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MORTARING BY A METHOD OF J.A. NITSCHE

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Abstract. *We apply a classical method of J.A. Nitsche* [9] *for the approximation of interface conditions in the Domain Decomposition of the Finite Element Method.*

1 INTRODUCTION

In engineering calculations it happens that one has a region consisting of subdomains with independent finite element meshes that do not match at the interfaces. A natural idea (cf. e.g. [4, 5, 8, 1, 11]) is to introduce Lagrange multipliers to "mortar" the subregions, i.e. to approximatively enforce the interface conditions. In order that this method should work, rather restrictive stability conditions are required. Hence, the different finite element meshes cannot be completely arbitrary.

Since a decade it is well known that much more freedom in designing a method for a saddle point problem is obtained by using so called stabilizing technique, cf. [7, 6] and the references therein. Recently, this approach has been proposed in connection with interface and boundary conditions [2, 13, 3].

In a previous paper [12] we discussed the technique of stabilizing boundary conditions as proposed by Barbosa–Hughes [2] and Verfürth [13], and we showed that it is closely related to a classical method of Nitsche [9]. It appears that Nitsches method is easily implemented and robust and hence it deserves to be revived. In this communcation we show how it can be used for mortaring.

2 THE MORTARING METHOD

Let us consider the simple Poisson model problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$.

For notational simplicity let us assume a decomposition of the domain into two disjont subdomains Ω_1 and Ω_2 , with $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$and the interface $\Upsilon = \bar{\Omega}_1 \cap \bar{\Omega}_2$. We then write the original problem as two equations and the interface conditions:

$$\begin{aligned} -\Delta u^i &= f \quad \text{in } \Omega_i, \quad i = 1, 2, \\ u^1 &= u^2 \quad \text{on } \Upsilon, \\ \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} &= 0 \quad \text{on } \Upsilon, \\ u^i &= 0 \quad \text{on } \partial\Omega \cap \bar{\Omega}_i, \quad i = 1, 2. \end{aligned} \tag{2}$$

Here n_i is the outward unit normal to $\partial\Omega_i$.

The two problems are clearly equivalent and it holds

$$u_{|\Omega_i} = u^i, \quad i = 1, 2. \tag{3}$$

Suppose next that we have finite element partitionings $\mathcal{C}_h^i$ of the subdomains Ω_i, $i = 1, 2$, into (say) simplices and we want to approximate the solution in each domain with independent finite element spaces:

$$V_h^i = \{ \ v \in H^1(\Omega_i) \mid v_{|K} \in P_k(K) \ \forall K \in \mathcal{C}_h^i, \quad v_{|\partial\Omega} = 0 \ \}. \tag{4}$$

We now give one alternative for using Nitsche's method for the approximate enforcement of the interface conditions. To this end we introduce a mesh (of intervals or triangles) $\mathcal{E}_h$ on Υ. Let h_E be the diagonal of $E \in \mathcal{E}_h$. Further, we let γ be a sufficiently large positive constant (see below) and let α_i be parameters satisfying

$$0 \le \alpha_i \le 1, \qquad \alpha_1 + \alpha_2 = 1. \tag{5}$$

The method is then defined as follows.

The Mortaring Method. *Find* $(u_h^1, u_h^2) = u_h \in V_h = V_h^1 \times V_h^2$ *such that*

$$\mathcal{B}_h(u_h; v) = \mathcal{F}_h(v) \quad \forall v \in V_h,$$

with

$$\begin{aligned} \mathcal{B}_h(w; v) \;\; = \;\; & \sum_{i=1}^{2} (\nabla w^i, \nabla v^i)_{\Omega_i} - \langle \alpha_1 \frac{\partial w^1}{\partial n_1} - \alpha_2 \frac{\partial w^2}{\partial n_2}, v^1 - v^2 \rangle_\Upsilon \\ & - \langle \alpha_1 \frac{\partial v^1}{\partial n_1} - \alpha_2 \frac{\partial v^2}{\partial n_2}, w^1 - w^2 \rangle_\Upsilon \quad + \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \langle w^1 - w^2, v^1 - v^2 \rangle_E , \end{aligned} \tag{6}$$

and

$$\mathcal{F}_h(v) = \sum_{i=1}^{2} (f, v^i)_{\Omega_i}. \qquad \square \tag{7}$$

First, we note that the formulation is consistent.

Lemma 1. *The exact solution* (u^1, u^2) *to* (2) *satisfies the discrete variational equations:*

$$\mathcal{B}_h(u; v) = \mathcal{F}_h(v) \quad \forall v \in V_h. \tag{8}$$

Proof: Since $u^1 = u^2$ on the interface we have

$$\begin{aligned} \mathcal{B}_h(u; v) &= \sum_{i=1}^{2} (\nabla u^i, \nabla v^i)_{\Omega_i} - \langle \alpha_1 \frac{\partial u^1}{\partial n_1} - \alpha_2 \frac{\partial u^2}{\partial n_2}, v^1 - v^2 \rangle_\Upsilon \\ &= \sum_{i=1}^{2} (\nabla u^i, \nabla v^i)_{\Omega_i} - \langle \alpha_1 \frac{\partial u^1}{\partial n_1} - \alpha_2 \frac{\partial u^2}{\partial n_2}, v^1 \rangle_\Upsilon + \langle \alpha_1 \frac{\partial u^1}{\partial n_1} - \alpha_2 \frac{\partial u^2}{\partial n_2}, v^2 \rangle_\Upsilon . \end{aligned}$$

Next, using the second interface condition, the relation $\alpha_1 + \alpha_2 = 1$ and integrating by parts, we get

$$\begin{aligned}\mathcal{B}_h(u;v) &= \sum_{i=1}^{2}(\nabla u^i, \nabla v^i)_{\Omega_i} - \langle \alpha_1 \frac{\partial u^1}{\partial n_1} + \alpha_2 \frac{\partial u^1}{\partial n_1}, v^1 \rangle_\Upsilon - \langle \alpha_1 \frac{\partial u^2}{\partial n_2} + \alpha_2 \frac{\partial u^2}{\partial n_2}, v^2 \rangle_\Upsilon \\ &= \sum_{i=1}^{2}(\nabla u^i, \nabla v^i)_{\Omega_i} - \langle \frac{\partial u^1}{\partial n_1}, v^1 \rangle_\Upsilon - \langle \frac{\partial u^2}{\partial n_2}, v^2 \rangle_\Upsilon \\ &= -\sum_{i=1}^{2}(\Delta u^i, v^i)_{\Omega_i} = \sum_{i=1}^{2}(f, v^i)_{\Omega_i} = \mathcal{F}_h(v). \qquad \square\end{aligned}$$

For the meshes we need the following natural condition.

Assumption. *There exists positive constants C_1, C_2, such that*

$$C_1 h_{K_i} \le h_E \le C_2 h_{K_i}$$

for all $K_i \in \mathcal{C}_h^i$ and $E \in \mathcal{E}_h$ with $K_i \cap E \neq \emptyset$, $i = 1, 2$. $\square$

From this assumption the following result follows by standard scaling arguments.

Lemma 2. *There exists a positive constant C_I such that*

$$\sum_{E \in \mathcal{E}_h} h_E \Big\| \alpha_1 \frac{\partial v^1}{\partial n_1} - \alpha_2 \frac{\partial v^2}{\partial n_2} \Big\|_{0,E}^2 \le C_I \sum_{i=1}^{2} \|\nabla v^i\|_{0,\Omega_i}^2. \quad \square$$

Next, let us discuss the choice of the interface mesh $\mathcal{E}_h$ and the parameters γ and α_i. The most natural choice would be to let $\mathcal{E}_h$ be equal to $\mathcal{E}_h^1$ or $\mathcal{E}_h^2$, with

$$\mathcal{E}_h^i = \{ \ E \mid E = K \cap \Upsilon, \ K \in \mathcal{C}_h^i \ \}.$$

In this case when we choose $\mathcal{E}_h = \mathcal{E}_h^i$, then the natural choice is to choose $\alpha_i = 1$. Then the constant C_I is easily estimated (especially for linear elements).

The stability and error estimates will be given in the following mesh dependent norm.

$$\|v\|_{1,h}^2 = \sum_{i=1}^{2} \|\nabla v^i\|_{0,\Omega_i}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \|v^1 - v^2\|_{0,E}^2.$$

The advantage of Nitsches method is the stability:

Lemma 3. *Suppose that $\gamma > C_I$. Then it holds*

$$\mathcal{B}_h(v;v) \ge C \|v\|_{1,h}^2 \quad \forall v \in V_h.$$

Proof: Using the Schwartz and the arithmetic-geometric-mean inequalites, and Lemma 2 we get

$$\begin{aligned}
\mathcal{B}_h(v;v) &= \sum_{i=1}^{2}\|\nabla v^i\|^2_{0,\Omega_i} - 2\langle \alpha_1\frac{\partial v^1}{\partial n_1} - \alpha_2\frac{\partial v^2}{\partial n_2}, v^1 - v^2\rangle_\Upsilon + \gamma\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v^1-v^2\|^2_{0,E} \\
&\geq \sum_{i=1}^{2}\|\nabla v^i\|^2_{0,\Omega_i} - \frac{1}{\varepsilon}\sum_{E\in\mathcal{E}_h} h_E\Big\|\alpha_1\frac{\partial v^1}{\partial n_1} - \alpha_2\frac{\partial v^2}{\partial n_2}\Big\|^2_{0,E} + (\gamma-\varepsilon)\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v^1-v^2\|^2_{0,E} \\
&\geq \Big(1-\frac{C_I}{\varepsilon}\Big)\sum_{i=1}^{2}\|\nabla v^i\|^2_{0,\Omega_i} + (\gamma-\varepsilon)\sum_{E\in\mathcal{E}_h} h_E^{-1}\|v^1-v^2\|^2_{0,E}) \\
&\geq C\|v\|^2_{1,h},
\end{aligned}$$

by choosing $\gamma > \varepsilon > C_I$. □

For a function v^i defined on the subdomain Ω_i we define the mesh dependent norm

$$\|v^i\|^2_{h,\Omega_i} = \|\nabla v^i\|^2_{0,\Omega_i} + \sum_{E\in\mathcal{E}_h}\Big(h_E^{-1}\|v^i\|^2_{0,E} + h_E\|\frac{\partial v^i}{\partial n_i}\|^2_{0,E}\Big), \quad i=1,2.$$

The interpolation estimate in this norm is proved by scaling, cf. [10]. For this we need the assumption on the meshes.

Lemma 4. *Suppose the assumption on the meshes is valid. Then it holds*

$$\inf_{v^i\in V_h^i}\|u-v^i\|_{h,\Omega_i} \leq Ch^k\|u\|_{k+1,\Omega_i}\,. \quad \square$$

We now have established the stability, consistency and the optimal interpolation estimates, and hence we arrive at the error estimate for the method.

Theorem. *Suppose that the assumption on the meshes is valid and that $\gamma > C_I$. Then it holds*

$$\|u-u_h\|_{1,h} \leq Ch^k\|u\|_{k+1}\,. \quad \square$$

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