Reduced Formulation of Steady Fluid-structure Interaction with Parametric Coupling

Toni Lassila*,○, Alfio Quarteroni†,×, Gianluigi Rozza†

* Department of Mathematics and Systems Analysis
School of Science and Technology
Aalto University

× MOX - Modellistica e Calcolo Scientifico
Dipartimento di Matematica “F. Brioschi”
Politecnico di Milano

† Chair of Modelling and Scientific Computing
Mathematics Institute of Computational Science and Engineering
École Polytechnique Fédérale de Lausanne

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“Towards reducing the geometric complexity of FSI problems”

- Motivation for fluid-structure interaction
  - Previous three days of this conference...

- Steady fluid-structure interaction problem
  - Incompressible Stokes equations for fluid
  - Generalized 1-d string model for the structure
  - Coupling between traction applied by fluid and structural displacement

- Parametric flow geometry
  - Parametric free-form deformation of geometry
  - Fluid equations on fixed domain with parametric coefficients

- Model reduction
  - Fluid-structure coupling variables are the geometric parameters
  - Iterative scheme in parameter space
  - Reduced basis method for approximation of parametric Stokes
Approaches to Reduced Modelling of Fluid-Structure Interaction

Classical model reduction applied to linear systems of ODEs and PDEs ⇒ not very useful for FSI with strong geometric nonlinearity. Some new approaches have been proposed:

- Proper Orthogonal Decomposition (review in [DH01])
  - Eigendecomposition-based method for approximating an ensemble of trajectories of a given dynamical system
  - Widely used in aeroelasticity simulations (not so much in hemodynamics)
  - Cons: Computationally expensive, error of reduced model difficult to estimate

- Geometrical multiscale (review in [FQV09])
  - Different fidelity models (0D vs. 1D vs. 3D-models) used in different parts of the cardiovascular system, coupled together with suitable boundary conditions
  - Combines modelling scales ranging from peripheral circulation all the way to the major arteries
  - Cons: Physically meaningful boundary conditions between 0D-1D-3D models are challenging (talk of C. Malossi)

- Reduced basis element method [LMR06]
  - Decomposition of complex flow network to a small collection of “simple elements” like T-junctions and straight pipes, combined with reduced basis method (talk by L. Iapichino)
Our Model Reduction Strategy for Fluid-Structure Interaction

Standard Fluid-Structure Interaction

- Geometry
- Fluid
- Structure
- Coupling

Reduced Fluid-Structure Interaction

- Parametric FFD
- Geometry
- Reduced basis
- Fluid
- Structure
- Parametric weak coupling

Iteration loop

Fluid-structure Interaction with Parametric Coupling
Steady Fluid-Structure Interaction Model Problem

\[ \Sigma(\eta) \]

\[ \begin{align*}
\Omega(\eta) & \quad \Delta \mathbf{u} + \nabla p = 0 \\
& \quad \nabla \cdot \mathbf{u} = 0
\end{align*} \]

Fluid:

\[ \begin{align*}
- \nu \Delta \mathbf{u} + \nabla p &= f, \quad \text{in } \Omega(\eta) \\

\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega(\eta) \\

\mathbf{u} &= \mathbf{u}_0, \quad \text{on } \Gamma_{in} \cup \Gamma_{out} \\

\mathbf{u} &= 0, \quad \text{on } \Sigma(\eta)
\end{align*} \]

Structure:

\[ \begin{align*}
\varepsilon \frac{d^4 \eta}{d x_1^4} - K \frac{d^2 \eta}{d x_1^2} + \eta &= \tau_\Sigma, \quad \text{for } x_1 \in (0, L) \\

\eta(0) &= \eta'(0) = \eta(L) = \eta'(L) = 0
\end{align*} \]

Coupling condition:

\[ \tau_\Sigma = [\rho \mathbf{n} - \nu (\nabla \mathbf{u} + \nabla \mathbf{u}^t) \mathbf{n}] \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{on } \Sigma(\eta) \]

Existence proved in [G98] with fixed point argument + some additional regularity assumptions.
Choice of Structural Model and Treatment of Boundary Conditions

Generalized 1-d string model for arterial wall [QTV00] in the steady case:

\[-kGh \frac{\partial^2 \eta}{\partial x_1^2} + \frac{Eh}{1 - \nu_P^2} \frac{\eta}{R_0(x_1)^2} = \tau_\Sigma, \quad \text{on } x_1 \in (0, L),\]

where \( h = \) wall thickness, \( k = \) Timoshenko shear correction factor, \( G = \) shear modulus, \( E = \) Young modulus, \( \nu_P = \) Poisson ratio, and \( R_0(x_1) = \) radius of the reference configuration at distance \( x_1 \) from inflow.

We choose to include a fourth order singular perturbation term \( \tilde{\varepsilon} \frac{d^4 \eta}{dx_1^4} \), which after nondimensionalizing the equations gives

\[\varepsilon \frac{d^4 \eta}{dx_1^4} - K \frac{d^2 \eta}{dx_1^2} + \eta = \tau_\Sigma, \quad \text{for } x_1 \in (0, L)\]

with the boundary conditions

\[\eta(0) = \eta'(0) = \eta(L) = \eta'(L) = 0.\]
Steady Fluid-Structure Interaction Problem (weak form)

Incompressible Stokes fluid + 1-d membrane structure

Find \((u, p, \eta) \in \mathcal{V}(\Omega(\eta)) \times \mathcal{Q}(\Omega(\eta)) \times \mathcal{S}(0, L)\) s.t.

\[
\begin{cases}
A(u, v) + B(p, v) = \langle F, v \rangle & \text{for all } v \in \mathcal{V}(\Omega(\eta)) \\
B(q, u) = 0 & \text{for all } q \in \mathcal{Q}(\Omega(\eta)) \\
C(\eta, \phi) = \langle R(u, p), \phi \rangle & \text{for all } \phi \in \mathcal{S}(0, L).
\end{cases}
\]

where we have the bilinear forms for the incompressible Stokes equations

\[
A(u, v) = \nu \int_{\Omega(\eta)} \nabla u \cdot \nabla v \, d\Omega,
B(q, u) = -\int_{\Omega(\eta)} q \nabla \cdot v \, d\Omega,
\]

and the linear form

\[
\langle F, v \rangle = \int_{\Omega(\eta)} f \cdot v \, d\Omega
\]

and the structural bilinear form

\[
C(\eta, \phi) = \varepsilon \int_{\Sigma_0} \frac{d^2 \eta}{dx_1^2} \frac{d^2 \phi}{dx_1^2} \, dx_1 + K \int_{\Sigma_0} \frac{d\eta}{dx_1} \frac{d\phi}{dx_1} \, dx_1 + \int_{\Sigma_0} \eta \phi \, dx_1.
\]

The fluid residual is the normal component of the normal Cauchy stress of the fluid.
Reduction Step #1: Free-form Deformations of the Fluid Domain

fixed reference domain

\[ \Omega_0 \]

deformed parametric domain

\[ T(\cdot, \mu) \]

FFD map

affine map \( \psi \)

\[ \hat{T}(\cdot, \mu) \]

parameter matrix \( \mu \)

FFD control points

\[ P_{\ell,m}^0 \]

parameters = displacements of control points

\[ P_{\ell,m}^0 + \mu_{\ell,m} \]

Recalling from the talk of A. Manzoni...
Parametric Fluid Equations on the Fixed Reference Domain

Parametric FFD deformation map $T(\cdot; \mu) : \Omega_0 \rightarrow \Omega(\mu)$ defined as $T = \Psi^{-1} \circ \hat{T} \circ \Psi$

where

$$
\hat{T}(\hat{x}; \mu) = \sum_{\ell=0}^{L} \sum_{m=0}^{M} b_{\ell,m}^L \left( \mathbf{p}_{\ell,m}^0 + \mu \mathbf{p}_{\ell,m} \right)
$$

and its Jacobian matrix $J_T(x; \mu) := \nabla_x T$ define the transformation tensors \cite{RV07}

$$
v_T(x; \mu) := J_T^{-t} J_T^{-1} \det(J_T), \quad \chi_T(x; \mu) := J_T^{-1} \det(J_T)
$$

used to map fluid problem back to reference domain: find $(\tilde{u}, \tilde{p}) \in \mathcal{V}(\Omega_0) \times \mathcal{Q}(\Omega_0)$ s.t.

\[
\begin{align*}
\int_{\Omega_0} \left( n \frac{\partial \tilde{u}_k}{\partial x_i} [v_{T \eta}]_{i,j} \frac{\partial \tilde{v}_k}{\partial x_j} + \tilde{p} [\chi_{T \eta}]_{k,j} \frac{\partial \tilde{v}_k}{\partial x_j} \right) \, d\Omega_0 &= \int_{\Omega_0} \det(J_{T \eta})[f^F + f_{\text{lift}}]_k \, d\Omega_0, \\
&\quad \text{for all } \tilde{v} \in \mathcal{V}(\Omega_0) \\
\int_{\Omega_0} \tilde{q} [\chi_{T \eta}]_{k,j} \frac{\partial \tilde{u}_k}{\partial x_j} \, d\Omega_0 &= 0, \\
&\quad \text{for all } \tilde{q} \in \mathcal{Q}(\Omega_0)
\end{align*}
\]

Recalling from the talk of A. Manzoni...
Parametric Coupling of Fluid and Structure

Standard iterative scheme for fluid-structure coupling

\[ \Omega(\eta^k) \rightarrow (u^k, p^k) \]

Stokes update

geometry \[ \eta^{k+1} \]

fluid residual \[ R(u^k, p^k) \]

structural equation

Our parametric coupling approach

\[ \mu^k \rightarrow \Omega(\mu^k) \rightarrow (\tilde{u}^k, \tilde{p}^k) \]

parametric domain Stokes

update parameters \[ \mu^{k+1} \]

least squares fit \[ \hat{\eta} \]

structure \[ R(\mu^k) \]
Fixed-point algorithm for weak parametric coupling

Start with initial guess for the parameter $\mu^0$ and set $k = 0$.

1. **[Fluid substep]** Solve the discretized fluid problem in $\Omega_0$ to obtain $(\tilde{u}_h(\mu^k), \tilde{p}_h(\mu^k))$

2. Form the discrete fluid residual

   $$ R(\mu^k) := G \left[ F - A\tilde{u}_h^k - B\tilde{p}_h^k \right] $$

   where $G(\tilde{v}_h)$ takes the boundary normal trace of any $\tilde{v}_h$ on $\Sigma_0$

3. **[Structure substep]** Solve for assumed structural displacement $\hat{\eta}(\mu^k) \in \mathcal{S}$ s.t.

   $$ C(\hat{\eta}_h, \phi) = \langle R(\mu^k), \phi \rangle \quad \text{for all } \phi \in \mathcal{S} $$

4. **[Parametric projection substep]** Solve “inverse problem” of finding parameter value that gives best fit to the assumed displacement $\hat{\eta}(\mu^k)$

   $$ \mu^{k+1} := \arg\min_{\bar{\mu}} \int_{\Sigma} |\eta_h(\bar{\mu}) - \hat{\eta}_h(\mu^k)|^2 \, d\Gamma $$

   to obtain next parameter value. Displacement $\eta_h(x; \bar{\mu}) = T(x; \bar{\mu}) - T(x; 0)$ is given by the FFD and requires no structural equation solutions.

5. Iterate until $|\mu^{k+1} - \mu^k| < \text{TOL}$.
Reduction Step #2: Reduced Basis Methods for Parametric PDEs

- **Problem:** FE solution \((u_h(\mu), p_h(\mu)) \in \mathcal{V}_h \times \mathcal{Q}_h\) too expensive to compute for many different values of \(\mu\).

- **Observation:** Dependence of the bilinear forms \(A(\cdot, \cdot; \mu)\) and \(B(\cdot, \cdot; \mu)\) on \(\mu\) is smooth \(\Rightarrow\) parametric manifold of solutions in \(\mathcal{V}_h \times \mathcal{Q}_h\) is smooth

- **Solution:** Choose a representative set of parameter values \(\mu^1, \ldots, \mu^N\) with \(N \ll \mathcal{N}\)

- Snapshot solutions \(u_h(\mu^1), \ldots, u_h(\mu^N)\) span a subspace \(\mathcal{V}^N_h\) for the velocity and \(p(\mu^1), \ldots, p(\mu^N)\) span a subspace \(\mathcal{Q}^N_h\) for the pressure

### Galerkin reduced basis formulation

For given parameter vector \(\mu \in \mathcal{D}\) find approximate solution \(u^N_h(\mu) \in \mathcal{V}^N_h\) and \(p^N_h(\mu) \in \mathcal{Q}^N_h\) in reduced spaces such that

\[
A(u^N_h(\mu), v; \mu) + B(p^N_h(\mu), v; \mu) = \langle F_h(\mu), v \rangle \quad \text{for all } v \in \mathcal{V}^N_h
\]

\[
B(q, u^N_h(\mu); \mu) = 0 \quad \text{for all } q \in \mathcal{Q}^N_h
\]

Recalling from the talk of **A. Manzoni**...
Comparison Between Finite Element and Reduced Basis Methods

FE basis functions
- Locally supported
- Generic, work for many problems
- A priori estimates readily available

RB basis functions
- Globally supported
- Constructed for specific problem
- A posteriori estimates to guarantee reliability and accuracy
Algorithm for Building the Reduced Basis

Greedy Algorithm \([GP05,RHP08,RV07]\)

1. Large (but finite) training set of parameters \(\Xi_{\text{train}} \subset \mathcal{D}\)

2. Choose first snapshot \(\mu^1\) and obtain first approximation space for velocity \(\mathcal{V}_h^1 = \text{span}(\mathbf{u}_h(\mu^1))\) and pressure \(\mathcal{Q}_h^1 = \text{span}(p_h(\mu^1))\)

3. Next snapshot is chosen as

\[
\mu^n = \arg\max_{\mu \in \Xi_{\text{train}}} \Delta_{n-1}(\mu),
\]

where \(\Delta_n(\mu)\) is an efficiently computable upper bound for the error

\[
\varepsilon_n(\mu) := \inf_{\mathbf{u}_h^n(\mu) \in \mathcal{V}_h^n} ||\mathbf{u}_h(\mu) - \mathbf{u}_h^n(\mu)||_1 \leq \Delta_n(\mu)
\]

4. Construct next spaces \(\mathcal{V}_h^n = \text{span}(\mathbf{u}_h(\mu^1), \ldots, \mathbf{u}_h(\mu^n))\) and \(\mathcal{Q}_h^n = \text{span}(p_h(\mu^1), \ldots, p_h(\mu^n))\).

Repeat from 3 until upper bound of error \(\Delta\) sufficiently small.

Finally we perform Gram-Schmidt to obtain a basis \(\{\xi_{Vn}^N\}_{n=1}^N\) for the velocity space \(\mathcal{V}_h^N\) and a basis \(\{\xi_{Pn}^N\}_{n=1}^N\) for the pressure space \(\mathcal{Q}_h^N\). To stabilize the reduced velocity-pressure pair it is necessary to add the so called “supremizer” solutions to the velocity space \([RV07]\). Total RB dimension is therefore \(3N\).
Are There Computational Savings in Practice?

- Assembly of RB system can depend on $\mathcal{N} \Rightarrow$ no computational savings are realized
- Assumption of **affine parameterization**

\[
A(v, w; \mu) = \sum_{m=1}^{M_a} \Theta^m_a(\mu) A^m(v, w), \quad B(p, w; \mu) = \sum_{q=1}^{M_b} \Theta^m_b(\mu) B^m(p, w)
\]

leads to a split

\[
A(\xi^v_n, \xi^v_n'; \mu) = \sum_{M=1}^{M_a} \Theta^m_a(\mu) A^m(\xi^v_n, \xi^v_n'), \quad B(\xi^p_n, \xi^v_n'; \mu) = \sum_{m=1}^{M_b} \Theta^m_b(\mu) B^m(\xi^p_n, \xi^v_n')
\]

so that the matrices $A^m$ and $B^m$ do not depend on $\mu$ and can be precomputed (offline stage)
- After precomputation, RB system assembly and solution independent from $\mathcal{N}$ (online stage)
- When parameterization is nonaffine, use Empirical Interpolation Method [BMNP04]

For any $\mu \in \mathcal{D}$ find reduced velocity $u_N(\mu)$ and reduced pressure $p_N(\mu)$ s.t.

\[
\left( \sum_{m=1}^{M_a} \Theta^m_a(\mu) A^m \right) u_N + \left( \sum_{m=1}^{M_b} \Theta^m_b(\mu) B^m \right) p_N = F(\mu)
\]

\[
\sum_{m=1}^{M_b} \Theta^m_b(\mu) [B^m]^T u_N = 0.
\]
Free-form Deformation Setup for the Model Problem

- 14 × 2 grid of control points (not all shown in figure)
- Ten control points in the top row move in $x_2$-direction, others fixed
- $P = 10$ parameters
- Each parameter varies in range $\mu_i \in [-0.25, 0.25]$ (small deformations)
Results of Model Reduction for the Model Problem

**Parametric coupling**

- Free-form deformation with $P = 10$ parameters
- Fixed point iteration until $|\mu^{k+1} - \mu^k| < 1e-3$
- Coupling accuracy measured with

$$J_k = \int_\Sigma |\eta(\mu^{k+1}) - \hat{\eta}(\mu^k)|^2 d\Gamma$$

- Nodes on the fluid-structure boundary $N_B = 41$
- Reduction in number of coupling variables 4:1

**Reduced basis approximation**

- Greedy algorithm picks $N = 20$ velocity basis functions $\Rightarrow$ total size of reduced basis $3N = 60$
- Basis functions computed using $P_2/P_1$ FEM with $N = 22227$ DOFs
- Reduction in size of Stokes system (FEM vs. RB) is 370:1.

| Iteration step # | Coupling error | Step size $|\mu^{k+1} - \mu^k|$ |
|------------------|----------------|-------------------------------|
| 0                | 3.77e-3        | 2.22e-1                       |
| 1                | 4.39e-3        | 1.72e-1                       |
| 5                | 2.28e-4        | 3.57e-2                       |
| 10               | 4.88e-5        | 1.56e-2                       |
| 14               | 3.08e-6        | 6.99e-4                       |

Fluid-structure Interaction with Parametric Coupling
Coupled reduced problem solved with $P = 3, \ldots, 9$ parameters free, others fixed

Coupling accuracy measured with

$$J_k = \int_{\Sigma} |\eta(\mu^{k+1}) - \hat{\eta}(\mu^k)|^2 d\Gamma$$

until convergence achieve to tolerance $|\mu^{k+1} - \mu^k| < 1e^{-4}$

Logarithmic best-fit line indicates convergence like $P^{-\alpha}$ with $\alpha \approx 0.495$

FFDs are spline-based and possess good general approximation properties

Using $P = 20$ parameters could obtain coupling up to $10^{-9}$ accuracy
Conclusion

- New model reduction approach to steady fluid-structure interaction
  - Parameterization of fluid domain with free-form deformations
  - Parametric coupling of applied traction and structural displacement
  - Reduced basis method for efficient approximation of parametric Stokes solutions

- Fluid system dimension reduction 370:1 compared to FEM
- Coupling variables reduction 4:1 compared to explicit nodal deformations
- Coupling error behaves like $P^{-1/2}$ as number of FFD parameters $P$ increases

- Future work
  - Navier-Stokes equations for the fluid
  - Coupling in stress space instead of displacement space
  - Unsteady problems, ALE formulation with parametric maps
  - Alternatives to free-form deformation parameterization
Thank you for your attention.
References


