Numerical conformal mappings and capacity computation

Tri Quach

Institute of Mathematics
Helsinki University of Technology

Seminar on numerical analysis and computational science
November 9, 2009
Motivation

Compute a capacitance of a condenser:
- analytically only for few condenser
- numerically
  - Schwarz–Christoffel mapping
  - Finite element methods
- Capacitance of a cylindrical condenser per unit length:
  \[ \frac{C}{L} = \frac{2\pi \varepsilon}{\ln (R/r)} \]

**Figure:** The cross-section of the cylinder.
Preliminaries - Generalized Quadrilateral

Definition (Generalized Quadrilateral)

A *generalized quadrilateral* is a Jordan domain $\Omega$ with four separate boundary points $z_1, z_2, z_3,$ and $z_4$ given in positive order on the boundary curve $\partial \Omega$ of $\Omega$. These points are called vertices of the generalized quadrilateral. They divide $\partial \Omega$ into four curves $\gamma_1, \gamma_2, \gamma_3,$ and $\gamma_4$ which are called the sides of the generalized quadrilateral and denoted by $(z_1, z_2), (z_2, z_3), (z_3, z_4),$ and $(z_4, z_1)$, respectively.

In this case, when we traverse $\partial \Omega$ such that $\Omega$ is on the left-hand side, the points $z_1, z_2, z_3,$ and $z_4$ occur in this order. We denote a generalized quadrilateral by $Q(\Omega; z_1, z_2, z_3, z_4)$. 
Figure: A generalized quadrilateral $Q(\Omega; z_1, z_2, z_3, z_4)$. 
A mapping $f : z \mapsto w$ is said to be *conformal* at $z_0$ if it preserves angles and their orientation between smooth curves through $z_0$.

**Figure:** An illustration of a conformal map.
Preliminaries - Riemann Mapping Theorem

Theorem (Riemann mapping theorem)

Given any simply connected domain $\Omega$ in $\mathbb{C}$, and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in $\Omega$, normalized by the conditions $f(z_0) = 0$, $f'(z_0) \in \mathbb{R}^+$, such that $f(z)$ defines a one-to-one mapping of $\Omega$ onto the disk $|w| < 1$.

- Riemann stated it in his doctoral dissertation in 1851.
- First rigorous proof due to Osgood (1900) and Koebe (1908)
Theorem (Schwarz–Christoffel mapping for a half plane)

Let $\Omega$ be the interior of a polygon $P$ in the $w$-plane with vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1 \pi, \ldots, \alpha_n \pi$ given in positive order. Let $f(z)$ be any conformal map from the upper half plane onto $\Omega$ with $f(\infty) = w_n$. Then the Schwarz–Christoffel mapping $f(z)$ is given by

$$w = f(z) = A + C \int_z^{w_n} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k-1} d\zeta,$$

for some complex constants $A$ and $C$, where $w_k = f(z_k)$ for $k = 1, \ldots, n-1$. 


Preliminaries - Schwarz–Christoffel

- Can be used for map a polygonal domain onto a rectangle

**Figure**: Illustration, how to conformally map a polygonal domain onto a rectangle.
The *conformal modulus* of a quadrilateral divides quadrilaterals into conformal equivalence classes.

**Definition (Geometric)**

Let \( Q(\Omega; a, b, c, d) \) be a quadrilateral. Let the function \( w = f(z) \), where \( w = u + iv \), be a one-to-one conformal mapping of the domain \( \Omega \) onto a rectangle \( 0 < u < 1, 0 < v < M \) such that the vertices \( a, b, c, \text{and} d \) correspond to the vertices \( 0, 1, 1 + iM, \text{and} iM \), respectively. The number \( M \) is called the (conformal) *modulus* of the quadrilateral \( Q(\Omega; a, b, c, d) \) and we will denote it by \( M(Q; a, b, c, d) \).
Figure: Exponential mapping from a rectangle onto an annulus.
• Consider the following Laplace equation

\[
\begin{align*}
\Delta u &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \gamma_2, \\
u &= 1, \quad \text{on } \gamma_4, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \gamma_1 \cup \gamma_3.
\end{align*}
\] (2)

**Theorem (Dirichlet–Neumann definition)**

Let \( u \) be the solution for the problem (2), then the modulus of the quadrilateral \( Q \) is given by

\[
M(Q; a, b, c, d) = \int_{\Omega} |\nabla u|^2 \, dx \, dy.
\] (3)
Modulus - Properties

For computation we use following properties.

- \( M(Q; c, d, a, b) = M(Q; a, b, c, d) \)
- \( M(Q; b, c, d, a) = \frac{1}{M(Q;a,b,c,d)} \)
- Every symmetric quadrilateral has modulus 1
Consider the following Laplace equation

\[
\begin{aligned}
\Delta u &= 0, \quad \text{in } \Omega, \\
    u &= 0, \quad \text{on } \gamma_2, \\
    u &= 1, \quad \text{on } \gamma_4, \\
\frac{\partial u}{\partial n} &= 0, \quad \text{on } \gamma_1 \cup \gamma_3.
\end{aligned}
\]

The corresponding variational problem is to find \( u \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} \nabla u \cdot \nabla \psi \, dx \, dy = 0, \quad \forall \psi \in H^1_0(\Omega).
\]
Legendre’s polynomials of degree $n$ can be defined recursively by

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0,$$

where $P_0(x) = 1$ and $P_1(x) = x$.

- Derivatives of Legendre’s polynomials:

$$(1 - x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x), \quad n \geq 1.$$
FEM - Polynomials

- Integrated Legendre’s polynomials:

  \[ \phi_n(x) = \frac{1}{\sqrt{2(2n-1)}} [P_n(x) - P_{n-2}(x)], \quad n \geq 2. \]

Properties of Legendre’s polynomials:
- orthogonal in interval \([-1, 1]\]
- form a hierarchic basis
Reference domain:
- $[-1, 1] \times [-1, 1]$

Shape functions:
- Nodal shape functions: 4 functions
- Side Nodes: $4(p - 1)$ functions
- Internal modes (full space): $(p - 1)(p - 1)$ functions
FEM - Nodal shape functions

- Nodal shape functions:

\[
N_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta),
\]

\[
N_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta),
\]

\[
N_3(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta),
\]

\[
N_4(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta).
\]
FEM - Side & Internal Nodes

• Side Nodes:

\[
N_i^1(\xi, \eta) = \frac{1}{2}(1 - \eta)\phi_i(\xi), \quad i = 2, 3, \ldots, p,
\]

\[
N_i^2(\xi, \eta) = \frac{1}{2}(1 + \xi)\phi_i(\eta), \quad i = 2, 3, \ldots, p,
\]

\[
N_i^3(\xi, \eta) = \frac{1}{2}(1 + \eta)\phi_i(\xi), \quad i = 2, 3, \ldots, p,
\]

\[
N_i^4(\xi, \eta) = \frac{1}{2}(1 - \xi)\phi_i(\eta), \quad i = 2, 3, \ldots, p.
\]

• Internal modes (full space):

\[
N_{ij}(\xi, \eta) = \phi_i(\xi)\phi_j(\eta), \quad i, j = 2, 3, \ldots, p.
\]
Figure: From the top left to the lower right the nodal shape functions are in a following order: \( N_1(\xi, \eta), N_2(\xi, \eta), N_3(\xi, \eta), N_4(\xi, \eta) \).
Figure: From the top left to the lower right the side mode functions are in a following order: $N_1^1(\xi, \eta), N_2^1(\xi, \eta), N_3^1(\xi, \eta), N_4^1(\xi, \eta)$. 
**Figure:** From the top left to the lower right the inner mode functions are in a following order: 
$N_{11}(\xi, \eta), N_{12}(\xi, \eta), N_{21}(\xi, \eta), N_{22}(\xi, \eta)$. 

**Motivation**

**Preliminaries**

**Conformal modulus**

**Finite Element Methods**

**Variational Problem**

**Legendre’s Polynomials**

**Reference Domain**

**Shape Functions**

**Numerical Results**

**References**
Numerical tests:

- Scaling factor:
- Nesting level:
- Refining vertices:

**Figure:** A quadrilateral $Q \left(0, 1, \frac{3}{10} + \frac{3}{10}i, i\right)$ with the initial mesh and the $(0.4, 2)$-mesh on the left- and right-hand side, respectively.
Symmetric Quadrilateral - Scaling Factor

Figure: Logarithmic error obtained by different scaling factor with the fixed nesting level of 12 with polynomial degree of 4, 6, · · ·, 18.
Symmetric Quadrilateral - Nesting Level

**Figure:** Logarithmic error of $p$-FEM with polynomial degree of 4, 5, ..., 18.
Symmetric Quadrilateral - Nesting Level

![Log10 Error Graph](image_url)

**Figure:** Logarithmic error obtained by different nesting level and polynomial degree of 4, 6, · · · , 18.
Figure: Logarithmic errors of computations of modulus by different mesh refinements on vertices. Note that the error of refining all vertices and refining the singular vertex alone are indistinguishable.
Convex Quadrilaterals

Figure: On top we have the reproduction of the modulus of the quadrilateral with vertices 0, 1, \(x + iy, i\) from [HVV] and the logarithmic error of the SC toolbox. On bottom we have logarithmic errors of the \(hp\)-FEM with \((0.15, 12)\)-meshes and the polynomial degree of 6 and 12, respectively.
Square in a square:

- \( \Omega = \Omega_1 \setminus \Omega_2 \), where \( \Omega_1 = [-1, 1] \times [-1, 1] \) and \( \Omega_2 = [-a, a] \times [-a, a] \), \( 0 < a < 1 \).

**Figure:** The domain of interest is on the left-hand side and on the right-hand side we have one of the quadrilateral decomposed from the original domain.
Table: The parameter $p$ refers to polynomial degree of hp-FEM. Reference values are obtained by HVV–iteration.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$p = 6$</th>
<th>$p = 12$</th>
<th>$p = 18$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$1.0 \cdot 10^{-4}$</td>
<td>$7.0 \cdot 10^{-8}$</td>
<td>$8.1 \cdot 10^{-11}$</td>
<td>2.817122196</td>
</tr>
<tr>
<td>0.2</td>
<td>$3.4 \cdot 10^{-6}$</td>
<td>$2.0 \cdot 10^{-10}$</td>
<td>$1.6 \cdot 10^{-14}$</td>
<td>1.934943792</td>
</tr>
<tr>
<td>0.3</td>
<td>$3.4 \cdot 10^{-7}$</td>
<td>$4.2 \cdot 10^{-12}$</td>
<td>$3.3 \cdot 10^{-15}$</td>
<td>1.420245745</td>
</tr>
<tr>
<td>0.4</td>
<td>$1.1 \cdot 10^{-7}$</td>
<td>$2.1 \cdot 10^{-12}$</td>
<td>$1.8 \cdot 10^{-15}$</td>
<td>1.057986726</td>
</tr>
<tr>
<td>0.5</td>
<td>$8.9 \cdot 10^{-8}$</td>
<td>$1.9 \cdot 10^{-12}$</td>
<td>$1.2 \cdot 10^{-15}$</td>
<td>0.781700961</td>
</tr>
<tr>
<td>0.6</td>
<td>$5.1 \cdot 10^{-8}$</td>
<td>$1.0 \cdot 10^{-12}$</td>
<td>$1.1 \cdot 10^{-16}$</td>
<td>0.561999833</td>
</tr>
<tr>
<td>0.7</td>
<td>$8.8 \cdot 10^{-8}$</td>
<td>$9.1 \cdot 10^{-13}$</td>
<td>$3.8 \cdot 10^{-15}$</td>
<td>0.382746154</td>
</tr>
<tr>
<td>0.8</td>
<td>$7.0 \cdot 10^{-7}$</td>
<td>$1.2 \cdot 10^{-10}$</td>
<td>$2.6 \cdot 10^{-14}$</td>
<td>0.233679562</td>
</tr>
<tr>
<td>0.9</td>
<td>$9.9 \cdot 10^{-6}$</td>
<td>$2.2 \cdot 10^{-8}$</td>
<td>$8.8 \cdot 10^{-11}$</td>
<td>0.107766002</td>
</tr>
</tbody>
</table>
Table: Table for values obtained with the side-length method by using the default and a custom $10^{-14}$ tolerance rate.

<table>
<thead>
<tr>
<th>$a$</th>
<th>default</th>
<th>custom</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>5.2 \times 10^{-9}</td>
<td>1.6 \times 10^{-14}</td>
</tr>
<tr>
<td>0.2</td>
<td>8.4 \times 10^{-13}</td>
<td>3.1 \times 10^{-15}</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1 \times 10^{-13}</td>
<td>1.6 \times 10^{-15}</td>
</tr>
<tr>
<td>0.4</td>
<td>3.8 \times 10^{-11}</td>
<td>1.3 \times 10^{-15}</td>
</tr>
<tr>
<td>0.5</td>
<td>2.7 \times 10^{-9}</td>
<td>1.1 \times 10^{-15}</td>
</tr>
<tr>
<td>0.6</td>
<td>7.1 \times 10^{-9}</td>
<td>1.1 \times 10^{-14}</td>
</tr>
<tr>
<td>0.7</td>
<td>3.5 \times 10^{-9}</td>
<td>4.9 \times 10^{-15}</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1 \times 10^{-9}</td>
<td>2.5 \times 10^{-15}</td>
</tr>
<tr>
<td>0.9</td>
<td>2.9 \times 10^{-10}</td>
<td>2.1 \times 10^{-14}</td>
</tr>
</tbody>
</table>
References


References

