WORKSHOP ON HARMONIC ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

June 8-12, 2015

Department of Mathematics and Systems Analysis

Aalto University

Aalto University
School of Science

MINI COURSES

PHILIP GRESSMAN

OSCILLATORY INTEGRALS AND GEOMETRIC INTERACTIONS

In these lectures we will survey recent developments in the study of the asymptotics of scalar oscillatory integrals and related oscillatory integral operators. In most cases, the goal is to establish higher-dimensional analogues of the classical van der Corput Lemma, which is robust under perturbations of the phase and exhibits a natural, though often overlooked, diffeomorphism invariance. While methods based on resolution of singularities have enjoyed success in this area, I will focus on the development of more geometrically-inspired alternatives which seek to avoid the difficulties that one encounters when considering the effects of small C^∞ perturbations of the phase function. One of the main results in this direction is the development of a smooth extension of the notion of spaces of homogeneous type pioneered by Coifman and Weiss.

GIUSEPPE MINGIONE

A PRIMER ON NONLINEAR CALDERÓN-ZYGMUND

The classical Calderón-Zygmund theory is bound to provide optimal integrability and differentiability results for solutions to linear elliptic and parabolic equations. In the last years there has been a great development on a nonlinear version of such theory. This applies to nonlinear equations and has large and natural intersection with nonlinear potential theory. In the mini course I will outline the main results of this new and yet developing topic.

INVITED LECTURES

MATTEO BONFORTE

A PRIORI ESTIMATES FOR FRACTIONAL NONLINEAR DEGENERATE DIFFUSION EQUATIONS ON BOUNDED DOMAINS

We investigate quantitative properties of nonnegative solutions $u(t,x) \geq 0$ to the nonlinear fractional diffusion equation, $\partial_t u + \mathcal{L}(u^m) = 0$, posed in a bounded domain, $x \in \Omega \subset \mathbb{R}^N$ for $t > 0$ and $m > 1$. As $\mathcal{L}$ we can take the most common definitions of the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$, in a bounded domain with zero Dirichlet boundary conditions, as well as more general classes of operators. We consider a class of very weak solutions for the equation at hand, that we call weak dual solutions, and we obtain a priori estimates in the form of smoothing effects, absolute upper bounds, lower bounds, and Harnack inequalities. We also investigate the boundary behaviour and we obtain sharp estimates from above and below. The standard Laplacian case $s = 1$ or the linear case $m = 1$ are recovered as limits. The method is quite general, suitable to be applied to a number of similar problems that will be briefly discussed as examples. As a consequence, we can prove existence and uniqueness of minimal weak dual solutions with data in $L^1_{\Phi_1}$, where Φ_1 is the first eigenfunction of $\mathcal{L}$. We also briefly show existence and uniqueness of H^{-s} solutions with a different approach. As a byproduct, we derive similar estimates for the elliptic semilinear equation $\mathcal{L}S^m = S$ and we prove existence and uniqueness of $H^{-s}(\Omega)$ solutions via parabolic techniques. Solutions to this elliptic problem represents the asymptotic profiles of the rescaled solutions, namely the stationary states of the rescaled equation $\partial_t v = -\mathcal{L}(v^m) + v$. Finally, we will study the asymptotic behaviour. We will prove sharp rates of decay of the rescaled solution to the unique stationary profile S and also for the relative error $v/S - 1$. The sharp rates of convergence can be obtained with two different methods: one is based on the above estimates, that guarantee existence of the "friendly giant". Another approach is given by a new entropy method, based on the so-called Caffarelli-Silvestre extension. This is a joint work with J. L. Vázquez (UAM, Madrid, Spain) and Y. Sire (Univ. Marseille, France).

GALIA DAFNI

SOME APPLICATIONS OF HARMONIC ANALYSIS IN FLUID FLOW

We look at the use of Hardy spaces and related function spaces in the study of the two-dimensional inviscid lake equations. In particular, we will study the weighted Biot-Savar law and the associated Calderon-Zygmund operator. This is joint work with D.-C. Chang and C. K. Lin.

YOSHIKAZU GIGA

WEIGHTED ESTIMATES IN L^∞ FOR THE NEUMANN PROBLEM AND ITS APPLICATIONS TO THE STOKES SEMIGROUP

We are interested in what domain admits a weighted L^∞ estimate for a solution u the Neumann problem for the Laplace equation when the Neumann data is given as the surface divergence of a bounded tangent vector g. If the domain is bounded and smooth, it has been proved a few years ago that L^∞-norm of d times ∇u is dominated by a constant multiple of L^∞-norm of g on the boundary, where d is the distance from the boundary. In this talk, we give several examples so that such an estimate is valid. This estimate has a wide application to the Stokes system, the linearized Navier-Stokes system. In particular, this estimate gives a control on Stokes pressure, which enables us to prove that the solution operator, the Stokes semigroup is analytic in spaces of bounded functions and moreover a BMO-type space. This is my joint work with K. Abe (Kyoto University), M. Bolkart (TU Darmstadt), K. Schade (TU Darmstadt) and T. Suzuki (University of Tokyo).

PAUL A. HAGELSTEIN

SOLYANIK ESTIMATES IN HARMONIC ANALYSIS

Let $\mathcal{B}$ be a collection of open sets in $\mathbb{R}^n$. Associated to $\mathcal{B}$ is the geometric maximal operator $M_{\mathcal{B}}$ defined by

$$M_{\mathcal{B}}f(x) = \sup_{x \in R \in \mathcal{B}} \int_R |f|.$$

For $0 < \alpha < 1$, the associated *Tauberian constant* $C_{\mathcal{B}}(\alpha)$ is given by

$$C_{\mathcal{B}}(\alpha) = \sup_{E \subset \mathbb{R}^n : 0 < |E| < \infty} \frac{1}{|E|} |\{x \in \mathbb{R}^n : M_{\mathcal{B}}\chi_E(x) > \alpha\}|.$$

A maximal operator $M_{\mathcal{B}}$ such that $\lim_{\alpha \to 1^-} C_{\mathcal{B}}(\alpha) = 1$ is said to satisfy a *Solyanik estimate*. In this talk we will prove that the uncentered Hardy-Littlewood maximal operator satisfies a Solyanik estimate. Moreover, we will indicate applications of Solyanik estimates to smoothness properties of Tauberian constants and to weighted norm inequalities. We will also discuss several fascinating open problems regarding Solyanik estimates. This research is joint with Ioannis Parissis.

TONI HEIKKINEN

HAJŁASZ-BESOV AND HAJŁASZ-TRIEBEL-LIZORKING SPACES ON METRIC MEASURE SPACES

In this talk I present some new results concerning extension and restriction properties of Hajłasz–Besov and Hajłasz–Triebel–Lizorkin functions on metric measure spaces. I will also discuss approximation by Lipschitz functions, quasicontinuity and existence of generalized Lebesgue points in this setting. The talk is based on joint works with Lizaveta Ihnatsyeva, Pekka Koskela and Heli Tuominen.

STEVE HOFMANN

QUANTITATIVE RECTIFIABILITY AND BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

A classical theorem of F. and M. Riesz states that for a simply connected domain in the complex plane with a rectifiable boundary, harmonic measure and arc length measure on the boundary are mutually absolutely continuous. On the other hand, an example of C. Bishop and P. Jones shows that the latter conclusion may fail, in the absence of some sort of connectivity hypothesis. In this talk, we discuss recent developments in an ongoing program to find scale-invariant, higher dimensional versions of the F. and M. Riesz Theorem, as well as converses. In particular, we discuss substitute results that continue to hold in the absence of any connectivity hypothesis.

TUOMAS HYTÖNEN

APPROXIMATE AND EXACT EXTENSIONS OF LEBESGUE BOUNDARY FUNCTIONS

I discuss the following problem and its solution obtained in my joint work with Andreas Rosén (Göteborg): Given a function in $L^p(\mathbb{R}^n)$, we would like to see it as the continuous boundary trace of a function in the upper half space of n+1 dimensions. We find such an extension in suitable space of locally bounded variation, defined by the Carleson functional, in such a way that the trace map is continuous and surjective. The extension map is constructed through a stopping time argument, which extends earlier work by Varopoulos in the end-point BMO case.

LIZAVETA IHNATSYEVA

HARDY INEQUALITIES IN TRIEBEL-LIZORKIN SPACES

In this talk we consider inequalities of Hardy type for functions in Triebel-Lizorkin spaces. In particular, we discuss these inequalities for functions defined on domains whose boundary has the Aikawa dimension strictly less than n-sp (the case of a 'thin' boundary). We also show the validity of Hardy inequalities on open sets under a combined fatness and visibility condition on the boundary (the case of a 'fat' set). In addition, we would like to give a short exposition of various fatness conditions related to the theory, and apply Hardy inequalities in connection to the boundedness of extension operators for Triebel-Lizorkin spaces. The talk is based on joint work with Antti Vähäkangas and joint work with Juha Lehrbäck, Heli Tuominen and Antti Vähäkangas.

JANNE KORVENPÄÄ

REGULARITY OF THE LOCAL FRACTIONAL MAXIMAL FUNCTION

In this talk we consider smoothing properties of the local fractional maximal operator, which is defined in a proper subset of $\mathbb{R}^n$. Our main results include pointwise estimates for the weak gradient of the maximal function, which imply norm estimates in Sobolev spaces. An unexpected feature is that, compared to the corresponding estimates for slightly simpler maximal functions, these estimates contain extra terms involving spherical and fractional maximal functions. We also give some explicit examples which show that our results are essentially optimal. This is based on a joint work with T. Heikkinen, J. Kinnunen and H. Tuominen.

ANDREI LERNER

ON THE BOUNDEDNESS OF THE MAXIMAL OPERATOR ON DUAL SPACES

We discuss the following question: what conditions should be imposed on a Banach function space X in order to the boundedness of the maximal operator M on X would imply its boundedness on the associate space X'? If X is the weighted L^p space, this implication follows immediately by the A_p-theory. If X is the variable L^p space, this implication holds by a very interesting and difficult result of L. Diening. In this talk, a particular attention will be given to the case when X is the weighted variable L^p space.

JOHN LEWIS

SIGMA FINITENESS AND HAUSDORFF DIMENSION OF CERTAIN A HARMONIC MEASURES

In this talk we discuss recent work with coauthors Akman and Vogel concerning a certain postive A harmonic function u vanishing on a portion of a given domain in Euclidean n space and the corresponding Borel measure. Our work generalizes previous work with the above coauthors when $p \geq n$ and u is p harmonic, which in turn generalized work of Jones and Wolff on harmonic measure in arbitrary domains of the plane.

OLLI SAARI

PARABOLIC WEIGHTS AND NORM INEQUALITIES

In this talk, we introduce a class of weights arising from parabolic equations. We characterize them through weighted norm inequalities and through parabolic BMO. Parabolic weights can be regarded as a multidimensional generalization of one-sided weights on the real line, first studied by Sawyer. This is joint work with Juha Kinnunen.

NAGESWARI SHANMUGALINGAM

TREES AND ULTRAMETRICS, SOBOLEV SPACES AND BESOV SPACES, ROUGH QUASIISOMETRIES AND QUSISYMMETRIES

Metric trees are the quintessential Gromov hyperbolic metric spaces, and their boundaries, under the visual metric, are ultrametric spaces. In this talk we will discuss connections between Sobolev functions on trees and Besov functions on the boundary of the trees. This is joint work with Anders Björn, Jana Björn and James Gill.

PAVEL SHVARTSMAN

ON PLANAR SIMPLY CONNECTED SOBOLEV EXTENSION DOMAINS

We characterize finitely connected bounded Sobolev W_p^m-extension domains in $\mathbf{R^2}$ for each $p > 2$ and $m \geq 1$. We present an explicit criterion for Sobolev extension domains expressed in terms of certain inner subhyperbolic metrics. Our approach to this extension problem is based on several novel results related to the existence of a special chain of squares joining given points $x, y \in \Omega$ where Ω is a simply connected bounded domain in $\mathbf{R^2}$. We show that a geometrical background of these results is a new ``Square Separation Theorem" which states that under certain natural assumptions on relative positions of x and y in Ω there exists a square $Q \subset \Omega$ such that x and y belong to distinct connected components of $\Omega \setminus Q$.

This is a joint work with Nahum Zobin.

JOSÉ MIGUEL URBANO

QUANTITATIVE REGULARITY RESULTS FOR SINGULAR AND DEGENERATE PROBLEMS

We will report on two recent developments concerning sharp regularity for singular and degenerate problems: the finding of the precise Hölder exponent for the solutions of the inhomogeneous p-Laplace equation in terms of p, the integrability of the source and the space dimension n (Anal. PDE 2014, joint with E. Teixeira) and the derivation of a quantitative modulus of continuity, which we conjecture to be optimal, for solutions of the p-degenerate two-phase Stefan problem (ARMA 2014, joint with P. Baroni and T. Kuusi).

JIE XIAO

VARIATIONAL CAPACITY VS SURFACE AREA VIA MEAN CURVATURE

This talk will address an optimal relationship between the variational capacity and the surface area in the Euclidean n-space which especially shows: if $\Omega\subset\mathbb{R}^n$ is a convex, compact, smooth set with its interior $\Omega^\circ\neq\emptyset$ and the mean curvature $H(\partial\Omega,\cdot)>0$ of its boundary $\partial\Omega$ then

$$\left(\frac{n(p-1)}{p(n-1)}\right)^{p-1}\le\frac{\left(\frac{\mathrm{cap}_p(\Omega)}{\left(\frac{p-1}{n-p}\right)^{1-p}\sigma_{n-1}}\right)}{\left(\frac{\mathrm{area}(\partial\Omega)}{\sigma_{n-1}}\right)^{\frac{n-p}{n-1}}}\le\left(\sqrt[n-1]{\int_{\partial\Omega}\left(H(\partial\Omega,\cdot)\right)^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}}\right)^{p-1}$$

$\forall p\in(1,n)$, whose limits $1\leftarrow p\ \&\ p\rightarrow n$ imply

$$1=\frac{\mathrm{cap}_1(\Omega)}{\mathrm{area}(\partial\Omega)}\ \&\ \int_{\partial\Omega}\left(H(\partial\Omega,\cdot)\right)^{n-1}\frac{d\sigma(\cdot)}{\sigma_{n-1}}\ge 1,$$

thereby not only finding a half-way to approach the 1945 Pólya-Szegő conjecture but also extending the 1945 Pólya-Szegő inequality, with both the conjecture and the inequality being stated for the electrostatic capacity of a convex solid in the Euclidean 3-space.