Introduction - Definitions

- A real-valued function \( u(x, y) \) is harmonic it satisfies the Laplace equation
  \[
  \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.
  \]

- A complex-valued function \( f(x + iy) = u(x, y) + iv(x, y) \) from a domain \( \mathbb{D} \to \mathbb{C} \) is harmonic if the two coordinate functions \( u, v \) are harmonic.

- A complex-valued harmonic function is a harmonic mapping if it is univalent (one-to-one).

References


Introduction – Compositions of Complex-Valued Harmonic Functions

- If \( f : \Omega \to \mathbb{C} \), \( g : f(\Omega) \to \mathbb{C} \) are harmonic functions, \( g \circ f \) is not necessarily harmonic.
- If \( f : \Omega \to \mathbb{C} \) is analytic and \( g : f(\Omega) \to \mathbb{C} \) is harmonic, then \( g \circ f \) is harmonic.
- If \( f : \Omega \to \mathbb{C} \) is harmonic and \( g : f(\Omega) \to \mathbb{C} \) is analytic, then \( g \circ f \) is not necessarily harmonic.
- In fact, even a square or the reciprocal of a harmonic function need not be harmonic.
- Inverse of a harmonic mapping need not be harmonic.

Introduction – Example 1

Square may not be a harmonic mapping:

\[
\begin{align*}
  f(x, y) &= x^2 - y^2 + ix. \\
  \Rightarrow \Delta \Im f^2 &= 16x \neq 0.
\end{align*}
\]

Simple harmonic mapping that is not necessarily conformal is an affine mapping:

\[
  f(z) = \alpha z + \gamma + \beta \bar{z}, \quad |\alpha| \neq |\beta|.
\]

In fact, we have:

- If \( f \) is harmonic, then \( \alpha f + \gamma + \beta \bar{f} \), is harmonic as well.

Introduction – Example 2 (1/2)

- Consider a mapping

\[
f(z) = z - \frac{1}{2} z^2,
\]

which maps the open \( \mathbb{D} \) onto a hypocycloid with three cusps.

- To verify its univalence, suppose \( f(z_1) = f(z_2) \) for \( z_1, z_2 \in \mathbb{D} \).

Then

\[
2(z_2 - z_1) = \bar{z}_2^2 - \bar{z}_1^2 = (\bar{z}_1 + \bar{z}_2)(\bar{z}_1 - \bar{z}_2).
\]

By taking absolute values on both sides. We see that this is impossible unless \( z_1 = z_2 \), because \( |z_1 + z_2| < 2 \).

- A similar argument shows that \( f(z) = z - \frac{1}{n} z^n \) is univalent for each \( n \geq 2 \).

Introduction – Example 2 (2/2)

(a) \( n = 2 \)

(b) \( n = 5 \)

Figure: \( f(z) = z - \frac{1}{n} z^n \).
Preliminaries – Canonical Representation

Theorem
In a simply connected domain \( \Omega \subset \mathbb{C} \), a complex-valued harmonic function \( f \) has a representation \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( \Omega \). The representation is unique up to an additive constant.

- For a harmonic mapping \( f \) of the unit disk \( \mathbb{D} \), it is convenient to choose the additive constant so that \( g(0) = 0 \).
- Then the representation \( f = h + \bar{g} \) is unique and is called the canonical representation of \( f \).

Notations and Jacobian
- Connection between the differential operators
  \[
  \frac{\partial f}{\partial z} = \frac{\partial \bar{f}}{\partial \bar{z}}.
  \]
- Suppose that \( f = h + \bar{g} \), then we have following notations:
  \[
  f_z = \frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} = h',
  \frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = g'.
  \]
- Jacobian of \( f \) is defined by \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 \).
- Function \( f \) is locally univalent and sense-preserving if \( J_f(z) > 0 \).

Shear Construction

Definition
A domain \( \Omega \subset \mathbb{C} \) is said to be convex in the horizontal direction (CHD) if its intersection with each horizontal line is connected (or empty).

Shear construction is based on the following Theorem:

Theorem
Let \( f = h + \bar{g} \) be a harmonic and locally univalent in the unit disk \( \mathbb{D} \). Then \( f \) is univalent in \( \mathbb{D} \) and its range is a CHD domain if and only if \( h - g \) is a conformal mapping of \( \mathbb{D} \) onto a CHD domain.

- A way to construct harmonic mappings from analytic functions.
- Introduced by Clunie and Sheil-Small in 1984 [CluShe].
Shear Construction

Information we have:
- Write \( f = h + \bar{g} \).
- Denote \( \varphi = h - g \).
- Dilatation \( \omega = g'/h' \), such that \( |\omega(z)| < 1 \), for \( z \in \mathbb{D} \).

Solve \( h, g \) from differential equations:
\[
\begin{align*}
\left\{ \begin{array}{l}
    h' - g' = \varphi', \\
    \omega h' - g' = 0.
\end{array} \right.
\]

Minimal Surfaces

- Minimal surface is a surface which minimizes the area with a fixed curve as its boundary.
- Harmonic mapping \( f = h + \bar{g} \) can be lifted to a minimal surface if and only if the dilatation is a square of an analytic function.
- Suppose that \( \omega = q^2 \). Then the corresponding minimal surface has the form
\[
\{ u, v, w \} = \{ \text{Re} \, f, \text{Im} \, f, 2 \text{Im} \, \psi \},
\]
where
\[
\psi(z) = \int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)} \, d\zeta.
\]

Shear Construction

We have
\[
\begin{align*}
h(z) &= \int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)} \, d\zeta, \\
g(z) &= \int_0^z \omega(\zeta) \frac{\varphi'(\zeta)}{1 - \omega(\zeta)} \, d\zeta.
\end{align*}
\]

Observe that
\[
f(z) = h(z) + \bar{g}(z) = 2 \text{Re} \left[ \int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)} \, d\zeta \right] - \overline{\varphi(z)}.
\]

Shear Construction & Minimal Surface

Algorithm

1. Supervisor gives you conformal mappings \( \varphi \).
2. Come up with easy dilatations \( \omega \).
3. Compute \( h \) and \( g \).
4. Construct \( f = h + \bar{g} \).
5. Compute \( \psi \) if possible for minimal surface.
6. Plot \( f \) and minimal surface.
7. Profit.
Example - Simple [Duren, pp. 39-40] (1/4)

- Suppose $\varphi(z) = z$, thus $\varphi'(z) = 1$.
- Let $\omega(z) = z^2$.
- We have:

$$
\begin{align*}
    h(z) &= \int_0^z \frac{d\zeta}{1 - \zeta^2} = \frac{1}{2} \log \frac{1 + z}{1 - z}, \\
    g(z) &= \int_0^z \frac{\zeta^2}{1 - \zeta^2} d\zeta = -z + \frac{1}{2} \log \frac{1 + z}{1 - z}.
\end{align*}
$$

Therefore:

$$
f(z) = h(z) + g(z) = -z + 2 \Re \left[ \log \frac{1 + z}{1 - z} \right] = -z + \log \left| \frac{1 + z}{1 - z} \right|.
$$

Example - Simple (3/4)

- For the minimal surface, let us compute

$$
\psi(z) = \int_0^z \frac{d\zeta}{1 - \zeta^2} = -\frac{1}{2} \log(1 - z^2).
$$

- Then we have

$$
\begin{align*}
    u(z) &= \Re \left[ -\bar{z} + \log \frac{1 + z}{1 - z} \right] = \arctan \left( \frac{2 \Re z}{1 + |z|^2} \right) - \Re z, \\
    v(z) &= \Im \left[ -\bar{z} + \log \frac{1 + z}{1 - z} \right] = \Im z, \\
    w(z) &= 2 \Im \left[ -\frac{1}{2} \log(1 - z^2) \right] = -\Arg(1 - z^2).
\end{align*}
$$

Example - Simple (4/4)

Figure: Shearing of the identity map with the dilatation $\omega(z) = z^2$.

Figure: Minimal surface of the identity map with the dilatation $\omega(z) = z^2$. 

Example - Simple [Duren, pp. 39-40] (2/4)
Example - Polygonal Mappings (1/2)

Lemma
The function \( \varphi \) defined by

\[
\varphi(z) = \int_0^z (1 - \zeta^n)^{-2/n} d\zeta
\]

maps the unit disk \( \mathbb{D} \) onto regular \( n \)-gons.

Dilatations considered:
- [DriDur]: \( z, z^n, z^{n/2} \). In the last dilatation \( n \) is even.
- [PonQuaRas]: \( z^{2n}, z^2 \). In the latter dilatation \( n \) is odd.

Figure: Polygonal mapping \( \varphi \).

Example - Polygonal Mappings (2/2)

Polygonal Mappings - Hypergeometric Functions

- Gaussian hypergeometric function:

\[
F(a, b; c; z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} z^n, \quad |z| < 1,
\]

where \((\alpha)_n\) is Pochhammer symbol defined as follows:

\[(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, \quad \alpha \in \mathbb{C}.
\]

- Euler integral presentation: For \( \Re c > \Re b > 0 \):

\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt.
\]

Polyen Mappings - Hypergeometric Functions

- Appel’s hypergeometric function:

\[
F_1(a, b_1, b_2; c, x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a)_{k+l} (b_1)_k (b_2)_l}{(c)_{k+l} k! l!} x^k y^l,
\]

where \((\alpha)_n\) is Pochhammer symbol.

- Euler integral presentation: For \( \Re c > \Re a > 0 \):

\[
F_1(a, b_1, b_2; c, x, y) = C \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b_1}(1-yt)^{-b_2} dt,
\]

where \(C = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\).
Polygons Mappings - $\omega(z) = z^n$ (1/5)

- Idea is to rewrite integrals as Gaussian hypergeometric functions.
- Chosen dilatation $\omega(z) = z^n$ leads to integrals:
  
  \[
  \begin{align*}
  h(z) &= \int_0^z (1 - \zeta^n)^{-1-2/n} d\zeta, \\
  g(z) &= \int_0^z \zeta^n (1 - \zeta^n)^{-1-2/n} d\zeta.
  \end{align*}
  \]

- By substituting $u = \zeta^n$, we have
  
  \[
  h(z) = \frac{1}{n} \int_0^{z^n} (1 - u)^{-1-2/n} u^{1/n-1} du.
  \]

Polygons Mappings - $\omega(z) = z^n$ (2/5)

- Substitution $u = z^n t$ leads to
  
  \[
  h(z) = \frac{1}{n} \int_0^1 (1 - z^n t)^{-1-2/n} (z^n t)^{1/n-1} z^n dt = zF_{1+2/n, 1/n; 1+1/n; z^n}.
  \]

- By similar procedure, we have
  
  \[
  g(z) = \frac{z^{n+1}}{n+1} F_{1+2/n, 1/n; 2+1/n; z^n}.
  \]

Polygons Mappings - $\omega(z) = z^n$ (3/5)

Figure: Harmonic shears of polygonal mappings with a dilatation $\omega(z) = z^n$.

Polygons Mappings - $\omega(z) = z^n$ (4/5)

- For minimal surfaces, we assume that $n$ is even, say $n = 2m$.
- Then $q(z) = z^m$ and $h'(z) = (1 - z^{2m})^{-1-1/m}$.
- Therefore
  
  \[
  \psi(z) = \int_0^z \zeta^m (1 - \zeta^{2m})^{-1-1/m} d\zeta = \frac{z^{m+1}}{2m} \int_0^1 t^{1/2m-1/2} (1 - z^{2m} t)^{-1-1/m} dt = \frac{z^{1+n/2}}{n+2} F_{1+2/n, 1/n; 2/n; z^n}.
  \]
Polygonal Mappings - $\omega(z) = z^n$ (5/5)

Figure: Minimal surface of the polygonal mappings with the dilatation $\omega(z) = z^n$, for $n = 4$.

Polygonal Mappings - $\omega(z) = z^{2n}$ (2/5)

- By the change of variable $\zeta = z^{1/n}$, we obtain
  
  $h(z) = \frac{z}{n} \int_0^1 \left(1 - z^n t\right)^{-1-2/n} \left(1 + z^n t\right)^{-1} t^{1/n-1} dt$.

- This can be rewritten by using Appel’s hypergeometric functions as follows:
  
  $h(z) = z F_1 \left(\frac{1}{n}, 1 + \frac{2}{n}, 1 + \frac{1}{n}; z^n, -z^n\right)$.

- Again in the same fashion, we have
  
  $g(z) = \frac{z^{2n+1}}{2n+1} F_1 \left(2 + \frac{1}{n}, 1 + \frac{2}{n}, 1 + \frac{1}{n}; z^n, -z^n\right)$.

Polygonal Mappings - $\omega(z) = z^{2n}$ (3/5)

- Chosen dilatation $\omega(z) = z^{2n}$ leads to integrals:
  
  \[
  \begin{align*}
  h(z) &= \int_0^z (1 - \zeta^n)^{-2/n} (1 - \zeta^{2n})^{-1} d\zeta, \\
  g(z) &= \int_0^z \zeta^{2n}(1 - \zeta^n)^{-2/n} (1 - \zeta^{2n})^{-1} d\zeta.
  \end{align*}
  \]

- This can be rewritten as:
  
  \[
  \begin{align*}
  h(z) &= \int_0^z (1 - \zeta^n)^{-1-2/n} (1 + \zeta^n)^{-1} d\zeta, \\
  g(z) &= \int_0^z \zeta^{2n}(1 - \zeta^n)^{-1-2/n} (1 + \zeta^n)^{-1} d\zeta.
  \end{align*}
  \]

Figure: Harmonic shears of polygonal mappings with a dilatation $\omega(z) = z^{2n}$.

Polygonal Mappings - $\omega(z) = z^{2n}$ (1/5)
Polygonal Mappings - $\omega(z) = z^{2n}$ (4/5)

- The minimal surface is determined by the integral

$$\psi(z) = \int_0^z \zeta^n (1 - \zeta^n)^{-1} d\zeta = \frac{z^{n+1}}{n} \int_0^1 t^{1/n} (1 - z^n t)^{-1} d\zeta$$

$$= z^{n+1} F_1 \left( 1 + \frac{1}{n}, 1 + \frac{2}{n}, 1; 2 + \frac{1}{n}, z^n, -z^n \right).$$

Figure: Minimal surface of the polygonal mappings with the dilatation $\omega(z) = z^{2n}$, for $n = 4$.

Polygonal Mappings - $\omega(z) = z^{2}$ (1/5)

- In this example, we assume that $n$ is odd.
- This time we have following integrals:

$$h(z) = \int_0^z (1 - \zeta^n)^{-2/n} (1 - \zeta^2)^{-1} d\zeta,$$

$$g(z) = \int_0^z \zeta^2 (1 - \zeta^n)^{-2/n} (1 - \zeta^2)^{-1} d\zeta.$$

Recall the following identities:

$$1 + \zeta^n = (1 + \zeta)(1 - \zeta + \cdots - \zeta^{n-2} + \zeta^{n-1}), \quad n \text{ is odd},$$

$$1 - \zeta^n = (1 - \zeta)(1 + \zeta + \cdots + \zeta^{n-2} + \zeta^{n-1}).$$

From above, we have

$$1 + \zeta^n \frac{1 - \zeta}{1 + \zeta} = 1 + \zeta^2 + \cdots + \zeta^{2(n-1)}.$$

Polygonal Mappings - $\omega(z) = z^{2}$ (2/5)

- The above identity leads to

$$h(z) = \int_0^z \frac{1 + \zeta^2 + \cdots + \zeta^{2(n-1)}}{(1 - \zeta^n)^{1+2/n}(1 + \zeta^n)} d\zeta = \sum_{k=0}^{n-1} \int_0^z \zeta^{2k} (1 - \zeta^n)^{1+2/n}(1 + \zeta^n) d\zeta.$$

- By computation, we have

$$h(z) = \sum_{k=0}^{n-1} \frac{z^{2k+1}}{2k+1} F_1 \left( \frac{2k+1}{n}, 1 + \frac{2}{n}, 1; 1 + \frac{2k+1}{n}, z^n, -z^n \right),$$

$$g(z) = \sum_{k=1}^{n} \frac{z^{2k+1}}{2k+1} F_1 \left( \frac{2k+1}{n}, 1 + \frac{2}{n}, 1; 1 + \frac{2k+1}{n}, z^n, -z^n \right).$$
Polyononal Mappings - $\omega(z) = z^2$ (3/5)

Figure: Harmonic shears of polygonal mappings with a dilatation $\omega(z) = z^2$.

$\omega(z) = z^2 (3/5)$

(a) $n = 3$

(b) $n = 5$

Polyononal Mappings - $\omega(z) = z^2$ (4/5)

- The minimal surface is determined by the integral

$$
\psi(z) = \int_0^z \zeta(1 - \zeta^n)^{-2/n}(1 - \zeta^2)^{-1} d\zeta \\
= \sum_{k=0}^{n-1} \int_0^z \zeta^{2k+1}(1 - \zeta^n)^{-1-2/n}(1 + \zeta^n)^{-1} d\zeta \\
= \sum_{k=0}^{n-1} \frac{2^{2(k+1)}}{n} \int_0^1 t^{2(k+1)/n - 1}(1 - z^nt)^{-1-2/n}(1 + z^nt)^{-1} dt \\
= \sum_{k=0}^{n-1} \frac{2^{2(k+1)}}{2(k+1)} F_1\left(\frac{2(k+1)}{n},1+\frac{2}{n};1,1+\frac{2(k+1)}{n};z^n,-z^n\right).
$$

Polyononal Mappings - $\omega(z) = z^2$ (5/5)

Figure: Minimal surface of the polygonal mappings with the dilatation $\omega(z) = z^2$, for $n = 5$.

Strip Mappings (1/2)

Lemma

The function $\varphi$ defined by

$$
\varphi(z) = \int_0^z \frac{\sqrt{1 + \zeta^4}}{1 - \zeta^4} d\zeta
$$

maps the unit disk $\mathbb{D}$ onto a CHD domain $\Omega$.

- $\Omega = \Omega_1 \cup \Omega_2$, where

$$
\Omega_1 = \left\{ x + iy : x \in \mathbb{R}, |y| \leq \text{Re} \left[ \frac{\sin \pi/4 \Gamma(5/4)}{\Gamma(1/4)} \right] \right\}, \\
\Omega_2 = \left\{ x + iy : |y| \leq \text{Re} \left[ \frac{\sin \pi/4 \Gamma(5/4)}{\Gamma(1/4)} \right], y \in \mathbb{R} \right\}.
$$
Strip Mappings (2/2)

Figure: Strip Mappings

Strip Mappings - $\omega(z) = z^4$ (1/4)

- Chosen dilatation $\omega(z) = z^4$ leads to integrals:
  \[
  \begin{align*}
  h(z) &= \int_0^z \frac{(1 + \zeta^4)^{1/2}}{(1 - \zeta^4)^2} d\zeta, \\
  g(z) &= \int_0^z \zeta^4 \frac{(1 + \zeta^4)^{1/2}}{(1 - \zeta^4)^2} d\zeta.
  \end{align*}
  \]

- By the substitution $\zeta = zt^{1/4}$, we have
  \[
  \begin{align*}
  h(z) &= z F_1 \left( \frac{1}{4}, 2, -\frac{1}{2}; \frac{5}{4}; z^4, -z^4 \right), \\
  g(z) &= \frac{z^5}{5} F_1 \left( \frac{5}{4}, 2, -\frac{1}{2}; \frac{9}{4}; z^4, -z^4 \right).
  \end{align*}
  \]

Strip Mappings - $\omega(z) = z^4$ (2/4)

Figure: Strip Mappings

Strip Mappings - $\omega(z) = z^4$ (3/4)

- For minimal surface, we compute
  \[
  \psi(z) = \int_0^z \zeta^2 (1 - \zeta^4)^{-2}(1 + \zeta^4)^{1/2} d\zeta.
  \]

- The substitution $\zeta = zt^{1/4}$ leads to
  \[
  \psi(z) = \frac{z^3}{4} \int_0^1 t^{-1/4}(1 - z^4 t)^{-2}(1 + z^4 t)^{1/2} dt \\
  = \frac{z^3}{3} F_1 \left( \frac{3}{4}, 2, -\frac{1}{2}; \frac{7}{4}; z^4, -z^4 \right)
  \]
Figure: Minimal surface associated with the harmonic shear of the strip mapping with dilatation $\omega(z) = z^4$. 