Conjugate Function Method for Numerical Conformal Mappings

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Introduction

We present a new method to approximate conformal mappings. Conformal mappings can be applied in numerous applications, e.g., electrostatics and aerodynamics. In applications equipotential lines can be interpreted physically as a potential, wind, heat flow [SL]. There are few examples where conformal mappings onto canonical domains, a rectangle or circle, can be given easily in analytical form. Thus use of computational methods is required.

The history of numerical computation of conformal mappings date back to 1950, see e.g. [Por]. The most popular algorithm is based on the Schwarz-Christoffel mapping and is implemented for MATLAB by Driscoll [Dri, DT, Tre]. Another method is due to Marshall [Mar]. For further references and discussion see also the recent book [PS].

Conformal Moduli and Mappings

The modulus of a quadrilateral can be given by the Dirichlet-Neumann boundary value problem. This quantity is closely related to the capacity of a condenser.

Definition. (Conformal Modulus of a Quadrilateral)
Let $Q(\Omega; z_1, z_2, z_3, z_4)$ be a generalized quadrilateral, where Ω is a simply connected domain and points z_1, z_2, z_3, z_4 are positively ordered on the boundary curve $\partial\Omega$ and let γ_j, $j = 1, 2, 3, 4$, be the arcs of $\partial\Omega$ between $(z_1, z_2), (z_2, z_3), (z_3, z_4)$, and (z_4, z_1), respectively. Suppose that u is the unique harmonic solution of the Dirichlet-Neumann problem with boundary conditions $u = 0$ on γ_2, $u = 1$ on γ_4, and $\frac{\partial u}{\partial n} = 0$ on $\gamma_1 \cup \gamma_3$. Then the conformal modulus of a generalized quadrilateral is given by

$$\mathrm{M}(\Omega; z_1, z_2, z_3, z_4) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

The problem with reversed boundary conditions is called the conjugate Dirichlet-Neumann problem. This problem can be solved by using standard numerical methods. In our examples the hp-FEM software by H. Hakula (see [HRV]) is used.

Theorem. (Conjugate Function Method)
Let $(\Omega; z_1, z_2, z_3, z_4)$ be a quadrilateral with modulus h and let u_1 satisfy the Laplace equation with Dirichlet-Neumann boundary conditions of above Definition. Let u_2 satisfy the conjugate problem. Then $f = u_1 + ihu_2$ is the conformal mapping that maps Ω onto a rectangle R such that the image of the points z_1, z_2, z_3, z_4 are $1 + ih, ih, 0, 1$, respectively. The mapping f maps the boundary curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ onto curves $\gamma_1', \gamma_2', \gamma_3', \gamma_4'$, respectively.

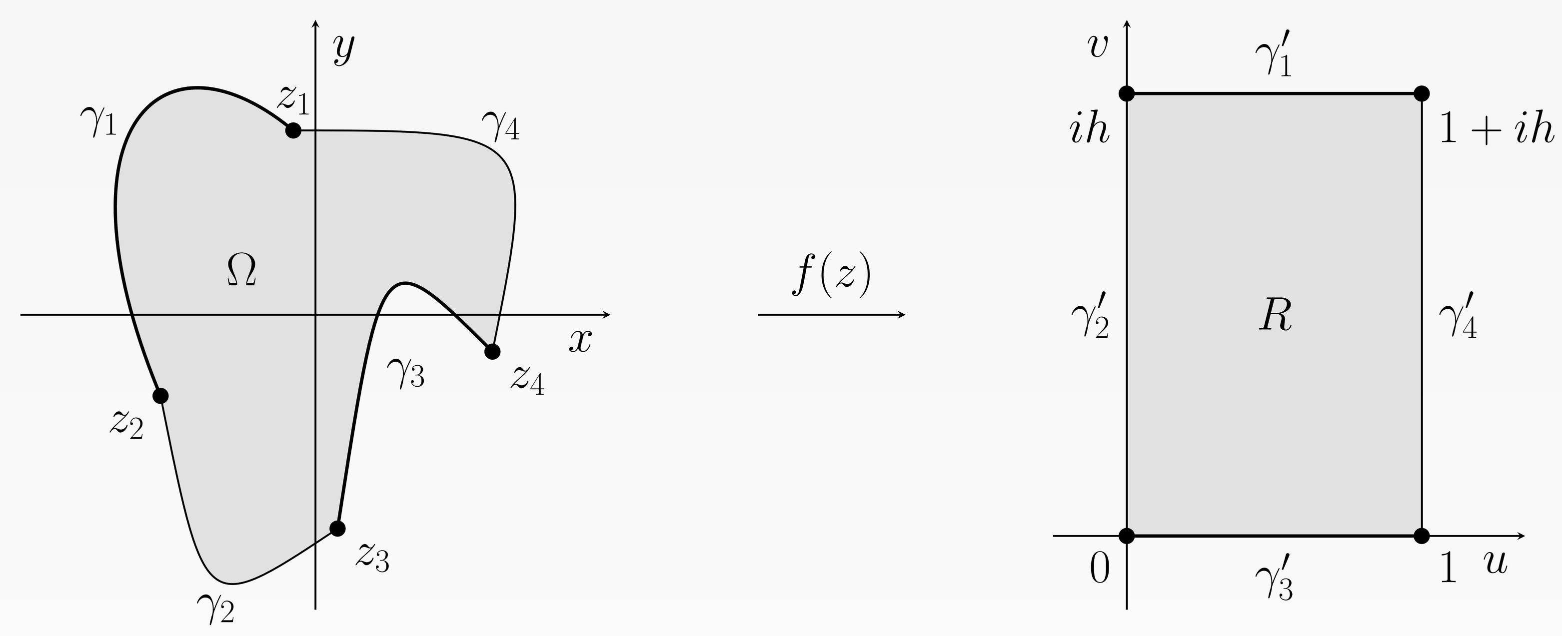


Figure: Dirichlet and Neumann boundary conditions are mark with thin and thick lines, respectively, and h is the modulus of the quadrilateral.

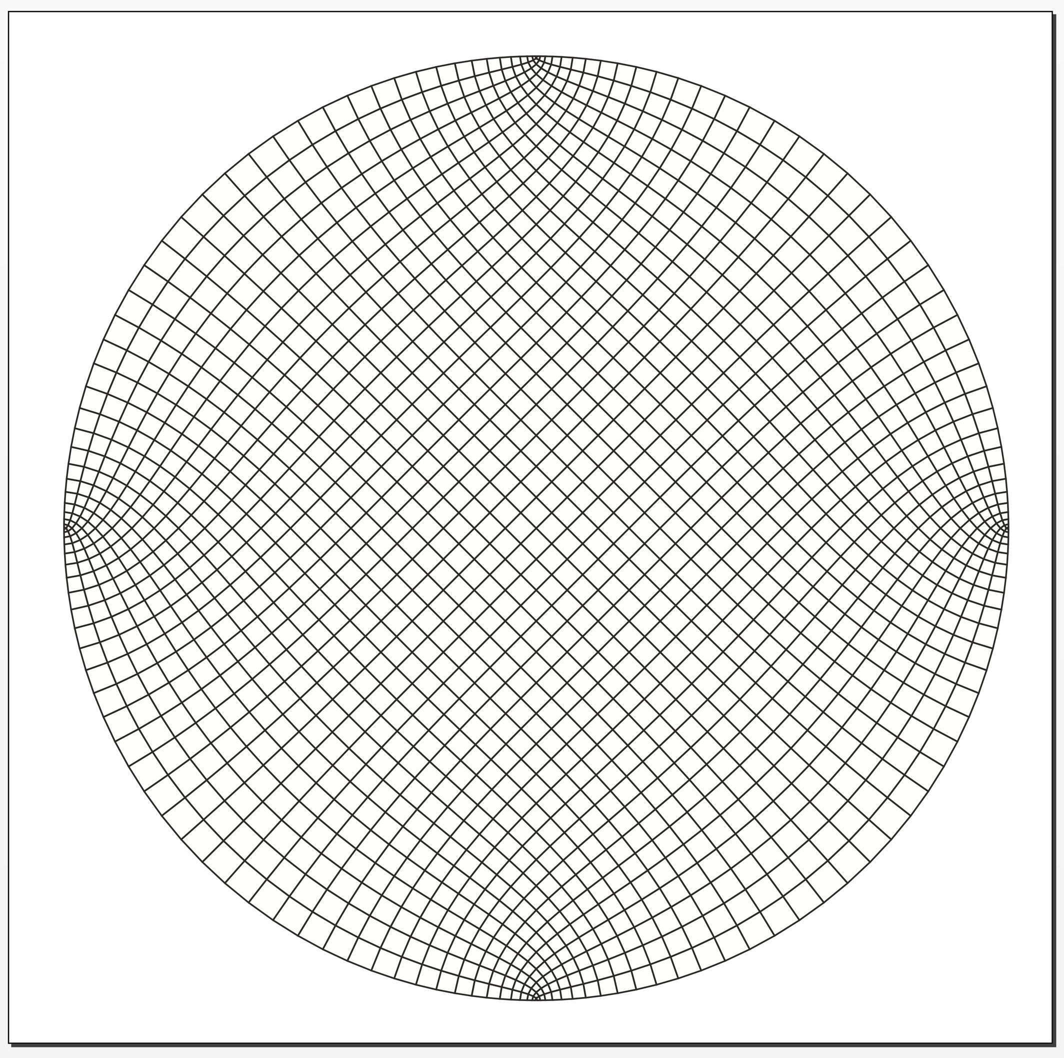
For construction of the conformal mapping and drawing equipotential lines we use the following algorithm, which construct a conformal mapping from the domain Ω onto a rectangle R. The boundary curves γ_i of Ω correspond to the sides of the rectangle R.

Algorithm. [HQR]

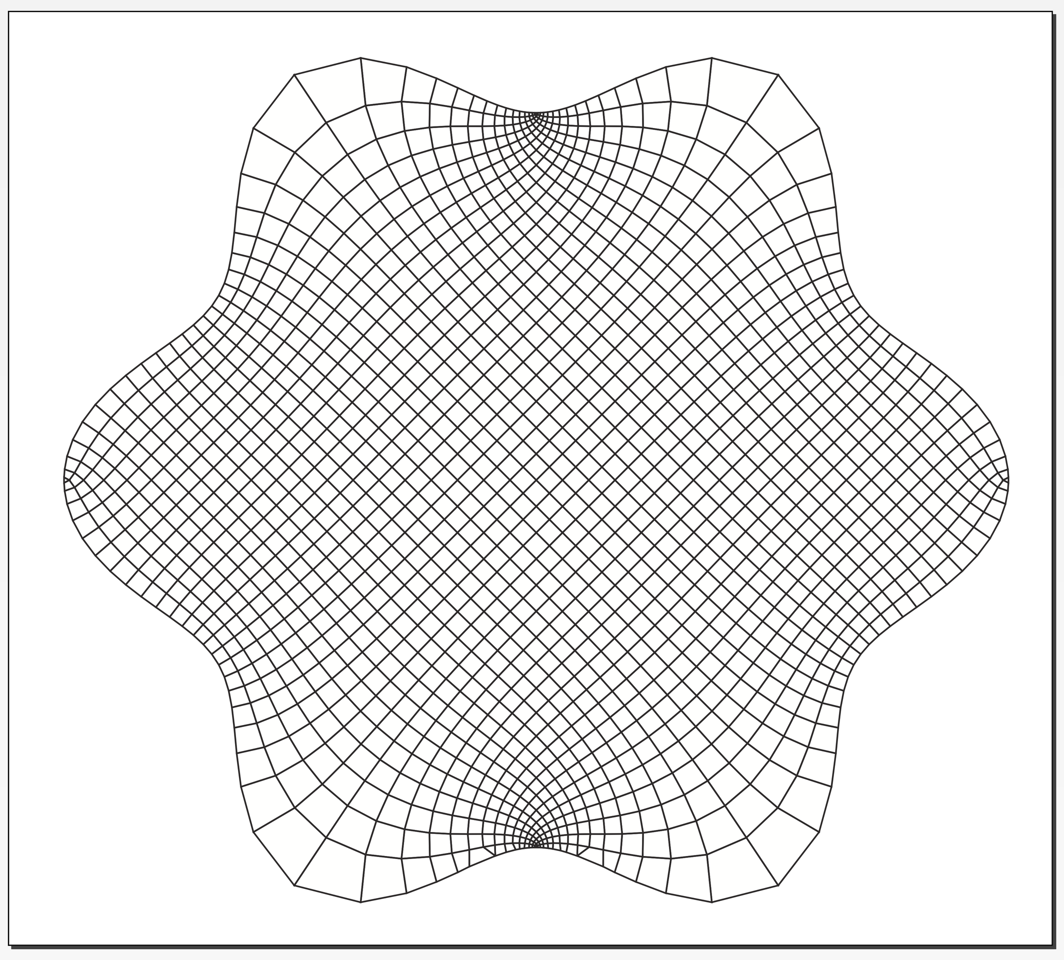
1. Solve the DN-problem to obtain u_1 and compute the modulus of the quadrilateral.
2. Solve the conjugate DN-problem for u_2.
3. Then the conformal mapping is given by $f = u_1 + ihu_2$.
4. Fix the equipotential line grid on the rectangle R and solve the pre-image of the equipotential lines.

Examples

Examples of numerical conformal mappings, obtained by using the algorithm described, are given below. The first example is a reproduction of a picture first obtained by Schwarz in 1869. The image of a rectangular grid in a conformal mapping of a square onto a disk is illustrated. In this case an analytic presentation of the mapping can also be obtained by using elliptic integrals.



Our second example deals with a quadrilateral with a curved boundary. In this case an analytic expression of the mapping is not known. Moreover, standard numerical methods, such as the Schwarz-Christoffel mapping, are not well suited for situations like this.



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