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THE STABILIZED MITC PLATE BENDING ELEMENTS

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Abstract. *Three new families of finite element methods for the Reissner-Mindlin plate bending model are described. The methods are based on a combination of the stabilized formulation presented in* [25] *and the MITC reduction technique* [6]*. The families use identical basis functions for the deflection and the rotation. Optimal order of convergence, independent of the plate thickness, is proved. The theoretical results are confirmed by numerical computations.*

1 INTRODUCTION

In this paper we present our stabilized MITC plate bending elements. In these methods we combine the shear projection technique of the original MITC elements [6] with recent stabilized formulations [16, 25]. The advantage of this, compared to both the MITC elements and the previous stabilized formulations, is that identical shape functions can be used for all unknows. Compared to more traditional methods, a stabilized formulation gives a more well conditioned stiffness matrix. Another big advantage of these new families of methods is that they include convergent triangular linear and quadrilateral bilinear elements. These lowest order elements were introduced in [9] in connection with a general analysis of the MITC elements. In that context, the modification was in the spirit of the "trick" introduced by Fried and Yang already in 1973 [13], and more recently analyzed by Pitkäranta [22]. This is, however, not more the case when the methods are viewed as stabilized formulations. Then, they arise from a very systematic approach, cf. [16, 25] and the presentation below. Recently we have used the same approach for designing methods for the Naghdi shell model in a bending dominated state [11].

Recently, Lyly has observed [19] (cf. [18] in these proceedings) that our linear triangular element is equivalent to an earlier formulation (which from the outset looks different) given by Tessler and Hughes [27]. Later the formulation of Hughes and Taylor has been rediscovered by Xu, Aurichhio and Taylor [29, 26, 4].

The plan of the paper is as follows: in the next two sections we introduce our notation and present the elements. In Section 4 we perform an error analysis of the methods. In the analysis we use a mesh dependent norm in which we show the methods to be uniformly stable with respect to the thickness of the plate. The convergence rate is therefore determined by the interpolation error and the consistency error due to the use of the MITC reduction technique. In the estimation of these we, for simplicity, assume that the solution is smooth. We then obtain optimal error estimates which are independent of the thickness of the plate. This means that the methods are completely free from locking. As it is known that the Reissner-Mindlin plate model give rise to strong boundary layers (cf. [3, 1]) the assumption of a smooth solution is of course unrealistic. The inclusion of an analysis of this is, however, out of the scope of the present paper (in a recent paper by Pitkäranta and Suri [23] this is done for the original MITC elements). Finally, in Section 5, we give the results of benchmark computations with the elements. We consider the worst case with respect to locking, i.e. a very thin plate. The numerical results show that the methods converge optimally.

2 NOTATION AND PRELIMINARIES

We consider the Reissner-Mindlin plate bending model and assume that the plate is clamped along its boundary.[1] Denoting the midsurface of the plate by $\Omega \subset \mathbb{R}^2$, the variational problem is: find the deflection $w \in H_0^1(\Omega)$ and the rotation vector $\beta \in [H_0^1(\Omega)]^2$

[1]Clamped boundary conditions are chosen only for notational convinience.

such that

$$Gt^3 a(\beta,\eta) + G\kappa t(\nabla w - \beta, \nabla v - \eta) = (f, v) \qquad \forall (v,\eta) \in [H_0^1(\Omega)]^3. \tag{1}$$

Here G is the shear modulus and κ denotes the shear correction factor. f is the transverse load and t is the thickness of the plate. The bilinear form a is defined as

$$a(\beta,\eta) = \frac{1}{6}\{(\varepsilon(\beta),\varepsilon(\eta)) + (\frac{\nu}{1-\nu})(\operatorname{div}\beta, \operatorname{div}\eta)\}, \tag{2}$$

where $\varepsilon(\cdot)$ is the small strain tensor and ν is the Poisson ratio. As usual, the L_2-inner products are denoted by $(\cdot,\cdot)_D$ and the corresponding norms by $||\cdot||_{0,D}$, with the subscript D dropped when $D = \Omega$.

The shear force $\boldsymbol{Q}$ and bending moment $\boldsymbol{M}$ are obtained from

$$\boldsymbol{Q} = G\kappa t(\nabla w - \beta) \tag{3}$$

and

$$\boldsymbol{M} = \frac{Gt^3}{6}\{\varepsilon(\beta) + (\frac{\nu}{1-\nu})\operatorname{div}\beta\ \boldsymbol{I}\}, \tag{4}$$

respectively.

For the theoretical analysis one assumes that the load is proportional to the third power of the plate thickness, i.e. $f = Gt^3 g$ with g fixed independent of t. With this assumption the problem (1) has a finite and non-trivial solution in limit when $t \to 0$ (cf. [8]). Hence, the problem becomes: find $(w,\beta) \in [H_0^1(\Omega)]^3$ such that

$$a(\beta,\eta) + \kappa t^{-2}(\nabla w - \beta, \nabla v - \eta) = (g, v) \qquad \forall (v,\eta) \in [H_0^1(\Omega)]^3. \tag{5}$$

Introducing the scaled shear force

$$\boldsymbol{q} = \kappa t^{-2}(\nabla w - \beta) \tag{6}$$

as an independent unknown, the mixed form of (5) is: find $(w,\beta,\boldsymbol{q}) \in [H_0^1(\Omega)]^3 \times [L_2(\Omega)]^2$ such that

$$\begin{aligned} a(\beta,\eta) + (\boldsymbol{q}, \nabla v - \eta) &= (g, v) \qquad \forall (v,\eta) \in [H_0^1(\Omega)]^3 \\ \kappa^{-1}t^2(\boldsymbol{q},\boldsymbol{s}) - (\nabla w - \beta, \boldsymbol{s}) &= 0 \qquad \forall \boldsymbol{s} \in [L_2(\Omega)]^2. \end{aligned} \tag{7}$$

The strong form corresponding to this system is obtained by integrating by parts:

$$\boldsymbol{L}\beta + \boldsymbol{q} = 0 \quad \text{in } \Omega, \tag{8}$$

$$-\operatorname{div}\boldsymbol{q} = g \quad \text{in } \Omega, \tag{9}$$

$$-\kappa^{-1}t^2\boldsymbol{q} + \nabla w - \beta = 0 \quad \text{in } \Omega, \tag{10}$$

$$w = 0 \quad \text{on } \partial\Omega, \tag{11}$$

$$\beta = 0 \quad \text{on } \partial\Omega. \tag{12}$$

Above the differential operator $\boldsymbol{L}$ is defined through

$$\boldsymbol{L}\eta = \frac{1}{6}\,\mathbf{div}\,\{\varepsilon(\eta) + (\frac{\nu}{1-\nu})\mathrm{div}\,\eta\ \boldsymbol{I}\}, \tag{13}$$

where $\mathbf{div}$ stands for the divergence of second order tensors and $\boldsymbol{I}$ is the unit tensor.

3 THE FINITE ELEMENT METHODS

We let $\mathcal{C}_h$ be the finite element partitioning of $\bar{\Omega}$ into triangles or convex quadrilaterals and define the finite element subspaces for the deflection and rotation vector with the index $k \geq 1$ as

$$W_h = \{v \in H^1_0(\Omega) \mid v_{|K} \in R_k(K),\ \forall K \in \mathcal{C}_h\}, \tag{14}$$

$$\boldsymbol{V}_h = \{\eta \in [H^1_0(\Omega)]^2 \mid \eta_{|K} \in [R_k(K)]^2,\ \forall K \in \mathcal{C}_h\}, \tag{15}$$

where $R_k(K)$ is a space of polynomials of degree $\leq k$ defined on K. We point out that this means that equal basis functions are used for the deflection and both components of the rotation.

The shear energy will be modified by interpolating with the MITC technique. For an element $K \in \mathcal{C}_h$ an auxiliary space $\boldsymbol{\Gamma}_k(K)$ and an MITC reduction operator $\boldsymbol{R}_{h|K} : [H^1(K)]^2 \to \boldsymbol{\Gamma}_k(K)$ are introduced.

The finite element methods for the problem (1) are then defined as follows: find $(w_h, \beta_h) \in W_h \times \boldsymbol{V}_h$ such that

$$\mathcal{B}_h(w_h, \beta_h; v, \eta) = (f, v) \quad \forall (v, \eta) \in W_h \times \boldsymbol{V}_h. \tag{16}$$

The bilinear form $\mathcal{B}_h$ is defined as

$$\mathcal{B}_h(w, \beta; v, \eta) = Gt^3\Big\{a(\beta, \eta) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{L}\beta, \boldsymbol{L}\eta)_K \tag{17}$$

$$+ \sum_{K\in\mathcal{C}_h} \kappa(t^2 + \kappa\alpha h_K^2)^{-1}(\nabla w - \boldsymbol{R}_h\beta - \alpha h_K^2\boldsymbol{L}\beta, \nabla v - \boldsymbol{R}_h\eta - \alpha h_K^2\boldsymbol{L}\eta)_K.\Big\}$$

Here h_K denotes the diameter of the element $K \in \mathcal{C}_h$ and α is a positive constant for which an upper bound will be defined below.

The different methods will then be defined by specifying the spaces $R_k(K)$ and $\boldsymbol{\Gamma}_k(K)$ together with the reduction operator $\boldsymbol{R}_h$.

From the solution (w_h, β_h) to (16), the approximations for the shear force and bending moment are obtained from

$$\boldsymbol{Q}_{h|K} = \Big(\frac{G\kappa t^3}{t^2 + \kappa\alpha h_K^2}\Big)(\nabla w_h - \boldsymbol{R}_h\beta_h - \alpha h_K^2\boldsymbol{L}\beta_h)_{|K} \quad \forall K \in \mathcal{C}_h \tag{18}$$

and

$$\boldsymbol{M}_h = \frac{Gt^3}{6}\{\varepsilon(\beta_h) + (\frac{\nu}{1-\nu})\mathrm{div}\,\beta_h\ \boldsymbol{I}\}, \tag{19}$$

respectively. An alternative way to determine the approximate shear force is to calculate it through the equilibrium equation

$$\boldsymbol{Q}_{e,h|K} = -Gt^3\boldsymbol{L}\beta_{h|K}, \quad \forall K \in \mathcal{C}_h. \tag{20}$$

This is, of course, reasonable only when $k \geq 2$. In the sequel we use the notation $\boldsymbol{Q}_{(e)h}$ for $\boldsymbol{Q}_h$ and $\boldsymbol{Q}_{e,h}$.

Since $\boldsymbol{Q}_{(e)h}$ and $\boldsymbol{M}_h$ are discontinuous, it is customary to "smooth" them e.g. by projecting onto some suitably chosen finite element space consisting of continuous functions. This we do by letting

$$S_h = \{s \in C(\bar{\Omega}) \mid s_{|K} \in R_k(K),\ \forall K \in \mathcal{C}_h\} \tag{21}$$

and then defining the smoothed shear force $\boldsymbol{Q}^*_{(e)h} \in [S_h]^2$ and bending moment $\boldsymbol{M}^*_h \in [S_h]^{2\times 2}$ through

$$(\boldsymbol{Q}^*_{(e)h}, \boldsymbol{s}) = (\boldsymbol{Q}_{(e)h}, \boldsymbol{s}), \quad \forall \boldsymbol{s} \in [S_h]^2, \tag{22}$$

and

$$(\boldsymbol{M}^*_h, \boldsymbol{r}) = (\boldsymbol{M}_h, \boldsymbol{r}), \quad \forall \boldsymbol{r} \in [S_h]^{2\times 2}. \tag{23}$$

In practice these smoothenings usally increase the accuracy. The asymtotic convergence rate is, however, not improved.

Next, let us define the different methods.

Method I

We let K be a triangle, $R_k(K) = P_k(K)$ with $k \geq 1$ and denote by

$$\boldsymbol{\Gamma}_k(K) = [P_{k-1}(K)]^2 \oplus (y, -x)\tilde{P}_{k-1}(K), \tag{24}$$

the rotated Raviart-Thomas space [24]. Here $\tilde{P}_{k-1}(K)$ is the space of homogeneous polynomials of degree $k-1$. The reduction operator is defined through the conditions

$$\int_E [(\boldsymbol{R}_h\eta - \eta)\cdot ø] v\ ds = 0,\ \forall v \in P_{k-1}(E), \quad \text{for every edge } E \text{ of } K, \tag{25}$$

and for $k \geq 2$

$$\int_K (\boldsymbol{R}_h\eta - \eta)\cdot \boldsymbol{r}\ dx\,dy = 0,\ \forall \boldsymbol{r} \in [P_{k-2}(K)]^2. \tag{26}$$

Above ø is the tangent to the edge E.

Remark 1 For linear elements with $k = 1$ it holds

$$\boldsymbol{L}\eta_{|K} = 0, \quad \forall K \in \mathcal{C}_h, \quad \forall \eta \in \boldsymbol{V}_h, \tag{27}$$

and so the bilinear form $\mathcal{B}_h$ reduces to

$$\mathcal{B}_h(w,\beta;v,\eta) = Gt^3a(\beta,\eta) + \sum_{K\in\mathcal{C}_h}\Big(\frac{G\kappa t^3}{t^2+\kappa\alpha h_K^2}\Big)(\nabla w - \boldsymbol{R}_h\beta, \nabla v - \boldsymbol{R}_h\eta)_K. \tag{28}$$

This gives our linear element (introduced in [9]), which in [19] is proved to be equivalent to the elements of Tessler-Hughes [27] and Xu et al. [29, 26, 4]. Taking $\alpha = 0$ we get an unstable element introduced by Hughes and Taylor [17]. In the above mentioned papers we have not found any remark showing the near relationship between this element and the elements later considered by the same authors. □

Remark 2 The MITC7 element [5] is obtained from Method I by choosing $\alpha = 0$, $k = 2$, and taking $R_2(K) = P_2(K)\oplus\text{span}\{\lambda_1\lambda_2\lambda_3\}$ in the rotation space $\boldsymbol{V}_h$. Here $\lambda_i,\ i = 1,2,3,$ denote the barycentric coordinates of K. □

Remark 3 With $\alpha = 0$ and $k = 2$ one obtains an element proposed in [21]. The element is unfortunately not optimally convergent. □

Method II

Now K is a quadrilateral and $R_k(K) = Q_k(K)$ with $k \geq 1$. We let $\boldsymbol{J}_K$ be the Jacobian matrix of the mapping $\boldsymbol{F}_K : \hat{K} \to K$ ($\hat{K}$ is the unit square with coordinates ξ and η) and define

$$\boldsymbol{\Gamma}_k(K) = \{\eta \mid \eta = \boldsymbol{J}_K^{-T}\hat{\eta}\circ\boldsymbol{F}_K^{-1},\ \hat{\eta}\in\boldsymbol{\Gamma}_k(\hat{K})\}, \tag{29}$$

where $\boldsymbol{J}_K^{-T}$ is the transpose of $\boldsymbol{J}_K^{-1}$, and

$$\boldsymbol{\Gamma}_k(\hat{K}) = P_{k-1,k}(\hat{K})\times P_{k,k-1}(\hat{K}). \tag{30}$$

This is the rectangular rotated Raviart-Thomas space with

$$P_{m,n}(\hat{K}) = \{v \mid v = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\xi^i\eta^j \text{ for some } a_{ij}\in\mathbb{R}\}. \tag{31}$$

The reduction operator $\boldsymbol{R}_h : [H^1(K)]^2 \to \boldsymbol{\Gamma}_k(K)$ is now defined through

$$\boldsymbol{R}_h\eta = \boldsymbol{J}_K^{-T}\boldsymbol{R}_{\hat{K}}\boldsymbol{J}_K^T\eta, \tag{32}$$

where $\boldsymbol{R}_{\hat{K}} : [H^1(\hat{K})]^2 \to \boldsymbol{\Gamma}_k(\hat{K})$ is an operator satisfying the conditions

$$\int_{\hat{E}}[(\boldsymbol{R}_{\hat{K}}\hat{\eta}-\hat{\eta})\cdot\boldsymbol{\phi}]v\,ds = 0,\ \forall v\in P_{k-1}(\hat{E}),\quad \text{for every edge } \hat{E} \text{ of } \hat{K}, \tag{33}$$

and in the case if $k \geq 2$

$$\int_{\hat{K}}(\boldsymbol{R}_{\hat{K}}\hat{\eta}-\hat{\eta})\cdot\boldsymbol{r}\,d\xi\,d\eta = 0,\quad \forall\boldsymbol{r}\in P_{k-1,k-2}(\hat{K})\times P_{k-2,k-1}(\hat{K}). \tag{34}$$

Remark 4 If $k = 1$ it is possible use the reduced bilinear form (28). By doing this we get the stabilized MITC4 element [20], and if we further choose $\alpha = 0$ we obtain the original MITC4 element of Bathe and Dvorkin [7]. □

Method III

Again, K is a quadrilateral but now we choose $R_k(K) = Q_k^r(K) = Q_k(K) \cap P_{k+1}(K)$ (isoparametric) with $k \geq 1$. For this method we define

$$\mathbf{\Gamma}_k(\hat{K}) = [P_k(\hat{K})]^2 \setminus \text{span}\{(\xi^k, 0), (0, \eta^k)\}, \tag{35}$$

which is the rotated rectangular Brezzi-Douglas-Fortin-Marini (BDFM) space [10]. The operator $\boldsymbol{R}_h$ is defined as in (32) with $\boldsymbol{R}_{\hat{K}}$ satisfying

$$\int_{\hat{E}} [(\boldsymbol{R}_{\hat{K}}\hat{\eta} - \hat{\eta}) \cdot \boldsymbol{\o}] v \; ds = 0, \; \forall v \in P_{k-1}(\hat{E}) \quad \text{for every edge } \hat{E} \text{ of } \hat{K}, \tag{36}$$

and for $k \geq 2$

$$\int_{\hat{K}} (\boldsymbol{R}_{\hat{K}}\hat{\eta} - \hat{\eta}) \cdot \boldsymbol{r} \; d\xi \, d\eta = 0, \; \forall \boldsymbol{r} \in [P_{k-2}(\hat{K})]^2. \tag{37}$$

Remark 5 The MITC9 element [5] is obtained from Method III by taking $\alpha = 0$, $k = 2$ and $R_2(K) = Q_2(K)$ in the rotation space $\boldsymbol{V}_h$. □

Remark 6 For all three methods it holds

$$\boldsymbol{R}_h \nabla v = \nabla v, \quad \forall v \in W_h.$$

This property is used in the analysis below. □

4 ERROR ANALYSIS

As mentioned the error analysis should be done for the scaled problem (5). Without any loss of generality we can also choose $\kappa = 1$. Therefore we consider the scaled finite element formulation: find $(w_h, \beta_h) \in W_h \times \boldsymbol{V}_h$ such that

$$\mathcal{S}_h(w_h, \beta_h; v, \eta) = (g, v), \quad \forall (v, \eta) \in W_h \times \boldsymbol{V}_h, \tag{38}$$

with

$$\begin{aligned} \mathcal{S}_h(w, \beta; v, \eta) \;\; &= \;\; a(\beta, \eta) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\beta, \boldsymbol{L}\eta)_K \\ &+ \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla w - \boldsymbol{R}_h \beta - \alpha h_K^2 \boldsymbol{L}\beta, \nabla v - \boldsymbol{R}_h \eta - \alpha h_K^2 \boldsymbol{L}\eta)_K. \end{aligned} \tag{39}$$

The approximation to the scaled shear force (6) is then defined by

$$\boldsymbol{q}_{h|K} = (t^2 + \alpha h_K^2)^{-1}(\nabla w_h - \boldsymbol{R}_h\beta_h - \alpha h_K^2 \boldsymbol{L}\beta_h)_{|K}. \tag{40}$$

The aim is now to derive error estimates which are independent of the plate thickness. To this end C will denote various positive constants which do not depend on the thickness t or the global mesh parameter

$$h = \max_{K\in\mathcal{C}_h} h_K. \tag{41}$$

We will use standard finite element notation with $|\cdot|_{m,D}$ and $\|\cdot\|_{m,D}$ denoting the seminorms and norms in $H^m(D)$ and $[H^m(D)]^2$. Again, the subscript D is dropped when $D = \Omega$.

Under some (minor) restrictive assumptions on the mesh (see [28]) we have the following result which states that the operator $\boldsymbol{R}_h$ has optimal interpolation properties.

Lemma 1 [24, 8] *There exist a positive constant C such that for $1 \le m \le k$ and $\boldsymbol{\eta} \in [H^m(K)]^2$ it holds*

$$\|\boldsymbol{\eta} - \boldsymbol{R}_h\boldsymbol{\eta}\|_{0,K} \le C h_K^m \|\boldsymbol{\eta}\|_{m,K}, \ \forall K \in \mathcal{C}_h. \ \square$$

We will also make use of the following inverse estimate which is valid since the space $\boldsymbol{V}_h$ consists of piecewise polynomials (cf. e.g. [14]).

Lemma 2 *There exists a constant $C_I > 0$ such that*

$$C_I \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \le a(\boldsymbol{\eta},\boldsymbol{\eta}), \quad \forall \boldsymbol{\eta} \in \boldsymbol{V}_h. \ \square$$

Remark 7 The constant C_I of Lemma 2 plays an important role, not only in the analysis of the methods, but also in numerical calculations. Hence, we refer to [15] where numerical techniques for estimating constants like C_I have been considered. □

The stability will be formulated using the following mesh dependent seminorm and norm:

$$|(v,\boldsymbol{\eta})|_h = \Big(\sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})\|_{0,K}^2\Big)^{1/2}, \tag{42}$$

$$|||(v,\boldsymbol{\eta})|||_h = \|v\|_1 + \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h. \tag{43}$$

We also define

$$\|\boldsymbol{q}\|_{-1,h} = (\sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{q}\|_{0,K}^2)^{1/2}, \tag{44}$$

and note that the following equivalence holds.

Lemma 3 *There exists a positive constant* C *such that*

$$C|||(v,\boldsymbol{\eta})|||_h \le \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h \le |||(v,\boldsymbol{\eta})|||_h, \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: The Poincaré inequality, Remark 6, Lemma 1 (with $m = 1$), and the inequality $(t^2 + \alpha h_K^2) \le C$ give

$$\begin{aligned} \|v\|_1^2 &\le C\|\nabla v\|_0^2 = C\|\boldsymbol{R}_h \nabla v\|_0^2 \\ &\le C(\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})\|_0^2 + \|\boldsymbol{R}_h\boldsymbol{\eta}\|_0^2) \\ &\le C(\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})\|_0^2 + \|\boldsymbol{\eta}\|_1^2) \\ &\le C(\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2), \end{aligned}$$

which proves the claim. □

With the aid of the previous auxiliary results we are now ready to prove that the methods are stable with respect to the norm $|||\cdot|||_h$.

Lemma 4 *There exists a constant* $C > 0$ *such that for* $0 < \alpha < C_I$ *it holds*

$$\mathcal{S}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \ge C|||(v,\boldsymbol{\eta})|||_h^2, \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the inverse estimate of Lemma 2 and the Korn inequality we get

$$\begin{aligned} \mathcal{S}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) &= a(\boldsymbol{\eta},\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \\ &\quad + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \\ &\ge (1-\alpha C_I^{-1})a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 \\ &\ge C(\|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2). \end{aligned} \tag{45}$$

The same inverse estimate and the boundedness of the bilinear form a also give

$$\begin{aligned} |(v,\boldsymbol{\eta})|_h^2 &= \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta})\|_{0,K}^2 \\ &\le C(\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \alpha \sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2) \\ &\le C(\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + a(\boldsymbol{\eta},\boldsymbol{\eta})) \\ &\le C(\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2). \end{aligned} \tag{46}$$

Combining (45), (46), and using Lemma 3 gives the desired result. □

Next, we note that in the bilinear form $\mathcal{S}_h$ is not consistent with the exact energy. In order to characterize the consistency error we define

$$\begin{aligned}\mathcal{E}_h(\boldsymbol{s}; v, \boldsymbol{\eta}) = (\boldsymbol{s}, (\boldsymbol{R}_h - \boldsymbol{I})(\nabla v - \boldsymbol{\eta}))& \\ +t^2 \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{s} - \boldsymbol{s}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K.&\end{aligned} \tag{47}$$

We then have

Lemma 5 *The solution* $(w, \boldsymbol{\beta})$ *to* (5) *satisfies*

$$\mathcal{S}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (g, v) + \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}), \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the constitutive relation (10) and the equilibrium equation (8), we get

$$\begin{aligned}
\mathcal{S}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta}, \boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2 (\boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(\boldsymbol{R}_h(\nabla w - \boldsymbol{\beta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 (\boldsymbol{q}, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(t^2 \boldsymbol{R}_h\boldsymbol{q} + \alpha h_K^2 \boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 (\boldsymbol{q}, \boldsymbol{L}\boldsymbol{\eta})_K + \sum_{K\in\mathcal{C}_h} (\boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad +t^2 \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} (\boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}))_K \\
&\quad +t^2 \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) + (\boldsymbol{q}, (\boldsymbol{R}_h - \boldsymbol{I})(\nabla v - \boldsymbol{\eta})) \\
&\quad +t^2 \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= (g, v) + \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}). \quad \square
\end{aligned}$$

Remark 8 Note that if we choose $\boldsymbol{R}_h = \boldsymbol{I}$, we get a pure consistent formulation:

$$\mathcal{S}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (g, v), \quad \forall (v, \boldsymbol{\eta}) \in\in W_h \times \boldsymbol{V}_h. \tag{48}$$

A family of methods of this kind has been introduced and analyzed in [25]. The only drawback of these consistent methods is that higher degree (i.e. $k + 1$) shape functions must be used for the deflection in order to obtain the right balance between the approximation properties of W_h and $\boldsymbol{V}_h$. $\square$

The following auxiliary result is needed in estimating the consistency error.

Lemma 6 *For* $\boldsymbol{s} \in [H^{k-1}(\Omega)]^2$ *it holds*

$$|(\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta})| \le C h^k \|\boldsymbol{s}\|_{k-1} \|\boldsymbol{\eta}\|_1, \quad \forall \boldsymbol{\eta} \in [H^1(\Omega)]^2.$$

Proof: If $k = 1$ the result follows directly from the Schwarz inequality and Lemma 1.

For $k \ge 2$, we let $\boldsymbol{P}_K : [L^2(\hat{K})]^2 \to [L^2(K)]^2$ be the Piola transformation defined through [8, page 97]

$$\boldsymbol{P}_K \hat{\boldsymbol{s}} = |\boldsymbol{J}_K|^{-1} \boldsymbol{J}_K \hat{\boldsymbol{s}}, \quad \hat{\boldsymbol{s}} \in [L^2(\hat{K})]^2, \tag{49}$$

and define the space $\boldsymbol{S}(\hat{K})$ by

$$\boldsymbol{S}(\hat{K}) = \begin{cases} [P_{k-2}(\hat{K})]^2 & \text{for Methods I and III,} \\ P_{k-1,k-2}(\hat{K}) \times P_{k-2,k-1}(\hat{K}) & \text{for Method II.} \end{cases} \tag{50}$$

Using the definition of the operator $\boldsymbol{R}_h$ and the properties (26), (34) and (37) we then get

$$\begin{aligned} &(\boldsymbol{P}_K \hat{\boldsymbol{s}}, \boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta})_K \\ &= \int_K (\boldsymbol{P}_K \hat{\boldsymbol{s}})(x,y) \cdot (\boldsymbol{\eta}(x,y) - \boldsymbol{R}_h \boldsymbol{\eta}(x,y)) dx dy \\ &= \int_{\hat{K}} |\boldsymbol{J}_K|^{-1} \boldsymbol{J}_K \hat{\boldsymbol{s}}(\xi,\eta) \cdot (\hat{\boldsymbol{\eta}}(\xi,\eta) - \boldsymbol{J}_K^{-T} \boldsymbol{R}_{\hat{K}} \boldsymbol{J}_K^{-1} \hat{\boldsymbol{\eta}}(\xi,\eta)) |\boldsymbol{J}_K| d\xi d\eta \\ &= \int_{\hat{K}} \hat{\boldsymbol{s}}(\xi,\eta) \cdot (\boldsymbol{J}_K^{-1} \hat{\boldsymbol{\eta}}(\xi,\eta) - \boldsymbol{R}_{\hat{K}} \boldsymbol{J}_K^{-1} \hat{\boldsymbol{\eta}}(\xi,\eta)) d\xi d\eta \\ &= 0, \qquad \forall \hat{\boldsymbol{s}} \in \boldsymbol{S}(\hat{K}). \end{aligned} \tag{51}$$

Next, we let $\boldsymbol{\Pi}_{\hat{K}} : [L_2(\hat{K})]^2 \to \boldsymbol{S}(\hat{K})$ be the L_2-projection and define the mapping $\boldsymbol{\Pi}_K$ through

$$\boldsymbol{\Pi}_K = \boldsymbol{P}_K \boldsymbol{\Pi}_{\hat{K}} \boldsymbol{P}_K^{-1}. \tag{52}$$

By using standard techniques [12, 8] for deriving interpolation estimates we get

$$\|\boldsymbol{s} - \boldsymbol{\Pi}_K \boldsymbol{s}\|_{0,K} \le C h_K^k \|\boldsymbol{s}\|_{k-1,K}. \tag{53}$$

Using (51) we then have

$$\begin{aligned} (\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta})_K &= (\boldsymbol{s} - \boldsymbol{\Pi}_K \boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta})_K \\ &\le \|\boldsymbol{s} - \boldsymbol{\Pi}_K \boldsymbol{s}\|_{0,K} \|\boldsymbol{\eta} - \boldsymbol{R}_h \boldsymbol{\eta}\|_{0,K} \le C h_K^k \|\boldsymbol{s}\|_{k-1,K} \|\boldsymbol{\eta}\|_{1,K}. \end{aligned} \tag{54}$$

The desired result follows from (54) by summing over the elements $K \in \mathcal{C}_h$. $\square$

For the consistency error we now have the following result.

Lemma 7 *Suppose that the exact shear force satisfies $\boldsymbol{q} \in [H^{k-1}(\Omega)]^2$ and $t\boldsymbol{q} \in [H^k(\Omega)]^2$. Then it holds*

$$|\mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta})| \leq Ch^k(\|\boldsymbol{q}\|_{k-1} + t\|\boldsymbol{q}\|_k)\ |||(v, \boldsymbol{\eta})|||_h,\ \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: We first note that the boundedness of the bilinear form a together with Lemmas 2 and 1 imply

$$\begin{aligned}
&\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v-\boldsymbol{\eta})-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2} \qquad (55)\\
&\quad\leq C\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\boldsymbol{R}_h(\nabla v-\boldsymbol{\eta})\|_{0,K}^2+\alpha\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\\
&\quad\leq C(|(v,\boldsymbol{\eta})|_h^2+\alpha C_I^{-1}a(\boldsymbol{\eta},\boldsymbol{\eta}))^{1/2}\leq C(|(v,\boldsymbol{\eta})|_h^2+\|\boldsymbol{\eta}\|_1^2)^{1/2}.
\end{aligned}$$

Hence, using Remark 6 Lemmas 6 and 1, we get

$$\begin{aligned}
\mathcal{E}_h(\boldsymbol{q}; v,\boldsymbol{\eta}) &= (\boldsymbol{q},(\boldsymbol{R}_h-\boldsymbol{I})(\nabla v-\boldsymbol{\eta})) \qquad (56)\\
&+t^2\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{q}-\boldsymbol{q},\boldsymbol{R}_h(\nabla v-\boldsymbol{\eta})-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta})_K\\
&=(\boldsymbol{q},\boldsymbol{\eta}-\boldsymbol{R}_h\boldsymbol{\eta})+t^2\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\boldsymbol{R}_h\boldsymbol{q}-\boldsymbol{q},\boldsymbol{R}_h(\nabla v-\boldsymbol{\eta})-\alpha h_K^2\boldsymbol{L}\boldsymbol{\eta})_K\\
&\leq(\boldsymbol{q},\boldsymbol{\eta}-\boldsymbol{R}_h\boldsymbol{\eta})+Ct^2\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\boldsymbol{q}-\boldsymbol{R}_h\boldsymbol{q}\|_{0,K}^2\Big)^{1/2}(|(v,\boldsymbol{\eta})|_h^2+\|\boldsymbol{\eta}\|_1^2)^{1/2}\\
&\leq(\boldsymbol{q},\boldsymbol{\eta}-\boldsymbol{R}_h\boldsymbol{\eta})+Ct\|\boldsymbol{q}-\boldsymbol{R}_h\boldsymbol{q}\|_0(|(v,\boldsymbol{\eta})|_h^2+\|\boldsymbol{\eta}\|_1^2)^{1/2}\\
&\leq Ch^k(\|\boldsymbol{q}\|_{k-1}+t\|\boldsymbol{q}\|_k)|||(v,\boldsymbol{\eta})|||_h.\quad\Box
\end{aligned}$$

For the rest of the error analysis we will next define a special interpolation operator $I_h : H_0^1(\Omega) \to W_h$ through the following three conditions:

$$((v - I_h v)\circ \boldsymbol{F}_K)(\hat{p}) = 0,\ \forall \text{ vertices } \hat{p} \text{ of } \hat{K}, \tag{57}$$

$$\int_{\hat{E}}((v - I_h v)\circ \boldsymbol{F_K})\hat{r}\ d\hat{s} = 0,\ \forall \hat{r}\in P_{k-2}(\hat{E}),\ \forall \text{ edges } \hat{E} \text{ of } \hat{K}, \tag{58}$$

and

$$\int_{\hat{K}}((v - I_h v)\circ \boldsymbol{F}_K)\hat{s}\ d\xi d\eta = 0,\ \begin{cases}\forall \hat{s}\in P_{k-3}(\hat{K}), & \text{for the Methods I and III,}\\ \forall \hat{s}\in Q_{k-2}(\hat{K}), & \text{for the Method II,}\end{cases} \tag{59}$$

for every element $K \in \mathcal{C}_h$.

The operator I_h has optimal interpolation properties:

Lemma 8 *There exists a positive constant C such that for $v \in H^m(\Omega)$ and $1 \leq m \leq k+1$ it holds*

$$\|v - I_h v\|_s \leq Ch^{m-s}\|v\|_m,\ s = 0, 1.$$

Proof: Clearly I_h is a polynomial preserving operator in the sense that $I_h v = v,\ \forall v \in W_h$. Hence, we can deduce the asserted estimate from [12, Sections 15 and 16]. $\Box$

The reason for introducing the operator I_h is the following technical result.

Lemma 9 *For $v \in H_0^1(\Omega)$ it holds*

$$\boldsymbol{R}_h \nabla (v - I_h v) = \boldsymbol{0}.$$

Proof: On each element $K \in \mathcal{C}_h$ we have (using (57) and (58))

$$\begin{aligned}\int_{\hat{E}} \hat{\nabla}((v - I_h v) \circ \boldsymbol{F}_K) \cdot \hat{\boldsymbol{\phi}}\ \hat{r}\ d\hat{s} &= \int_{\hat{E}} \frac{\partial}{\partial \hat{s}}((v - I_h v) \circ \boldsymbol{F}_K)\ \hat{r}\ d\hat{s} \\ &= \int_{\partial \hat{E}} ((v - I_h v) \circ \boldsymbol{F}_K)\hat{r} - \int_{\hat{E}} ((v - I_h v) \circ \boldsymbol{F}_K) \frac{\partial \hat{r}}{\partial \hat{s}}\ d\hat{s} = 0, \end{aligned} \tag{60}$$

for every edge $\hat{E}$ of $\hat{K}$ if $\hat{r} \in P_{k-1}(\hat{E})$ and (using (58) and (59))

$$\begin{aligned}\int_{\hat{K}} \hat{\nabla}((v - I_h v) \circ \boldsymbol{F}_K) \cdot \hat{\boldsymbol{s}}\ d\xi d\eta &= \int_{\partial \hat{K}} ((v - I_h v) \circ \boldsymbol{F}_K) \hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}}\ d\hat{s} \\ &- \int_{\hat{K}} ((v - I_h v) \circ \boldsymbol{F}_K)\ \widehat{\mathrm{div}} \hat{\boldsymbol{s}}\ d\xi d\eta = 0, \end{aligned} \tag{61}$$

if $k \geq 2$ and $\hat{\boldsymbol{s}} \in \boldsymbol{S}(\hat{K})$. (See Lemma 6 for the definition of the space $\boldsymbol{S}(\hat{K})$). Here $\hat{\nabla}$ and $\widehat{\mathrm{div}}$ stand for the gradient and divergence operators with respect to the ξ and η variables of $\hat{K}$ and $\hat{\boldsymbol{n}}$ is the unit outward normal to $\partial \hat{K}$.

Hence, using (60), (61), and recalling the definition of the operator $\boldsymbol{R}_{\hat{K}}$, we get

$$\boldsymbol{R}_{\hat{K}} \hat{\nabla}((v - I_h v) \circ \boldsymbol{F}_K) = \boldsymbol{0},\ \forall K \in \mathcal{C}_h, \tag{62}$$

and since it holds $\boldsymbol{J}_K^{-1}(\nabla v \circ \boldsymbol{F}_K) = \hat{\nabla}(v \circ \boldsymbol{F}_K),\ \forall K \in \mathcal{C}_h$, we conclude that

$$\begin{aligned}\boldsymbol{R}_h \nabla (v - I_h v)_{|K} &= \boldsymbol{J}_K^{-T} \boldsymbol{R}_{\hat{K}} \boldsymbol{J}_K^{-1} \nabla ((v - I_h v) \circ \boldsymbol{F}_K) \\ &= \boldsymbol{J}_K^{-T} \boldsymbol{R}_{\hat{K}} \hat{\nabla}((v - I_h v) \circ \boldsymbol{F}_K) = \boldsymbol{0},\ \forall K \in \mathcal{C}_h.\ \square \end{aligned} \tag{63}$$

We will next state our main result.

Theorem 1 *Suppose that the solution to the problem (5) satisfies $w \in H^{k+1}(\Omega)$, $t\boldsymbol{\beta} \in [H^{k+2}(\Omega)]^2$ and $\boldsymbol{\beta} \in [H^{k+1}(\Omega)]^2$. For $0 < \alpha < C_I$ it then holds*

$$\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + t\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \leq Ch^k(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}).$$

Proof: Let $\tilde{\boldsymbol{\beta}} \in \boldsymbol{V}_h$ be the usual Lagrange interpolant to $\boldsymbol{\beta}$ and $\tilde{w} = I_h w \in W_h$ the interpolant to w. From Lemmas 4 and 5, there exists a pair $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$ such that

$$|||(v, \boldsymbol{\eta})|||_h \leq C, \tag{64}$$

and

$$
\begin{aligned}
|||(w_h - \tilde{w}, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h &\leq \mathcal{S}_h(w_h - \tilde{w}, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) \\
&= \mathcal{S}_h(w - \tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) - \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}). \qquad (65)
\end{aligned}
$$

For the consistency error term in (65) we directly obtain (using (64), (8) and Lemma 7)

$$
|\mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta})| \leq Ch^k(\|\boldsymbol{q}\|_{k-1} + t\|\boldsymbol{q}\|_k) \leq Ch^k(\|\boldsymbol{\beta}\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2}). \qquad (66)
$$

Next, let us write out the bilinear form $\mathcal{S}_h$ on the the right hand side of (65). Due to the definition of $\tilde{w}$ we have (using Lemma 9)

$$
\begin{aligned}
\mathcal{S}_h(w - \tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}, \boldsymbol{\eta}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \boldsymbol{L}\boldsymbol{\eta})_K \qquad (67) \\
&+ \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{R}_h(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) - \alpha h_K^2 \boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K.
\end{aligned}
$$

From (64) and Lemma 2 it follows that

$$
(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2)^{1/2} \leq C, \qquad (68)
$$

and

$$
(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \|\boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2)^{1/2} \leq C. \qquad (69)
$$

Hence, for the first and second terms in (67) we get (using (68), Lemma 2 and continuity of the bilinear form a)

$$
a(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}, \boldsymbol{\eta}) \leq C\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_1 \leq Ch^k \|\boldsymbol{\beta}\|_{k+1}, \qquad (70)
$$

and

$$
\sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \boldsymbol{L}\boldsymbol{\eta})_K \leq C(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K}^2)^{1/2} \leq Ch^k \|\boldsymbol{\beta}\|_{k+1}. \qquad (71)
$$

Using the same estimates and (69) we obtain for the third term

$$
\begin{aligned}
&\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{R}_h(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) - \alpha h_K^2 \boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \boldsymbol{R}_h(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad \leq C(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \|\boldsymbol{R}_h(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) + \alpha h_K^2 \boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K}^2)^{1/2} \\
&\quad \leq C(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \|\boldsymbol{R}_h(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K}^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K}^2)^{1/2} \\
&\quad \leq C(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\|(\boldsymbol{I} - \boldsymbol{R}_h)(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K}^2 + \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{0,K}^2) + \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_1^2)^{1/2} \\
&\quad \leq C(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (h_K^2 \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{1,K}^2 + \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{0,K}^2) + \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_1^2)^{1/2} \\
&\quad \leq Ch^k \|\boldsymbol{\beta}\|_{k+1}. \qquad (72)
\end{aligned}
$$

The estimate

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \le Ch^k(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}) \tag{73}$$

follows now by combining (66), (70)-(72), using the triangle inequality and the interpolation estimate (here we need Lemma 9)

$$|||(w - \tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})|||_h \le Ch^k(\|w\|_{k+1} + \|\boldsymbol{\beta}\|_{k+1}). \tag{74}$$

After this, the H^1-estimates for the errors $w - w_h$ and $\boldsymbol{\beta} - \boldsymbol{\beta}_h$ follow directly from (73) and from the definition of the norm $|||\cdot|||_h$.

Next, let us derive the asserted estimates for the shear. Recalling the definitions (6) and (40) of the quantities $\boldsymbol{q}$ and $\boldsymbol{q}_h$, we get

$$\begin{aligned}(\boldsymbol{q} - \boldsymbol{q}_h)_{|K} = (t^2 + \alpha h_K^2)^{-1}(&\boldsymbol{R}_h(\nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h)) \\ &-\alpha h_K^2 \boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h) + t^2(\boldsymbol{q} - \boldsymbol{R}_h\boldsymbol{q}))_{|K}, \ \forall K \in \mathcal{C}_h.\end{aligned} \tag{75}$$

From this it follows that

$$\begin{aligned}&\Big(\sum_{K\in\mathcal{C}_h}(t^2 + \alpha h_K^2)\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,K}^2\Big)^{1/2} \\ &\quad\le C(|(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|_h + \|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{-1,h} + t\|\boldsymbol{q} - \boldsymbol{R}_h\boldsymbol{q}\|_0).\end{aligned} \tag{76}$$

Now, since an inverse estimate, an interpolation estimate and the estimate for $|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h$ imply that

$$\|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{-1,h} \le Ch^k(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}), \tag{77}$$

both estimates for the shear follow from (76) and Lemma 1. □

Remark 9 By defining the scaled moment tensor $\boldsymbol{m}$, its approximation $\boldsymbol{m}_h$ and the smoothed approximation by $\boldsymbol{m}_h^*$

$$\boldsymbol{M} = Gt^3\boldsymbol{m}, \quad \boldsymbol{M}_h = Gt^3\boldsymbol{m}_h, \quad \text{and} \quad \boldsymbol{M}_h^* = Gt^3\boldsymbol{m}_h^*, \tag{78}$$

respectively, the above estimate for the rotation contains the following estimate

$$\|\boldsymbol{m} - \boldsymbol{m}_h\|_0 + \|\boldsymbol{m} - \boldsymbol{m}_h^*\|_0 \le Ch^k(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}).$$

□

For a quasiuniform mesh we get the following estimates for the shear approximations.

Corollary 1 *Suppose that the mesh is quasiuniform, i.e. such that $h_K \ge Ch$, $\forall K \in \mathcal{C}_h$. Then it follows from Theorem 1 that*

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 + +\|\boldsymbol{q} - \boldsymbol{q}_{(e)h}^*\|_0 \le Ch^{k-1}(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}). \ \square \tag{79}$$

Here $\boldsymbol{q}^*_{(e)h} = t^{-3}\boldsymbol{Q}^*_{(e)h}$ denotes the scaled smoothened shear approximations computed from the constitutive and equilibrium equations, respectively.

We close this section by mentioning that for a convex domain optimal L^2-estimates can be derived by duality techniques. The rather technical proof of this result is omitted. This is the only result of the paper for which the clamped boundary conditions are needed (for the necessary shift theorem to be valid, cf. [2]).

Theorem 2 *Suppose, in addition to the assumptions of Theorem 1, that the region Ω is convex. Then it holds*

$$\|w - w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 \leq Ch^{k+1}(\|w\|_{k+1} + t\|\boldsymbol{\beta}\|_{k+2} + \|\boldsymbol{\beta}\|_{k+1}). \quad \square$$

5 NUMERICAL EXAMPLES

The numerical examples will be given for a clamped square plate subject to a uniform load $f = 1$. The side length of the plate equals to unity and the thickness is $t = 0.01$. In the calculations we choose $E = 1$, $\nu = 0.3$ (which gives $G = 5/13$) and $\kappa = 5/6$. Since the thickness of the plate is "small", we will calculate the exact solution to the problem using the classical Kirchhoff plate model.

Due to the symmetry, only one quadrant of the plate is discretized. The computational domain is divided uniformly into $N \times N$ quadrilaterals or $2N \times N$ triangles. Examples of the finite element meshes (for the case $N = 4$) are shown in Figure 1.

We will consider Methods I-III with $k = 1, 2$ and rename them according to Table 1 below. Note that both Methods II & III give the same element, namely the STAB4, when $k = 1$.

Table 1. Abbreviations for the different methods.

	Method I	Method II	Method III
$k = 1$	STAB3	STAB4	STAB4
$k = 2$	STAB6	STAB9	STAB8

In all test cases we choose $\alpha = 0.2$ for the STAB3 element and $\alpha = 0.1$ for the STAB4 element. For the STAB6, STAB8 and STAB9 elements we choose $\alpha = 0.05$. As h_K we take the largest side length of the elements.

In Table 2 below the normalized centerpoint deflections for different elements are shown. Table 3 gives the normalized L_2-errors for the deflection.

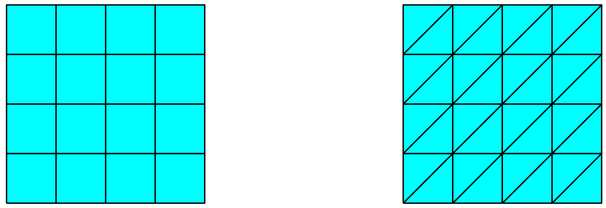

Figure 1. The square plate mesh with $N = 4$ for quadrilateral and triangular elements.

Table 2. The normalized center point deflection $w_h(\text{center})/w(\text{center})$.

N	STAB3	STAB4	STAB6	STAB8	STAB9
4	1.0682	1.0484	1.0234	1.0176	1.0144
8	1.0191	1.0125	1.0055	1.0031	1.0006
16	1.0064	1.0032	1.0014	1.0007	1.0001

Table 3. The errors $||w - w_h||_0/||w||_0$.

N	STAB3	STAB4	STAB6	STAB8	STAB9
4	0.0809	0.0360	0.0350	0.0286	0.0224
8	0.0225	0.0096	0.0071	0.0047	0.0030
16	0.0074	0.0025	0.0017	0.0009	0.0004

Table 4 give the normalized L_2-errors for the smoothened bending moment.

Table 4. The errors $||\boldsymbol{M} - \boldsymbol{M}_h^*||_0/||\boldsymbol{M}||_0$.

N	STAB3	STAB4	STAB6	STAB8	STAB9
4	0.1713	0.1811	0.1110	0.1171	0.1150
8	0.0632	0.0711	0.0301	0.0332	0.0307
16	0.0228	0.0265	0.0078	0.0082	0.0081

In Table 5 the normalized L_2-errors for the smoothened shear force are shown. The shear force for the quadratic elements with $k = 2$ has been calculated from (20) and for linear elements with $k = 1$ from (18). Eventhough the error estimates for the two different approximate shear forces are the same, it is our experience that one gets better results from (20) when $k \geq 2$.

Table 5. The errors $||\boldsymbol{Q} - \boldsymbol{Q}^*_{(e)h}||_0/||\boldsymbol{Q}||_0$.

N	STAB3	STAB4	STAB6	STAB8	STAB9
4	0.1415	0.2319	0.3371	0.3360	0.3439
8	0.0774	0.1567	0.1586	0.1471	0.1471
16	0.0561	0.1096	0.0683	0.0586	0.0594

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