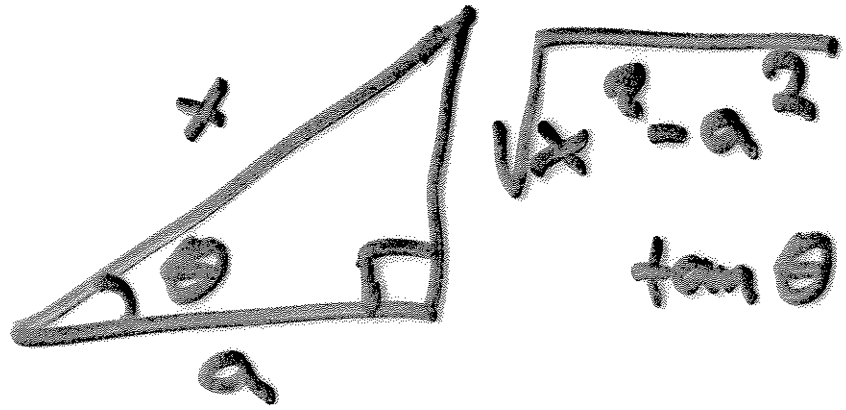
(3.) Integraalit jotka sisältävät lausekkeen
$\sqrt{x^2 - a^2}$, $|x| > a$

Liittyvät hyperbeleihin
$x^2 - y^2 = a^2$



Käänteinen sijoitus

$$x = \frac{a}{\cos\theta}$$



$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$

$$\cos\theta = \frac{a}{x}$$

$$\sin\theta = \frac{\sqrt{x^2-a^2}}{x}$$

__Esim:__

$$\int \frac{dx}{\sqrt{x^2-a^2}}, \quad |x|>a$$

$$x = \frac{a}{\cos\theta}$$

$$dx = -\frac{a}{\cos^2\theta}\cdot(-\sin\theta)$$

$$= \frac{a\sin\theta}{\cos^2\theta}$$

$$x^2 - a^2$$

$$= \frac{a^2}{\cos^2\theta} - a^2$$

$$= a^2\left(\frac{1 - \cos^2\theta}{\cos^2\theta}\right)$$

$$= a^2 \frac{\sin^2\theta}{\cos^2\theta} = a^2\tan^2\theta.$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{\sin\theta \, d\theta}{\cos^2\theta \cdot a\tan\theta}$$

$$= \frac{1}{a}\int \frac{d\theta}{\cos\theta}$$

$$= \frac{1}{a} \ln\left|\frac{1}{\cos\theta} + \tan\theta\right|$$

$$= \frac{1}{a} \ln\left|\frac{x}{a} + \frac{1}{a}\sqrt{x^2-a^2}\right|.$$

Neliöksi täydentämisen integraalimerkin alla

Esim: $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$

$$2x - x^2 = 1 - \underbrace{1 + 2x - x^2}$$

$$= 1 - (x-1)^2$$

Siis

$$\int \frac{dx}{\sqrt{2x-x^2}}$$

$$\boxed{|x-1|<1}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$u = x-1$$
$$du = dx$$

$$= \int \frac{du}{\sqrt{1-u^2}} \qquad |u|<1$$

[illegible] 10:

$$= \arcsin u + C$$

$$= \arcsin (x-1) + C.$$

<u>Ekim</u>: $\displaystyle\int \frac{x\,dx}{4x^2+12x+13}$

$\qquad\qquad\qquad\qquad\qquad\qquad 9+4$

$$= \int \frac{x\,dx}{4\big(\underbrace{x^2+3x+\left(\tfrac{3}{2}\right)^2}+1\big)}$$

$$= \frac{1}{4}\int \frac{x\,dx}{1+\left(x+\frac{3}{2}\right)^2}$$

$$u = x+\frac{3}{2}$$
$$du = dx$$

$$= \frac{1}{4}\int \frac{\left(u-\frac{3}{2}\right)du}{1+u^2}$$

$$= \frac{1}{8}\int \frac{2u\,du}{1+u^2} - \frac{3}{8}\int \frac{du}{1+u^2}$$

$$\int \frac{f'(u)}{f(u)}\,du = \ln|f(u)| + C$$

Tällöin

$$\int \frac{2u}{1+u^2} = \ln|1+u^2| + C,$$

$$\frac{d}{du}\,\overline{\arctan}\,u = \frac{1}{1+u^2}.$$

Siis integraali on

$$= \frac{1}{8}\ln|1+u^2|$$

$$- \frac{3}{8}\,\overline{\arctan}\,u + C$$

$$= \frac{1}{8}\ln\left|1 + x^2 + 3x + \frac{9}{4}\right|$$

$$- \frac{3}{8}\,\overline{\arctan}\left|x + \frac{3}{2}\right| + C$$

$$= \frac{1}{8}\ln\frac{1}{4}\left|4x^2 + 12x + 9\right| + $$

$$- \frac{3}{\ }\,\overline{\arctan} + \cdots \overline{\ }$$

C

Ehm: Jaurilaukekan perstaman integaaehih

$$\int \frac{dx}{1+\sqrt{2x}}$$

$u^2 = 2x$

$\sqrt{2x} = u$

$2u \cdot \frac{du}{dx} = 2$

$2u\,du = 2dx$

$u\,du = dx$

Saadaan

$$\int \frac{dx}{1+\sqrt{2x}} = \int \frac{u\,du}{1+\sqrt{u^2}}$$

$$= \int \frac{u\,du}{1+u}$$

$$= \int \frac{1+u-1}{1+u}\,du$$

$$= \int \left(1 - \frac{1}{1+u}\right) du$$

$$= u - \int \frac{du}{1+u}$$

$$\begin{cases} v = 1+u \\ dv = du \end{cases}$$

$$\int \frac{du}{1+u} = \int \frac{dv}{v}$$

$$= \ln|v| + C = \ln|1+u| + C.$$

Güis [illegible]

$$\int \frac{dx}{1+\sqrt{x}} = u + \ln|1+u| + C$$

$$= \sqrt{2x} + \ln|1+\sqrt{2x}| + C.$$

<u>Ehm:</u>

$$\int \frac{x\,dx}{\sqrt[3]{3x+2}}$$

$u^3 = 3x+2$

$3u^2du = 3xd$

$xdx = u^2du$

$$= \int \frac{u^2\,du}{u} = \int u\,du$$

$$= \frac{1}{2}u^2 + C = \frac{1}{2}(3x+2)^{\frac{2}{3}} + C.$$