LOCAL TO GLOBAL RESULTS FOR SPACES OF BMO TYPE

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Abstract. We study a class of spaces, JN_p, related to BMO in the abstract setting of a metric space with a doubling measure. We obtain a Reimann–Rychener type local to global result and also show that a Boman type condition is sufficient to make the embedding of JN_p into weak L^p to hold. We discuss certain necessary condition. Our abstract setting covers various situations at once and, in particular, includes all geodesic spaces.

1. Introduction

In addition to the classical space BMO, John and Nirenberg introduced a class of larger spaces, which we refer to as JN_p, in their article [19]. Whereas BMO has been studied a lot, these JN_p spaces have not been investigated so systematically. In the original definition a function $u \in L^1(Q)$ is in the class $JN_p(Q)$, where Q is a Euclidean cube, if it satisfies

$$K_u^p(Q) := \sup_{\mathcal{W}} \sum_{P \in \mathcal{W}} |P| \left(\fint_P |u - u_P| \, dx \right)^p < \infty, \tag{1.1}$$

where the supremum was taken over all partitions of Q into pairwise disjoint cubes P. John and Nirenberg proved that the functions in $JN_p(Q)$ satisfy the weak L^p estimate

$$\lambda^p \left| \{ x \in Q : |u(x) - u_Q| > \lambda \} \right| \leq C K_u^p(Q),$$

for each $\lambda > 0$, and $C > 0$ is a dimensional constant.

The definitions of JN_p and BMO seem similar, and in fact $\lim_{p \to \infty} K_u = \|u\|_{\mathrm{BMO}}$. Even if this may be the original reason that resulted in their discovery, it is not the only thing to make JN_p interesting. In [8] Campanato gave a proof of the Stampacchia interpolation theorem using the weak L^p estimate of JN_p functions (see also [12] and [11]).

In this paper we are interested in local to global properties of JN_p. The motivation comes from corresponding results about BMO. Given

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a domain $\Omega \subset \mathbb{R}^n$, the classical space $\mathrm{BMO}(\Omega)$ has a local counterpart $\mathrm{BMO}_{\mathrm{loc}}(\Omega)$ defined by

$$\|u\|_{\mathrm{BMO}_{\mathrm{loc}}} := \sup_{\tau B \subset \Omega} \fint_B |u - u_B| \, dx < \infty,$$

where $\tau > 1$. This kind of local conditions arise naturally in context of partial differential equations, but usually the global case $\tau = 1$ is equally interesting. A well-known result by Reimann and Rychener [22] tells that local BMO conditions imply the global one. For further local to global type results we refer to Staples [24] and [25].

Another local to global question is to characterize the domains in which the John–Nirenberg inequality holds globally. For BMO it was proved by Smith and Stegenga [23] and Hurri-Syrjänen [17] that the answer is Hölder domains. The results about both the local to global property of norms and that of the John–Nirenberg inequality have also been generalized to metric measure spaces with some additional structure. See [7] and [20].

A recent paper by Hurri-Syrjänen et al. [18] was probably the first one to discuss local to global questions related to JN_p. They used definition (1.1) on domains $\Omega \subset \mathbb{R}^n$ more general than Euclidean cubes. It was not completely clear what the local counterpart of (1.1) should be, but in [18] it was defined by asking the cubes in (1.1) to have bounded overlap and to have their concentric τ-dilates contained in the domain Ω. With this definition, a Reimann–Rychener type local to global result could be proved. In addition to that, the authors of [18] managed to show that a weak L^p estimate holds at least for John domains.

The first generalization of JN_p spaces from Euclidean to metric spaces was done by Aalto et al. in [1]. Their definition and results were later improved by Berkovits et al. in [2]. The main problem in generalizing the definition (1.1) is to decide how partitions should be interpreted. The clearest definition can be found in [2]. It replaces the partitions into cubes by collections of disjoint balls and also gives a local version where τ-dilates of the balls should still form a collection that satisfies the global defining criterion. See Definition 2.1 in Section 2.

In this paper we will continue the work with JN_p spaces on metric measure spaces by proving some results of [18] in the metric setting. We use the definitions and results of [2]. The proof of the Reimann–Rychener type result is a modification of the original proof in [18]. In order to prove the weak L^p estimate, we will apply a proof described by Franchi et al. in [10], that is, we use Chua's [9] chaining argument in context of weak L^p spaces. This will, together with the weak L^p estimate for balls proved in [2], yield an inequality corresponding to the one proved in [18].

As a short introduction to the subject, we briefly sketch the idea of chaining argument. Domain Ω is covered with balls that have bounded overlap. The distribution function of

$$|u - u_\Omega| \leq |u - u_B| + |u_B - u_{B_0}| + |u_{B_0} - u_\Omega|$$

is controlled by the distribution functions of the three terms on the right-hand side. Difficulties usually have to do with the second term. In case the domain satisfies an appropriate chaining condition, it can be brought back to the first one, for which we already have an estimate.

If u is a BMO function, then $|u_B - u_{B_0}|$ is controlled by the quasihyperbolic distance, which is actually a BMO function itself. This makes it possible to characterize domains with good properties in terms of the quasihyperbolic metric. See [25], [17] and [23]. For JN_p, however, no analogous phenomenon is known.

The paper is organized as follows. In Section 2, we recall the basic definitions. In Section 3, we define Boman sets and prove a lemma needed in the proof of a global John–Nirenberg inequality. In Section 4, we prove the Reimann–Rychener type result, and in Section 5 we prove the global John–Nirenberg type inequality in Boman sets and discuss a necessary condition.

2. Notation and Definitions

Throughout the paper, $X = (X, d, \mu)$ will be a metric measure space with metric d and doubling measure μ. A Borel regular measure is said to be doubling if there exists a constant c_μ such that given any open ball $B \subset X$, we have

$$0 < \mu(2B) \leq c_\mu \mu(B) < \infty.$$

Every open metric ball $B = B(z, r) = \{x \in X : d(x, z) < r\}$ in X always comes with a fixed center $z \in X$ and a radius $r > 0$. In what follows, we write a concentric dilate as $\lambda B = B(z, \lambda r)$, and Ω is always a bounded open subset of X. The integral average of a function u over a set E with positive μ-measure is written as u_E, i.e.

$$u_E = \fint_E u \,\mathrm{d}\mu = \frac{1}{\mu(E)} \int_E u \,\mathrm{d}\mu.$$

Generic constants will be denoted by C without subscript. Their values may vary even within a line. The notation $\lesssim$ means that an inequality holds up to generic constant C.

We follow the definition introduced in [2].

Definition 2.1. Let $\Omega \subset X$ be open, $\tau \geq 1$ and $1 < p < \infty$. An $L^1(\Omega)$ function u is in the space $JN_{p,\tau}(\Omega)$ if

$$K^p_{u,\tau}(\Omega) := \sup_{\mathcal{F}} \sum_{\tau B \in \mathcal{F}} \mu(B) \left(\fint_B |u - u_B| \,\mathrm{d}\mu \right)^p < \infty$$

where $\mathcal{F}$ denotes any collection of pairwise disjoint balls contained in Ω. If $\tau = 1$, it can be omitted in the notation.

We will consider functions that are in weak L^p on certain sets. Thus the standard notation

$$\|u\|_{L^{p,\infty}(E)} = \sup_{\lambda>0} \lambda\mu\left(\{x \in E : |u(x)| > \lambda\}\right)^{1/p}$$

will be used to clarify the arguments.

3. Boman Sets

We begin by defining a class of sets that satisfy a condition closely related to the Boman chain condition introduced in his unpublished paper [4]. That, in turn, is known to characterize John domains in Euclidean and many metric measure spaces (see [6]). Thus the following class of sets will be a natural substitute to John domains as we generalize results of [18] to metric measure spaces.

Definition 3.1 (Boman set)**.** A set Ω is called *Boman* if there are constants $C_2 > C_1 > 1$, $C_3 > 1, \lambda > 1$ and $M \in \mathbb{N}$ and a collection of pairwise disjoint balls $\mathcal{F}$ such that

(i) $\Omega = \bigcup_{B\in\mathcal{F}} C_1 B = \bigcup_{B\in\mathcal{F}} C_2 B$.
(ii) If $B \in \mathcal{F}$, there are at most M balls $V \in \mathcal{F}$ with $C_2 V \cap C_2 B \neq \varnothing$.
(iii) There is a central ball $B_* \in \mathcal{F}$ such that for each $B \in \mathcal{F}$ there exists a finite *chain* of balls $\mathcal{C}(B) = \{B_i\}_{i=1}^{k_B} \subset \mathcal{F}$ with $B_1 = B_*$ and $B_{k_B} = B$.
(iv) In a chain $\mathcal{C}(B)$, for each pair of balls B_i and B_{i-1} corresponding to consecutive indices there exists a ball $D_i \subset C_1 B_i \cap C_1 B_{i-1}$ such that $\mu(D_i) \geq C_3(\mu(B_i) + \mu(B_{i-1}))$.
(v) If $V \in \mathcal{C}(B)$, then $B \subset \lambda V$.

The constants C_1, C_2, C_3, M and λ will be called Boman parameters of Ω, and any chain $\mathcal{C}(B)$ satisfying the conditions above will be called a Boman chain of B.

The main difference between the original Boman chain condition and our definition is the way the C_2-dilates are allowed to overlap. In the classical version a point could belong to at most M balls, but here we ask every ball to meet at most M balls. For instance, in a geodesic metric space with a doubling measure both definitions yield the same class of domains. This can be seen essentially by applying the results in [6] Lemma 3.2 there replaced by a modified version of a Whitney type decomposition.

The John–Nirenberg lemma for JN_p is an embedding into weak L^p. As we study its local to global properties, the interplay between $L^{p,\infty}$ and geometry of Boman sets will play an important role. The following lemma is a version of an argument used by Chua in [9].

Lemma 3.2. *Let Ω be a Boman set and $1 < p < \infty$. Then there exists a constant C depending only on p, c_μ and the Boman parameters of Ω so that*

$$\sum_{B\in\mathcal{F}} \|u_{C_1B} - u_{C_1B_*}\|^p_{L^{p,\infty}(C_1B)} \leq C \sum_{V\in\mathcal{F}} \|u - u_{C_1V}\|^p_{L^{p,\infty}(C_1V)}.$$

Proof. It is well-known that even if $\|\cdot\|_{L^{p,\infty}(E)}$ is merely a quasinorm, there exists a norm equivalent to it. See e.g. [13]. This fact will allow us to iterate the (quasi-)triangle inequality so that the number of applications does not affect the constant we get.

Denote the collection of Boman balls of Ω by $\mathcal{F}$. Take any $B \in \mathcal{F}$ and form its Boman chain $\{B_i\}_{i=1}^{k_B}$. Then, because of the preceding remark, we have

$$\|u_{C_1B} - u_{C_1B_*}\|_{L^{p,\infty}(C_1B)} \leq C \sum_{i=2}^{k_B} \left\|u_{C_1B_i} - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B)}$$

with C not depending on choice of $B \in \mathcal{F}$.

Each term in the sum is $L^{p,\infty}$ quasinorm of a constant so we may directly compute

$$\begin{aligned}
&\left\|u_{C_1B_i} - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B)} = \left|u_{C_1B_i} - u_{C_1B_{i-1}}\right| \mu\left(C_1B\right)^{1/p} \\
&= \left|u_{C_1B_i} - u_{C_1B_{i-1}}\right| \mu\left(C_1B_i \cap C_1B_{i-1}\right)^{1/p} \left(\frac{\mu\left(C_1B\right)}{\mu\left(C_1B_i \cap C_1B_{i-1}\right)}\right)^{1/p}
\end{aligned}$$

and further

$$\begin{aligned}
&\left|u_{C_1B_i} - u_{C_1B_{i-1}}\right| \mu\left(C_1B_i \cap C_1B_{i-1}\right)^{1/p} \\
&= \left\|u_{C_1B_i} - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B_i\cap C_1B_{i-1})} \\
&\leq C\left(\|u - u_{C_1B_i}\|_{L^{p,\infty}(C_1B_i\cap C_1B_{i-1})} + \left\|u - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B_i\cap C_1B_{i-1})}\right) \\
&\leq C\left(\|u - u_{C_1B_i}\|_{L^{p,\infty}(C_1B_i)} + \left\|u - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B_{i-1})}\right).
\end{aligned}$$

Now, using part (iv) of the Boman condition, we get

$$\begin{aligned}
&\sum_{i=2}^{k_B} \left\|u_{C_1B_i} - u_{C_1B_{i-1}}\right\|_{L^{p,\infty}(C_1B)} \\
&\qquad\qquad \leq C \sum_{i=1}^{k_B} \left(\frac{\mu\left(B\right)}{\mu\left(C_1B_i\right)}\right)^{1/p} \|u - u_{C_1B_i}\|_{L^{p,\infty}(C_1B_i)}.
\end{aligned}$$

Part (v) of the Boman condition states that $B \subset \lambda B_i$ for all chain balls $B_i \in \mathcal{C}(B)$. Thus $\chi_B \leq \chi_{\lambda B_i}$, and

$$\begin{aligned}
&\frac{\chi_B(x)}{\mu(B)^{1/p}} \|u_{C_1B} - u_{C_1B_*}\|_{L^{p,\infty}(C_1B)} \\
&\qquad \leq C \sum_{i=1}^{k_B} \frac{\chi_B(x)}{\mu(C_1B_i)^{1/p}} \|u - u_{C_1B_i}\|_{L^{p,\infty}(C_1B_i)} \\
&\qquad \leq C \sum_{V \in \mathcal{F}} \frac{\chi_{\lambda V}(x)}{\mu(V)^{1/p}} \|u - u_{C_1V}\|_{L^{p,\infty}(C_1V)},
\end{aligned}$$

where C depends only on p, c_μ and the Boman parameters of Ω.

Plugging these estimates into the original quantity to be controlled and using the fact that Boman balls are pairwise disjoint, we get

$$\begin{aligned}
&\sum_{B \in \mathcal{F}} \|u_{C_1B} - u_{C_1B_*}\|^p_{L^{p,\infty}(C_1B)} \\
&\quad = \sum_{B \in \mathcal{F}} \int_B \frac{\chi_B(x)}{\mu(B)} \|u_{C_1B} - u_{B_*}\|^p_{L^{p,\infty}(C_1B)} \, \mathrm{d}\mu \\
&\quad \leq C \sum_{B \in \mathcal{F}} \int_B \left(\sum_{V \in \mathcal{F}} \frac{\chi_{\lambda V}(x)}{\mu(V)^{1/p}} \|u - u_{C_1V}\|_{L^{p,\infty}(C_1V)} \right)^p \mathrm{d}\mu \\
&\quad \leq C \int_\Omega \left(\sum_{V \in \mathcal{F}} \frac{\chi_{\lambda V}(x)}{\mu(V)^{1/p}} \|u - u_{C_1V}\|_{L^{p,\infty}(C_1V)} \right)^p \mathrm{d}\mu.
\end{aligned}$$

In order to conclude the proof, we make use of a classical fact. The inequality

$$\left\| \sum_{B \in \mathcal{F}} a_B \chi_{\lambda B} \right\|_p \lesssim_{\lambda,p} \left\| \sum_{B \in \mathcal{F}} a_B \chi_B \right\|_p \tag{3.1}$$

holds for any collection of balls $\mathcal{F}$ and any collection of positive numbers $\{a_B\}_{B \in \mathcal{F}}$ whenever $1 < p < \infty$ and $\lambda \geq 1$. See [9] or [25]. By (3.1) and Hölder's inequality,

$$\begin{aligned}
&\int_\Omega \left(\sum_{V \in \mathcal{F}} \frac{\chi_{\lambda V}(x)}{\mu(V)^{1/p}} \|u - u_{C_1V}\|_{L^{p,\infty}(C_1V)} \right)^p \mathrm{d}\mu \\
&\leq C \int_\Omega \left(\sum_{V \in \mathcal{F}} \frac{\chi_V(x)}{\mu(V)^{1/p}} \|u - u_{C_1V}\|_{L^{p,\infty}(C_1V)} \right)^p \mathrm{d}\mu \\
&\leq C \int_\Omega \left(\sum_{V \in \mathcal{F}} \frac{\chi_V(x)}{\mu(V)} \|u - u_{C_1V}\|^p_{L^{p,\infty}(C_1V)} \right) \left(\sum_{V \in \mathcal{F}} \chi_V(x) \right)^{p/p'} \mathrm{d}\mu \\
&\leq C \sum_{V \in \mathcal{F}} \|u - u_{C_1V}\|^p_{L^{p,\infty}(C_1V)},
\end{aligned}$$

which is the desired inequality. □

4. A Local to Global Result

In their paper [18] Hurri-Syrjänen et al. formulated a chain decomposition lemma for John domains. Its essential content was roughly equivalent to the definition of Boman sets. Using the decomposition, they managed to prove a local to global result for JN_p spaces. They could, in addition, prove that functions in JN_p on John domains satisfy a weak L^p estimate.

Using the chain condition 3.1 instead of the chain decomposition in [18], we will generalize both results to metric measure spaces with doubling measure. The embedding into weak L^p will work without further assumptions about the underlying measure space, but the equality of global and local spaces $JN_{p,\tau}(\Omega)$ requires something more. One sufficient condition is that all metric balls are Boman sets with uniform parameters. It should be noted that this requirement is a condition on the space instead of the domain. The uniform Boman condition on metric balls is satisfied, for example, by all *geodesic spaces*. See [16].

Whereas Lemma 3.2 was a statement about Boman sets and $L^{p,\infty}$, the following lemma is about corresponding phenomena between Boman sets and JN_p. It is together with its proof almost identical to Lemma 3.2 in [18].

Lemma 4.1. *Let Ω be a Boman set with parameters C_1, C_2, C_3, λ and M as in definition 3.1. Let $1 < p < \infty$ and $u \in L^1(\Omega)$. Let $\mathcal{F}$ be the collection of Boman balls of Ω, and denote the central Boman ball by B_*. Then*

$$\left(\fint_\Omega |u - u_\Omega| \,\mathrm{d}\mu\right)^p + \left(\fint_\Omega |u - u_{C_1 B_*}| \,\mathrm{d}\mu\right)^p \le \frac{1}{\mu(\Omega)} \sum_{P \in \mathcal{F}} \mu(C_1 P) \left(\fint_{C_1 P} |u - u_{C_1 P}| \,\mathrm{d}\mu\right)^p,$$

where C depends only on p, c_μ and the Boman parameters of Ω.

Proof. Clearly,

$$\begin{aligned}
\int_\Omega |u - u_\Omega| \,\mathrm{d}\mu &\le 2 \int_\Omega |u - u_{C_1 B^*}| \,\mathrm{d}\mu \\
&\le 2 \sum_{B \in \mathcal{F}} \left(\int_{C_1 B} |u - u_{C_1 B}| \,\mathrm{d}\mu + \mu(C_1 B) |u_{C_1 B} - u_{C_1 B^*}| \right) \\
&= 2 \sum_{B \in \mathcal{F}} \int_{C_1 B} |u - u_{C_1 B}| \,\mathrm{d}\mu + 2 \sum_{B \in \mathcal{F}} \mu(C_1 B) |u_{C_1 B} - u_{C_1 B^*}| \\
&=: 2S_1 + 2S_2. \qquad (4.1)
\end{aligned}$$

In order to estimate S_2, we form a Boman chain $\mathcal{C}(B) = \{B_i\}_{i=1}^{k_B}$ for every ball in the sum. Using (iv) of Definition 3.1, we get

$$\begin{aligned}
\sum_{B\in\mathcal{F}} \mu(C_1B)\,|u_{C_1B} - u_{C_1B^*}| &\le \sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{i=2}^{k_B} \left|u_{C_1B_i} - u_{C_1B_{i-1}}\right| \\
&= \sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{i=2}^{k_B} \fint_{C_1B_i\cap C_1B_{i-1}} \left|u_{C_1B_i} - u_{C_1B_{i-1}}\right| \,\mathrm{d}\mu \\
&\le \sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{i=2}^{k_B} \Bigg(\fint_{C_1B_i\cap C_1B_{i-1}} \left|u - u_{C_1B_{i-1}}\right| \,\mathrm{d}\mu \\
&\qquad\qquad + \fint_{C_1B_i\cap C_1B_{i-1}} \left|u - u_{C_1B_i}\right| \,\mathrm{d}\mu\Bigg) \\
&\le C \sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{i=2}^{k_B} \left(\fint_{C_1B_{i-1}} \left|u - u_{C_1B_{i-1}}\right| \,\mathrm{d}\mu + \fint_{C_1B_i} \left|u - u_{C_1B_i}\right| \,\mathrm{d}\mu\right) \\
&\le C \sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{V\in\mathcal{C}(B)} \fint_{C_1V} |u - u_{C_1V}| \,\mathrm{d}\mu. \qquad (4.2)
\end{aligned}$$

Part (v) of the chain condition states that if $V \in \mathcal{C}(B)$, then $B \subset \lambda V$. Thus the collection $\mathcal{S}(V) = \{B \in \mathcal{F} : V \in \mathcal{C}(B)\}$ satisfies

$$\bigcup_{B\in\mathcal{S}(V)} B \subset \lambda V. \qquad (4.3)$$

Changing order of summation in (4.2) and using (4.3) together with the doubling condition, we obtain

$$\begin{aligned}
&\sum_{B\in\mathcal{F}} \mu(C_1B) \sum_{V\in\mathcal{C}(B)} \fint_{C_1V} |u - u_{C_1V}| \,\mathrm{d}\mu \\
&\le C \sum_{V\in\mathcal{F}} \left(\sum_{B\in\mathcal{S}(V)} \mu(B)\right) \fint_{C_1B} |u - u_{C_1B}| \,\mathrm{d}\mu \\
&\le C \sum_{V\in\mathcal{F}} \int_{C_1V} |u - u_{C_1V}| \,\mathrm{d}\mu,
\end{aligned}$$

and hence $S_1 + S_2 \lesssim S_1$.

It remains to estimate S_1. Applying Hölder's inequality, we see that

$$\begin{aligned}
&\sum_{B\in\mathcal{F}}\int_{C_1B}|u-u_{C_1B}|\,\mathrm{d}\mu\\
&=\sum_{V\in\mathcal{F}}\mu(C_1B)^{1/p'}\mu(C_1B)^{-1/p'}\int_{C_1B}|u-u_{C_1B}|\,\mathrm{d}\mu\\
&\le\left(\sum_{B\in\mathcal{F}}\mu(C_1B)\right)^{1/p'}\left(\sum_{B\in\mathcal{F}}\mu(C_1B)^{1-p}\left(\int_{C_1B}|u-u_{C_1B}|\,\mathrm{d}\mu\right)^p\right)^{1/p}\\
&\lesssim\mu(\Omega)^{1/p'}\left(\sum_{B\in\mathcal{F}}\mu(C_1B)\left(\fint_{C_1B}|u-u_{C_1B}|\,\mathrm{d}\mu\right)^p\right)^{1/p}
\end{aligned}$$

which proves the claim. □

The equality of local and global JN_p spaces is just a simple corollary provided that all metric balls are Boman sets with uniform parameters. C_1-dilates of Boman balls will do in the definition of local JN_p.

Theorem 4.2. *Let X be a metric measure space with a doubling measure μ such that all balls are Boman sets with parameters C_1, C_2, C_3, M and λ. Then for every open $\Omega\subset X$ we have $JN_{p,\tau}(\Omega)=JN_p(\Omega)$ provided $\tau\le C_2/C_1$.*

Proof. We define $JN_{p,\tau,M}(\Omega)$ as the space defined by the same condition as in Definition 2.1 but by replacing $\mathcal{F}$ with a collection of balls included in Ω such that each ball meets at most M balls in the same collection. A collection of balls satisfying this can be divided into at most M collections of pairwise disjoint balls. Thus we have that $JN_{p,\tau}(\Omega)\subset JN_{p,\tau,M}(\Omega)$.

In order to prove that $JN_{p,\tau,M}(\Omega)\subset JN_p(\Omega)$, take $u\in JN_{p,\tau,M}(\Omega)$ and an arbitrary collection of pairwise disjoint balls contained in Ω. Denote it by $\mathcal{D}$. For each $B\in\mathcal{D}$ form its collection of Boman balls $\mathcal{F}(B)$. The C_1-dilates of these Boman balls form a collection accessible in the defining condition of $JN_{p,\tau,M}(\Omega)$. Using Lemma 4.1, we get

$$\begin{aligned}
&\sum_{B\in\mathcal{D}}\mu(B)\left(\fint_B|u-u_B|\,\mathrm{d}\mu\right)^p\\
&\le C\sum_{B\in\mathcal{D}}\mu(B)\frac{1}{\mu(B)}\sum_{P\in\mathcal{F}(B)}\mu(C_1P)\left(\fint_{C_1P}|u-u_{C_1P}|\,\mathrm{d}\mu\right)^p\\
&=C\sum_{B\in\mathcal{J}}\mu(B)\left(\fint_B|u-u_B|\,\mathrm{d}\mu\right)^p\\
&\le CK^p_{u,\tau,M}(\Omega).
\end{aligned}$$

Thus $JN_{p,\tau,M}(\Omega)\subset JN_p(\Omega)$. The remaining inclusion $JN_p(\Omega)\subset JN_{p,\tau}(\Omega)$ is clear. □

5. A John–Nirenberg Inequality

In their paper [2] Berkovits et al. generalized the weak type estimate of functions belonging to JN_p to metric measure spaces. More specifically, they proved that in case a function is in $JN_p(\hat{B})$ with $\hat{B}$ a ball, then it is in $L^{p,\infty}(B)$ where B is a proper concentric subball of $\hat{B}$.

Theorem 5.1 (Berkovits–Kinnunen–Martell [2])**.** *Let $u \in JN_p(\Omega)$. If $\hat{B} = \tau B \subset \Omega$ for some $\tau > 1$, then for every $\lambda > 0$*

$$\lambda^p \mu\left(\{x \in B : |u - u_B| > \lambda\}\right) \leq C K_u^p(\hat{B}),$$

where C depends only on τ and the doubling constant.

5.1. **A Sufficient Condition.** Theorem 5.1 together with Lemma 3.2 constitutes the proof of a global weak L^p estimate in Boman sets. To give a detailed proof of the weak L^p estimate, we will apply an argument first described by Franchi et al. in [10, see pp. 133–134], that is, we use Chua's [9] chaining argument in context of weak L^p spaces.

Theorem 5.2. *Let Ω be a Boman set with parameters C_1, C_2, C_3, λ and M as in definition 3.1. Let $u \in JN_p(\Omega)$. Then*

$$\lambda^p \mu\left(\{x \in \Omega : |u - u_\Omega| > \lambda\}\right) \leq C K_u^p(\Omega),$$

where C depends only on p, the doubling constant and the Boman parameters of Ω.

Proof. Let $\mathcal{F}$ be the collection of Boman balls. Denote the central ball by B_*. If $\lambda > 0$, then

$$\begin{aligned}
\lambda^p \mu(\{x \in \Omega : |u - u_{C_1 B_*}| > \lambda\}) &\leq \sum_{B \in \mathcal{F}} \lambda^p \mu\left(\{x \in C_1 B : |u - u_{C_1 B_*}| > \lambda\}\right) \\
&\lesssim \sum_{B \in \mathcal{F}} \left(\frac{\lambda}{2}\right)^p \mu\left(\{x \in C_1 B : |u - u_{C_1 B}| > \frac{\lambda}{2}\}\right) \\
&\quad + \sum_{B \in \mathcal{F}} \left(\frac{\lambda}{2}\right)^p \mu\left(\{x \in C_1 B : |u_{C_1 B} - u_{C_1 B_*}| > \frac{\lambda}{2}\}\right) \\
&=: S_1 + S_2.
\end{aligned}$$

Notice that the balls $C_1 B$ satisfy the assumption of Theorem 5.1. Thus using the definition of weak L^p norm, Lemma 3.2 and Theorem 5.1, we may estimate

$$\begin{aligned}
S_2 &\leq \sum_{B \in \mathcal{F}} \|u_{C_1 B} - u_{C_1 B_*}\|^p_{L^{p,\infty}(C_1 B_1)} \leq C \sum_{B \in \mathcal{F}} \|u - u_{C_1 B}\|^p_{L^{p,\infty}(C_1 B_1)} \\
&= C \sum_{B \in \mathcal{F}} \sup_{\lambda > 0} \lambda^p \mu\left(\{x \in C_1 B : |u - u_{C_1 B}| > \lambda\}\right) \leq C \sum_{B \in \mathcal{F}} K_u^p(C_2 B)
\end{aligned}$$

and

$$S_1 \leq C \sum_{B \in \mathcal{F}} K_u^p(C_2 B).$$

The collection $\{C_2B\}_{B\in\mathcal{F}}$ consists of balls that meet at most M balls of the same collection. We may decompose it into at most M collections of pairwise disjoint balls $\mathcal{D}_i, i = 1, \dots, M$ such that $\{C_2B\}_{B\in\mathcal{F}} = \bigcup_{i=1}^{M} \mathcal{D}_i$. This leads to the estimate

$$S_1 + S_2 \le C \sum_{i=1}^{M} \sum_{B\in\mathcal{D}_i} K_u^p(B) \le M K_u^p(\Omega). \tag{5.1}$$

On the other hand, we have that

$$\begin{aligned} \lambda^p \mu(\{x \in \Omega : |u_{C_1B_*} - u_\Omega| > \lambda\}) &\le \lambda^p \int_\Omega \left(\frac{|u_{C_1B_*} - u_\Omega|}{\lambda} \right)^p \mathrm{d}\mu \\ &\le \mu(\Omega) \left(\fint_\Omega |u - u_{C_1B_*}| \right)^p \mathrm{d}\mu \le C K_u^p(\Omega), \end{aligned} \tag{5.2}$$

where we used Lemma 4.1 and the fact that the collection of dilated Boman balls can be decomposed into at most M collections of pairwise disjoint balls. With estimates (5.1) and (5.2) we conclude that

$$\begin{aligned} \lambda^p \mu(\{x \in \Omega : |u - u_\Omega| > \lambda\}) &\le \lambda^p \mu(\{x \in \Omega : |u - u_{C_1B_*}| > \lambda/2\}) \\ &\quad + \lambda^p \mu(\{x \in \Omega : |u_{C_1B_*} - u_\Omega| > \lambda/2\}) \\ &\le C(S_1 + S_2 + K_u^p(\Omega)) \le C K_u^p(\Omega), \end{aligned}$$

which proves the claim. □

5.2. **A Necessary Condition.** We close the paper by a discussion on a necessary condition. It was shown in [18, Theorem 5.1] that if the global weak L^p estimate holds, then the domain necessarily supports a Poincaré inequality. In [18, Corollary 5.6], on the other hand, a necessary condition is formulated in terms of the John condition. This result is strongly based on a separation property and the results in [5] that do not seem to carry over to the generality of the present paper. However, provided that the underlying metric space X is assumed to support a (p, p)-Poincaré inequality with suitable dilatation constant, the aforementioned result in [18, Theorem 5.1] has an analogue in more general metric measure spaces.

Let $q \ge 1$. A metric space X is said to support a (q, p)-Poincaré inequality if there exist constants $C_{PI} > 0$ and dilatation constant $\tau \ge 1$ such that for every ball $B \subset X$, all integrable functions u on X, and all upper gradients g of u,

$$\left(\fint_B |u - u_B|^q \,\mathrm{d}\mu \right)^{1/q} \le C_{PI} \operatorname{diam}(B) \left(\fint_{\tau B} g^p \,\mathrm{d}\mu \right)^{1/p}$$

with C_{PI} independent of the ball B. We refer to [3] for a detailed account of upper gradients and metric spaces supporting Poincaré inequalities.

In his article [14] Hajłasz discussed an interesting consequence of the Maz'ya truncation argument presented in [21]. A Poincaré inequality can be deduced from a corresponding distributional inequality where the L^q norm on the left hand side is replaced by the $L^{q,\infty}$ norm. More precisely, let $1 < p \le q < \infty$ be fixed. If u is in weak $L^q(\Omega)$ and g in $L^p(\Omega)$ then the condition

$$\inf_{c\in\mathbb{R}} \sup_{\lambda>0} \lambda^q \mu\left(\{x \in \Omega : |u(x) - c| > \lambda\}\right) \le C \left(\int_\Omega g^p \,\mathrm{d}\mu\right)^{q/p}$$

is equivalent to

$$\inf_{c\in\mathbb{R}} \left(\int_\Omega |u-c|^q \,\mathrm{d}\mu\right)^{1/q} \le C \left(\int_\Omega g^p \,\mathrm{d}\mu\right)^{1/p}.$$

We refer also to [16, Section 2] and [15].

We close this paper with the following observation which gives a necessary condition for a weak type inequality.

Proposition 5.3. *Let $1 < p < \infty$ and $\tau_0 \ge 1$ be fixed and let X be a metric measure space with a doubling measure μ and which supports a (p,p)-Poincaré inequality with dilatation constant $\tau \le \tau_0$. Suppose that for every $u \in L^1(\Omega)$*

$$\inf_{c\in\mathbb{R}} \sup_{\lambda>0} \lambda^p \mu\left(\{x \in \Omega : |u(x) - c| > \lambda\}\right) \le C K^p_{u,\tau_0}(\Omega).$$

If g is an upper gradient of u, then

$$\inf_{c\in\mathbb{R}} \left(\int_\Omega |u-c|^p \,\mathrm{d}\mu\right)^{1/p} \le C \left(\int_\Omega g^p \,\mathrm{d}\mu\right)^{1/p}.$$

Proof. Let $\mathcal{F}$ be a collection of pairwise disjoint balls contained in Ω (as in Definition 2.1). Then by Hölder's inequality and the (p,p)-Poincaré inequality

$$\begin{aligned}\inf_{c\in\mathbb{R}} \sup_{\lambda>0} \lambda^p \mu\left(\{x \in \Omega : |u(x) - c| > \lambda\}\right) &\lesssim \sum_{\tau_0 B\in\mathcal{F}} \mu(B)\operatorname{diam}(B)^p \fint_{\tau B} g^p \,\mathrm{d}\mu \\ &\lesssim \operatorname{diam}(\Omega)^p \sum_{\tau_0 B\in\mathcal{F}} \int_{\tau B} g^p \,\mathrm{d}\mu \le C \int_\Omega g^p \,\mathrm{d}\mu,\end{aligned}$$

where C depends on C_{PI} and on the diameter of Ω. This together with the above discussion proves the claim. □

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