

HOMEOMORPHISMS OF THE HEISENBERG GROUP PRESERVING BMO

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ABSTRACT. We provide a new geometric proof of Reimann's theorem characterizing quasiconformal mappings as the ones preserving functions of bounded mean oscillation. While our proof is new already in the Euclidean spaces, it is applicable in Heisenberg groups as well as in more general stratified nilpotent Carnot groups.

1. INTRODUCTION

It is well known that composition with quasiconformal mappings of the Euclidean spaces preserves functions of bounded mean oscillation, and with an additional differentiability assumption the preservation property implies that the mapping is quasiconformal. This result was proved by Reimann [13] in \mathbb{R}^n , $n \geq 2$. The corresponding problem on the real-line was solved by Jones [9]. Later analogous results were proved for maximal functions and Muckenhoupt weights, see Uchiyama [17].

Astala [1] showed that by localizing the preservation property, the differentiability assumptions could be removed. Using the approach initiated by Astala, Staples [14, 15] proved analogues of Uchiyama's theorems, and finally, following Astala's idea of localizing the preservation property, Vodop'yanov and Greshnov generalized the Reimann type characterization of quasiconformal mappings to Carnot groups and Carnot–Carathéodory spaces in [16]. However, even in Euclidean spaces, it remains open whether preservation of BMO alone implies quasiconformality. The known results assume either differentiability or *local* BMO invariance.

In this paper, we take another, more geometric, point of view and give a new proof of Reimann's theorem that applies in the Heisenberg group. Reimann's original proof relied on a construction of suitable functions of bounded mean oscillation, whereas our approach makes use of a general characterization of BMO-preserving mappings due to

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Gotoh [4], or more precisely its generalization to doubling metric measure spaces by Kinnunen et al. [10]. We do not assume local BMO invariance, but we will use assumptions more close to Reimann's original approach instead. We mention that our proof of Reimann's theorem can also be applied in more general Carnot groups, we refer the reader to the discussion in Remark 3.4. But more importantly, it provides a new simple proof in the Euclidean setting with obvious changes.

2. PRELIMINARIES

We denote by \mathbb{H}^n the n th Heisenberg group, that is, the set $\mathbb{C}^n \times \mathbb{R}$ endowed with the group operation

$$(z, t)(z', t') = (z + z', t + t' + 2\operatorname{Im} \sum_j z_j \overline{z'_j}).$$

The inverse of (z, t) is $(z, t)^{-1} = (-z, -t)$. For $z \in \mathbb{C}$, we denote $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$. The horizontal distribution of the Heisenberg group is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$$

as $j = 1, \dots, n$, and the only non-trivial commutator relation of the tangent bundle is $[X_j, Y_j] = -4T$ where $T = \partial/\partial t$ is the vertical direction.

A path $\gamma : [0, 1] \rightarrow \mathbb{H}^n$ is horizontal if $\gamma'(t)$ is horizontal for all t . We define the Carnot–Carathéodory distance as the length of the shortest horizontal path joining two points in \mathbb{H}^n , and it will be denoted by d . Using this metric, the left-translations are isometries of the Heisenberg group, i.e. $d(p, q) = d(lp, lq)$, where $p, q, l \in \mathbb{H}^n$.

The Lebesgue measure on \mathbb{R}^{2n+1} is the bi-invariant Haar measure for \mathbb{H}^n , and it will be denoted by $|\cdot|$. Moreover, we define the dilations by positive reals δ as $\delta(z, t) = (\delta z, \delta^2 t)$. The dilations satisfy $|\delta E| = \delta^{2n+2} |E|$ and $d(\delta z, \delta z') = \delta d(z, z')$.

There is another metric on \mathbb{H}^n that is bi-Lipschitz equivalent to the Carnot–Carathéodory metric, namely, the one induced by the Koranyi norm. For $p = (z, t) \in \mathbb{H}^n$, we define

$$d_K(p, 0) = (|z|^4 + |t|^2)^{1/4}.$$

Hence the metric notions can be defined using whichever metric.

A homomorphism $L : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is homogeneous if it commutes with dilations. A mapping $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is said to be Pansu differentiable at $p \in \mathbb{H}^n$ if there is a homogeneous homomorphism L such that

$$\frac{d(f(p)^{-1}f(hp), L(h))}{d(h, 0)} \rightarrow 0$$

as $h \rightarrow 0$. The homogeneous homomorphism L satisfying the limit is unique and is called Pansu differential of f at p . Note that the

homogeneous homomorphism L is continuous if f is. We refer the reader, for example, to [12] for a detailed discussion on differential calculus on general stratified groups.

Some results in this paper are most conveniently stated in more general setting of doubling metric measure spaces. The Heisenberg group and more general Carnot groups discussed in Remark 3.4, however, are known to be particular examples of such general spaces. For brevity, we refer the reader to [5] and [2] for this concept of a doubling metric measure space (X, d, μ) and also for all the necessary definitions.

A locally integrable function u on a metric measure space with a doubling measure μ is said to be of bounded mean oscillation, abbreviated to $\text{BMO}(X)$, if its mean oscillations on metric balls $B \subset X$ are uniformly bounded, that is,

$$\sup_B \int_B |u - u_B| d\mu =: \|u\|_{\text{BMO}} < \infty.$$

Here both u_B and the barred integral mean the integral average of u over a ball B . One of the most fundamental properties of the class BMO is the following John–Nirenberg lemma. For a proof, we refer the reader to [2].

Lemma 2.1. *Let X be a metric measure space with a doubling measure μ . Suppose that $u \in \text{BMO}(5B)$. Then for every $\lambda > 0$*

$$\mu(\{x \in B : |u(x) - u_B| > \lambda\}) \leq 2\mu(B) \exp\left(-\frac{A\lambda}{\|u\|_{\text{BMO}}}\right),$$

where A depends only on the doubling constant of the measure μ .

A homeomorphism of metric spaces $f : X \rightarrow X$ respecting null sets is said to be BMO -preserving if $\|u \circ f^{-1}\|_{\text{BMO}} \leq C_f \|u\|_{\text{BMO}}$. This property has a characterization in terms of densities, and the following theorem was first found by Gotoh in [4]. The metric measure space version of the theorem can be found in [10].

Theorem 2.2. *A homeomorphism $f : X \rightarrow X$ on a metric measure space with a doubling measure μ such that $f^{-1}(E)$ is a null set for all null sets $E \subset X$ is BMO -preserving if and only if the following holds: For any pair of measurable sets E_1, E_2 of X we have*

$$\sup_B \min_i \frac{\mu(E_i \cap B)}{\mu(B)} \leq K \left(\sup_B \min_i \frac{\mu(f(E_i) \cap B)}{\mu(B)} \right)^\alpha$$

for universal constants $\alpha, K > 0$. The supremum is taken over all metric balls in X .

Given a homeomorphism $f : X \rightarrow X$, then for every $x \in X$ and $r > 0$ set

$$K_f(x, r) = \frac{\sup\{d(f(x), f(y)) : d(x, y) = r\}}{\inf\{d(f(x), f(y)) : d(x, y) = r\}}.$$

We recall that f is called quasiconformal if

$$\limsup_{r \rightarrow 0} K_f(x, r) \leq K$$

for all $x \in X$ and for some uniform $K < \infty$. A homeomorphism f is quasi-symmetric, if the inequality above is satisfied without \limsup , that is, it is satisfied for all $x \in X$ and all $r > 0$. We refer the reader to [6, 7, 8] for more on quasiconformal mappings between metric spaces.

3. CHARACTERIZATION VIA BMO-MAPS

In this section we prove that BMO preserving maps are quasiconformal and vice versa, provided that one assumes some additional regularity. The proof of the necessity part is new, and it shows that Gotoh's characterization of BMO-maps is one of the underlying phenomena connecting quasiconformal and BMO-preserving maps. The following is our main result, see also Remark 3.4.

Theorem 3.1. *Let $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ be an almost everywhere Pansu differentiable homeomorphism such that $f^{-1}(E)$ is a null set for all null sets E . Then f is quasiconformal if and only if it is BMO-preserving.*

The proof of Theorem 3.1 is divided into Lemma 3.2 and Lemma 3.3. The proof is new already in the Euclidean spaces, see [13].

3.1. Sufficiency. The sufficiency part is most conveniently stated in a much more general setting than the Heisenberg group. It is certainly well known among specialists that this part is true but nevertheless it seems to lack references. We state it in full generality and note that the Heisenberg group satisfies the assumptions of the following lemma. This fact and all the necessary definitions can be checked in [7, 8].

Lemma 3.2. *Let X be a linearly locally connected unbounded metric measure space of Q -bounded geometry. Suppose that $f : X \rightarrow X$ is K -quasiconformal and $u \in \text{BMO}(X)$. Then*

$$\|Fu\|_{\text{BMO}} \leq C\|u\|_{\text{BMO}},$$

where we write $Fu = u \circ f^{-1}$ and C is a positive constant depending only on K and the data of X .

Proof. By [7] (see also [5, Theorem 9.10]) f is quasisymmetric, and the pull-back measure $\mu(f(E))$ and μ are known to be A_∞ -related. As a consequence of quasisymmetry (see [5, Proposition 10.8], for each ball $B' \subset f(X) \subseteq X$ there is a ball $B \subset X$ such that $B' \subset f(B)$ and $\mu(f(B)) \leq C\mu(B')$ for some positive constant C .

By Lemma 2.1,

$$\begin{aligned}
\int_{B'} |Fu - u_B| \, d\mu &= \frac{1}{\mu(B')} \int_0^\infty \mu(\{x \in B' : |Fu(x) - u_B| > \lambda\}) \, d\lambda \\
&\leq \frac{C}{\mu(f(B))} \int_0^\infty \mu(\{x \in f(B) : |Fu(x) - u_B| > \lambda\}) \, d\lambda \\
&\leq \frac{C}{\mu(B)^\delta} \int_0^\infty \mu(\{x \in B : |u(x) - u_B| > \lambda\})^\delta \, d\lambda \\
&\leq C \int_0^\infty \exp(-\delta A\lambda / \|u\|_{\text{BMO}}) \, d\lambda \leq C\|u\|_{\text{BMO}},
\end{aligned}$$

where C is independent of B . It is straightforward to verify that

$$\int_{B'} |Fu - (Fu)_{B'}| \, d\mu \leq 2 \int_{B'} |Fu - u_B| \, d\mu,$$

and hence the claim follows. \square

3.2. Necessity. The proof of the necessity part proceeds through approximation by linear mappings. In order to do that, one has to assume differentiability. In Heisenberg group, we have to assume Pansu differentiability in order to approximate by homogeneous homomorphisms.

Reimann's original proof in [13] constructed an explicit BMO-function such that a differentiable map keeping it in BMO had to be quasiconformal. Our proof is less constructive, and one motivation for it is to point out that the interaction between BMO-preserving and quasiconformality is closely related to densities. Indeed, both classes of mappings have a characterization in terms of densities. For the quasiconformal case, we refer the reader to [11] and [3]. For the BMO-maps the characterization is stated in Theorem 2.2. These characterizations are, however, slightly different.

Our proof will work in Euclidean spaces and also in more general Carnot groups as is discussed in Remark 3.4.

Lemma 3.3. *Let $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ be an almost everywhere Pansu differentiable BMO-preserving homeomorphism such that $f^{-1}(E)$ is a null set for all null sets E . Then f is quasiconformal.*

Proof. Take any point of differentiability $z \in \mathbb{H}^n$ and let $B(z, r)$ be a ball centred at it. Since left-translations are isometries, we may assume that $z = 0$. Let L be the homogeneous homomorphism given by Pansu differentiability. For any points $x, y \in \mathbb{H}^n$ there is $v \in \partial B(0, 1)$ such that

$$\begin{aligned}
d(L(x), L(y)) &= d(L(y)^{-1}L(x), 0) = d(L(y^{-1}x), 0) \\
&= d(y^{-1}x, 0)d(L(v), 0) = d(x, y)d(L(v), 0).
\end{aligned}$$

We denote by λ_{\max} the maximum that the functional $v \mapsto d(L(v), 0)$ attains in $\partial B(0, 1)$. Recall that L is continuous.

Take $x \in \partial B(0, \frac{15}{16}r)$ such that

$$d(L(x), L(0)) = \lambda_{\max} d(x, 0) = \frac{15}{16} r \lambda_{\max}.$$

Let $E_1 = B(x, \frac{1}{16}r)$ and $E_2 = B(0, \frac{1}{16}r)$, and take $a \in E_1, b \in E_2$. Then

$$\begin{aligned} d(L(a), L(b)) &\geq d(L(x), L(0)) - d(L(x), L(a)) - d(L(0), L(b)) \\ &\geq \frac{15}{16} r \lambda_{\max} - \frac{1}{16} r \lambda_{\max} - \frac{1}{16} r \lambda_{\max} > \frac{3}{4} r \lambda_{\max}. \end{aligned}$$

Hence

$$\text{dist}(L(E_1), L(E_2)) \geq \frac{3}{4} r \lambda_{\max} \geq \frac{3}{8} \text{diam } L(B(0, r)).$$

In Gotoh's Theorem 2.2, we may choose $B(0, r)$ to be the ball in the left hand side in order to obtain a lower bound for the supremum. The balls in the right hand side must meet both E_1 and E_2 , so their radii have to exceed $\text{dist}(E_1, E_2)$. Altogether we get

$$\begin{aligned} 1 &\lesssim \sup_B \min_i \frac{|E_i \cap B|}{|B|} \leq K \left(\sup_B \min_i \frac{|L(E_i) \cap B|}{|B|} \right)^\alpha \\ &\lesssim \left(\frac{|L(B(0, r))|}{(\text{dist}(E_1, E_2))^{2n+2}} \right)^\alpha \lesssim \left(\frac{|L(B(0, r))|}{(\text{diam } L(B(0, r)))^{2n+2}} \right)^\alpha. \end{aligned}$$

This estimate holds for all $r > 0$. Letting $r \rightarrow 0$, we get the same estimate for f , because near origin f is L up to an epsilon. So

$$1 \lesssim \liminf_{r \rightarrow 0} \frac{|f(B(0, r))|}{(\text{diam } f(B(0, r)))^{2n+2}}.$$

and it follows by standard arguments that f is quasiconformal. \square

Remark 3.4. It is easy to check that Theorem 3.1 is valid also in more general Carnot groups \mathbb{G} (of step k). By these groups we mean simply connected Lie groups whose Lie algebra \mathfrak{g} admits a nilpotent stratification up to step $k \geq 2$, i.e. $\mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_k$ and $[V_1, V_j] = V_{j+1}$ for $j = 1, \dots, k-1$ and $[V_1, V_k] = \{0\}$.

A Carnot group is equipped with a family of non-isotropic dilations $\delta_\lambda : \mathfrak{g} \rightarrow \mathfrak{g}$ defined as $\delta_\lambda(\xi) = \lambda^j \xi$ whenever $\xi \in V_j$, $j = 1, \dots, k$, where λ is a positive real. The metric structure on \mathbb{G} is given by the Carnot–Carathéodory distance d for which $d(\delta_\lambda z, \delta_\lambda z') = \lambda d(z, z')$ for every $z, z' \in \mathbb{G}$. Denoting by dg the bi-invariant Haar measure on \mathbb{G} , obtained by lifting via exponential map the Lebesgue measure on \mathfrak{g} , we see that $(d \circ \delta_\lambda)(g) = \lambda^Q dg$, where $Q = \sum_{j=1}^k j \dim(V_j)$ is the homogeneous dimension of \mathbb{G} . We refer the reader to [12] for more on calculus on stratified groups.

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