HELSINKI UNIVERSITY OF TECHNOLOGY
Faculty of Information and Natural Sciences

Tri Quach

Numerical conformal mappings and capacity computation

Master's thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Technology in Degree Programme in Automation and Systems Technology.

Espoo, November 3, 2009

Supervisor: Professor Timo Eirola
Instructor: Ph.D. Antti Rasila

TEKNILLINEN KORKEAKOULU
INFORMAATIO- JA LUONNONTIETEIDEN TIEDEKUNTA
TEKNILLISEN FYSIIKAN JA MATEMATIIKAN OSASTO

DIPLOMITYÖN TIIVISTELMÄ

Tekijä:	Tri Quach
Osasto:	Automaatio- ja systeemitekniikan osasto
Pääaine:	Automaation tietämystekniikat (2271)
Sivuaine:	Laskennallinen tiede ja tekniikka (2254)
Työn nimi:	Numeeriset konformikuvaukset ja kapasiteettilaskenta
Professuuri:	Mat-1, Matematiikka
Työn valvoja:	Professori Timo Eirola
Työn ohjaaja:	FT Antti Rasila

Tiivistelmä:

Tämän diplomityön tarkoituksena on perehdyttää numeerisiin konformikuvauksiin ja nelikulmion konformisen modulin laskentaan. Konformikuvauksia tutkitaan sekä niiden kytköksistä fysikaalisiin sovellutuksiin, että matemaattisesta tärkeydestä. Laskennallista etua saavutetaan kuvaamalla konformisesti yhdesti yhtenäinen alue joko yksikkökiekolle, ylemmälle puolitasolle tai suorakaiteelle.

Toisaalta kondensaattorin kapasitanssi voidaan karakterisoida nelikulmion konformisen modulin avulla. Tässä diplomityössä annamme kaksi tapaa modulin laskemiseen:

- Schwarz–Christoffelin kuvaukset,
- elementtimenetelmät.

Saatuja tuloksia verrataan sekä keskenään, että tunnettuihin kirjallisuuden analyyttisiin ja referenssiarvoihin.

Sivumäärä: 100	Avainsanat: konformikuvaukset, nelikulmion moduli, numeeriset menetelmät, Schwarz–Christoffelin kuvaus, elementtimenetelmät
Täytetään osastolla	
Hyväksytty:	Kirjasto:

HELSINKI UNIVERSITY OF TECHNOLOGY — ABSTRACT OF MASTER'S THESIS

FACULTY OF INFORMATION AND NATURAL SCIENCES

DEPARTMENT OF ENGINEERING PHYSICS AND MATHEMATICS

Author:	Tri Quach
Programme:	Degree Programme in Automation and Systems Technology
Major subject:	Knowledge Engineering in Automation (2271)
Minor subject:	Computational Science and Engineering (2254)
Title:	Numerical conformal mappings and capacity computation
Chair:	Mat-1, Mathematics
Supervisor:	Professor Timo Eirola
Instructor:	Ph.D. Antti Rasila

Abstract:

The purpose of this thesis is to give an introduction to numerical conformal mappings and the computation of a conformal modulus of a quadrilateral. The theory of conformal mapping is studied because of its connections to physical applications and for its significance in mathematics. The computational advantage can be gain by conformally mapping a simply connected domain onto the unit disk, the upper half plane, or a rectangle.

On the other hand the capacitance of a condenser can be characterized by the conformal modulus of a quadrilateral. In this thesis, we give two ways to compute the modulus, namely

- Schwarz–Christoffel mappings,
- finite element methods.

The obtained results are compared to each other and to known analytic and reference values from literatures.

Pages: 100	Keywords: conformal mappings, conformal modulus of a quadrilateral, numerical methods, Schwarz–Christoffel mapping, finite element methods
Department fills	
Approved:	Library code:

Preface

This thesis was done in the Department of Mathematics and Systems Analysis at the Helsinki University of Technology during the years 2008 and 2009. Writing this thesis had been a long journey which was anything but smooth. I would like to thank my instructor Ph.D. Antti Rasila for his guidance through the obstacles and for his valuable suggestions and corrections to the thesis. He was the one who introduced me to this fascinating subject which I otherwise would not have considered. Also I wish to thank my supervisor professor Timo Eirola for comments on the thesis. Furthermore I like to thank Ph.D. Harri Hakula for discussions about mathematics in general and especially for advice and comments on finite element methods. I warmly thank professor Matti Vuorinen for his comments and suggestions to the work.

Lastly I wish to thank my friends for listening my mathematical jabbering all these years and for playing and keeping me up-to-date on the Pokémon trading card game.

Tri Quach
Espoo, November 3, 2009

Contents

List of Symbols

$\mathbb{C}$	Complex plane
$\hat{\mathbb{C}}$	Extended complex plane, $\mathbb{C} \cup \{\infty\}$
$\mathbb{R}_+$	Set of positive real numbers
γ	Curve
Γ	Curve family
Ω	Domain
$\partial\Omega$	Boundary of Ω
$\mathbb{D}$	Unit disk, $\mathbb{D} = \{z \in \mathbb{C} : \lvert z \rvert < 1\}$
$\mathbb{H}^+$	Upper half plane, $\mathbb{H}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
$B(z_0, r)$	Disk with the center z_0 and the radius $r > 0$
$Q(\Omega, z_1, z_2, z_3, z_4)$	Quadrilateral
$M(Q)$	Conformal modulus of the quadrilateral Q
k	Elliptic modulus
k'	Complementary elliptic modulus, $k = \sqrt{1-k^2}$
$F(k, z)$	Elliptic integral of the first kind
$K(k)$	Complete elliptic integral of the first kind, $K(k) = F(k, 1)$
C^m	Space of m times continuously differentiable functions
L^p	Lebesgue space
$H^{m,p}, W^{m,p}$	Sobolev space

Chapter 1

Introduction

The theory of conformal mappings are studied because of their close relation to physical applications in, for example, electrostatistics and aerodynamics, as well as their theoretical significance in mathematics. In applications numerical computations are usually required. For example, the analytical computation of the capacitance can be carried out only for few condensers. This is illustrated by the following simple example. Let us consider a cylindrical condenser, see Figure 1.1. Then the capacitance per unit length is given by

$$\frac{C}{L} = \frac{2\pi\varepsilon}{\ln{(R/r)}},$$

where ε is the permittivity factor and L is the length of the cylinder. The connection between the capacitance and the conformal modulus of a quadrilateral is shown in Example 4.1.2.

We consider mappings that map conformally simply connected domains onto simplier domains like the unit disk, the upper half plane, or rectangles. In physical applications partial differential equations usually arise

$$-a\Delta u + b\nabla u + cu = f.$$

By mapping the domain onto simplier one, the computational advantage is clear. In particular, the Laplace equation $\Delta u = 0$ is one of the the most important partial differential equations in engineering mathematics. For Laplace equations, we may use the complex analysis to represent $f(x+iy) = u(x,y)+iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are harmonic functions.

There are many old and new applications of conformal mappings, for example in cartography. Historically, Mercator's cylindrical map projection was the first

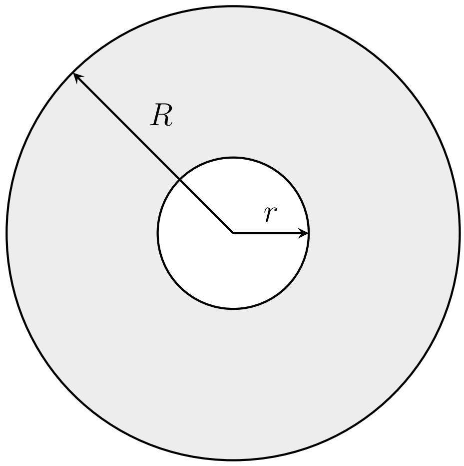


Figure 1.1: The cross-section of the cylinder.

conformal mapping studied because of this property. It maps conformally the Earth's surface onto the plane. The projection distort the area and length near the poles, for example Greenland and Africa have approximately the same size at the projection, but in the real world, Africa is about 10 times as large as Greenland is. Furthermore, in the last century conformal mappings have been used in wide range of applications such as integrated and printed circuits, nuclear reactors, airfoils, pattern recognitions, and condensers [SL]. Applications on vortex dynamics have been studied in [SC] and further applications to fluids and flows are described in [Cro1, Cro2, Cro3, TD].

In this thesis we are interested on a quantity called the conformal modulus of a quadrilateral. For computations we use mainly two different approaches

1. Schwarz–Christoffel mappings,

2. finite element methods.

The former methods give the conformal modulus as well as the auxiliary conformal mapping of the quadrilateral onto a rectangle.

Schwarz–Christoffel mappings are closely related to the Riemann mapping theorem which states that any simply connected domain except the whole complex plane can be map onto the unit disk. It is noteworthy that even simply connected domains with, for example, fractal boundaries such as Koch's snowflake (Figure 1.2) can be conformally mapped onto the unit disk [Pom]. Another important result is a theorem of Carathéodory, which gives a condition for the continuous boundary extensions. This is crucial in our applications.

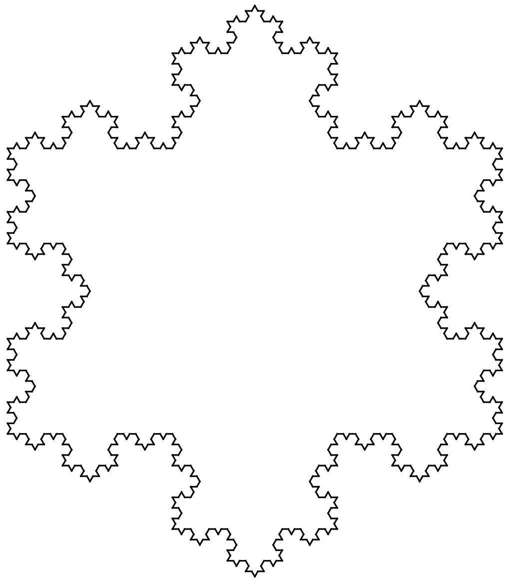


Figure 1.2: Koch's snowflake with 4 consecutive iterations.

The other technique we will be using, finite element methods, does not immediately arise from the complex analysis. They can be used for solving partial differential equations by decomposing the domain of interest into elements and approximating the solution on each of the elements. In this thesis, finite element methods are used for solving the conformal modulus of quadrilaterals. This is possible because the conformal modulus can be characterized by means of the Laplace equation with Dirichlet–Neumann boundary conditions. By this methods we minimize the Dirichlet integral

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y,$$

the value, which equals the capacitance of a condenser, where Ω is a domain in the complex plane. This approach gives another way to characterize the conformal modulus of a quadrilateral.

Besides the Schwarz–Christoffel mapping, there exists many other numerical methods that can be used for obtaining conformal mappings. Some of these techniques are overviewed in [Por] and for more details, see [Hen2].

This thesis is organized as follows. In Chapter 2 we give background material to understand this thesis. In Chapter 3 we state the Riemann mapping theorem and give a proof through a normal family argument. Definitions and properties of the conformal modulus of quadrilaterals are given in Chapter 4. In Chapter 5 we study the Schwarz–Christoffel mappings which can be used for mapping a polygonal domain onto the unit disk, the upper half plane, or a rectangle. The theory of finite element methods is developed in Chapter 6. Numerical methods related to the Schwarz–Christoffel mapping and finite element methods are studied in Chapter 7. Finally, in Chapter 8, we consider examples of quadrilaterals and compute the modulus by both the Schwarz–Christoffel toolbox [Dri] and hp-version of finite element methods. The results are compared to each other and to known reference results. In Chapter 9 we discuss about the results obtained in Chapter 8 and we give some ideas for further research.

This thesis is closely related to earlier work in the same research group, see for example theses [Num, Vuo, Yrj] and research papers [BSV, DuVu, HRV, RV].

Chapter 2

Preliminaries

In this chapter we give basic definitions and results used in the theory of conformal mappings. Presented results are well known, so the reader familiar with the topic may glance through it quickly and begin with the next chapter, referring to this chapter when necessary.

2.1 Curves and domains

Definition 2.1.1. (Curve)
A *curve* is a continuous function $\gamma\colon [a,b] \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane, the so called one point compactification of $\mathbb{C}$.

A curve is said to be *smooth* if it is continuously differentiable and $\gamma(t) \neq 0$. We denote a set of curves by Γ and call it a *curve family*.

Definition 2.1.2. (Length of a curve)
Let γ be a curve, $\gamma\colon [a,b] \to \hat{\mathbb{C}}$, and let $T_k\colon a = t_0 < t_1 < \cdots < t_k = b$ be a partition of the closed interval $[a,b]$. The set $\gamma([a,b])$ is called the *locus* of γ. Then, by denoting $\gamma_i = \gamma(t_i)$, the *length* of the curve γ is defined by the supremum of sums

$$l(\gamma) = \sup_{T_k} \sum_{i=1}^{k} |\gamma_i - \gamma_{i-1}| \, .$$

If the curve γ is piecewise differentiable, then we can define, alternatively, the length of the curve by

$$l(\gamma) = \int_a^b |y'(t)| \, \mathrm{d}t.$$

A curve γ is said to be *rectifiable* if its length is finite. A curve γ with parametric interval $[a, b]$ such that $\gamma(a) = \gamma(b)$ is called a *closed curve*. That is, the starting point and the end point of γ are the same. Furthermore, γ is said to be *simple* if it does not intersect itself, that is, if $\gamma(c) \neq \gamma(d)$, for all $c \neq d$, where $c, d \in (a, b)$. Note that the exception $\gamma(a) = \gamma(b)$ is allowed. If γ is both simple and closed, then it is called a *Jordan curve* [Pon, p. 117].

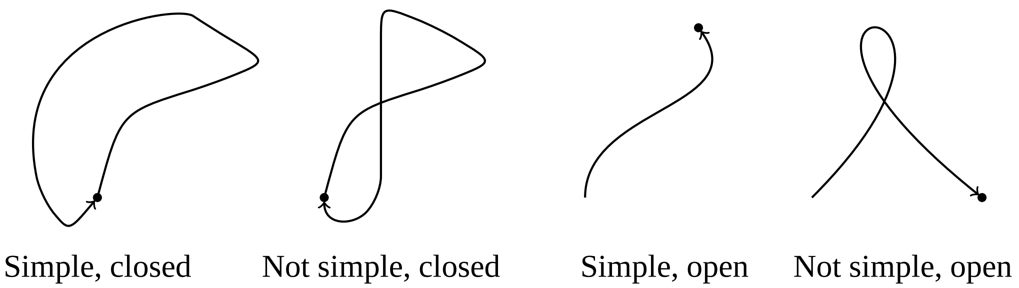


Figure 2.1: Different types of curves.

Definition 2.1.3. (Domain)
A *domain* Ω is a non-empty open connected set in $\mathbb{C}$. In particular, a domain Ω is path-wise connected, that is for each pair of points z_1 and z_2 in Ω can be connected with a curve γ such that the γ lies entirely in Ω.

A domain together with some, none, or all of its boundary points is called a *region*. The closure of a domain Ω is denoted by $\overline{\Omega}$ and is the union of the domain Ω and the boundary curve $\partial\Omega$. A domain Ω is said to be *simply connected* if its complement with respect to the extended plane is connected. For reasons of convenience we do not consider the whole complex plane $\mathbb{C}$ as simply connected. If Ω is not simply connected, then we say that Ω is *multiply connected*. A domain Ω bounded by a Jordan curve is called a *Jordan domain*. Note that every Jordan domain is simply connected.

Definition 2.1.4. (Generalized quadrilateral)
A *generalized quadrilateral* is a Jordan domain Ω with four separate boundary points z_1, z_2, z_3, and z_4 given in positive order on the boundary curve $\partial\Omega$ of Ω. These points are called vertices of the generalized quadrilateral. They divide $\partial\Omega$ into four curves γ_1, γ_2, γ_3, and γ_4 which are called the sides of the generalized quadrilateral and denoted by (z_1, z_2), (z_2, z_3), (z_3, z_4), and (z_4, z_1), respectively.

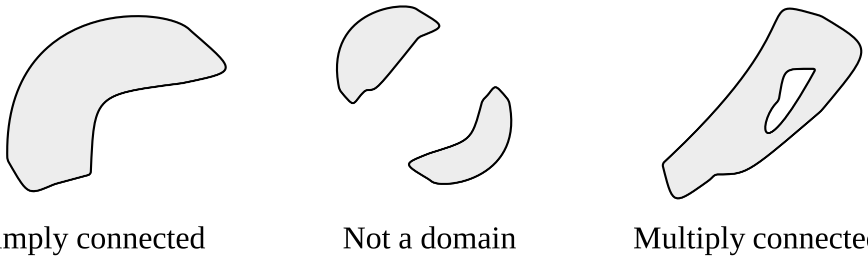


Figure 2.2: Different types of sets.

In this case, when we traverse $\partial\Omega$ such that Ω is on the left-hand side, the points z_1, z_2, z_3, and z_4 occur in this order. We denote a generalized quadrilateral by $Q(\Omega; z_1, z_2, z_3, z_4)$.

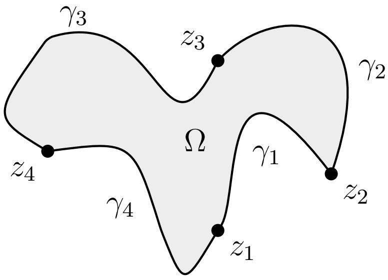


Figure 2.3: A generalized quadrilateral $Q(\Omega; z_1, z_2, z_3, z_4)$.

If the sides γ_k, where $k = 1, 2, 3, 4$, are line segments, then the generalized quadrilateral will be a quadrilateral in the usual geometric sense. In what follows the word quadrilateral is used to describe both geometric and generalized quadrilaterals.

2.2 Complex analysis

The origin of complex numbers lies in the problem of finding roots of polynomial equations. Already in the early 16th century Cardano, Tartaglia, and Ferro found a long sought general solution for the cubic equation. In some cases Cardano–Tartaglia–Ferro formula gives a solution which seemingly looks like a complex number even though the solution is, for example a positive real number. This puzzled Cardano who acknowledged the existence of these bizarre numbers even though he could not make any use of them. Bombelli showed in 1572 that the roots of a negative number have a great utility by manipulating a seemingly complex solution obtained by Cardano–Tartaglia–Ferro formula into a real number solution. In the 18th century Euler introduced the modern notation $i = \sqrt{-1}$ [Lau, pp. 64–69].

We assume that the reader is familiar with complex numbers. If this is not the case, please see, for example, [Ahl1, Gam, MH, NP, Pon] for basic concepts of complex numbers.

2.2.1 Derivative

Suppose that for every value z in a domain Ω there corresponds a definite complex value w. Then the function $f\colon z \mapsto w$ is said to be a complex function defined in Ω. A function $f(z)$ is said to be *single-valued* if $f(z)$ satisfies

$$f(z) = f(z(r,\varphi)) = f(z(r,\varphi + 2\pi)).$$

Otherwise, $f(z)$ is said to be *multiple-valued*.

Definition 2.2.1. (Derivative)
A complex function $f(z)$ defined in a domain Ω is *differentiable* at a point $z_0 \in \Omega$ if the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is independent of the path along which $\Delta z \to 0$. The limit is denoted by $f'(z_0)$ and is called the complex derivative of the function $f(z)$ at the point z_0.

The complex derivative shares many of the properties of the real derivative. See [Ahl1] for a further reference.

Definition 2.2.2. (Analytic function)
A function $f(z)$ is said to be *analytic*, or *holomorphic*, at a point $z_0 \in \mathbb{C}$ if it is differentiable at every point of some neighborhood of the point z_0. Similarly, a function $f(z)$ is said to be analytic in a set E if it is differentiable at every point of some open set Ω such that $E \subset \Omega$.

A function $f(z)$, which is analytic in the whole complex plane $\mathbb{C}$ is called an *entire function*. Suppose that a function $f(z)$ is analytic in a neighborhood of a point z_0, except perhaps at z_0 itself. If $\lim\limits_{z\to z_0} f(z) = \infty$, the point z_0 is said to be a *pole* of $f(z)$, and we set $f(z_0) = \infty$. A function $f(z)$ which is analytic in a domain Ω, except for poles, is said to be *meromorphic* in Ω. Furthermore, a meromorphic function $f(z)$ at a point z_0 is said to have an order N at z_0 if $f(z) = (z - z_0)^N g(z)$ for some analytic function $g(z)$ at z_0 such that $g(z_0) \neq 0$.

Determining whether a given function $f(z)$ is analytic or not directly from the definition is not usually practical. Fortunately, the *Cauchy–Riemann equations* give us a convenient characterization of analytic functions.

Theorem 2.2.3. (Cauchy–Riemann equations)
Let a function $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of a point $z_0 = x_0 + iy_0$ and differentiable at z_0. Then $f(z)$ is analytic if partial derivatives[1] of $u(x,y)$ and $v(x,y)$ exists at z_0 and satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.1}$$

in the neighborhood of z_0.

Furthermore, if $f(z)$ is analytic in a domain Ω, then $f(z)$ satisfies the Cauchy–Riemann equations for every point $z \in \Omega$.

Definition 2.2.4. (Harmonic function) [Ahl1, p. 162]
A real-valued function $u(z) = u(x,y)$ defined and single-valued in a domain Ω, is said to be *harmonic* in Ω if it is continuous together with its partial derivatives of the first two orders and satisfies *Laplace's equation*

$$\Delta u = u_{xx} + u_{yy} = 0.$$

It is easy to see that for analytic function , $f(z) = u(z) + iv(z)$, $f\colon \Omega \to \mathbb{C}$, the functions $u(z)$ and $v(z)$ are harmonic in Ω. A function $v(z) = v(x,y)$ is called a *conjugate harmonic function* for a harmonic function $u(z)$ in Ω whenever $f(z) = u(z) + iv(z)$ is analytic in Ω.

To construct a conjugate harmonic function $v(z)$, we use the information that $f(z)$ is analytic. In particular, $f(z)$ satisfies the Cauchy-Riemann equations (2.1). So $v(z)$ can be expressed by

$$v(x,y) = \int u_x(x,y)\,\mathrm{d}y + C(x), \tag{2.2}$$

[1] Notation: $u_x(x,y) = \frac{\partial u}{\partial x}(x,y)$.

where $C(x)$ is a function of x alone to be determined. Differentiating (2.2) with respect to x, we will get

$$\begin{aligned} v_x(x,y) &= \frac{\partial}{\partial x}\int u_x(x,y)\,\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}x}C(x) \\ \overset{\text{C-R}}{\Rightarrow} -u_y(x,y) &= \frac{\partial}{\partial x}\int u_x(x,y)\,\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}x}C(x). \end{aligned}$$

The function $C(x)$ can now be solved from the last equation by integrating with respect of x. Note that the conjugate harmonic function $v(z)$ is unique up to an addition of a real constant.

We also use the following result:

Theorem 2.2.5. (Liouville's Theorem) [Ahl1, p. 122]
A bounded and entire function is constant.

2.2.2 Integral

Arithmetic operations and calculus of differentiations generalize from the real to a complex variable without difficulties. But defining a *complex integral*, also known as *contour integral*, the transition is not as straightforward as it could be imagined. For some historical remarks see [Lau, pp. 73–75].

Let a curve γ and a partition T_k of an interval $[a,b]$ be as in Definition 2.1.2. For each interval (t_{i-1},t_i), where $i = 1,\cdots,k$, we choose an arbitrary point $t=\tau_i$. Suppose that $f(z)$ is defined and continuous on γ. Setting $\xi_i=\gamma(\tau_i)$ and $\gamma_i=\gamma(t_i)$, we consider the expression

$$\Sigma_{T_k} = \sum_{i=1}^{k} f(\xi_i)(\gamma_i-\gamma_{i-1}). \tag{2.3}$$

Suppose that the length of each interval (t_{i-1},t_i) of the partition T_k is bounded, then the sum (2.3) will tend to a finite limit when the partition T_k is refined so that $k\to\infty$ and the length of the longest interval $|t_i-t_{i-1}|$, $i=1,2,\cdots,k$, tends to zero. Σ_{T_k} tends to a limit which is called the integral of $f(z)$ along the curve γ and denoted by $\displaystyle\int_\gamma f(z)\,\mathrm{d}z$. Thus,

$$\int_\gamma f(z)\,\mathrm{d}z = \sup_{T_k}\sum_{i=1}^{k} f(\xi_i)(\gamma_i-\gamma_{i-1}).$$

The value of the integral is independent of the way the refining process of T_k is carried out [Neh, pp. 81–83], [NP, pp. 108–109]. For a proof, see [NP, pp. 109–110].

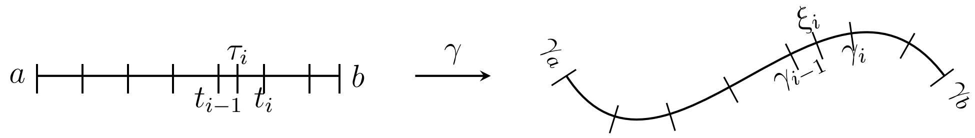


Figure 2.4: Partition of a curve in a complex integral.

If the curve γ is a line segment $[a, b]$ of the real line, then the integral of the continuous complex valued function $f(t) = u(t) + iv(t)$ is defined by

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

Suppose that a curve γ is piecewise differentiable with a parametrization $\gamma = \gamma(t)$, $a \leq t \leq b$. If the function $f(z)$ is defined and continuous on γ then $f(\gamma(t))$ is continuous as well. Then we define the integral of $f(z)$ over the curve γ by

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\,\gamma'(t)\,\mathrm{d}t.$$

The complex integral has the usual properties of the real integral. For further reference see [Ahl1].

For analytic functions we have following theorem.

Theorem 2.2.6. (Cauchy's integral theorem) [Ahl1, p. 109]
Let Ω be simply connected domain and suppose that $f(z)$ is analytic on Ω. Then

$$\int_{\partial\Omega} f(z) = 0.$$

By Theorem 2.2.6, the contour integral of the analytic function $f(z)$ is path independent.

2.2.3 Winding number and Argument principle

The concepts of winding numbers and argument principles tell us how many times the given curve γ wind up a given point z. The theory is based on calculus of residues and Cauchy theorems. We will only give the necessary definitions and theorems to understand the proof of Theorem 4.1.3.

Definition 2.2.7. (Branch) [Pon, p. 105]
Suppose $F(z)$ is a multiple-valued function defined in Ω. A *branch* of $F(z)$ is a single-valued analytic function $f(z)$ in some domain $U \subset \Omega$ obtained from $F(z)$ in such a way that at each point of U, $f(z)$ assumes exactly one of the possible values of $F(z)$.

Definition 2.2.8. (Winding number) [Ahl1, p. 115]
Let γ be a piecewise smooth closed curve. Suppose a point $a \notin \gamma$. Then the *winding number* of the point a respect to the curve γ is given by

$$n(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - a}.$$

The winding number can be interpreted intuitively as the number of times γ wraps around the point a in positive order. The following theorem states all the possible winding numbers for a Jordan curve.

Theorem 2.2.9. (Jordan curve theorem) [NP, pp. 178–179]
A Jordan curve γ separates the complex plane into two domains Ω_1 and Ω_2, both of which are bounded by γ. One of the domains is bounded and the other is unbounded. Without loss of generality we may assume that Ω_1 is bounded and Ω_2 is unbounded.

Then the winding number of each point $a \in \Omega_2$ respect of γ is zero and the winding number of each point $a \in \Omega_1$ respect of γ is either $+1$ or -1 depending on the orientation of γ, positive or negative, respectively.

In calculations of the argument principle, we will be using the above property of the winding number.

Theorem 2.2.10. (Argument principle) [Gam, pp. 224–225]
Let Ω be bounded domain with a piecewise smooth boundary $\partial\Omega$, and let $f(z)$ be a meromorphic function on Ω that extends analytically on $\partial\Omega$, such that $f(z) \neq 0$ on $\partial\Omega$. Then

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} \,\mathrm{d}z = N_0 - N_\infty,$$

where N_0 and N_∞ denote, respectively, the numbers of zeros and poles of $f(z)$ in Ω, counted according the orders.

The integral in Theorem 2.2.10 is often referred to as a *logarithmic integral* of $f(z)$ along γ. In case of a Jordan curve the Argument principle can be stated as following theorem:

Theorem 2.2.11. (Argument principle for a Jordan curve) [Hen1, p. 278]
Let $f(z)$ be analytic in a simply connected domain Ω and let γ be a positively oriented Jordan curve in Ω not passing through any zero of $f(z)$. Then the number of zeros of $f(z)$ in the interior of γ, each zero counted according to its multiplicity, equals the winding number of the image curve $f(\gamma)$ with respect to 0.

2.3 Conformal mappings

The history of conformal mappings can be dated back to the 16th century. In 1569 Mercator presented a cylindrical map projection which is a conformal mapping from a sphere onto the plane. It was not until 1820 that Gauss gave the formal definition to conformal mappings. Thus, Mercator preceded Gauss by nearly three centuries.

A heuristic way to define a conformal mapping is the following. A mapping $f\colon z \mapsto w$ is said to be *conformal* at z_0 if it preserves angles and their orientation between smooth curves through z_0. Obviously, such mappings are very useful in cartography.

More precisely, let f be an analytic function in the domain Ω and let z_0 be a point in Ω. If $f'(z_0) \neq 0$, then f can be expressed by

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

where $\eta(z) \to 0$ as $z \to z_0$. Whenever z is in a sufficiently small neighborhood of z_0, the transformation $w = f(z)$ can be approximated by

$$\begin{aligned} S(z) &= f(z_0) + f'(z_0)(z - z_0) \\ &= f(z_0) - f'(z_0)z_0 + f'(z_0)z. \end{aligned}$$

Here the mapping $S(z)$ can be represented as a following composite map. First apply a rotation of the plane through the angle $\mathrm{Arg} f'(z_0)$, then a scaling by the factor $|f'(z_0)|$. Finally use the translation $f(z_0) - f'(z_0)z_0$.

Let $\gamma(t)$ be a smooth curve that passes through a point z_0 and $f'(z_0) \neq 0$. Then the tangent to the curve

$$\tilde{\gamma}(t) = (f \circ \gamma)(t),$$

at $w_0 = f(z_0)$ is given by

$$(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0).$$

A mapping $f\colon \Omega \to \mathbb{C}$ is said to be conformal at $z_0 \in \Omega$ if, for any two parameterized curves γ_1 and γ_2 intersecting at the point $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ with non-zero tangents, the following conditions hold:

(i) the transformed curves $\tilde{\gamma}_1 = f \circ \gamma_1$ and $\tilde{\gamma}_2 = f \circ \gamma_2$ have non-zero tangents at the point t_0, and

(ii) the angle between $\tilde{\gamma}_1'(t_0) = (f \circ \gamma_1)'(t_0)$ and $\tilde{\gamma}_2'(t_0) = (f \circ \gamma_2)'(t_0)$ is same as the angle between $\gamma_1'(t_0)$ and $\gamma_2'(t_0)$.

If the function $f(z)$ is conformal at each point of a domain Ω, then $f(z)$ is said to be locally conformal in Ω [Pon, pp. 194–195]. If $f(z)$ is also a bijection, $f(z)$ is a conformal mapping in Ω.

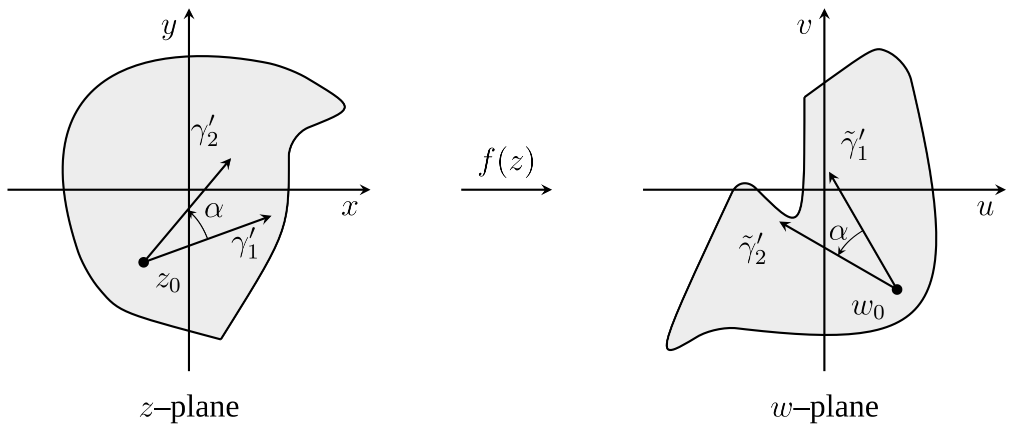


Figure 2.5: An illustration of a conformal map.

Example 2.3.1. Let us consider a mapping

$$f(z) = \left(\frac{1+z}{1-z}\right)^2,$$

which is analytic for every $z \in \mathbb{C} \backslash \{1\}$. Thus, it maps conformally the upper part of the unit disk onto the upper half plane. See Figure 2.6 for an illustration of $f(z)$.

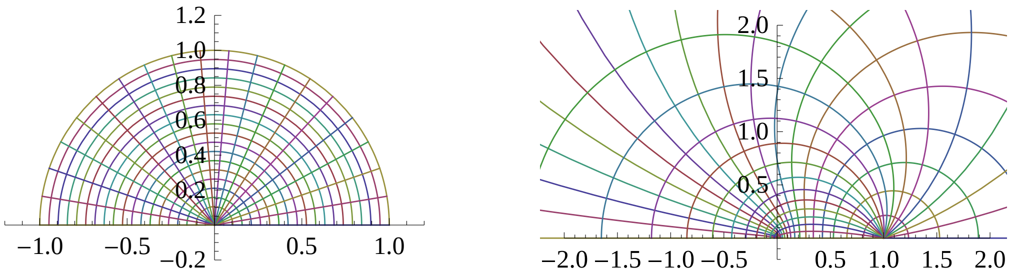


Figure 2.6: Example of a conformal mapping that maps the upper part of the unit disk onto the upper half plane.

2.4 Möbius transformations

Möbius transformations are one class of conformal mappings. Thus we may try to construct conformal mappings of one domain onto another using Möbius transformations.

Möbius transformations are essentially compositions of one or more of the simpler types of transformations.

- **Translation:** A mapping of the form $z \mapsto z + c$, where $c \in \mathbb{C}$. If $c = 0$ then the mapping is the identity map.

- **Magnification:** A mapping of the form $z \mapsto rz$, where $r \in \mathbb{R} \backslash \{0\}$. If $r = 1$ then the mapping is the identity map. If $r < 0$ then the mapping is also a reflection with respect to the origin.

- **Rotation:** A mapping $z \mapsto e^{i\varphi} z$, where $\varphi \in \mathbb{R}$ produces a rotation through the angle φ about the origin in positive sense if $\varphi > 0$.

- **Inversion:** A mapping $z \mapsto \frac{1}{z}$ produces a geometric inversion.

Möbius transformations are always rational functions of the first order and thus sometimes are referred to as *fractional linear transformations* as well.

Definition 2.4.1. (Möbius transformations)
Let parameters $a, b, c, d \in \mathbb{C}$ be chosen so that $ad - bc \neq 0$. Then a Möbius transformation is defined by

$$w = f(z) = \frac{az + b}{cz + d},$$

where $z \in \hat{\mathbb{C}}$.

Since the derivative of $f(z)$ is given by

$$f'(z) = \frac{ad - bc}{(cz + d)^2},$$

the condition $ad - bc \neq 0$ ensures that a Möbius transformation is not constant. In addition a Möbius transformation $f(z)$ is analytic for all $z \in \mathbb{C} \backslash \{-\frac{d}{c}\}$. If $c = 0$, then the Möbius transformation will reduce to the form

$$f(z) = \frac{a}{d} z + \frac{b}{d} = a'z + b',$$

which is called an *affine mapping*. It is convenient to define Möbius transformation as a mapping from $\hat{\mathbb{C}}$ onto itself. If $f(z)$ is an affine mapping, then we define $f(\infty) = \infty$. Otherwise when $c \neq 0$ we define $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$.

Möbius transformations can be associated with a 2×2 matrix via the map

$$z \mapsto A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim f(z).$$

Let mappings $f(z)$ and $g(z)$ be Möbius transformations as follows

$$f(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad \text{and} \qquad g(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then computing the composition $(f \circ g)$

$$\begin{aligned} (f \circ g)(z) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}, \end{aligned}$$

and a derivation gives us

$$(f \circ g)'(z) = \frac{(a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (c_1 a_2 + d_1 c_2)(a_1 b_2 + b_1 d_2)}{[(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)]^2}.$$

Then by simplifying the numerator, we have

$$\begin{aligned}
&(a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (c_1a_2 + d_1c_2)(a_1b_2 + b_1d_2) \\
&= a_1a_2d_1d_2 + b_1c_2c_1b_2 - c_1a_2b_1d_2 + d_1c_2a_1b_2 \\
&= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2). \qquad (2.4)
\end{aligned}$$

Since $f(z)$ and $g(z)$ are Möbius transformations, the factors in (2.4) are not zero. This implies that a composition of Möbius transformations is a Möbius transformation as well. In addition the inverse of a Möbius transformation is also a Möbius transformation and is given by

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

The composition and the inverse of Möbius transformations correspond to product and inverse of the matrices, respectively. The analogy is following, the derivative of a Möbius transformation $f'(z) \neq 0$ if and only if the $\det(A_f) \neq 0$.

Möbius transformations map circles in $\hat{\mathbb{C}}$ onto circles in $\hat{\mathbb{C}}$, where a straight line is considered as a circle with an infinite radius [Gam, pp. 63–66], [Pon, pp. 200–206]. In particular we have Möbius transformations which map the unit disk onto itself.

Lemma 2.4.2. (Mapping of the unit disk onto itself) [Kre, p. 740]
The mapping

$$w = \frac{z - z_0}{\overline{z}_0 z - 1},$$

where $|z_0| < 1$ maps the unit disk onto the unit disk such that the point z_0 maps onto the origin.

Proof: We take $|z| = 1$ and calculate

$$\begin{aligned}
|z - z_0| &= |\overline{z} - \overline{z}_0| \\
&= |z| \cdot |\overline{z} - \overline{z}_0| \\
&= |1 - \overline{z}_0 z| \\
&= |\overline{z}_0 z - 1|.
\end{aligned}$$

Hence

$$|w| = \frac{|\overline{z} - \overline{z}_0|}{|\overline{z}_0 z - 1|} = 1,$$

so that the unit circle maps onto the unit circle. Noting that z_0 maps onto the origin, implies the claim [Kre, p. 740].

□

Example 2.4.3. Let $z_0 = \frac{1}{2}$. Then we have

$$w = \frac{2z-1}{z-2}.$$

In Figure 2.7 we have an illustration of the above mapping.

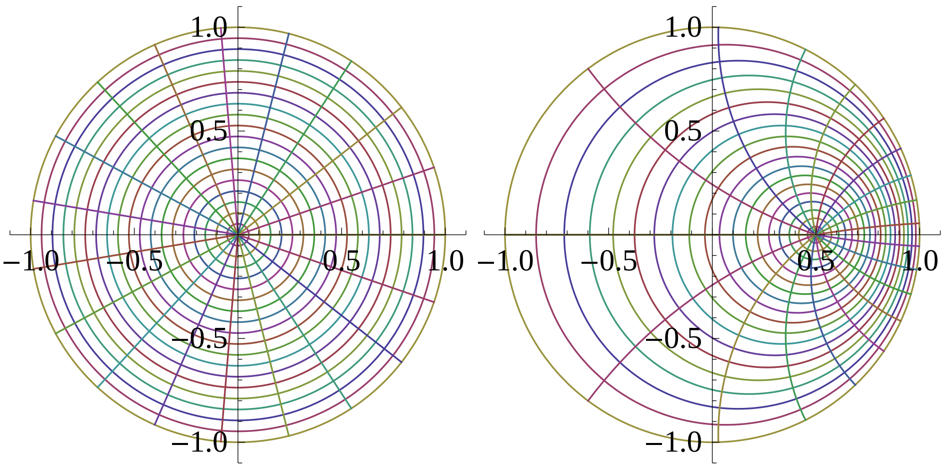


Figure 2.7: Mapping of the unit disk onto the unit disk.

Definition 2.4.4. (Cross ratio) [Ahl1, p. 78]
Fix points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$. Then a *cross ratio* (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \frac{(z_4 - z_2)(z_1 - z_3)}{(z_4 - z_3)(z_1 - z_2)}.$$

By the cross ratio, we may construct a Möbius transformation as follows

Theorem 2.4.5. If $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ such that $w_j = f(z_j)$, $j = 1, 2, 3$. Then the Möbius transformation $w = f(z)$ can be solved from the cross ratio

$$\frac{(w - w_2)(w_1 - w_3)}{(w - w_3)(w_1 - w_2)} = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}.$$

This is true, because the cross ratio is invariant under Möbius transformations [Ahl1, p. 79].

2.5 Elliptic integrals

There is a vast number of interesting integrals that cannot be expressed in terms of elementary functions. One type of such integrals is known as *elliptic integrals*. Such integrals arise from many elementary questions in the natural science. For example when *Kepler's laws* became known, the first natural aim was to compute the orbit of a planet. Wallis attempted to compute the arc length of an ellipse in 1655. Series expansion for elliptic integrals were given by Newton and Euler.

Elliptic integrals were extensively studied by Legendre, Gauss, Abel and Jacobi in the early 19th century. Legendre showed that every elliptic integral can be reduced by a suitable substitution to one of the three normal forms. These normal forms are called elliptic integral of the first, second and third kind [Cay]. We are only interested in the elliptic integrals of the first kind, because these provide a way to conformally map the upper half plane onto a rectangle.

Definition 2.5.1. (Elliptic integral of the first kind) [Cay, pp. 2–3]
The *elliptic integral of the first kind* is defined by

$$F(k,z) = \int_0^z \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}},$$

for $0 < k < 1$, where the parameter k is called the *elliptic modulus*. The complementary elliptic modulus is given by $k' = \sqrt{1-k^2}$.

Substituting $z = \sin\phi$ and $\zeta = \sin\theta$, we will get

$$F(k,\sin\phi) = \int_0^{\sin\phi} \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{(1-\sin^2\theta)(1-k^2\sin^2\theta)}}.$$

By the identity $\cos^2\theta = 1-\sin^2\theta$, we eliminate the cosine term and the elliptic integral of the first kind can also be expressed by

$$F(k,\sin\phi) = \int_0^{\sin\phi} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}},$$

where ϕ is called the amplitude. If the integral is taken up to the amplitude $\frac{\pi}{2}$, then it is called the complete elliptic integral of the first kind.

Definition 2.5.2. (Complete elliptic integral of the first kind) [Cay, p. 4]
The *complete elliptic integral of the first kind* is defined by

$$K(k) = F(k,1) = \int_0^1 \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}$$
$$= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

for $0 < k < 1$, where the parameter k is the elliptic modulus.

The complete complementary elliptic integral of the first kind is denoted by

$$K'(k) = K(k').$$

The inverse of the elliptic integrals is called Jacobi's elliptic functions. To simplify the notation we give a following definition.

Definition 2.5.3. (Jacobi's elliptic sine function) [Cay, p. 8]
Let $u = F(k,z)$. Then the *Jacobi's elliptic sine function* $\mathrm{sn}(u,k)$ is defined by

$$\mathrm{sn}(u,k) = z.$$

Jacobi's elliptic sine function maps conformally a rectangle onto upper half plane. This result is proved in Section 5.4.

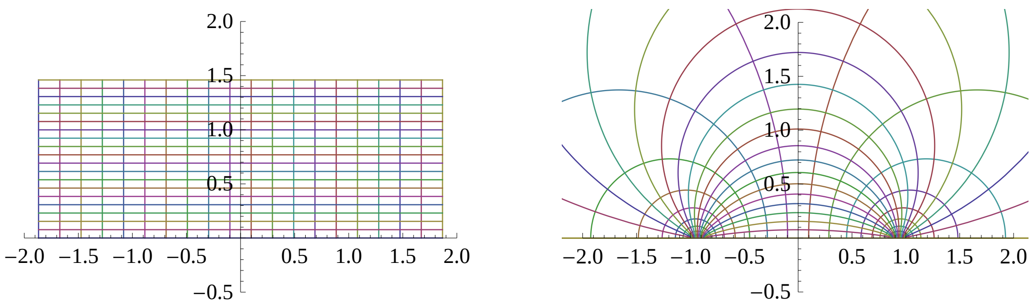


Figure 2.8: Example of a conformal mapping that maps a rectangle onto the upper half plane.

2.6 Lebesgue and Sobolev spaces

In finite element applications often arise situations where the functions in general are, strictly speaking, not differentiable, but can be well approximated with differentiable functions.

Sobolev spaces are vector spaces whose elements are functions defined on domain of $\mathbb{R}^n$ and whose partial derivatives satisfy certain integrability properties. A solution of partial differential equations are sought from Sobolev spaces.

We assume that the reader is familiar with basic concepts of a norm, the Lebesgue measure, and Lebesgue integration, for a reference see [Rud1, Rud2]. Following definitions and theorems are given in $\mathbb{R}^2$ even though generalizations to higher dimensions could be done, naturally.

Definition 2.6.1. (Compact support) [AF, p. 2]
Suppose $\Omega \subset \mathbb{R}^2$ is non-empty. The *support* of u is defined by

$$\operatorname{supp}(u) = \overline{\{x \in \Omega : u(x) \neq 0\}}.$$

We say u has a *compact support* in Ω if $\operatorname{supp}(u) \subset \Omega$ and $\operatorname{supp}(u)$ is compact.

In $\mathbb{R}^n$, compactness is equivalent to closedness and boundedness. This result is known as the Heine–Borel Theorem. For a proof, see [Rud1, p. 40].

Definition 2.6.2. (Space of continuous function) [AF, p. 10]
Let Ω be a domain. For any non-negative integer m let $C^m(\Omega)$ denote the vector space consisting of all functions ψ which, together with their partial derivatives $D^\alpha\psi$ of orders $|\alpha| \leq m$, are continuous on Ω, where

$$D^\alpha\psi = \frac{\partial^{\alpha_1}}{\partial_x^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial_y^{\alpha_2}}\psi,$$

and $\alpha = (\alpha_1, \alpha_2)$ is a pair of non-negative integers α_1, α_2 and $|\alpha| = \alpha_1 + \alpha_2$ is called a degree of α.

We abbreviate $C^0(\Omega) = C(\Omega)$ and $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. The family of functions of spaces $C(\Omega)$ and $C^\infty(\Omega)$ that have a compact support in Ω are denoted by $C_0(\Omega)$ and $C_0^\infty(\Omega)$, respectively.

Definition 2.6.3. (L_p–norm) [Rud2, p. 65]
Let $E \subset \mathbb{R}^2$ be a Lebesgue measurable set and let $f\colon E \to [-\infty, \infty]$ be a

Lebesgue measurable function. If $1 \leq p < \infty$, then the L_p*–norm* of f is defined by

$$\|f\|_p = \left(\int_E |f(x,y)|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

In some situations when confusion about domains may occur we write $\|\cdot\|_{p,E}$ instead of $\|\cdot\|_p$. Also the norm is denoted by $\|\cdot\|_{L^p(E)}$ if there is a confusion about the actual space.

Definition 2.6.4. (Lebesgue space) [AF, pp. 23]
Let p be a positive real number. We denote by $L^p(\Omega)$ the class of all measurable functions $f(x,y)$ defined on domain Ω for which

$$\|f\|_p < \infty.$$

Note that $L^1(\Omega)$ is a family of functions which are Lebesgue integrable on Ω. For $p \in [1,\infty)$ the space $L^p(\Omega)$ is not a normed space in a classical sense, since $\|f\|_p = 0$ does not imply that $f \equiv 0$. For this reason, we define $L^p(\Omega)$ as the space of equivalence classes

$$f \sim g \quad \Leftrightarrow \quad f = g, \text{ almost everywhere on } \Omega.$$

Then it follows that $L^p(\Omega)$ is a normed space [Rud2, pp. 65–69].

Suppose a function u is defined almost everywhere on a domain Ω and suppose $u \in L^1(U)$ for every compact $U \subset \Omega$. Then u is said to be *locally integrable* on Ω and we denote $u \in L^1_{\mathrm{loc}}(\Omega)$ [AF, p. 20].

We now proceed to define a concept of a function being the *weak derivative* of another function.

Definition 2.6.5. (Weak derivative) [AF, p. 22]
Let $u, v_\alpha \in L^1_{\mathrm{loc}}(\Omega)$. The function v_α is called weak partial derivative of u and denoted by

$$D^\alpha u = v_\alpha,$$

if it satisfies

$$\int_\Omega u(x,y) D^\alpha \psi(x,y) \,\mathrm{d}x\,\mathrm{d}y = (-1)^{|\alpha|} \int_\Omega v_\alpha(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

for all $\psi \in C_0^\infty(\Omega)$.

Note that such v_α may not exists. In this case we say that u does not have a weak αth partial derivative. On the other hand if such v_α exists then it is uniquely defined up to sets of measure zero. See [Eva, p. 243] for a proof.

Example 2.6.6. (1 dimensional) [Eva, p. 243]
Let $\Omega = (0,2)$ and let

$$u(x) = \begin{cases} x, & \text{if } 0 < x \le 1, \\ 1, & \text{if } 1 \le x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1, & \text{if } 0 < x \le 1, \\ 0, & \text{if } 1 \le x < 2. \end{cases}$$

Let us show $u'(x) = v(x)$ in the weak sense. We take any $\psi \in C_0^\infty(D)$ and we must show that

$$\int_0^2 u(x)\psi'(x)\,\mathrm{d}x = -\int_0^2 v(x)\psi(x)\,\mathrm{d}x.$$

By calculating the left-hand side we have

$$\begin{aligned} \int_0^2 u(x)\psi'(x)\,\mathrm{d}x &= \int_0^1 x\psi'(x)\,\mathrm{d}x + \int_1^2 \psi'(x)\,\mathrm{d}x \\ &= \Big/_0^1 x\psi(x) - \int_0^1 \psi(x)\,\mathrm{d}x + \Big/_1^2 \psi(x) \\ &= \psi(1) - \int_0^1 \psi(x)\,\mathrm{d}x + \psi(2) - \psi(1) \\ &= -\int_0^1 v(x)\psi(x)\,\mathrm{d}x + \psi(2) \end{aligned} \tag{2.5}$$

Since $\psi \in C_0^\infty(D)$ it follows that $\psi(2) = 0$. By adding the term

$$-\int_1^2 v(x)\psi(x)\,\mathrm{d}x = 0$$

to the equation (2.5) we obtain

$$\int_0^2 u(x)\psi'(x)\,\mathrm{d}x = -\int_0^2 v(x)\psi(x)\,\mathrm{d}x$$

as required. □

Definition 2.6.7. (Sobolev norm) [AF, p. 59]
Let m be a positive integer and let $1 \leq p < \infty$. Then we define the Sobolev norm by

$$\|f\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_p^p \right)^{1/p},$$

where the norm $\|\cdot\|_p$ is the corresponding norm in $L^p(\Omega)$.

Definition 2.6.8. (Sobolev spaces) [AF, pp. 59–60]
For any positive integer m and $1 \leq p < \infty$ we consider following vector spaces

(i) $H^{m,p}(\Omega)$ is the space of completion of $\{f \in C^m(\Omega) : \|f\|_{m,p} < \infty\}$ with respect to the norm $\|\cdot\|_{m,p}$,

(ii) $W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega)$ for $0 \leq |\alpha| \leq m\}$, where $D^\alpha f$ is the weak partial derivative of f.

The completion is understood as every Cauchy sequence in a given space converges to a limit in the same space. In case of $p = 2$ we abbreviate $H^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ by $H^m(\Omega)$ and $W^m(\Omega)$, respectively.

Even though the definition of of $H^{m,p}(\Omega)$ and $W^{m,p}(\Omega)$ differs, Meyers and Serrin [MS] showed in 1964 that $H^{m,p}(\Omega) = W^{m,p}(\Omega)$ for every Ω.

Chapter 3

Riemann mapping theorem

Riemann stated the *Riemann mapping theorem* in his doctoral dissertation in 1851. The theorem says that a disk can be conformally transformed onto any simply connected domain, which implies that any two simply connected domains can be conformally mapped onto each other, see Figure 3.1. In particular, the theorem applies to polygonal domains.

Riemann's own proof considered an extremal problem related to the Dirichlet problem. Riemann's argument was flawed since he assumed that the extremal problem always has a solution. Numerous mathematicians, for example, Schwarz, Harnack and Poincaré, sought after a proof until around 1908 a rigorous proof was given by Koebe. It should be mentioned that in 1900 Osgood gave a proof for a related theorem from which the Riemann mapping theorem can be proved [Ahl1, pp. 229–230], [Wal].

3.1 Preliminary concepts

Before stating and proving the Riemann mapping theorem, let us work through the preliminaries results on convergences, function sequence $\{f_n(z)\}$, and the family of functions $\mathcal{F}$.

Definition 3.1.1. (Pointwise convergence) [Rud1, pp. 143–144]
Suppose that $\{f_n(z)\}$ is a sequence of functions defined on a set E, and suppose that the sequences of values $\{f_n(z)\}$ converges for every $z \in E$. We can then define a function $f(z)$ by

$$f(z) = \lim_{n\to\infty} f_n(z),$$

for $z \in E$ and $f_n(z)$ is said to converge *pointwisely* to $f(z)$.

Definition 3.1.2. (Uniform Convergence) [Rud1, p. 147]
We say that a sequence of functions $\{f_n(z)\}$ converges *uniformly* on E to a function $f(z)$ if for every $\varepsilon > 0$ there exists an integer N such that

$$|f_n(z) - f(z)| < \varepsilon,$$

for $n \geq N$ and for every $z \in E$.

Let us emphasize the subject of convergence with a simple example.

Example 3.1.3. For instance, it is true that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)z = z,$$

for all z. But in order to have

$$\left|\left(1+\frac{1}{n}\right)z - z\right| = \frac{|z|}{n} < \varepsilon$$

for $n \geq N$ it is necessary that $N \geq \frac{|z|}{\varepsilon}$. Such an integer N exists for every fixed z, but the requirement cannot be met simultaneously for all z.

The above example showed that the sequence of functions defined by

$$f_n(z) = \left(1+\frac{1}{n}\right)z$$

is pointwise convergent and is not uniformly convergent.

Theorem 3.1.4. (Hurwitz's Theorem) [Ahl1, p. 178]
If the functions $f_n(z)$ are analytic and $f_n(z) \neq 0$ in a domain Ω, and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω. Then $f(z)$ is either identically zero or never equal to zero in Ω.

Proof: See [Ahl1, p. 178].

The following definition characterizes a regular behavior of families.

Definition 3.1.5. (Normal family) [Ahl1, p. 220]
A family $\mathcal{F}$ is said to be *normal* in Ω if every sequence $\{f_n(z)\}$ of functions $f_n(z) \in \mathcal{F}$ contains a subsequence which converges uniformly on every compact subset of Ω.

This definition does not require the limit function of the convergent subsequences to be members of $\mathcal{F}$.

Theorem 3.1.6. (Montel's Theorem) [Pon, p. 440]
Suppose that $\mathcal{F}$ is a family in domain Ω such that $\mathcal{F}$ is locally uniformly bounded in Ω. Then $\mathcal{F}$ is a normal family.

Proof: See [Pon, p. 440].

3.2 Statement and proof

In this section we state and proof the Riemann mapping theorem and discuss about the boundary regularity which is crucial in applications.

Theorem 3.2.1. (Riemann mapping theorem) [Ahl1, p. 230]
Given any simply connected domain Ω in $\mathbb{C}$, and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) \in \mathbb{R}_+$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $|w| < 1$.

Proof: We have to prove that the mapping $f(z)$ exists and it is unique. Let us start by showing the uniqueness, since it is easier to prove. Suppose that functions $f_1(z)$ and $f_2(z)$ satisfy the Riemann mapping theorem. Then the composite function $(f_1 \circ f_2^{-1})(w)$ defines a one-to-one mapping of $|w| < 1$ on to itself. The mapping is Möbius transformation since it maps the unit circle onto itself. The conditions $f(0) = 0$ and $f'(0) \in \mathbb{R}_+$, imply $f(z) = z$, hence $f_1(z) = f_2(z)$.

Second part of the proof is to show that the $f(z)$ with desired properties exists. Let $g(z)$ be an analytic function in Ω and let $z_1, z_2 \in \Omega$. Then $g(z)$ is said to be *univalent* in Ω if $g(z_1) = g(z_2)$ only for $z_1 = z_2$. That is $g(z)$ is one-to-one. Let us consider a family $\mathcal{F}$ consist of all functions $g(z)$ with following properties:

(i) function $g(z)$ is analytic and univalent in Ω,

(ii) $|g(z)| \leq 1$ for every $z \in \Omega$,

(iii) $g(z_0) = 0$ and $g'(z_0) \in \mathbb{R}_+$.

Let us define $f(z)$ in $\mathcal{F}$ such that the derivative $f'(z_0)$ is maximal. The existence proof will consist three part:

1. the family $\mathcal{F}$ is not empty set,

2. there exists a function $f(z)$ with maximal derivative,

3. the function $f(z)$ has the desired properties.

First we prove that $\mathcal{F}$ is not empty. By assumptions there exists a point $a \neq \infty$ such that $a \notin \Omega$. Simply connectedness of Ω implies that it is possible to define a single-valued branch of $\sqrt{z-a} \in \Omega$ and denote it by $h(z)$. Note that $h(z)$ does not take the same value twice since if there were two distinct points $z_1, z_2 \in \Omega$ such that

$$\sqrt{z_1 - a} = \sqrt{z_2 - a},$$

it would follow that

$$z_1 - a = z_2 - a,$$

which is only possible for $z_1 = z_2$. Also $h(z)$ cannot take both the values c and $-c$, $c \in \mathbb{C}$ for every $z \in \Omega$. From

$$\sqrt{z_1 - a} = c, \quad \sqrt{z_2 - a} = -c,$$

it would follow that

$$c^2 = z_1 - a = z_2 - a,$$

which is again only possible for $z_1 = z_2$. Suppose that $h\colon \Omega \to \Omega'$ and $z_0 \in \Omega$. Then we have a disk $B(\rho, h(z_0)) \in \Omega'$. By above discussion $h(z) \neq -h(z_0)$ for every $z \in \Omega \backslash \{z_0\}$. This implies that $-h(z_0) \notin \Omega'$. Then we have a disk $B(\rho, -h(z_0)) \notin \Omega'$ such that it does not intersect with $B(\rho, h(z_0))$. This implies that $|h(z) + h(z_0)| \geq \rho$, for $z \in \Omega$, and in particular we have $2 \cdot |h(z_0)| \geq \rho$. Next we will show that the function

$$g_0(z) = \frac{\rho}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$$

belongs to $\mathcal{F}$.

(i) The function $g_0(z)$ is obviously analytic, since $h(z)$ is analytic. Also $g_0(z)$ is univalent, because it is a constructed by means of a Möbius transformation of $h(z)$.

(ii) By the estimate

$$\begin{aligned}\left|\frac{h(z)-h(z_0)}{h(z)+h(z_0)}\right| &= |h(z_0)|\left|\frac{h(z)}{h(z_0)[h(z)+h(z_0)]}+\frac{1-2}{h(z)+h(z_0)}\right|\\ &= |h(z_0)|\left|\frac{h(z)+h(z_0)}{h(z_0)[h(z)+h(z_0)]}-\frac{2}{h(z)+h(z_0)}\right|\\ &= |h(z_0)|\left|\frac{1}{h(z_0)}-\frac{2}{h(z)+h(z_0)}\right|\\ &\le |h(z_0)|\left(\underbrace{\left|\frac{1}{h(z_0)}\right|}_{\le 2/\rho}+\underbrace{\left|-\frac{2}{h(z)+h(z_0)}\right|}_{\le 2/\rho}\right)\\ &\le \frac{4|h(z_0)|}{\rho},\end{aligned}$$

we have an estimate for $z \in \Omega$

$$\begin{aligned}|g_0(z)| &= \frac{\rho}{4}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{|h(z_0)|}{|h'(z_0)|}\cdot\left|\frac{h(z)-h(z_0)}{h(z)+h(z_0)}\right|\\ &\le \frac{\rho}{4|h(z_0)|}\cdot\frac{4|h(z_0)|}{\rho}\\ &= 1.\end{aligned}$$

(iii) We start by noting that $g_0(z_0) = 0$, because the factor $h(z) - h(z_0)$ vanishes for $z = z_0$. For the derivative, we have

$$\begin{aligned}g_0'(z) &= \frac{\rho}{4}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{h(z_0)}{h'(z_0)}\cdot\frac{h'(z)[h(z)+h(z_0)]-h'(z)[h(z)-h(z_0)]}{[h(z)+h(z_0)]^2}\\ &= \frac{\rho}{4}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{h(z_0)}{h'(z_0)}\cdot\frac{2h'(z)h(z_0)}{[h(z)+h(z_0)]^2}\\ &= \frac{\rho}{2}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2}\cdot\frac{h^2(z_0)}{[h(z)+h(z_0)]^2}.\end{aligned}$$

Then by evaluation at the point z_0 gives

$$g_0(z_0) = \frac{\rho}{8}\cdot\frac{|h'(z_0)|}{|h(z_0)|^2} \in \mathbb{R}_+.$$

This prove that $\mathcal{F}$ is not empty and end the first part of the existence proof.

Let us denote the least upper bound of $g'(z_0)$, $g(z) \in \mathcal{F}$ by M which a priori can be infinite. Since $g_n \in \mathcal{F}$ are bounded, then by Montel's theorem (Theorem 3.1.6) the family $\mathcal{F}$ is normal. Thus there exists a subsequence $\{g_{n_k}\}$ which converges to an analytic limit function $f(z)$, uniformly on compact sets. By the properties of g_n, it follows that $|f(z)| \leq 1$ in Ω, $f(z_0) = 0$ and $f'(z_0) = M < \infty$. We still have to show that $f(z)$ is univalent.

First of all we note that $f(z)$ is not a constant function, since $f'(z) = B \in R_+$. Let $z_1 \in \Omega$ be a arbitrary point and consider the function $g_1(z) = g(z) - g(z_1)$, $g(z) \in \mathcal{F}$. Now $g_1(z) \neq 0$ for all $z \neq z_1$. Then by Hurwitz's theorem (Theorem 3.1.4) every limit function is either identically zero or never equal to zero. But $f(z) - f(z_1)$ is the limit function and it is not identically zero. It follows that $f(z) \neq f(z_1)$ for $z \neq z_1$. Since $z_1 \in \Omega$ was arbitrary, we have proved that $f(z)$ is univalent.

Now it remains to show that $f(z)$ takes every value w with $|w| < 1$. Suppose it is true that $f(z) \neq w_0$ for some w_0, $|w_0| < 1$. Because Ω is simply connected, it is possible to define a single-valued branch

$$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w}_0 f(z)}}.$$

The function $F(z)$ is univalent since it is obtained by means of a Möbius transformation of $f(z)$. If $|f(z)| = 1$, then $\frac{1}{f(z)} = \overline{f(z)}$ and so $|F(z)| = 1$. By assumption $|F(z)| < 1$, we have

$$\begin{aligned} |F(z)| < 1 &\iff |f(z) - w_0|^2 < |1 - \overline{w}_0 f(z)|^2 \\ &\iff \underbrace{(1 - |w_0|^2)}_{>0}(1 - |f(z)|^2) > 0 \\ &\iff |f(z)| < 1. \end{aligned}$$

That is, $|F(z)| \leq 1$ in Ω. For the normalized form we have

$$G_0(z) = \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)}F(z)},$$

which vanishes at z_0. For the derivative, we have

$$\begin{aligned} G_0'(z) &= \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F'(z)[1 - \overline{F(z_0)}F(z)] + \overline{F(z_0)}F'(z)[F(z) - F(z_0)]}{[1 - \overline{F(z_0)}F(z)]^2} \\ &= \frac{|F'(z_0)|}{F'(z_0)} \cdot F'(z) \cdot \frac{1 - |F(z_0)|^2}{[1 - \overline{F(z_0)}F(z)]^2}. \end{aligned}$$

Evaluating at the point z_0 gives us

$$G_0'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2}.$$

By differentiating $F(z)$ we have

$$\begin{aligned} F'(z) &= \frac{1}{2} \cdot \frac{1}{\sqrt{\frac{f(z)-w_0}{1-\overline{w}_0 f(z)}}} \cdot \frac{f'(z) \cdot [1-\overline{w}_0 f(z)] + [f(z)-w_0] \cdot \overline{w}_0 f'(z)}{[1-\overline{w}_0 f(z)]^2} \\ &= \frac{1}{2} \cdot \sqrt{\frac{1-\overline{w}_0 f(z)}{f(z)-w_0}} \cdot \frac{f'(z) \cdot (1-|w_0|^2)}{[1-\overline{w}_0 f(z)]^2}. \end{aligned}$$

Now we proceed to evaluate $F(z)$ and its derivative at the point z_0. Remark that $f(z_0) = 0$, then

$$\begin{aligned} F(z_0) &= \sqrt{\frac{f(z_0)-w_0}{1-\overline{w}_0 f(z_0)}} \\ &= \sqrt{-w_0}\,, \end{aligned}$$

and

$$\begin{aligned} F'(z_0) &= \frac{1}{2} \cdot \sqrt{\frac{1-\overline{w}_0 f(z_0)}{f(z_0)-w_0}} \cdot \frac{f'(z_0) \cdot (1-|w_0|^2)}{[1-\overline{w}_0 f(z_0)]^2} \\ &= \frac{1}{2} \cdot \frac{1-|w_0|^2}{\sqrt{-w_0}} \cdot f'(z_0). \end{aligned}$$

Then

$$\begin{aligned} G'(z_0) &= \frac{|F'(z_0)|}{1-|F(z_0)|^2} \\ &= \frac{\frac{1}{2} \cdot \frac{1-|w_0|^2}{\sqrt{|w_0|}} \cdot |f'(z_0)|}{1-|w_0|} \\ &= \frac{1+|w_0|}{2\sqrt{|w_0|}} \cdot B, \end{aligned}$$

where $B = f'(z_0)$. It remains to show that

$$\frac{1+|w_0|}{2\sqrt{|w_0|}} > 1.$$

By a brief computation we have

$$\begin{aligned} 1+|w_0| &= 2\sqrt{|w_0|} + (1-\sqrt{w_0})^2 \\ 1+|w_0| &> 2\sqrt{|w_0|} \\ \Longrightarrow \frac{1+|w_0|}{2\sqrt{|w_0|}} &> 1, \end{aligned}$$

for $|w_0| < 1$. Which implies that $G_0'(z_0) > B$ and contradicting our assumptions. We conclude that $f(z)$ assumes all the values w, $|w| < 1$. This complete the last part of the proof [Ahl1, pp. 229–232], [Neh, pp. 173–178].

□

The Riemann mapping theorem is an existence theorem. So it does not say anything about how to construct the mapping $f(z)$. The proof given above does not give us a way to construct the desire conformal mapping either. This is due to the following reasons:

(i) There is no prescription for constructing a sequence $\{g_n(z)\}$ such that $g_n'(z_0) \to f'(z_0) = B$.

(ii) The process of selecting a convergent subsequence from the sequence $\{f_n(z)\}$ cannot actually be carried out.

The proof given by Koebe is actually a constructive proof which can be use to construct the desired conformal mapping given by Riemann mapping theorem. The discussion and the proof can be found in [Hen2, pp. 328–328]. Yet another proof through potential theory and a discussion of Riemann's own flawed proof and its correction, we refer to [Wal].

By Theorem 3.2.1 any simply connected domain can be mapped onto any simply connected domain conformally as illustrated in Figure 3.1.

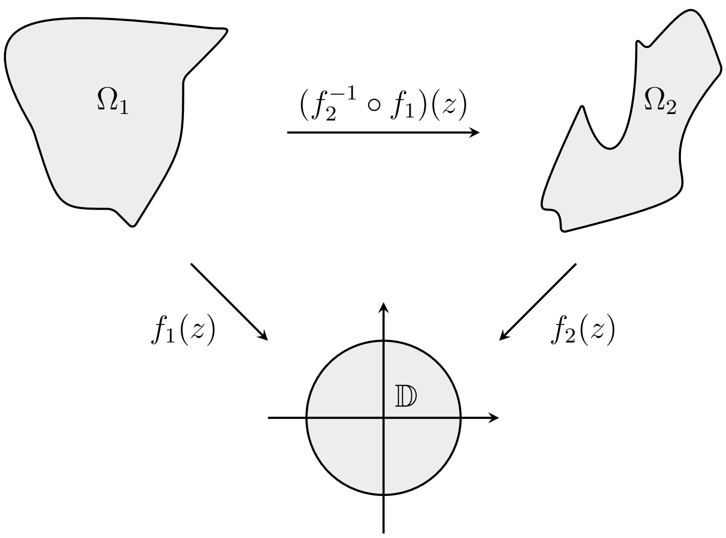


Figure 3.1: Riemann mapping theorem.

On the other hand, the Riemann mapping theorem does not say anything about the boundary regularity of conformal mappings either. In general, a conformal mapping of the unit disk onto a simply connected domain, not the entire complex plane $\mathbb{C}$, cannot be extended continuously to the boundary. A counterexample is a so called comb domain, see Figure 3.2, because the boundary of a comb domain is not a Jordan curve and there are portions of the boundary of infinite length in arbitrarily small neighborhoods of the origin.

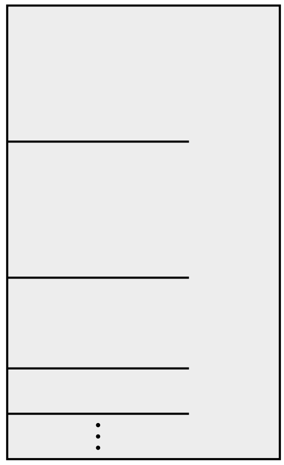


Figure 3.2: The comb domain.

Fortunately, there are ways to construct a conformal mapping $f(z)$ onto the unit disk for a large class of domains. In Chapter 5 we shall give a way to construct the mapping $f(z)$ for polygonal domains. Under some hypotheses the continuous boundary extension is known to exist. The following theorem states the required conditions for the continuity of the boundary extension.

Theorem 3.2.2. (Carathéodory) [Kra, p. 110]
Let Ω_1, Ω_2 be Jordan domains. If $f\colon \Omega_1 \to \Omega_2$ is a conformal mapping, then $f(z)$ extends continuously and one-to-one to $\partial\Omega_1$. That is, there is a continuous, one-to-one function $\hat{f}\colon \overline{\Omega}_1 \to \overline{\Omega}_2$ such that $\hat{f}(z)|_{\Omega_1} = f(z)$.

Proof: See [Kra, pp. 111–118].

Boundary extensions are used later in the connection with the conformal modulus of quadrilaterals.

Chapter 4

Conformal modulus of a quadrilateral

The concern of a conformal modulus of a quadrilateral arisen from the studies of quasiconformal mappings, which was introduced by Grötzsch in 1928. Grötzsch showed that there does not exists a conformal mapping from a square onto a rectangle, not a square, which maps vertices onto vertices. The terminology of quasiconformal is due to Ahlfors [Ahl3, pp. 5–7], [AIM, pp. 27–31].

In section 4.1 we give a proof to Theorem 4.1.3 which cannot be found in usually reference books.

4.1 Definitions of a conformal modulus

We call a *conformal modulus* of a quadrilateral in the complex plane a non-negative real number which divides quadrilaterals into conformal equivalence classes. The conformal modulus can be defined in many equivalent ways.

Definition 4.1.1. (Geometric) [Küh]
Let $Q(\Omega; a, b, c, d)$ be a quadrilateral. Let the function $w = f(z)$, where $w = u + iv$, be a one-to-one conformal mapping of the domain Ω onto a rectangle $0 < u < 1, 0 < v < M$ such that the vertices $a, b, c,$and d correspond to the vertices $0, 1, 1+iM$, and iM, respectively. The number M is called the (conformal) *modulus* of the quadrilateral $Q(\Omega; a, b, c, d)$ and we will denote it by $M(Q; a, b, c, d)$.

Example 4.1.2. Let us go back to the example given in Introduction, where we considered a cylindrical capacitor. Consider a rectangle with vertices $0, a, a +$

$2\pi i, 2\pi i$ and the exponential function e^z. The exponential function maps the rectangle onto an annulus, see Figure 4.1. By Definition 4.1.1 the conformal modulus equal $\frac{2\pi}{a}$. The capacitance per unit length is given by

$$\frac{C}{L} = \frac{2\pi\varepsilon}{\ln(R/r)} = \frac{2\pi\varepsilon}{a},$$

which differs from the definition of conformal modulus by a constant factor ε.

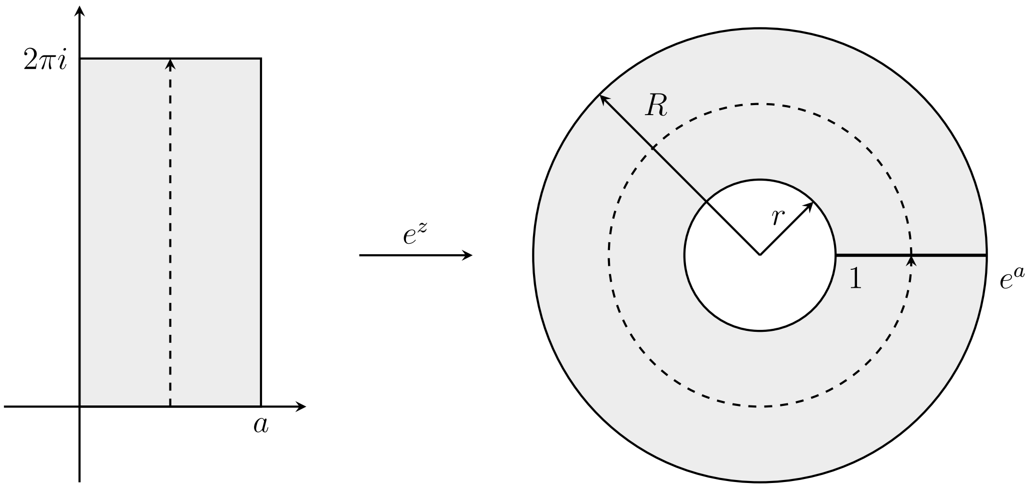


Figure 4.1: Exponential mapping from a rectangle onto a annulus.

Let us consider the following Laplace equation with *Dirichlet–Neumann* boundary conditions on a quadrilateral $Q(\Omega; a, b, c, d)$:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \gamma_2, \\ u = 1, & \text{on } \gamma_4, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases} \tag{4.1}$$

Theorem 4.1.3. (Conformal mapping from Ω onto a rectangle)
Let $Q(\Omega; a, b, c, d)$ be a quadrilateral and let $u(z)$ satisfy the equation (4.1) and let $v(z)$ be a conjugate harmonic function for $u(z)$. Then there exists a conformal mapping $f(z) = u(z) + iv(z)$ that maps Ω onto a rectangle such that the images of the points a, b, c, and d are $1 + iM, iM, 0$, and 1, respectively. The mapping $f(z)$ maps the boundary curves $\gamma_1, \gamma_2, \gamma_3$, and γ_4 onto curves $\gamma_1', \gamma_2', \gamma_3'$, and γ_4', respectively.

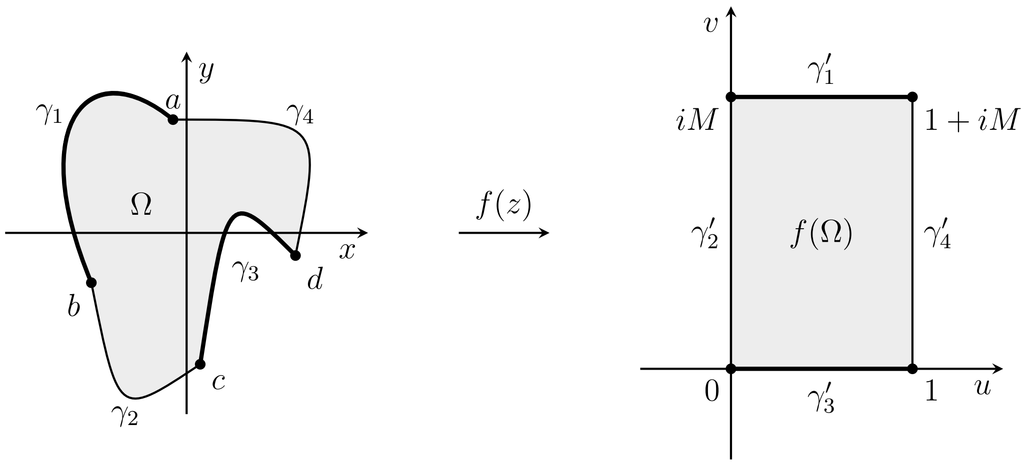


Figure 4.2: Dirichlet–Neumann boundary value problem. Dirichlet and Neumann boundary conditions are mark with thin and thick lines, respectively.

Proof: Riemann Mapping Theorem (Theorem 3.2.1) ensures that there exists a conformal mapping from quadrilateral $Q(\Omega; a, b, c, d)$ onto a rectangle. We have to show that there exists a conformal mapping $f(z)$ which satisfies our Dirichlet–Neumann boundary conditions.

Suppose that $u(x, y)$ is a solution to problem (4.1). Then there exists a conjugate harmonic function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic. So $u(x, y)$ is the real part of $f(z)$. By assuming that $f(z)$ is a mapping from Ω onto a rectangle as sketched in Figure (4.2). Then Dirichlet boundary conditions on γ_2' and γ_4' are readily satisfied. To get the Neumann boundary conditions right, we use the Cauchy-Riemann equations. Since

$$\frac{\partial u}{\partial n}(x, y) = \langle \nabla u(x, y), n(x, y) \rangle = 0$$

on γ_1 and γ_3, where the notation $\langle \cdot\,, \cdot \rangle$ stands for an inner product. This implies that $u_x(x, y) = 0$ and $u_y(x, y) = 0$. Then by the Cauchy-Riemann equations we have $v_y(x, y) = 0$ and $v_x(x, y) = 0$, which imply that $v(x, y)$ is constant on γ_1' and γ_3'. By translation we may assume that $v(x, y) = 0$ on γ_3'.

The discussion of Garnett and Marshall [GM, p. 50] implies that $f(z)$ is conformal. We still have to show that $f(z)$ maps Ω onto the rectangle only once. To prove this, we take a Jordan curve γ in Ω which is sufficiently close to the boundary $\partial\Omega$. Then the image of γ under mapping $f(z)$ is also a Jordan curve, since the image on the boundary is fixed and Ω is Jordan domain and by Theorem 3.2.2

we can extend γ continuously to the boundary of Ω. By applying Theorems 2.2.9 and 2.2.11, we conclude that the winding number of any point in the rectangle respect to γ is $+1$. Furthermore this implies that the winding number of any Jordan curve in Ω equals $+1$. Finally we conclude that $f(z)$ does not have branches. This shows that $f(z)$ does not map Ω onto the rectangle more than once and proves the claim.

$\square$

Theorem 4.1.4. (Dirichlet–Neumann definition) [Ahl2, p. 65]
Let u be the solution for the problem (4.1), then the modulus of the quadrilateral Ω is given by

$$M(Q; a, b, c, d) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}y. \tag{4.2}$$

Proof: By Theorem 4.1.3 there exists a conformal mapping $f(z)$ from Ω onto a rectangle of a width one. This implies that the modulus of the quadrilateral is given by the area of $f(\Omega)$, which is given by an integral

$$\int_{f(\Omega)} 1 \,\mathrm{d}u \,\mathrm{d}v.$$

By changing the variables to the original domain Ω, we need to calculate a determinant of the Jacobian J_f. By the Cauchy-Riemann equations, the Jacobian of f is given by

$$J_f(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \overset{\text{C–S}}{=} \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}.$$

Therefore the determinant of the Jacobian J_f can be written by

$$\det(J_f) = u_x^2 + u_y^2 = |\nabla u|^2.$$

Finally the modulus of the quadrilateral is given by

$$\begin{aligned} M(Q; a, b, c, d) &= \int_{f(\Omega)} 1 \,\mathrm{d}u \,\mathrm{d}v \\ &= \int_\Omega |\det(J_f)| \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_\Omega |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}y. \end{aligned}$$

$\square$

A proof through the modulus of the curve family (Definition 4.1.8) can be found in [Ahl2, pp. 65–70].

Corollary 4.1.5. The modulus of the quadrilateral can also be given by

$$M(Q;a,b,c,d) = \int_{\gamma_4} \frac{\partial u}{\partial n} \mathrm{d}s.$$

Proof: Let us consider the *Green's formula*

$$\int_\Omega \psi \Delta\varphi \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \psi \frac{\partial u}{\partial n}\,\mathrm{d}s - \int_\Omega \nabla\psi \cdot \nabla u \,\mathrm{d}x\,\mathrm{d}y$$

and by setting $\psi = \varphi = u$, we get the following identity

$$\int_\Omega u\Delta u \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\,\mathrm{d}s - \int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{4.3}$$

Since u is the solution to the Laplace problem (4.1), the left-hand side of the identity (4.3) equals 0. Likewise an integral over the boundary $\partial\Omega$ will reduce to an integral

$$\int_{\gamma_4} \frac{\partial u}{\partial n}\mathrm{d}s.$$

This proves the corollary

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_{\gamma_4} \frac{\partial u}{\partial n}\,\mathrm{d}s.$$

□

The third way to define the modulus of a quadrilateral is through the *curve family* Γ, which has influenced the theory of conformal mappings and the more general theory of quasiconformal mappings [Ahl2, p. 50].

Suppose that $\rho(z)$ is a non-negative, real valued, continuous and integrable function in some domain Ω of the complex plane $\mathbb{C}$. We call $\rho(z)$ a metric in Ω and define it by $\rho := \rho(z)|dz|$. Then let us define concepts of ρ-length and ρ-area.

Definition 4.1.6. (ρ-length) [LV, p. 21]
Let Ω is a domain and let γ be a curve in Ω. The integral defined by

$$L_\rho(\gamma) = \int_\gamma \rho(z)\,|\mathrm{d}z|$$

is called the *ρ-length* of the curve γ.

A metric ρ is called *admissible* if $L_\rho(\gamma) \geq 1$.

Definition 4.1.7. (ρ-area) [Ahl2, p. 51]
Suppose that Ω is a domain and suppose that γ be a curve in Ω. Then the *ρ-area* of Ω is defined by an integral

$$A_\rho(\Omega) = \int_\Omega \rho^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Definition 4.1.8. (The modulus of the curve family) [Ahl2, p. 51]
Let Ω be a domain and let Γ be a curve family in Ω. Then the *modulus of the curve family* is given by

$$M(\Omega,\Gamma) = \inf_\rho \frac{A_\rho(\Omega)}{L_\rho^2(\gamma)},$$

where the infimum is taken over all metrics ρ in Ω and ρ is subject to the condition $0 < A_\rho(\Omega) < \infty$.

Suppose that Ω is a rectangle in Definition 4.1.8. Then it can be shown that the modulus of Ω coincide with Definition 4.1.1 [LV, pp. 19–22], [Ahl2, pp. 50–53].

4.2 Properties of a modulus

In computations of the modulus of a quadrilateral we try to exploit as many properties as possible. By the geometry we have following useful properties

$$\begin{cases} M(Q;c,d,a,b) = M(Q;a,b,c,d), \\ M(Q;b,c,d,a) = \dfrac{1}{M(Q;a,b,c,d)}. \end{cases}$$

The latter is called the reciprocal identity. In [HVV] some identities were given for $M(Q;a,b,0,1)$. For the numerical tests we use the following reciprocal identity

$$M(Q;a,b,0,1)\cdot M\left(Q;\frac{b-1}{a-1},\frac{1}{1-a},0,1\right) = 1.$$

Let us consider symmetric quadrilaterals.

Definition 4.2.1. (Symmetric quadrilateral) [Hen2, p. 433]
The quadrilateral $Q(\Omega;a,b,c,d)$ is called *symmetric* if the domain Ω is symmetric with respect to the straight line γ through a and c, and if the points b and d are symmetric with respect to γ.

Theorem 4.2.2. (Modulus of a symmetric quadrilateral) [Hen2, p. 433]
Every symmetric quadrilateral has modulus 1.

Proof: See [Hen2, p. 433].

Chapter 5

Schwarz–Christoffel mapping

After Riemann had stated his mapping theorem, mathematicians started to seek for a way to construct the function given by the Riemann mapping theorem. Soon Christoffel and Schwarz independently discovered the Schwarz–Christoffel mapping in 1867 and 1869 respectively, which provides a conformal mapping of the upper half plane onto a polygon. Besides the Schwarz–Christoffel mapping, there exists many other numerical methods for conformal mappings. For details of these techniques, see [Hen2].

5.1 Schwarz–Christoffel idea

The basic idea behind the Schwarz–Christoffel mapping is that a conformal mapping $f(z)$ may have a derivative which can be expressed by

$$f'(z) = \prod_{k=1}^{n-1} f_k(z) \tag{5.1}$$

for certain canonical functions $f_k(z)$. Geometrically speaking the equation (5.1) means that

$$\operatorname{Arg} f'(z) = \sum_{k=1}^{n-1} \operatorname{Arg} f_k(z).$$

Each function $f_k(z)$ is defined in the way that $\operatorname{Arg} f_k'(z)$ is a step function. So the $\operatorname{Arg} f'(z)$ is a piecewise constant function. Let us analyze the situation more carefully. Suppose that Ω is the interior of a polygon P with vertices $w_1, \cdots, w_n$ given in positive order, and interior angles $\alpha_1\pi, \cdots, \alpha_n\pi$, $\alpha_k \in (0, 2)$ for each k.

Let $f(z)$ be a conformal mapping from the upper half plane onto the polygon P, and let $z_k = f^{-1}(w_k)$ be the kth prevertex [DT, pp. 1–2].

As with all conformal mappings, the main effort is in getting the boundary right. In this case it requires the *Schwarz reflection principle*.

Theorem 5.1.1. (Schwarz reflection principle) [Ahl1, p. 172]
Let Ω^+ be the part in the upper half plane of a symmetric domain Ω, and let γ be the part of the real axis which is contained in Ω. Suppose that $v(x)$ is continuous in $\Omega^+ \cup \gamma$, harmonic in Ω^+, and zero on γ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\overline{z}) = -v(z)$. In the same situation, if $v(z)$ is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\overline{z})}$.

By Theorem 5.1.1 the mapping $f(z)$ can be analytically continued across the segment (z_k, z_{k+1}). In particular, if $f'(z)$ exists on this segment then $\operatorname{Arg} f'(z)$ must be a constant there. At a point $z = z_k$, $\operatorname{Arg} f(z)$ must undergo a specific jump

$$(1 - \alpha_k)\pi = \beta_k \pi, \tag{5.2}$$

which implies that $\operatorname{Arg} f(z)$ is a piecewise constant function. Now we can write function $f_k(z)$ such that it is analytic in the upper half plane, satisfies the equation (5.2), and $\operatorname{Arg} f_k(z)$ is a constant on the real axis:

$$f_k(z) = (z - z_k)^{-\beta_k}.$$

Then the arguments suggest that

$$f'(z) = C \prod_{k=1}^{n-1} f_k(z)$$

for some constant C. And the second derivative can be expressed by

$$f''(z) = C f'(z) \sum_{k=1}^{n-1} \frac{\alpha_k - 1}{z - z_k}.$$

5.2 Map from the upper half plane onto a polygon

The above discussions lead us to the theorem of Schwarz–Christoffel mapping.

Theorem 5.2.1. (Schwarz–Christoffel mapping for a half plane) [DT, p. 10] Let Ω be the interior of a polygon P in the w-plane with vertices $w_1, \cdots, w_n$ and interior angles $\alpha_1\pi, \cdots, \alpha_n\pi$ given in positive order. Let $f(z)$ be any conformal map from the upper half plane onto Ω with $f(\infty) = w_n$. Then the Schwarz–Christoffel representation for the mapping $f(z)$ is given by

$$w = f(z) = A + C\int^z \prod_{k=1}^{n-1}(\zeta - z_k)^{\alpha_k - 1}\mathrm{d}\zeta, \tag{5.3}$$

for some complex constants A and C, where $w_k = f(z_k)$ for $k = 1, \cdots, n-1$.

Proof: For simplicity, let us assume that all prevertices are finite and the product range from 1 to n instead from 1 to $n-1$. By the Schwarz reflection principle (Theorem 5.1.1) the mapping $f(z)$ can be analytically continued into the lower half-plane. The image continues into the reflection of Ω about one side of Ω'. Reflecting again, we can return analytically to the upper half plane. So any even number of reflections of Ω will create a new branch of $f(z)$. And the image of each branch must be a translated and rotated copy of Ω.

Now, let A and C be any complex constants, then

$$\frac{(A + Cf(z))''}{(A + Cf(z))'} = \frac{Cf''(z)}{Cf'(z)} = \frac{f''(z)}{f'(z)}.$$

By continuation, we can define a function $\frac{f''(z)}{f'(z)}$ to be a single-valued analytic function in the closure of the upper half plane, except at the prevertices z_k. Similarly, odd number of reflections lead to a fact that $\frac{f''(z)}{f'(z)}$ is a single-valued analytic function in the lower half-plane. At the prevertex z_k, we have

$$f'(z) = (z - z_k)^{\alpha_k - 1}\psi(z),$$

where $\psi(z)$ is analytic in a neighborhood of z_k. That is, $f(z)$ has a simple pole at z_k with residue $\alpha_k - 1$. Since

$$\frac{f''(z)}{f'(z)} = C\sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k},$$

we have

$$C\sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k} - \sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k} = (C - 1)\sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k}, \tag{5.4}$$

which is an entire function, because all the prevertices are finite and $f(z)$ is analytic at $z = \infty$. Thus the Laurent expansion implies that

$$(C-1)\sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k} \to 0, \text{ as } z \to \infty.$$

Then the expression (5.4) is bounded and by Liouville's Theorem (Theorem 2.2.5) it is identically zero, because

$$\frac{f''(z)}{f'(z)} \to 0, \text{ as } z \to \infty.$$

Then we have

$$\frac{f''(z)}{f'(z)} = \sum_{k=1}^{n}\frac{\alpha_k - 1}{z - z_k}.$$

To obtain the formula (5.3), we integrate twice the function $\frac{\mathrm{d}}{\mathrm{d}z}\log(f'(z)) = \frac{f''(z)}{f'(z)}$.

$$\begin{aligned}
\log(f'(z)) &= \int^{z}\sum_{k=1}^{n}\frac{\alpha_k - 1}{\zeta - z_k}\,\mathrm{d}\zeta + C \\
&= \sum_{k=1}^{n}\int^{z}\frac{\alpha_k - 1}{\zeta - z_k}\,\mathrm{d}\zeta + C \\
&= \sum_{k=1}^{n}(\alpha_k - 1)\log|z - z_k| + C \\
\Rightarrow f'(z) &= \exp\left(\sum_{k=1}^{n}(\alpha_k - 1)\log|z - z_k| + C\right) \\
&= C\prod_{k=1}^{n}(z - z_k)^{\alpha_k - 1} \\
\Rightarrow f(z) &= A + C\int^{z}\prod_{k=1}^{n}(\zeta - z_k)^{\alpha_k - 1}\mathrm{d}\zeta.
\end{aligned}$$

□

5.3 Map from the upper half plane onto the unit disk

An alternative version of the formula (5.3) applies the conformal mapping onto the unit disk.

Theorem 5.3.1. (Schwarz–Christoffel mapping for a disk) [DT, p. 11]
Let Ω be the interior of a polygon P in the w-plane with vertices $w_1, \cdots, w_n$ and interior angles $\alpha_1\pi, \cdots, \alpha_n\pi$ given in positive order. Let $f(z)$ be any conformal map from the unit disk onto Ω. Then the Schwarz–Christoffel mapping $f(z)$ can be given by

$$w = f(z) = A + C\int^{z}\prod_{k=1}^{n}\left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} \mathrm{d}\zeta, \tag{5.5}$$

for some complex constants A and C, where $w_k = f(z_k)$ for $k = 1, \cdots, n$.

The main difference between formulas (5.3) and (5.5) is that the product runs over all n prevertices in the latter case. Otherwise the integrand is in fact a constant multiple of the original form (5.3). Note that the quantities $\left(1 - \frac{\zeta}{z_k}\right)$ lie in the disk $|w - 1| < 1$ for $|z| < 1$. Therefore, choosing a branch of $\log(z)$ with branch cut on the negative real axis, $w = f(z)$ defines an analytic function in the disk $|z| < 1$ and it is continuous on $|z| \leq 1$ with a possible exception at vertices z_k. This will help us to avoid later troubles in numerical computations [DT, p. 12].

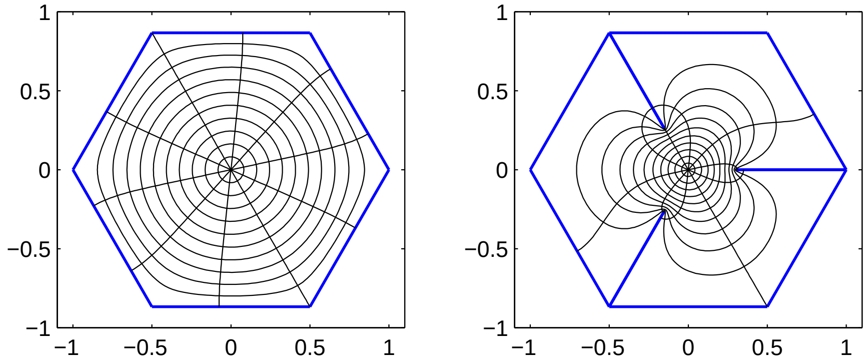


Figure 5.1: Example of mappings of the unit circle onto a regular polygon and a regular polygon with slits.

5.4 Map from the upper half plane onto a rectangle

Let us consider the case where we want to map the upper half plane onto a rectangle. The symmetry of a rectangle allows an explicit solution to the Schwarz–Christoffel mapping. Using symmetries we choose the prevertices as $z_1 = -\frac{1}{k}$,

$z_2 = -1$, $z_3 = 1$ and $z_4 = \frac{1}{k}$, where k, the elliptic modulus, present the degree of freedom in the prevertices. The Schwarz–Christoffel mapping can be expressed by an elliptic integral of the first kind

$$\begin{aligned}
f(z) &= A + C_1 \int^z \prod_{j=1}^{4} \frac{\mathrm{d}\zeta}{\sqrt{\zeta - z_j}} \\
&= C_1 \int_0^z \frac{\mathrm{d}\zeta}{\sqrt{(\zeta^2 - k^{-2})(\zeta^2 - 1)}} \\
&= k^2 C_1 \int_0^z \frac{\mathrm{d}\zeta}{\sqrt{(k^2\zeta^2 - 1)(\zeta^2 - 1)}} \\
&= C \int_0^z \frac{\mathrm{d}\zeta}{\sqrt{(1 - k^2\zeta^2)(1 - \zeta^2)}} \\
&= C \int_0^{\sin\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}} \\
&= CF(k, z).
\end{aligned}$$

By rotating, translating, and scaling the rectangle we get that $w_3 = f(z_3) = F(k, 1)$, which is a complete elliptic integral of the first kind. Furthermore the normalization ensures that the constant C equals 1. By denoting $w_3 = K$ and computing $w_4 = f(z_4) = f\left(\frac{1}{k}\right)$, we have

$$\begin{aligned}
w_4 &= \int_0^{\frac{1}{k}} \frac{\mathrm{d}\zeta}{\sqrt{(1 - k^2\zeta^2)(1 - \zeta^2)}} \\
&\overset{0<k<1}{=} \underbrace{\int_0^1 \frac{\mathrm{d}\zeta}{\sqrt{(1 - k^2\zeta^2)(1 - \zeta^2)}}}_{=K(k)} + \int_1^{\frac{1}{k}} \frac{\mathrm{d}\zeta}{\sqrt{(1 - k^2\zeta^2)(1 - \zeta^2)}}.
\end{aligned}$$

To transform the latter integral

$$\int_1^{\frac{1}{k}} \frac{\mathrm{d}\zeta}{\sqrt{(1 - k^2\zeta^2)(1 - \zeta^2)}}, \tag{5.6}$$

we make the change of variable as follows

$$\eta = \frac{\sqrt{\zeta^2 - 1}}{k'\zeta} \iff \zeta = \frac{1}{\sqrt{1 - k'^2\eta^2}}.$$

Hence

$$\mathrm{d}\zeta = \frac{k'^2 \eta \,\mathrm{d}\eta}{(1 - k'^2\eta^2)^{\frac{3}{2}}},$$

where k' is the complementary elliptic modulus given by $k' = \sqrt{1-k^2}$. Then new integral boundaries are

$$\begin{cases} \zeta = 1 \Rightarrow \eta = 0, \\ \zeta = \dfrac{1}{k} \Rightarrow \eta = 1. \end{cases}$$

The integrand (5.6) can be written as a product of the following two factors

$$\begin{aligned}
\frac{1}{\sqrt{1-\zeta^2}} &= \frac{1}{\sqrt{1-\frac{1}{1-k'^2\eta^2}}} \\
&= \frac{1}{\sqrt{-\frac{k'^2\eta^2}{1-k'^2\eta^2}}} \\
&= \frac{i\sqrt{1-k'^2\eta^2}}{k'\eta}, \\
\frac{1}{\sqrt{1-k^2\zeta^2}} &= \frac{1}{\sqrt{1-k^2\frac{1}{1-k'^2\eta^2}}} \\
&= \frac{1}{\sqrt{\frac{1-k'^2\eta^2-(1-k'^2)}{1-k'^2\eta^2}}} \\
&= \frac{\sqrt{1-k'^2\eta^2}}{k'\sqrt{1-\eta^2}}.
\end{aligned}$$

So the integral (5.6) can be written by

$$\begin{aligned}
\int_1^{\frac{1}{k}} \frac{\mathrm{d}\zeta}{\sqrt{(1-k^2\zeta^2)(1-\zeta^2)}} &= \int_0^1 \frac{\sqrt{1-k'^2\eta^2}}{k'\sqrt{1-\eta^2}} \frac{i\sqrt{1-k'^2\eta^2}}{k'\eta} \frac{k'^2\eta}{(1-k'^2\eta^2)^{\frac{3}{2}}} \mathrm{d}\eta \\
&= i\int_0^1 \frac{\mathrm{d}\eta}{\sqrt{(1-k'^2\eta^2)(1-\eta^2)}} \\
&= iK(k') \\
&= iK'(k).
\end{aligned}$$

That is, the prevertice $z_4 = \frac{1}{k}$ is mapped onto the point $w_4 = K(k) + iK'(k)$. By the symmetry of a rectangle, the prevertices $z_1 = -\frac{1}{k}$ and $z_2 = -1$ are mapped onto points $w_1 = -K(k) + iK'(k)$ and $w_2 = -K(k)$, respectively. This also implies that the modulus of a quadrilateral can be given by

$$M(Q; w_1, w_2, w_3, w_4) = \frac{K'(k)}{2K(k)}.$$

Figure 5.2 illustrates the correspondence of vertices under a conformal map of a polygonal domain onto a rectangle.

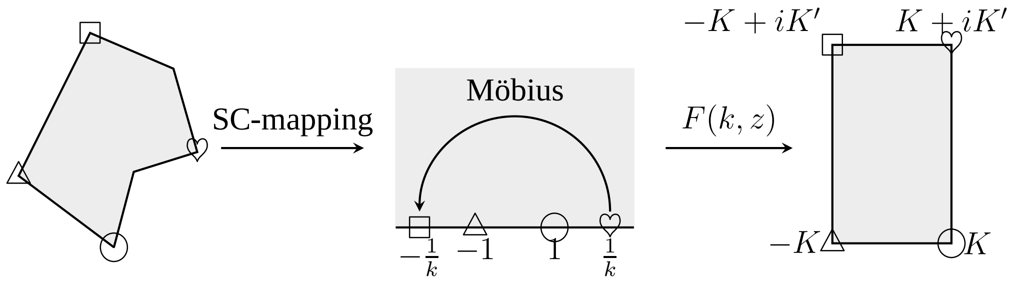


Figure 5.2: Illustration, how to conformally map a polygonal domain onto a rectangle.

Chapter 6

Finite element methods

There are two popular ways to approximate the solution of a partial differential equation (PDE), namely the finite difference method (FDM) and the finite element method (FEM). The former dominated the early development of numerical analysis. In finite difference methods an approximation to the solution is obtained by finite mesh of points where derivatives of the differential equation are replaced by appropriate difference quotients. This procedure reduces the problem to a finite linear system [LT, p. 43].

Finite element methods were introduced in 1960s and are probably the most used method in engineering. Finite element methods are based on a variational form of partial differential equations and involve an approximation of the exact solution by piecewise polynomial functions. This makes finite element methods to be more easily adapted to the underlying domain than finite difference methods. For symmetric positive definite elliptic equations such as Laplace equation, the problem reduces to a finite linear system with a positive definite matrix [LT, p. 51].

The rest of the chapter is devoted to the analysis of finite element methods since available numerical packages are revolved around it, even though finite difference methods could be used as well.

6.1 Variational formulation of Laplace equation

Let us consider the following Dirichlet boundary value problem of Laplace equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

By assuming $u \in C^2(\Omega)$, we multiply the above equation by a test function $\psi \in C_0^1(\Omega)$ and integrate over the domain Ω. Then by applying Green's formula

$$\int_\Omega \psi \Delta\varphi \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial n}\,\mathrm{d}s - \int_\Omega \nabla\psi\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y$$

we have for the left-hand side

$$\int_\Omega \Delta u\psi\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} \psi\frac{\nabla u}{\partial n}\,\mathrm{d}s - \int_\Omega \nabla u\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y, \quad \forall\psi \in C_0^1(\Omega).$$

Recall that $\psi \in C_0^1(\Omega)$ and we get

$$-\int_\Omega \Delta u\psi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla u\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f\psi\,\mathrm{d}x\,\mathrm{d}y, \quad \forall\psi \in C_0^1(\Omega)$$

Denote

$$\begin{cases} a(u,\psi) = \displaystyle\int_\Omega \nabla u\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}y, \\ \ (f,\psi) = \displaystyle\int_\Omega f\psi\,\mathrm{d}x\,\mathrm{d}y. \end{cases}$$

Since C_0^1 is dense in H_0^1, it follows

$$a(u,\psi) = (f,\psi), \quad \forall\psi \in H_0^1(\Omega). \tag{6.2}$$

The variational problem corresponding to (6.1) is to find $u \in H_0^1(\Omega)$ such that (6.2) holds and such a solution u is called a *weak* or a *variational solution* of (6.1). Thus a *classical* solution of (6.1) is also a weak solution. On the other hand, by assuming that $u \in H_0^1(\Omega)$ and $u \in C^2(\Omega)$, then it follows that u is a classical solution of (6.1) as well.

For a non-zero Dirichlet boundary value problem we have to define a concept of a *trace operator*. By a trace we means a way to assigning boundary values along $\partial\Omega$ to a function $u \in H^1(\Omega)$, assuming that $\partial\Omega \in C^1$. Obviously if $u \in C^1(\overline{\Omega})$ then u has boundary values in a natural sense. The problem arises when $u \in H^1(\Omega)$ and therefore it is not generally continuous, and even worse it is only defined almost everywhere in Ω [Eva, pp 257–259].

Theorem 6.1.1. (Trace theorem) [Eva, p. 258]
Assume Ω is bounded and $\partial\Omega \in C^1$. Then there exists a trace operator

$$Tr\colon H^1 \to L^2(\partial\Omega)$$

such that

$$\|Tr(u)\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

where the constant C depends only on Ω. We call $Tr(u)$ the trace of u on $\partial\Omega$.

Now consider a non-zero Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = g, & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Suppose that $\partial\Omega \in C^1$ and $u \in H^1(\Omega)$ is a weak solution of (6.3). This implies that $u = g$ on $\partial\Omega$ in the trace sense which means that g has to be the trace of some H^1 function, say h. Then $\tilde{u} = u - h$ belongs to $H_0^1(\Omega)$, and it is a weak solution of the boundary value problem

$$\begin{cases} -\Delta \tilde{u} = \tilde{f}, & \text{in } \Omega, \\ \quad \tilde{u} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\tilde{f} = f + \Delta h$.

For the Neumann boundary value problem of Laplace equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

we use the same argument as above to obtain the variational formulation corresponding to (6.4). Because $\frac{\partial u}{\partial n} = 0$, we take $\psi \in H^1(\Omega)$ instead of taking $\psi \in H_0^1(\Omega)$. Therefore we have

$$a(u,\psi) = (f,\psi), \quad \forall \psi \in H^1(\Omega). \tag{6.5}$$

The variational problem corresponding to (6.4) is to find $u \in H^1(\Omega)$ such that (6.5) holds.

Finally considering the Laplace equation with the Dirichlet–Neumann boundary value problem (4.1)

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \gamma_2, \\ \quad u = 1, & \text{on } \gamma_4, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \gamma_1 \cup \gamma_3. \end{cases}$$

By above arguments we obtain the following variational formulation:

$$a(u, \psi) = 0, \quad \forall \psi \in H_0^1(\Omega).$$

The above discussions can be stated in more general manner. See [Eva, pp. 293–297] for more details.

6.2 Finite element mesh

The first task in finite element methods is to define a finite element to work with. We have to give some properties that the finite element have to satisfy in order to be useful. After defining a finite element, we need a way to connect finite elements together which requires more from the elements.

Definition 6.2.1. (Finite element) [SSD, p. 1]
A *finite element* is a triad $\mathcal{K} = (K, P, \Sigma)$, where

- K is a domain in $\mathbb{R}^2$. It can be either a triangle or a quadrilateral.
- P is a space of polynomials on K of the dimension $\dim(P) = N_P$.
- $\Sigma = \{L_1, L_2, \cdots, L_{N_P}\}$ is a set of linear forms

$$L_i \colon P \to \mathbb{R}, \quad i = 1, 2, \cdots, N_P.$$

The elements of Σ are called *degrees of freedom*.

Definition 6.2.2. (Unisolvency of finite element) [SSD, p. 2]
The finite element $\mathcal{K} = (K, P, \Sigma)$ is said to be *unisolvent* if for every function $p \in P$ it holds that

$$L_1(g) = L_2(g) = \cdots = L_{N_P}(g) = 0 \Rightarrow g = 0.$$

In other words, every vector of numbers

$$L(g) = (L_1(g), L_2(g), \cdots, L_{N_P}(g))^{\mathsf{T}} \in \mathbb{R}^{N_P}$$

identifies a unique polynomial g in the space P.

Unisolvency is a way to express a compatibility of the set of degrees of freedom in Σ with the polynomial space P.

Theorem 6.2.3. (Characterization of unisolvency) [SSD, p. 3]
Consider a finite element $\mathcal{K} = (K, P, \Sigma)$, $\dim(P) = N_P$. The finite element $\mathcal{K}$ is unisolvent if and only if there exists a unique basis $\mathcal{B} = \{\theta_1, \theta_2, \cdots, \theta_{N_P}\} \subset P$ satisfying a following δ*–property*

$$L_i(\theta_j) = \delta_{ij}, \quad 1 \le i, j \le N_P,$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Proof:
"$\Rightarrow$" Let $\{g_1, g_2, \cdots, g_{N_P}\} \subset P$ an arbitrary basis and we express sought functions θ_j, $j = 1, \cdots, N_P$, by

$$\theta_j = \sum_{k=1}^{N_P} a_{kj} g_k.$$

For the δ-property, we require that

$$L_i(\theta_j) = L_i\left(\sum_{k=1}^{N_P} a_{kj} g_k\right) = \sum_{k=1}^{N_P} a_{kj} L_i(g_k) = \delta_{ij}, \quad 1 \le i, j \le N_P.$$

This yields a linear system of N_P variables and can be given by matrix equation as

$$LA = I,$$

where column of matrix A contains the coefficients of the functions $\theta_1, \theta_2, \cdots, \theta_{N_P}$. Since the element $\mathcal{K}$ is unisolvent it follows that L is invertible and that $\theta_1, \theta_2, \cdots, \theta_{N_P}$ are uniquely determined. Next we show that $\theta_1, \theta_2, \cdots, \theta_{N_P}$ are linearly independent, that is

$$\sum_{k=1}^{N_P} \beta_k \theta_k = 0 \quad \Rightarrow \quad \beta_1 = \beta_2 = \cdots = \beta_{N_P} = 0.$$

By evaluating the functions L_i with the above linear combinations we get

$$0 = L_i\left(\sum_{k=1}^{N_P} \beta_k \theta_k\right) = \sum_{k=1}^{N_P} \beta_k L_i(\theta_k) = \beta_i, \quad \forall\, i = 1, 2, \cdots, N_P.$$

"$\Leftarrow$" Let $\mathcal{B} = \{\theta_1, \theta_2, \cdots, \theta_{N_P}\}$ a basis of the space P satisfying the δ-property. Then every function $p \in P$ can be expressed by

$$p = \sum_{k=1}^{N_P} \beta_k \theta_k.$$

Assuming that

$$L_1(g) = L_2(g) = \cdots = L_{N_P}(g) = 0,$$

immediately we have

$$0 = L_i(g) = L_i\left(\sum_{k=1}^{N_P} \beta_k \theta_k\right) = \beta_i, \quad \forall\, i = 1, 2, \cdots, N_P.$$

Hence it follows that $g = 0$ and the finite element $\mathcal{K}$ is unisolvent [SSD, pp. 3–4].

$\square$

The proof of Theorem 6.2.3 gives us a convenient way to check the unisolvency of the finite element $\mathcal{K}$ in form of invertibility of the matrix L. Here we have discussed about the unisolvent of a single finite element $\mathcal{K}$. Next we will address the compatibility of finite elements of function spaces which are used for an approximation. We called this compatibility by *conformity* of finite elements to function spaces.

Suppose that a domain Ω is bounded with the boundary $\partial\Omega \in C^1$, where the underlying partial differential equation is considered. Then Ω is approximated by a computational domain Ω_h whose boundary is a piecewise polynomial.

Definition 6.2.4. (Finite element mesh) [SSD, p. 7]
A *finite element mesh* $\mathcal{T}_{h,p} = \{K_1, K_2, \cdots, K_M\}$ over a domain Ω_h with a piecewise polynomial boundary is a geometrical partition of Ω_h into a finite number of non-overlapping open polygonal K_i such that

$$\Omega_h = \bigcup_{i=1}^{M} \overline{K}_i,$$

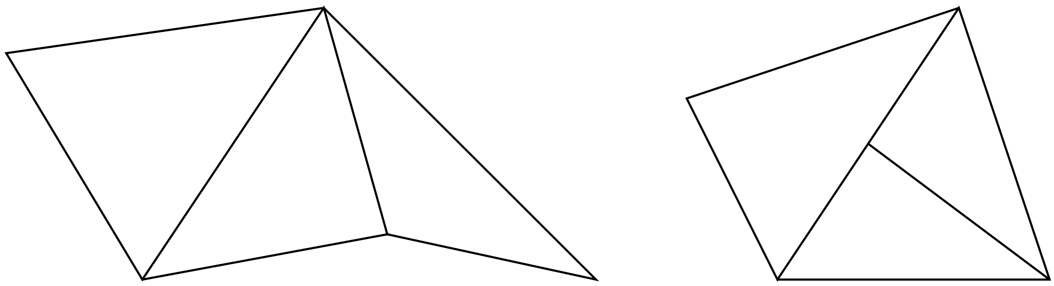
and each K_i, $1 \leq i \leq M$, is equipped with a polynomial order $p(K_i) = p_i \geq 1$.

A finite element mesh is called *regular* if for any two elements K_i and K_j, $i \neq j$, only one of the following statements hold

- $\overline{K}_i$ and $\overline{K}_j$ are disjoint,

- $\overline{K}_i$ and $\overline{K}_j$ have only one common vertex,
- $\overline{K}_i$ and $\overline{K}_j$ have only one common edge.

By a regular mesh we avoid *hanging nodes* which complicate the discretization process. A mesh with hanging nodes is called *irregular*.



Regular mesh Irregular mesh

Figure 6.1: Finite element meshes, on the left a regular and on the right an irregular mesh with one hanging node.

Definition 6.2.5. (Interpolant) [SSD, p. 9]
Given a unisolvent finite element (K, P, Σ), let $\mathcal{B} = \{\theta_1, \cdots, \theta_{N_P}\}$ be the unique basis of P satisfying the δ-property. Let $v \in V$, where $P \subset V$, be a function for which all $L_1, \cdots, L_{N_P}$ are defined. Then we define a local *interpolant* by

$$\mathcal{I}_K(v) = \sum_{i=1}^{N_P} L_i(v)\theta_i.$$

The global interpolant $\mathcal{I}$ over a finite mesh $\mathcal{T}_{h,p}$ is defined by means of local interpolants by

$$\mathcal{I}(v)|_{K_i} \equiv \mathcal{I}_{K_i}(v), \quad i = 1, \cdots, M.$$

Definition 6.2.6. (Conformity of finite elements) [SSD, p.10]
Let $\mathcal{T}_{h,p}$ be a finite element mesh consisting M unisolvent finite elements (K_i, P_i, Σ_i), $i = 1, 2, \cdots, M$. Let $V(\Omega_h)$ be H^1 and let $\mathcal{I}_{K_i} \colon V(K_i) \to P_i$ be the local finite element interpolation operator. We say $\mathcal{T}_{h,p}$ is conforming to the space H^1 if for each common edge of elements K_i and K_j, $K_i, K_j \in \mathcal{T}_{h,p}$ the trace of $v|_{K_i}$ equals to the trace of $v|_{K_j}$ on the edge.

In higher order finite elements methods we have lots of overlapping informations on each element and an effective management is needed. Therefore the usage of the reference element K_{ref} is encouraged. The task is to find a *reference mapping* $F_{K_i}\colon K_{ref} \to K_i$ that is smooth and bijective. The polynomial space with the reference map F_{K_i} is used for define the function space $V_{h,p}(\Omega_h)$, where the finite element solution is sought [SSD, pp. 16–17].

6.3 Shape functions

Shape functions are functions used for approximating the solution to partial differential equations. There are numerous ways to select shape functions but we will be using *Legendre's polynomials*, since they possess many useful properties. Though there are many ways to define Legendre's polynomials, the definition through a recursive formula is the most useful way to implement the higher order shape functions.

Definition 6.3.1. (Legendre's polynomial) [Leb, p. 46]
Legendre's polynomials of degree n can be defined recursively by

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, .$$

where $P_0(x) = 1$ and $P_1(x) = x$.

First few Legendre's polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}, \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x. \end{aligned}$$

The derivatives of Legendre's polynomials can also be given by a recursive formula as follow

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x), \quad n \geq 1.$$

By manipulating the derivatives further we have

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0, \quad n \geq 0. \tag{6.6}$$

One of the most important properties of Legendre's polynomials is the orthogonality in the interval $[-1,1]$. This can be shown from the differential equation (6.6). First we multiply the nth differential equation by $P_m(x)$ and subtract it from the mth differential equation multiplied by $P_n(x)$, which leads to

$$\begin{aligned} 0 &= \left[(1-x^2)P_m'(x)\right]' P_n(x) + m(m+1)P_m(x)P_n(x) \\ &\quad - \left[(1-x^2)P_n'(x)\right]' P_m(x) - n(n+1)P_n(x)P_m(x). \end{aligned} \tag{6.7}$$

By the identity

$$\left[(1-x^2)P_m'(x)P_n(x)\right]' = \left[(1-x^2)P_m'(x)\right]' + (1-x^2)P_m'(x)P_n'(x),$$

and the symmetry between indices m and n, we rewrite the equation (6.7) in form

$$\left[(1-x^2)\left(P_m'(x)P_n(x) - P_n'(x)P_m(x)\right)\right]' + (m-n)(m+n+1)P_m(x)P_n(x) = 0.$$

Then by integrating over the interval $[-1,1]$, the first term vanishes because $(1-x^2)$ vanishes and we have

$$\int_{-1}^{1} (m-n)(m+n+1)P_m(x)P_n(x)\,\mathrm{d}x = 0,$$

which yields the orthogonality of Legendre's polynomials [Leb, pp. 47–50]. It can also be shown that the norm is given by

$$\|P_n\|_2 = \left(\int_{-1}^{1} P_n^2(x)\,\mathrm{d}x\right)^{1/2} = \sqrt{\frac{2}{2n+1}}.$$

Let us define functions ϕ_j as the integrated Legendre's polynomials. Then we have

$$\begin{aligned} \phi_0(\xi) &= \frac{1-x}{2}, \\ \phi_1(\xi) &= \frac{x-1}{2}, \\ \phi_j(\xi) &= \frac{1}{\|P_{j-1}\|_2} \int_{-1}^{\xi} P_{j-1}(t)\,\mathrm{d}t \\ &= \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t)\,\mathrm{d}t, \quad j \geq 2, \end{aligned}$$

and we can rewrite them by mean of Legendre's polynomials

$$\phi_j(\xi) = \frac{1}{\sqrt{2(2j-1)}} \left[P_j(\xi) - P_{j-2}(\xi)\right], \quad j \geq 2.$$

Note that $\phi_j(-1) = 0$, $k \geq 1$, and they are orthogonal

$$\int_{-1}^{1} \frac{\mathrm{d}\phi_i(\xi)}{\mathrm{d}\xi} \frac{\mathrm{d}\phi_j(\xi)}{\mathrm{d}\xi} \,\mathrm{d}\xi = \delta_{ij}, \quad i, j \geq 2.$$

The integrated Legendre's polynomials form a shape function basis to be used in finite elements methods. In particular, they play an essential role in a construction of a hierarchic basis [SSD, p. 25], [SB, pp. 38–39].

For other definitions and for more details regarding Legendre's polynomials see [Leb, Sze].

Definition 6.3.2. (Hierarchic basis) [SB, p. 96]
Suppose that the basis $\mathcal{B}^p$ consist of polynomials of order p. Then $\mathcal{B}^p$ is said to be *hierarchic* if $\mathcal{B}^p \subset \mathcal{B}^{p+1}$.

In case of quadrilateral elements, we use $[-1, 1] \times [-1, 1]$ as the reference element. In order to satisfy the conformity requirements of the finite element mesh, we divide shape functions into three categories: *nodal shape functions*, *side modes*, and *internal modes*. We follow a presentation of Szabó and Babuška [SB, pp. 98–100] and present the shape functions as follows.

1. **Nodal shape functions.** Nodal shape functions are defined so that they get value one at only one vertex and vanish on the other vertices. There are total of 4 nodal shape functions and they are defined by

$$\begin{aligned} N_1(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 - \eta), \\ N_2(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 - \eta), \\ N_3(\xi, \eta) &= \frac{1}{4}(1 + \xi)(1 + \eta), \\ N_4(\xi, \eta) &= \frac{1}{4}(1 - \xi)(1 + \eta). \end{aligned}$$

2. **Side Nodes.** There are $4(p-1)$ side modes and they associate with only one edge of the finite element and vanish on the other edges. For side modes we

have ($p \geq 2$)

$$\begin{aligned}
N_i^1(\xi,\eta) &= \frac{1}{2}(1-\eta)\phi_i(\xi), \quad i = 2,3,\cdots,p, \\
N_i^2(\xi,\eta) &= \frac{1}{2}(1+\xi)\phi_i(\eta), \quad i = 2,3,\cdots,p, \\
N_i^3(\xi,\eta) &= \frac{1}{2}(1+\eta)\phi_i(\xi), \quad i = 2,3,\cdots,p, \\
N_i^4(\xi,\eta) &= \frac{1}{2}(1-\xi)\phi_i(\eta), \quad i = 2,3,\cdots,p,
\end{aligned}$$

where the superscript determines the edge of the finite element which the side mode function associate with.

3. **Internal modes.** For internal modes we have two different options to choose from. The *trunk* space which has $\frac{(p-2)(p-3)}{2}$ shape functions and they are defined by

$$N_{ij}(\xi,\eta) = \phi_i(\xi)\phi_j(\eta), \quad i,j \geq 2, \quad 4 \leq i+j \leq p.$$

On the other hand the *full* space has $(p-1)(p-1)$ shape functions, which are defined by

$$N_{ij}(\xi,\eta) = \phi_i(\xi)\phi_j(\eta), \quad i,j = 2,3,\cdots,p.$$

Note that internal modes vanish on the boundary of the reference element. This is the reason why they are sometimes referred to as *bubble functions*.

See Appendix A for an illustration of some of the shape functions.

6.4 Higher-order finite element methods

There are three different kinds of higher order finite element methods, namely h-, p-, and hp-version. The h-version of finite element methods is considered as the most popular version. In h-version the degree of freedom of elements are fixed by fixing the order of the polynomial space and a convergence is obtained by refining the mesh where errors are large. This can be done by computing the estimated error for each element. On the other hand, p-version of the finite element methods uses fixed elements and the convergence is obtained by increasing the degree of elements that is the order of the polynomial space. Lastly the hp-version simultaneously refines the mesh and increases the degree of elements. For more detailed discussion of the error analysis of higher order finite element methods, see series of papers by Gui and Babuška [GB1, GB2, GB3].

Chapter 7

Numerics of the modulus of a quadrilateral

There are two natural approaches to compute the modulus of a quadrilateral. Through the definition of the modulus and use of the conformal mapping from given domain onto a rectangle and methods that will only give the modulus. The former methods give the conformal mappings as well and usually involves solving a parameter problem for the Schwarz–Christoffel mapping. While the latter gives only the modulus, which we are interested in. Also the latter methods usually depends on solving the Dirichlet–Neumann boundary value problem for the Laplace equation.

Methods for numerical computation of the Schwarz–Christoffel mapping date back to around 1960. In 1980 Trefethen [Tre] introduced the *side-length* method based on works by Relly and others. Driscoll and Vavasis [DrVa] proposed in 1998 an algorithm *CRDT* (cross-ratios of the Delaunay triangulation) to overcome crowding [DT, p. 23]. Recently Banjai [Ban] gave modifications to Trefethen's algorithm to improve the accuracy of the computation of some elongated domains, which are the main cause of the crowding phenomenon.

Finite element methods can be applied to solve the Dirichlet–Neumann boundary value problem for the Laplace equation (4.1). In 2004 Samuelsson [BSV] described an AFEM (adaptive finite element method) software package based on an h-version finite element method. Recently Hakula [HRV] introduced an algorithm to an hp-version of finite element method for Mathematica. Both of the above methods can be applied to compute the modulus of a quadrilateral.

For side-length and CRDT methods a numerical integration is needed. Usual numerical integration methods give poor results since the Schwarz–Christoffel integral is usually singular at prevertices. To compute the integral, we use a *Gauss–Jacobi* quadrature, which is too lengthy to be described here. See [DT, pp. 28–30] for more details about Gauss–Jacobi quadrature.

The rest of the chapter, we discuss above the methods in more details and discuss the crowding phenomenon.

7.1 Side-length

To compute the Schwarz–Christoffel mapping using the side-length method, we must solve a parameter problem, which is in general non-linear. For Schwarz–Christoffel mappings we have three degrees of freedom. By choosing prevertices z_{n-2}, z_{n-1} and z_n from the boundary of the domain, we are left with $n-3$ quantities to be determinate by the following system of equations

$$\begin{cases} \dfrac{\left| \displaystyle\int_{z_j}^{z_{j+1}} f'(\zeta)\,\mathrm{d}\zeta \right|}{\left| \displaystyle\int_{z_1}^{z_2} f'(\zeta)\,\mathrm{d}\zeta \right|} = \dfrac{|w_{j+1}-w_j|}{|w_2-w_1|}, \qquad j = 2,3,\cdots,n-2, \\ \dfrac{\displaystyle\int_{z_{J-1}}^{z_{J+1}} f'(\zeta)\,\mathrm{d}\zeta}{\left| \displaystyle\int_{z_1}^{z_2} f'(\zeta)\,\mathrm{d}\zeta \right|} = \dfrac{w_{J+1}-w_{J-1}}{|w_2-w_1|}, \quad \text{if } w_J = \infty \text{ for } J < n, \end{cases} \tag{7.1}$$

where $f'(z)$ can be obtained from (5.3). In case of the unit circle, using the formula (5.5) instead of (5.3) is a better option. In addition, we must require that no two infinite vertices are adjacent. This can be achieved by introducing a degenerated vertex with the interior angle π on the straight line between infinite adjacent vertices [DT, pp. 23-25].

For details how this can be done in case of the unit circle see [DT, pp. 25-27] or the original paper by Trefethen [Tre].

7.2 CRDT

The CRDT algorithm has several phases in order to construct the Schwarz–Christoffel mapping. First we have to triangulate the given simple polygon P by using one kind of Delaunay triangulation process. Then we split the edges using edge splitting algorithm. Finally we use cross-ratios and solve the emerging parameter problem to determine the prevertices of P.

The Delaunay triangulation

A triangulation of P is a partition of P into non-degenerate triangles, whose vertices are vertices of P. If triangles intersect, they must intersect on a vertex or on an entire edge. It can be proved by induction, that a triangulation of P consists exactly of $n-2$ triangles and $n-3$ diagonals, which are edges of triangles that are not also the edges of P. Furthermore, if d is a diagonal, let $Q(d)$ be the quadrilateral, the union of the two triangles on either side of d. Remark that the triangulation of P is not unique.

Splitting edges

To avoid quadrilaterals that are long and narrow, we split the edges so that the quadrilaterals in the Delaunay triangulation are well conditioned. By ”well conditioned” we mean that the prevertices of the quadrilaterals are not too crowded.

The splitting procedure has two phases. In the first phase, we are looking for vertices w with an interior angle less than or equal to $\frac{\pi}{4}$. For every such a prevertices w, find a largest isosceles triangle T that can be formed by w with its adjacent edges such that T is contained in P. Next we introduced new vertices at midpoints of the two sides of T that are the edges adjacent to w and we call these adjacent edges to w as *protected*. That is, we do not allow them to be split during the second phase. Let P' denote the polygon obtained after the first phase of the splitting procedure.

On the second phase we iteratively split the polygon P' into partitions. Let e be an unprotected edge of some polygon during the splitting procedure. Let $l(e)$ be its length and let $d(e)$ be the smallest distance from e to any vertex except the endpoints of e. The distance is measured along the shortest piecewise linear path that remains inside the polygon. The edge e is said to be ill separated if

$$d(e) < \frac{l(e)}{3\sqrt{2}}.$$

Then the ill conditioned edges are split into three equal parts. It can be proved that the second phase of the splitting procedure will end in a finite number of steps.

Cross-ratios

Like side-length method, CRDT also have $n-3$ variable to be determined. These variables are determined using cross ratios of prevertices. Let $d_1, \cdots, d_{n-3}$ be diagonals and let $Q_1 = Q(d_1), \cdots, Q_{n-3}$ be quadrilaterals of the Delaunay triangulation of P. Denote the vertices of Q_i by $w_{\kappa(i,1)}, w_{\kappa(i,2)}, w_{\kappa(i,3)}, w_{\kappa(i,4)}$ for each $i = 1, \cdots, n-3$, where $\kappa(i,1), \kappa(i,2), \kappa(i,3), \kappa(i,4)$ is distinct indices in $\{1, \cdots, n\}$.

For a given list of prevertices $z_1, \cdots, z_n$, we compute variables σ_i, which are defined by

$$\sigma_i = \ln(-(z_{\kappa(i,1)}, z_{\kappa(i,2)}, z_{\kappa(i,3)}, z_{\kappa(i,4)})), \quad i = 1, \cdots, n-3. \tag{7.2}$$

Notice that there are n real variables and only $n-3$ real constraints. By choosing a Delaunay triangular T_0 in the way that its prevertices are arbitrarily placed on the unit circle in a manner of preserving the order. Then there exists a unique way to determine the remaining $n-3$ z_i's on the unit circle satisfying (7.2). It turns out that the choice for T_0 does not matter, we will end up with the same polygon, up to a similarity transformation [DrVa].

7.3 Adaptive finite element methods

AFEM

In order the *a posteriori* estimate to work well, AFEM triangulate the domain to have certain properties. For example, the lower bound of the smallest angle of the triangulation must be fixed. The refined elements must have these properties as well. For detailed properties and algorithms, see [BSV].

***hp*-FEM**

In hp-FEM, the mesh of a polygonal domain is generated in two phase algorithm:

1. Generate a minimal mesh where the vertices are isolated with a fixed number of triangle depending on the interior angle α:

   - $\alpha \le \frac{\pi}{2}$: one triangle,
   - $\frac{\pi}{2} \le \alpha < \pi$: two triangles,
   - $\alpha \ge \pi$: three triangles.

2. Every triangle attached to vertices is replaced by refined triangles where the edges adjacent to the vertex are split as specified by the scaling factor r. This process is repeated recursively until the the nesting level ν is reached [HRV].

The refined mesh is referred as (r,ν)-mesh. An example is shown in Figure 7.1. The actual numerical computation for the solution of (4.1) is done as described in Chapter 6.

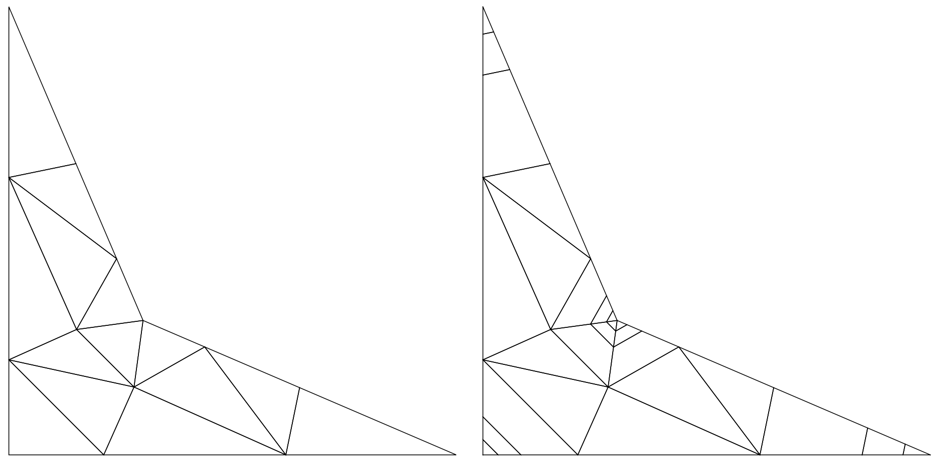


Figure 7.1: A quadrilateral $Q\left(0,1,\frac{3}{10}+\frac{3}{10}i,i\right)$ with the initial mesh and the $(0.4,2)$-mesh on the left- and right-hand side, respectively.

7.4 Heikkala–Vamanamurthy–Vuorinen iteration

Heikkala, Vamanamurthy, and Vuorinen [HVV] proposed an iteration which can be used for compute the modulus of a quadrilateral. The iteration consists of evaluations of hypergeometric functions, beta functions, and elliptic integrals.

Definition 7.4.1. (Gaussian hypergeometric function) [HVV, p. 1]
Given complex numbers a, b,and c with $c \neq 0, -1, -2, \cdots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C}\backslash[1,\infty)$ of the series

$$F(a,b;c;z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z| < 1.$$

Here $(a,0)=1$ for $a\neq 0$, and (a,n) is the *shifted factorial function* or the *Appell symbol*

$$(a,n)=a(a+1)(a+2)\cdots(a+n-1)$$

for $n\in\mathbb{N}$.

Definition 7.4.2. (Beta function) [HVV, p. 10]
Let $\mathrm{Re}(x)>0$ and $\mathrm{Re}(y)>0$. Then the *beta function* is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

Theorem 7.4.3. Let $0<a,b<1$, $\max\{a+b,1\}\leq c\leq 1+\min\{a,b\}$. Suppose that Q is a quadrilateral in the upper half plane with vertices $0,1,A$, and B, the interior angles at which are, respectively, $b\pi$, $(c-b)\pi$, $(1-a)\pi$, and $(1+a-c)\pi$. Then the modulus of Q is given by

$$M(Q;0,1,A,B)=\frac{K'(k)}{K(k)},$$

where the elliptic modulus k satisfies the equation

$$A-1=\frac{Lk'^{2(c-a-b)}F(c-a,c-b;c+1-a-b;k'^2)}{F(a,b;c;r^2)},$$

and

$$L=\frac{B(c-b,1-a)}{B(b,c-b)}e^{(b+1-c)i\pi}.$$

Proof: See [HVV, p. 7].

Note that the quadrilateral Q in Theorem 7.4.3 is convex. There is a slight chance that the iteration works for a non-convex quadrilateral as well.

7.5 Crowding

The biggest obstacle for numerical methods of conformal mappings is a so called *crowding*. Crowding is a form of ill-conditioning which is present in virtually all numerical methods of conformal mappings. Crowding occurs when two prevertices are too close to each other. It might be numerically impossible to distinguish the two prevertices from each other, because of the limited accuracy of the numerical floating point arithmetics [DT, pp. 20–21].

The situation can be illustrated by a long and thin rectangles. Consider a map $f(z)$ that maps a quadrilateral $Q(\Omega, 1+iM, iM, 0, 1)$ onto the unit disk $\mathbb{D}$ such that $f(1+iM) = -f(0)$ and $f(iM) = -f(1)$. Then the minimal distance of the image points $f(1+iM)$, $f(iM)$, $f(0)$, and $f(1)$ is less than $3.4 \cdot 10^{-16}$ for $M = \frac{1}{24}$ [HRV], [Pap, pp. 131–132].

Chapter 8

Numerical results

In this chapter we consider quadrilaterals and compute its modulus by different numerical tools introduced in Chapter 7. Then the results are compared to the reference value as well as with each other.

8.1 Symmetric quadrilateral

In this section we use hp-version of finite element methods [HRV] to run series of tests on a symmetric quadrilateral with vertices $0, 1, 0.3 + 0.3i, i$. See Figure 7.1 for an illustration. By Theorem 4.2.2 we know that the modulus is exactly one.

8.1.1 Scaling factor

Let us consider the scaling factor. We use the $(r, 12)$-meshes, where $r \in [0.1, 0.5]$ with polynomial degree of $4, 6, \cdots, 18$. Using different polynomial degrees we identify the effect of the scaling factor. The result of the logarithmic error is shown in Figure 8.1. The result shows that for a fixed nesting level ν and different polynomial degree there exists a optimal scaling factor. Generally the higher polynomial degree is, the smaller the scaling factor should be chosen. We emphasize that the result is only valid for this particular quadrilateral. For other configurations the optimal scaling factor may vary. There might also be an analytical formula for relations between the polynomial degree and the optimal scaling factor.

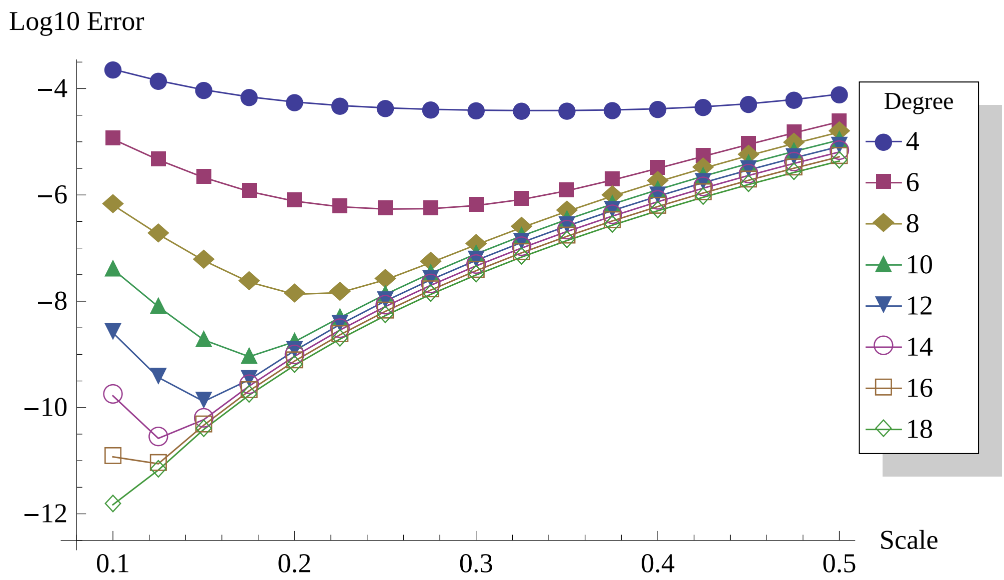


Figure 8.1: Logarithmic error obtained by different scaling factor with fixed the nesting level of 12 with polynomial degree of $4, 6, \cdots, 18$.

8.1.2 Nesting levels

We consider different nesting level to see the effect of the nesting. Before actually refining the mesh, let us first consider the nesting level ν equals zero. This means that we are working with the initial mesh and hp-FEM reduces to p-FEM. In Figure 8.2 shows the logarithmic error of the p-FEM for various polynomial degree ranging from 4 to 18.

The results for p-version of finite element methods are quite astonishing since the error should get smaller when the polynomial degree increases. The opposite happens once we have past the polynomial degree of 10, the error start to increase.

Let us move on with the computation of different nesting levels. For this test we compute the modulus of a quadrilateral with (r, ν)-meshes, with polynomial degree of $4, 6, \cdots, 12$. The scaling factor r is chosen from the above example (Section 8.1.1). The logarithmic errors are shown in Figure 8.3.

From the result we conclude that the error decreases exponentially and for lower polynomial degrees the error saturates more quickly than for higher polynomial degrees. So for higher polynomial degrees, we should use higher nesting levels to maintain the exponential rate of convergence.

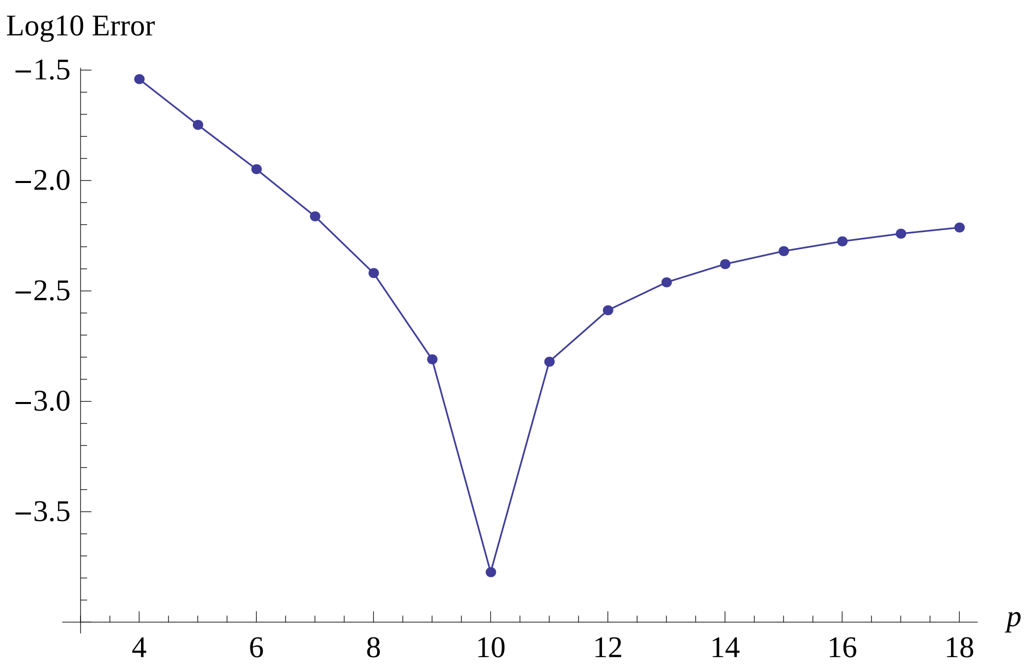


Figure 8.2: Logarithmic error of p-FEM with polynomial degree of $4, 5, \cdots, 18$.

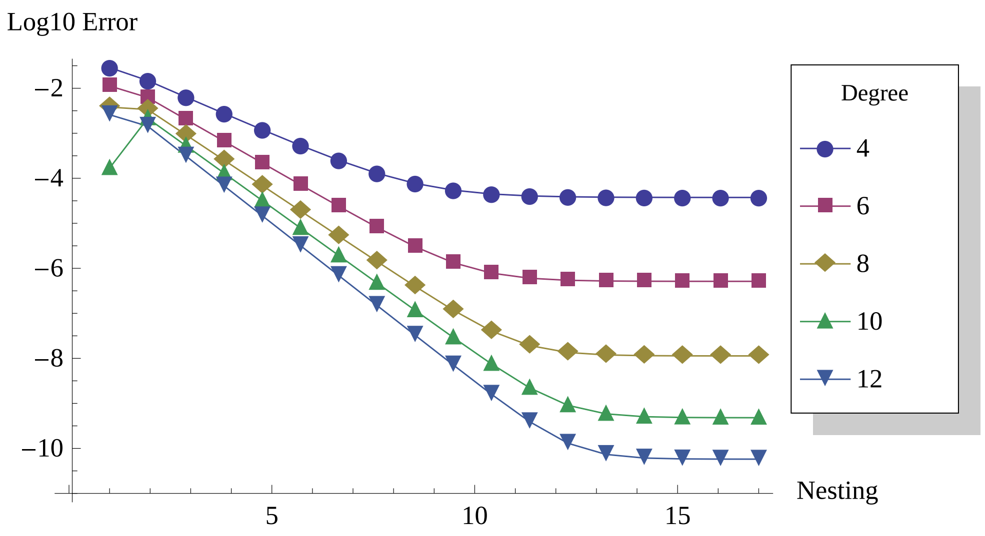


Figure 8.3: Logarithmic error obtained by different nesting level with the scaling factor obtained in Section 8.1.1 and polynomial degree of $4, 6, \cdots, 18$.

8.1.3 Refining vertices

In [BS] basic principles and properties are given to p- and hp-versions of finite element methods. One of the properties says that most of the error comes from the so called singularity component which is the vertex with a largest interior angle. In this test we use three different kind of mesh to verify the result. We use $(0.15, 12)$-meshes and refine the meshes

1. to vertices with interior angles $0 < \alpha \leq \frac{\pi}{2}$ (regular vertices),
2. to the singular vertex only,
3. to all vertices.

The polynomial degree is varied from 4 to 18. Results of the logarithmic errors are shown in Figure 8.4.

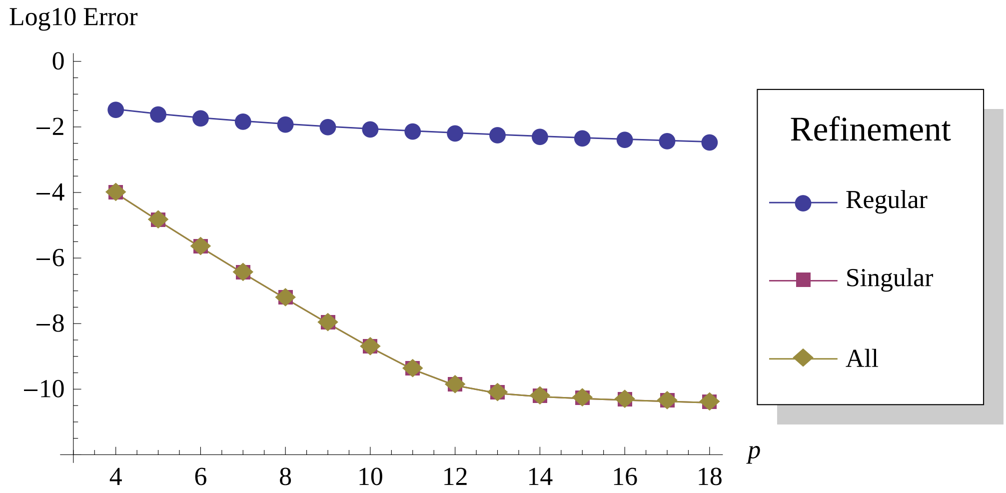


Figure 8.4: Logarithmic errors of computations of modulus by different mesh refinements on vertices. Notice that the error of refining all vertices and refining the singular vertex alone are indistinguishable.

The result suggests that the we cannot get more than 3 correct digits for this example by refining only the regular vertices. While refining only the singular vertex, we may obtain 11 correct digits at most. The result of refinement of all the vertices does not give any significant improvement over the case where we refine only the singular vertex. It should be notice that refining all vertices, the computing time is much longer than compared to the case we refine only the singular vertex.

8.2 Modulus of the convex quadrilateral

In this example we compute the modulus of convex quadrilaterals with vertices $0, 1, x + iy, i$. This test have been carried out with the Schwarz–Christoffel toolbox [Dri] and hp-version of finite element methods. We are using [HVV] as the reference result, since its analytic numeric presentation is mathematically exact. To be able to compare the result, we compute the difference of the obtained results against the reference result to gain the error for the used method and plot the logarithm of the errors.

First of all we have computed the modulus of the quadrilaterals. The computation of the Schwarz–Christoffel toolbox is carried by using the side-length method (`rectmap`). For the hp-FEM we are using $(0.15, 12)$-meshes and the test is carried out with a polynomial degree of 6 and 12. The logarithmic errors along with the reproduction of the moduli surface from [HVV] are shown in Figure 8.5.

The value of the z-axis on the error graphs tells the accuracy of the methods at the corresponding point. The Schwarz-Christoffel toolbox gives us $8 - 14$ correct digits. While hp-version of finite element methods gives us $5 - 8$ and $10 - 13$ correct digits when the degree of the polynomial is 6 and 12, respectively. It should be noted that for hp-version of finite element methods even better can be achieved if a more optimal scaling factor is chosen.

8.3 Modulus of the ring domains

In this section we will consider symmetric ring domains. The ring domains are studied in [BSV, Gai, HRV]. In [HRV] AFEM and hp-FEM are used for computing the capacitance of the condenser. In this section we are comparing the result obtained by hp-FEM to the Schwarz–Christoffel toolbox. By Schwarz reflection theorem (Theorem 5.1.1) the domain can be decomposed and the actual computation can be carried out on the decomposed domain.

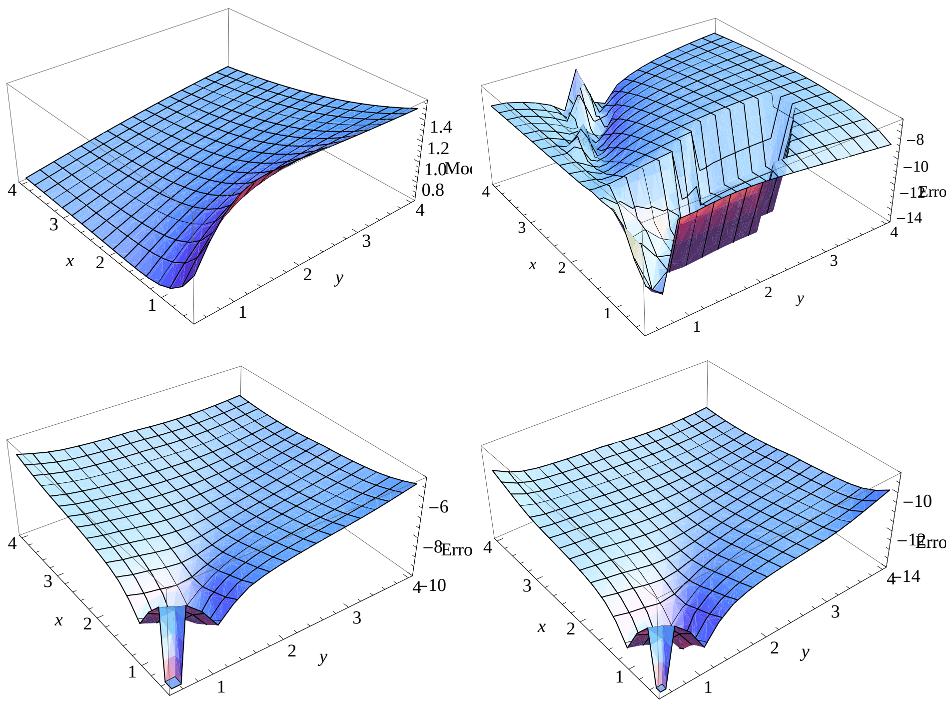


Figure 8.5: On top we have the reproduction of the modulus of the quadrilateral with vertices $0, 1, x+iy, i$ from [HVV] and the logarithmic error of the Schwarz–Christoffel toolbox. On bottom we have logarithmic errors of the hp-FEM with $(0.15, 12)$-meshes and the polynomial degree of 6 and 12 respectively.

Square in a square

We compute the modulus of the ring domain $\Omega = \Omega_1 \backslash \Omega_2$, where $\Omega_1 = [-1, 1] \times [-1, 1]$ and $\Omega_2 = [-a, a] \times [-a, a]$, $0 < a < 1$. For the computation the domain is decomposed into 8 quadrilaterals, see Figure 8.6. It can be proved that the modulus of the quadrilateral is doubled if the reflection boundary is Neumann boundary. This implies that the modulus of the original domain equals to 8 times the modulus of the decomposed quadrilateral.

The computation is carried out by the Schwarz–Christoffel toolbox and hp-FEM. In this case we computed the modulus of the decomposed quadrilateral with the side-length method (`rectmap`) and CRDT (`crrectmap`). For hp-FEM we have used $(0.15, 12)$-meshes with polynomial degrees of $6, 12$, and 18. Like in previous examples, the results are compared to the values obtained from HVV

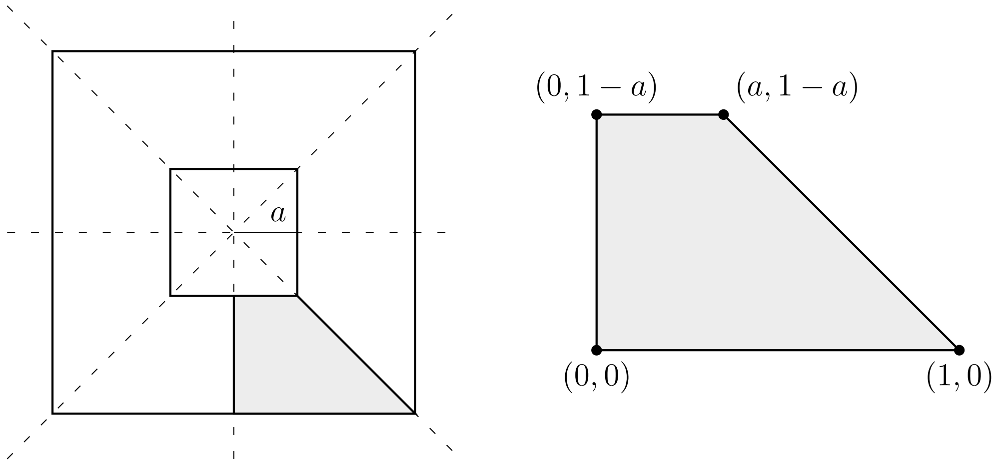


Figure 8.6: The domain of interest is on the left-hand side and on the right-hand side we have one of the quadrilateral decomposed from the original domain.

iteration. The errors are listed in Table 8.1 along with the modulus obtained by HVV iteration.

Table 8.1: Table for the exact value of the modulus and the error for different methods in a square in a square. The parameter p refers to polynomial degree of hp-FEM. Reference values are obtained by HVV–iteration.

a	rectmap	crrectmap	$p=6$	$p=12$	$p=18$	Reference
0.1	$5.2\cdot10^{-9}$	$5.3\cdot10^{-7}$	$1.0\cdot10^{-4}$	$7.0\cdot10^{-8}$	$8.1\cdot10^{-11}$	2.817122196
0.2	$8.4\cdot10^{-13}$	$4.1\cdot10^{-8}$	$3.4\cdot10^{-6}$	$2.0\cdot10^{-10}$	$1.6\cdot10^{-14}$	1.934943792
0.3	$1.1\cdot10^{-13}$	$6.3\cdot10^{-10}$	$3.4\cdot10^{-7}$	$4.2\cdot10^{-12}$	$3.3\cdot10^{-15}$	1.420245745
0.4	$3.8\cdot10^{-11}$	$5.7\cdot10^{-12}$	$1.1\cdot10^{-7}$	$2.1\cdot10^{-12}$	$1.8\cdot10^{-15}$	1.057986726
0.5	$2.7\cdot10^{-9}$	$2.4\cdot10^{-11}$	$8.9\cdot10^{-8}$	$1.9\cdot10^{-12}$	$1.2\cdot10^{-15}$	0.781700961
0.6	$7.1\cdot10^{-9}$	$7.2\cdot10^{-12}$	$5.1\cdot10^{-8}$	$1.0\cdot10^{-12}$	$1.1\cdot10^{-16}$	0.561999833
0.7	$3.5\cdot10^{-9}$	$1.6\cdot10^{-9}$	$8.8\cdot10^{-8}$	$9.1\cdot10^{-13}$	$3.8\cdot10^{-15}$	0.382746154
0.8	$1.1\cdot10^{-9}$	$5.2\cdot10^{-10}$	$7.0\cdot10^{-7}$	$1.2\cdot10^{-10}$	$2.6\cdot10^{-14}$	0.233679562
0.9	$2.9\cdot10^{-10}$	$7.4\cdot10^{-10}$	$9.9\cdot10^{-6}$	$2.2\cdot10^{-8}$	$8.8\cdot10^{-11}$	0.107766002

Since the quadrilaterals are not elongated the CRDT does not stand out from the side-length method. For some cases CRDT perform even worse than side-length method and vice versa. In case of hp-FEM we get $2-3$ more correct digits by increasing the polynomial degree by 6. For $0.3 < a < 0.7$, hp-FEM with the $(0.15, 12)$-mesh and polynomial degrees of $9 - 11$ gives the same performance as the Schwarz–Christoffel toolbox.

We have only emphasized how to improve the performance of hp-version of finite element methods. In the Schwarz–Christoffel toolbox we can change the tolerance rate to obtain much better result. In Table 8.2 we have the result obtained by the side-length method with the default and a custom 10^{-14} tolerance rate.

Table 8.2: Table for values obtained with the side-length method by using the default and a custom 10^{-14} tolerance rate.

a	default	custom
0.1	$5.2 \cdot 10^{-9}$	$1.6 \cdot 10^{-14}$
0.2	$8.4 \cdot 10^{-13}$	$3.1 \cdot 10^{-15}$
0.3	$1.1 \cdot 10^{-13}$	$1.6 \cdot 10^{-15}$
0.4	$3.8 \cdot 10^{-11}$	$1.3 \cdot 10^{-15}$
0.5	$2.7 \cdot 10^{-9}$	$1.1 \cdot 10^{-15}$
0.6	$7.1 \cdot 10^{-9}$	$1.1 \cdot 10^{-14}$
0.7	$3.5 \cdot 10^{-9}$	$4.9 \cdot 10^{-15}$
0.8	$1.1 \cdot 10^{-9}$	$2.5 \cdot 10^{-15}$
0.9	$2.9 \cdot 10^{-10}$	$2.1 \cdot 10^{-14}$

Lastly Figure 8.7 illustrates the potential function of one of the decomposed quadrilateral.

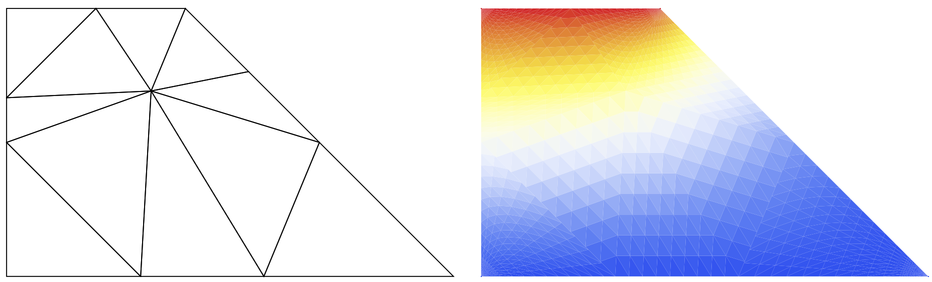


Figure 8.7: The initial mesh and the potential function of a square in a square with $a = 0.4$ on left- and right-hand side, respectively.

Chapter 9

Conclusion and further research

In this thesis we have given an introduction on the theory of a computation of the modulus of a quadrilateral. We have develop an extensive amount of theory in order to give numerical examples.

By looking closely to the error graphs and tables in Chapter 8, it seems that hp-version of finite element methods produces $3-4$ more correct digits whenever the degree of the polynomial is doubled. Of course this cannot be generalized because of the computational precision. While hp-version of finite element methods perform better than the Schwarz–Christoffel toolbox with the default error tolerance rate, it does not come for free. Since the computation with hp-version finite element methods is more time-consuming than with the Schwarz-Christoffel toolbox.

Lastly we want to give some ideas for further studies.

- In the Section 8.1 we considered a symmetric quadrilateral with one singular vertex to study hp-version of finite element methods. Following the example we could choose a different symmetric quadrilateral, for example, a parallelogram and try to find out how the geometry affect the connection between the scaling factor and the degree of polynomial.

- The natural continuation would be considering quadrilaterals with curved boundaries. In this case the comparison between hp-version of finite element methods and the Schwarz–Christoffel toolbox cannot be carried out unless we use a piecewise polynomial approximation to curved boundary segments.

- Constructing the conformal mapping from the potential function u obtained from hp-version of finite element methods and comparing the result obtained by the Schwarz–Christoffel toolbox.

Appendix A

Hierarchic shape functions

Let us illustrate some of the shape functions defined in Section 6.3 by plotting them to give an intuition what they look like. First in Figure A.1 we have all the four nodal shape functions. Side mode functions, which associate with the first edge, are shown in Figure A.2. Lastly in Figure A.3 we have plotted inner mode functions $N_{11}(\xi, \eta)$, $N_{12}(\xi, \eta)$, $N_{21}(\xi, \eta)$, and $N_{22}(\xi, \eta)$.

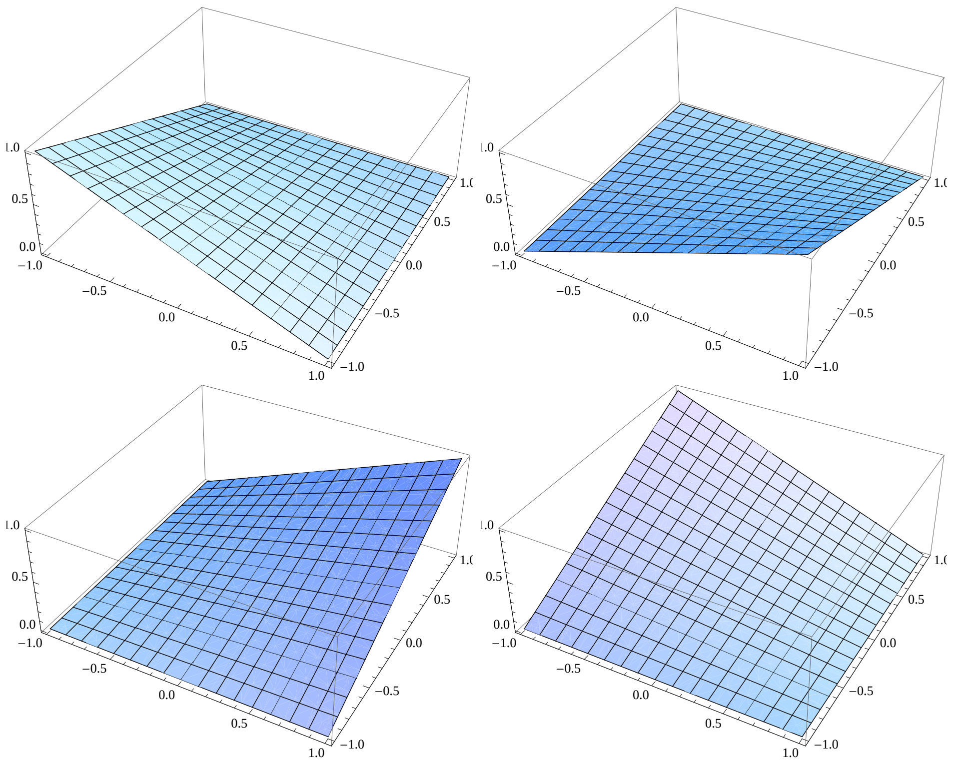


Figure A.1: From the top left to the lower right the nodal shape functions are in a following order: $N_1(\xi, \eta), N_2(\xi, \eta), N_3(\xi, \eta), N_4(\xi, \eta)$.

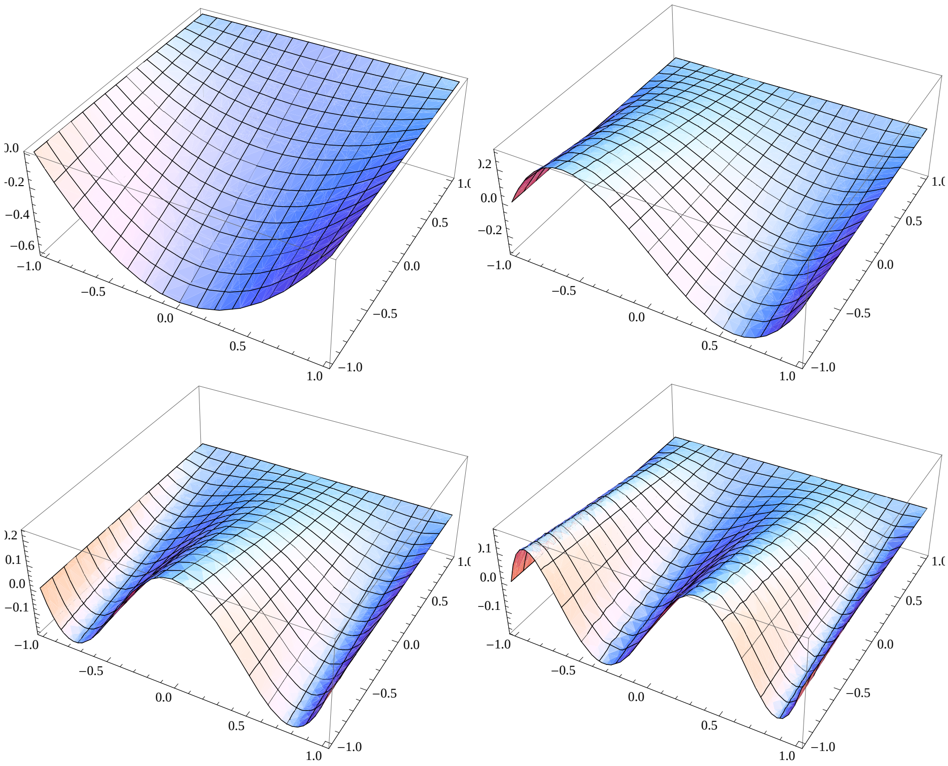


Figure A.2: From the top left to the lower right the side mode functions are in a following order: $N_1^1(\xi,\eta), N_2^1(\xi,\eta), N_3^1(\xi,\eta), N_4^1(\xi,\eta)$.

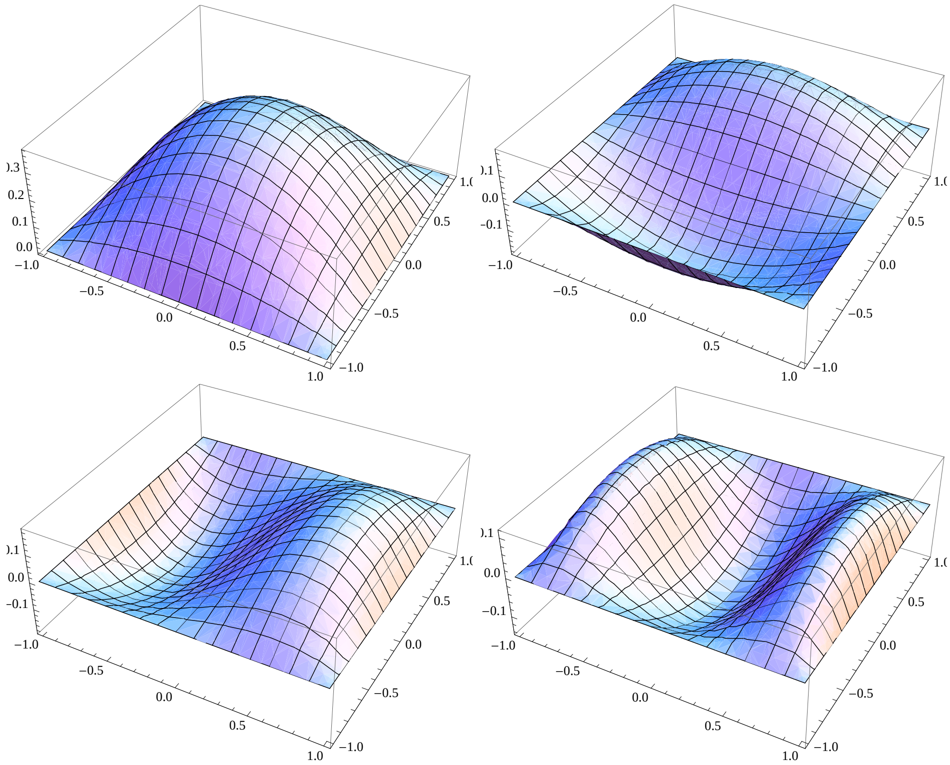


Figure A.3: From the top left to the lower right the inner mode functions are in a following order: $N_{11}(\xi, \eta), N_{12}(\xi, \eta), N_{21}(\xi, \eta), N_{22}(\xi, \eta)$.

Bibliography

[AF] R.A. ADAMS AND J.J FOURNIER, *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[Ahl1] L.V. AHLFORS, *Complex Analysis*. An introduction to the theory of analytic functions of one complex variable, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[Ahl2] L.V. AHLFORS, *Conformal invariants: topics in geometric function theory*. McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[Ahl3] L.V. AHLFORS, *Lectures on quasiconformal mappings*. Second edition. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. University Lecture Series, 38. American Mathematical Society, Providence, RI, 2006.

[AIM] K. ASTALA, T. IWANIEC, AND G. MARTIN, *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

[BS] I. BABUŠKA AND M. SURI, *The p and h-p versions of the finite element method, basic principles and properties*. SIAM Rev. 36 (1994), no. 4, 578–632.

[Ban] L. BANJAI, *Revisiting the crowding phenomenon in Schwarz-Christoffel mapping*. SIAM J. Sci. Comput. 30 (2008), no. 2, 618–636.

[BSV] D. BETSAKOS, K. SAMUELSSON, AND M. VUORINEN, *The computation of capacity of planar condensers*. Publ. Inst. Math. (Beograd) (N.S.) 75(89) (2004), 233–252.

[Cay] A. CAYLEY, *An elementary treatise on elliptic functions*. second edition. Dover Publications, Inc., New York 1961, (First published in 1895)

[Cro1] D. CROWDY, *Exact solutions for the viscous sintering of multiply-connected fluid domains*. J. Engrg. Math. 42 (2002), no. 3-4, 225–242.

[Cro2] D. CROWDY, *Exact solutions for two steady inviscid bubbles in the slow viscous flow generated by a four-roller mill*. J. Engrg. Math. 44 (2002), no. 4, 311–330.

[Cro3] D. CROWDY, *Explicit solution for the potential flow due to an assembly of stirrers in an inviscid fluid*. J. Engrg. Math. 62 (2008), no. 4, 333–344.

[Dri] T.A. DRISCOLL, *Schwarz-Christoffel toolbox for MATLAB*, `http://www.math.udel.edu/~driscoll/software/SC/`

[DT] T.A. DRISCOLL AND L.N. TREFETHEN, *Schwarz-Christoffel Mapping*. Cambridge Monographs on Applied and Computational Mathematics, 8. Cambridge University Press, Cambridge, 2002.

[DrVa] T.A. DRISCOLL AND S.A. VAVASIS, *Numerical conformal mapping using cross-ratios and Delaunay triangulation*. SIAM J. Sci. Comput. 19 (1998), no. 6, 1783–1803.

[DuVu] V.N. DUBININ AND M. VUORINEN, *On conformal moduli of polygonal quadrilaterals*. Israel J. Math. 171 (2009), 111–125.

[Eva] L.C. EVANS, *Partial differential equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[Gai] D. GAIER, *Ermittlung des konformen Moduls von Vierecken mit Differenzenmethoden*. Numer. Math. 19 (1972), 179–194.

[Gam] T.W. GAMELIN, *Complex analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[GM] J.B. GARNETT AND D.E MARSHALL, *Harmonic measure*. New Mathematical Monographs, 2. Cambridge University Press, Cambridge, 2005.

[GB1] W. GUI, I. BABUŠKA, *The $h,\ p$ and h-p versions of the finite element method in 1 dimension. I. The error analysis of the p-version*. Numer. Math. 49 (1986), no. 6, 577–612.

[GB2] W. GUI, I. BABUŠKA, *The $h,\ p$ and h-p versions of the finite element method in 1 dimension. II. The error analysis of the h- and h-p versions*. Numer. Math. 49 (1986), no. 6, 613–657.

[GB3] W. GUI, I. BABUŠKA, *The $h,\ p$ and h-p versions of the finite element method in 1 dimension. III. The adaptive h-p version*. Numer. Math. 49 (1986), no. 6, 659–683.

[HRV] H. HAKULA, A. RASILA, AND M. VUORINEN, *On moduli of rings and quadrilaterals: algorithms and experiments*. arXiv math.NA 0906.1261, 2009. TKK-A575 (2009)

[HVV] V. HEIKKALA, M.K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals*. Comput. Methods Funct. Theory 9 (2009), no. 1, 75–109.

[Hen1] P. HENRICI, *Applied and computational complex analysis. Vol. 1*. Power series—integration—conformal mapping—location of zeros. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[Hen2] P. HENRICI, *Applied and computational complex analysis. Vol. 3*. Discrete Fourier analysis—Cauchy integrals—construction of conformal maps—univalent functions. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1986.

[Kra] S.V. KRANTZ, *Geometric Function Theory*. Explorations in complex analysis. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2006.

[Kre] E. KREYSZIG, *Advanced engineering mathematics*. Ninth edition. John Wiley & Sons, Inc., 2005.

[Küh] R. KÜHNAU, *The conformal module of quadrilaterals and of rings*. Handbook of complex analysis: geometric function theory. Vol. 2, 99–129, Elsevier, Amsterdam, 2005.

[LT] S. LARSSON AND V. THOMÉE, *Partial differential equations with numerical methods*. Texts in Applied Mathematics, 45. Springer-Verlag, Berlin, 2003.

[Lau] D. LAUGWITZ, *Bernhard Riemann 1826–1866*. Turning points in the conception of mathematics. Translated from the 1996 German original by Abe Shenitzer with the editorial assistance of the author, Hardy Grant, and Sarah Shenitzer. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Leb] N.N. LEBEDEV, *Special functions and their applications*. Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication. Dover Publications, Inc., New York, 1972.

[LV] O. LEHTO AND K.I. VIRTANEN, *Quasiconformal mappings in the plane*. Second edition. Translated from the German by K. W. Lucas. Die Grundlehren der mathematischen Wissenschaften, Band 126. Springer-Verlag, New York-Heidelberg, 1973.

[MH] J.H. MATHEWS AND R.W. HOWEL, *Complex Analysis for Mathematics and Engineering*. Fifth edition, Narosa Publishing House, New Delhi, 2006.

[MS] N.G. MEYERS AND J. SERRIN, $H = W$. Proc. Nat. Acad. Sci. U.S.A. 51 1964 1055–1056.

[Neh] Z. NEHARI, *Conformal mapping*. Reprinting of the 1952 edition. Dover Publications, Inc., New York, 1975.

[NP] R. NEVANLINNA AND V. PAATERO, *Introduction to complex analysis*. Translated from the German by T. Kövari and G. S. Goodman Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[Num] M. NUMMELIN, *Konformikuvausten konstruoiminen yhdesti yhtenäisiltä kompleksitason alueilta kanonisille alueille*. Master's Thesis, Helsinki University of Technology, Department of Engineering Physics and Mathematics, 2007.

[Pap] N. PAPAMICHAEL, *Lectures on Numerical Conformal Mapping*. `http://194.42.1.1/˜nickp/numericalcn.htm`

[Pom] CH. POMMERENKE, *Conformal maps at the boundary*. Handbook of complex analysis: geometric function theory, Vol. 1, 37–74, North-Holland, Amsterdam, 2002.

[Pon] S. PONNUSAMY, *Foundations of Complex Analysis*. Second edition, Narosa Publishing House, New Delhi, 2005.

[Por] R.M. PORTER, *History and recent developments in techniques for numerical conformal mapping*. Quasiconformal mappings and their applications, 207–238, Narosa, New Delhi, 2007.

[RV] A. RASILA AND M. VUORINEN, *Experiments with moduli of quadrilaterals*. Rev. Roumaine Math. Pures Appl. 51 (2006), no. 5-6, 747–757.

[Rud1] W. RUDIN, *Principles of mathematical analysis*. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.

[Rud2] W. RUDIN, *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York, 1987.

[SL] R. SCHINZINGER AND P. A. A. LAURA, *Conformal mapping: methods and applications.*, Revised edition of the 1991 original. Dover Publications, Inc., Mineola, NY, 2003.

[SSD] P. ŠOLÍN, K. SEGETH, AND I. DOLEŽEL, *Higher-order finite element methods*. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[SC] A. SURANA AND D. CROWDY, *Vortex dynamics in complex domains on a spherical surface*. J. Comput. Phys. 227 (2008), no. 12, 6058–6070.

[SB] B. SZABÓ AND I. BABUŠKA, *Finite element analysis*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1991.

[Sze] G. SZEGÖ, *Orthogonal polynomials*. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975.

[Tre] L.N. TREFETHEN, *Numerical computation of the Schwarz-Christoffel transformation*. SIAM J. Sci. Statist. Comput. 1 (1980), no. 1, 82–102.

[TD] L.N. TREFETHEN AND T.A. DRISCOLL, *Schwarz-Christoffel mapping in the computer era*. Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998).

[Vuo] J. VUORISTO, *Schwarzin-Cristoffelin kaava Mathematicalla*. Master's Thesis, University of Turku, Faculty of Mathematics and Natural Sciences, Department of Mathematics, 2007.

[Wal] J.T. WALSH, *History of the Riemann mapping theorem*. American Mathematical Monthly, Vol. 80, No. 3, (1973), pp. 270–276.

[Yrj] J. YRJÖLÄ, *Numerical computation of the module of a quadrilateral*. Master's Thesis, University of Helsinki, Faculty of Science, Department of Mathematics and Statistics, 2007.