Workshop on Nonlinear, nonlocal Problems and Stochastic Methods

Aalto University, December 7–9, 2016

Abstracts of the workshop talks

Regularization by noise for the stochastic transport equation

Lisa Beck (University of Augsburg)

Abstract: We discuss several aspects of regularity and uniqueness for weak (L^∞-) solutions to the (deterministic and stochastic) transport equation

$$du = b \cdot Du\, dt + \sigma Du \circ dW_t.$$

Here, b is a vector field (the drift), u is the unknown, σ is a real number, W_t is a Brownian motion, and the stochastic term is interpreted in the Statonovich sense. For the deterministic equation ($\sigma = 0$) it is well-known that multiple solutions may exist and that solutions may blow up from smooth initial data in finite time if the drift is not regular enough. For the stochastic equation ($\sigma \neq 0$) instead, it turns out that a suitable integrability condition (known from fluid dynamics as the Ladyzhenskaya–Prodi–Serrin condition) on the drift is sufficient to prevent the formation of non-uniqueness and of singularities. After a short review of some techniques for the deterministic equation we explain how this regularization phenomenon, namely the conservation of Sobolev regularity of the initial data and the restoration of uniqueness, is obtained by means of PDE techniques (as opposed to stochastic characteristics).

The results presented in this talk are part of a joint project with F. Flandoli, M. Gubinelli and M. Maurelli.

On the master equation in mean field game theory

Pierre Cardaliaguet (Université Paris - Dauphine)

Abstract: In mean field game theory, the master equation is a kind of nonlinear, nonlocal transport equation stated on the space of measure. Based on a joint work with Delarue, Lasry and Lions, I will discussed the well-posedness of the equation and its applications to mean field games.

On function spaces and extension results for nonlocal Dirichlet problems

Bartłomiej Dyda (Politechnika Wrocławska)

Abstract: Let us consider the question, for which functions $g : \mathbb{R}^d \setminus \Omega \to \mathbb{R}$ there is a function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$Lu(x) := \text{p.v.} \int_{\mathbb{R}^d} \frac{u(y) - u(x)}{|y - x|^{1+2s}} \mathrm{d}y = 0 \qquad \text{for } x \in \Omega\,, \tag{0.1}$$

$$u(x) = g(x) \quad \text{for } x \in \mathbb{R}^d \setminus \Omega\,. \tag{0.2}$$

For open $\Omega, G \subset \mathbb{R}^d$ define two vector spaces by

$$V(\Omega, G) = \big\{ v \in L^2_{\text{loc}}(\mathbb{R}^d) \cap L^2(\Omega) \big| \int_\Omega \int_G \frac{\big(v(y) - v(x)\big)^2}{|x - y|^{d+2s}}\, dx\, dy < \infty \big\},$$

$$H_\Omega(\mathbb{R}^d) = \{ v \in V(\Omega, \mathbb{R}^d) | v = 0 \text{ on } \Omega^c \}\,.$$

Let us define the notion of a variational solution.

Definition 1. [cf. Definition 2.5 in [1]] Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Let $g \in V(\Omega, \mathbb{R}^d)$. Then $u \in V(\Omega, \mathbb{R}^d)$ is called a variational solution to (0.1)–(0.2), if $u - g \in H_\Omega(\mathbb{R}^d)$ and for every $\varphi \in H_\Omega(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\big(u(y) - u(x)\big)\big(\varphi(y) - \varphi(x)\big)}{|x - y|^{d+2s}}\, dy\, dx = 0\,. \tag{0.3}$$

In [1] it is proved that such a variational solution u exists. However, in order to apply Definition 1 one needs to prescribe the data function g in the vector space $V(\Omega, \mathbb{R}^d)$, i.e. in particular one needs to prescribe all values of g in $\mathbb{R}^d$. This leads to two obvious questions:

Questions: For which space of functions $g : \Omega^c \to \mathbb{R}$ is there an extension operator $g \mapsto \text{ext}(g) \in V(\Omega, \mathbb{R}^d)$?
Do elements of $V(\Omega, \mathbb{R}^d)$ have a trace in this space?

In the talk we will answer these questions.

Joint work with Moritz Kassmann (Universität Bielefeld).

References

[1] M. Felsinger, M. Kassmann, and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.

On continuous time tug-of-war with noise and p-harmonic functions

STEFAN GEISS (UNIVERSITY OF JYVÄSKYLÄ)

Abstract: Joint work with C. Geiss and M. Parviainen.

We consider the p-Laplacian

$$\Delta_p^N u := \frac{p-2}{|Du|^2}\Big(\sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} u_{x_j}\Big) + \Big(\sum_{i=1}^{n} u_{x_i x_i}\Big),$$

where $Du := (u_{x_1}, ..., u_{x_n})'$, and the boundary value problem

$$\Delta_p^N u = 0 \text{ in } U \text{ and } u(x) = g(x) \text{ on } \partial U$$

where $U \subseteq \mathbb{R}^n$, $n \geq 2$, is an open set satisfying certain regularity conditions and $2 < p < \infty$. We relate this problem to a two-player stochastic game - called tug-of-war game with noise - consider space-time discretized controls, and a corresponding approximation problem. The talk is based on joint work in progress.

References

[1] K. Nyström and M. Parviainen: Tug-og-War, market manipulation, and option pricing. *Mathematical Finance.*

[2] A. Swiech: Another approach to the existence of value functions of stochastic differential games. *J. Math. Anal. Appl.*, 204(3):884–897, 1996.

First order derivations for Dirichlet forms and applications

MICHAEL HINZ (BIELEFELD UNIVERSITY)

Abstract: We discuss some recent results on abstract first order derivations associated with Dirichlet forms. In classical Euclidean or Riemannian

context they are nothing but the usual gradients, on graphs they give the usual difference operators. On more complicated spaces such as fractals or abstract metric measure spaces they provide access to equations involving first order terms. We will discuss some applications to nonlinear PDE on fractals and to fractal boundary value problems. We will also mention some connections to nonlocal operators and stochastic analysis.

Rough Gronwall Lemma and weak solutions to RPDEs

Martina Hofmanová (Technische Universität Berlin)

Abstract: In this talk, I will present recent results that give the necessary mathematical foundation for the study of rough path driven PDEs in the framework of weak solutions. The main tool is a new rough Gronwall Lemma argument whose application is rather wide: among others, it allows to derive the basic energy estimates leading to the proof of existence. Besides, we develop a suitable tensorization method which is the key for establishing uniqueness.

The talk is based on a joint work with Aurelien Deya, Massimiliano Gubinelli and Samy Tindel.

Weak Harnack inequality for the Boltzmann equation without cut-off

Cyril Imbert (CNRS)

Abstract: In this talk, we present a result about the Hölder regularity of bounded solutions of the Boltzmann equation without cut-off. Such a result is a special case of a more general one about kinetic integro-differential equation.

Joint work with Luis Silvestre (Chicago).

Weighted Sobolev regularity of weak solutions of degenerate/singular elliptic problems

Tadele Mengesha (University of Tennessee)

Abstract: Global weighted estimates are obtained for the gradient of solutions to a class of linear degenerate/singular elliptic problems over a bounded, possibly non-smooth, domain.The class consists of those with coefficient matrix that symmetric, nonnegative definite, and both its smallest and largest eigenvalues are proportion to a weight that belongs to a Muckenhoupt class. The weighted estimates are obtained under a smallness condition on the

mean oscillation of the coefficients with a weight and a flatness condition on the boundary of the domain. To demonstrate the necessity of the smallness condition on the coefficients a counterexample will be provided. The motivation for this work comes from the characterization of fractional nonlocal problems via local degenerated elliptic problems. Connections between our Sobolev estimates for degenerate/singular problems and estimates for solutions of fractional elliptic problems will be established.

This is a joint work with D. Cao and T. Phan.

Regularity for the normalized p-Poisson problem

Eero Ruosteenoja (University of Jyväskylä)

Abstract: The normalized p-Poisson problem

$$-\Delta_p^N u := -|Du|^{2-p}\operatorname{div}(|Du|^{p-2}Du) = f$$

is a nonlinear PDE in non-divergence form. The normalized p-Laplacian Δ_p^N arises for example from stochastic games and has applications in image processing. When $p > 1$ and f is continuous, we show that viscosity solutions of the problem are of class $C^{1,\alpha}_{\text{loc}}$ for some $\alpha > 0$ depending on p and the dimension n.

The talk is based on a joint work with Amal Attouchi and Mikko Parviainen.

O'Hara's knot energies and $W^{1/p,p}$-harmonic maps into spheres

Armin Schikorra (Albert-Ludwigs-Universität Freiburg)

Abstract: I will report on advances in the regularity theory for minimizers and critical points of a class of knot energies defined by Jun O'Hara. When parametrized by arclength the tangent field of these knots are critical points of a $W^{1/p,p}$-type energy, and we employ arguments from the regularity theory of $W^{1/p,p}$-harmonic maps into the sphere.

Joint work with S. Blatt, Ph. Reiter.

Stable nonlocal phase transitions and minimal surfaces

Joaquim Serra (ETH Zurich)

Abstract: We will describe new results in collaborations with X. Cabre, E. Cinti, S. di Pierro, and E. Valdinoci on nonlocal phase transitions. The talk will orbit around a new result stating that the one dimensional profiles are,

in the three dimensional Euclidean space, the only entire stable solutions of the fractional Allen Cahn equation when the order of the diffusion operator is slightly smaller than one. Different steps in the proofs of this result ---motivated by the long standing (and still open) analogous question for the classical Allen Cahn equation--- required interesting developments in the theory of nonlocal phase transitions and nonlocal minimal surfaces. We will explain the main new theorems (of independent interest) that have been developed in order to prove it.

Branching diffusion representation for nonlinear PDEs

Nizar Touzi (École Polytechnique)
Abstract: We provide a general reprentation result of semilinear PDEs in terms of branching diffusions. This is an extension of the well-known corresponding result for the KPP equation. We show that our representation result extends to stochastic parabolic PDEs and to a large class of non-elliptic PDEs, including hyperbolic ones.

Li-Yau and Harnack inequalities on graphs

Rico Zacher (Ulm University)
Abstract: In the celebrated paper [2], Li and Yau proved the parabolic Harnack inequality for Riemannian manifolds with Ricci curvature bounded from below. The key step in their proof was a completely new type of Harnack estimate, namely a pointwise gradient estimate, called *differential Harnack inequality*, which, by integration along a path, yields the classical parabolic Harnack estimate. If one tries to apply this method to discrete structures (graphs) one is faced with two big obstacles. The main difficulty is that the chain rule for the Laplace operator fails on graphs. Another problem is that in the graph setting, it is a priori not clear how to define a proper notion of curvature, or more precisely the concept of lower bounds for the Ricci curvature. A first successful attempt to circumvent these difficulties was made in the very recent paper [1] and is based on the square-root approach. In my talk, I will present a different approach, which, as in the classical case ([2]), leads to logarithmic Li-Yau inequalities, and also significantly improves the results from [1].

This is joint work with D. Dier (Ulm) and M. Kassmann (Bielefeld).

References

[1] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi, S.-T. Yau: Li-Yau inequality on graphs. J. Differential Geom. **99** (2015), 359–405.

[2] P. Li, S.-T. Yau: On the parabolic kernel of the Schrödinger operator. Acta. Math. **156** (1986), 153–201.

Martingale solutions for a pseudomonotone evolution equation with multiplicative noise

Aleksandra Zimmermann (Universität Duisburg-Essen)

Abstract: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical Wiener space endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $T > 0$ with the usual assumptions, $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $Q := (0,T) \times D$ and $p > 2$. Our aim is the study of the problem

$$(P) \begin{cases} du - \operatorname{div}(|\nabla u|^{p-2}\nabla u + F(u))\ dt = H(u)\ dW & \text{in } \Omega \times (0,T) \times D \\ u = 0 & \text{on } \Omega \times (0,T) \times \partial D \\ u(0,\cdot) = u_0 & \text{in } \Omega \times D \end{cases}$$

for a cylindrical Wiener process in $L^2(D)$ and $F : \mathbb{R} \to \mathbb{R}^d$ Lipschitz continuous. We consider the case of multiplicative noise with $H : L^2(D) \to HS(L^2(D))$, $HS(L^2(D))$ being the space of Hilbert-Schmidt operators, satisfying appriopriate regularity conditions. By an implicit time discretization of (P), we obtain approximate solutions. Using the theorems of Skorokhod and Prokhorov, we are able to pass to the limit and show existence of martingale solutions.