Stabilized finite element methods for Reissner-Mindlin plates

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Abstract

In this paper we consider the design of stabilized finite element methods for the Reissner-Mindlin plate bending model. First, we discuss a mixed formulation introduced by Hughes and Franca [23]. Next, we show how that formulation leads to a new method in which the shear force does not appear explicitely, i.e. it is formulated in the displacement variables alone. In the new formulation, as well as in the formulation by Hughes and Franca, unequal basis functions have to be used for the deflection and rotation, respectively. We then show that a formulation with equal basis functions can be obtained if our stabilization technique is combined with a covariant interpolation of the shear strain (the MITC reduction technique [8]). For all methods we also give an alternative formulation in which the moment tensor is approximated indepently, which leads to a better robustness with regards to the choice of stability parameters. Extensive numerical examples are given.

1 Introduction

One of the original motivations for the engineering community to turn to the Reissner-Mindlin plate bending model was to avoid the use of C^1 basis functions needed in the classical fourth order Kirchhoff model. However, when an engineer uses a plate model he implicitely assumes that the plate is "thin". Otherwise a full three dimensional analysis is more natural. Therefore, the problem associated with Kirchhoff elements now reappears in a disguised form as the Kirchhoff constraint that the Reissner-Mindlin model leads to when the thickness of the plate approach zero. Numerically this is seen as a "locking" or "overstiffness" of the numerical model. For avoiding this numerous remedies have been developed, cf. [10, 44] for state of the art surveys. Still most of the methods lack a rigorous error analysis.

Mathematically, the Reissner-Mindlin model can be written as a saddle point problem and hence Brezzis theory [11] provides a theoretical frame for both designing and analysing stable (i.e. locking free) and optimally convergent methods. Elements that in this way have been shown to be optimal are the MITC elements, introduced in [9] and [8] and analyzed in [8, 13], a nonconforming element of Arnold and Falk [2] and a low order MITC element by Duran and Liberman [19]. In a recent paper [34] Pitkäranta and Suri introduces an analysis technique for Reissner-Mindlin plate elements which is not based on the saddle point approach. Although they only apply their theory on MITC elements it is likely that also other elements could be analyzed by it.

Since a decade it is now known that the conditions on the finite element subspaces dictated by the Brezzi theory can largely be avoided if the discrete problem is reformulated in various stabilized forms, cf. e.g. [15, 20, 23] for the problem of incompressibility and [27, 26] for beams and arches. For the Reissner-Mindlin problem a stabilized modification of the Arnold-Falk element has been presented in [18] and [21]. A general formulation was proposed by Hughes and Franca [24].

The purpose of this paper is to propose several new stabilized formulations which are stable and optimally convergent. We start by studuing the method of Hughes and Franca [24]. By an inspection of this formulation we then arrive at a new method in which the shear force is not approximated independently; we obtain a discretization involving only the deflection and the rotation. The drawback with our new formulation and with the one by Hughes and Franca is that the polynomial degree for the approximation of the deflection is one order higher than that of the approximation for the rotation. This is quite natural insofar as for a thin plate the Kirchhoff condition is approximately valid. We, however, show that by combining the new stabilized formulation with a covariant interpolation of the shear strain we obtain methods using equal interpolation. Compared to the original MITC elements, which use the same covariant interpolation, our methods have the advantage that identical basis functions are used for both the deflection and the rotation. In the original MITC elements the spaces are not identical due to the conditions imposed by the Brezzi theory. Furthermore, all the stabilized formulations studied in this paper have the advantage over classical methods that the condition number of the stiffness matrix is improved. In contrast to the original MITC elements our new families of methods also include convergent linear triangular and bilinear quadrilateral elements. The linear and bilinear elements have earlier been introduced in [13], and in [30] we have given the results of extensive numerical tests with the quadrilateral element. Recently, we have also observed that the linear triangular element is essentially equivalent to two other other linear elements, the "anistropic interpolation" by Tessler and Hughes [40], and the "linked interpolation" by Xu [43], Taylor and Auriccio [6, 39]. The proof of this equivalence has been reported in a separate paper [28].

We give our results only with respect to the meshlength h, but it is clear that all methods can be used as general hp-methods. For part of the methods the analysis could quite easily be performed with respect to p as well. In our analysis we, however, use some nonstandard interpolation estimates, and as we do not for all cases know these with respect to p, we have chosen to give a uniform presentation in a h-setting. Let us mention that so far very few methods for the Reissner-Mindlin model have been analyzed with respect to both h and p, cf. [38, 37].

Part of our results have also been presented in the earlier communications [29, 36]. In a parallel work [16] we study stabilized formulations for the Naghdi shell model.

The plan of the paper is the following. In the next section we collect our main notation and recall some basic results. Then we consider the methods with unequal approximation. In Section 4 we show that by combining the new stabilized formulation with the covariant interpolation technique on obtains methods with equal order apporximation. In the final section we report on extensive numerical benchmark studies.

2 Preliminaries

2.1 The plate model

Let $\Omega \subset \mathrm{I\!R}^2$ be the midsurface of the plate and suppose that the plate is clamped along the boundary part Γ_C and free along the complementary part Γ_F. The variational formulation of Reissner-Mindlin model is: find the deflection $w \in W$ and the rotation vector $\boldsymbol{\beta} = (\beta_x, \beta_y) \in \boldsymbol{V}$ such that

$$Gt^3 a(\boldsymbol{\beta}, \boldsymbol{\eta}) + G\kappa t(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (f, v) \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}. \tag{2.1}$$

Here G is the shear modulus and t is the thickness of the plate. f is the transverse load acting on Ω and κ is the usual shear correction factor. The bilinear form a represents bending energy and is defined as

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{1}{6}\Big\{(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div} \boldsymbol{\beta}, \operatorname{div} \boldsymbol{\eta})\Big\}, \tag{2.2}$$

where ν is the Poisson ratio, $\boldsymbol{\varepsilon}(\cdot)$ is the small strain tensor and "div" stands for the divergence, viz.

$$\boldsymbol{\varepsilon}(\boldsymbol{\beta}) = \frac{1}{2}\Big\{\nabla \boldsymbol{\beta} + (\nabla \boldsymbol{\beta})^T\Big\}, \tag{2.3}$$

$$\operatorname{div} \boldsymbol{\beta} = \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}. \tag{2.4}$$

The spaces for the deflection and rotation are

$$W = \{\ v \in H^1(\Omega) \mid v_{|\Gamma_C} = 0\ \} \tag{2.5}$$

and

$$\boldsymbol{V} = \{\ \boldsymbol{\eta} \in [H^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|\Gamma_C} = \boldsymbol{0}\ \}, \tag{2.6}$$

respectively. Here and below we for $D \subset \mathbb{R}^2$ define the Sobolev spaces $H^s(D)$, with $s \geq 0$, in the usual way, i.e. first for integral values s and then for nonintegral values by interpolation, cf. [25]. The norms and seminorms will be denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. The L_2-inner products in $L_2(D)$, $[L_2(D)]^2$ or $[L_2(D)]^{2\times 2}$ are denoted by $(\cdot,\cdot)_D$. The subscript D will be dropped when $D = \Omega$.

For the analysis of the finite element methods we scale the problem by assuming that the load is proportional to the third power of the plate thickness, i.e. $f = Gt^3 g$ where g is fixed independent of t. Then the problem has a finite non-trivial solution in the limit when $t \to 0$ ([12, Proposition 3.1, p. 297]). Moreover, we will assume that $\kappa = 1$ and hence the problem we consider is: find $(w, \boldsymbol{\beta}) \in W \times \boldsymbol{V}$ such that

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}. \tag{2.7}$$

By taking the scaled shear force

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}) \tag{2.8}$$

as an independent unknown in the space $\boldsymbol{S} = [L_2(\Omega)]^2$ one gets the following mixed formulation: find $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in W \times \boldsymbol{V} \times \boldsymbol{S}$ such that

$$\begin{aligned} a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) &= (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}, \\ (\nabla w - \boldsymbol{\beta}, \boldsymbol{s}) - t^2(\boldsymbol{q}, \boldsymbol{s}) &= 0 \qquad \forall \boldsymbol{s} \in \boldsymbol{S}. \end{aligned} \tag{2.9}$$

The distributional differential equations of this system are obtained by integrating by parts:

$$\begin{aligned} \boldsymbol{L}\boldsymbol{\beta} + \boldsymbol{q} &= \boldsymbol{0} \quad \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{q} &= g \quad \text{in } \Omega, \\ -t^2\boldsymbol{q} + \nabla w - \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{in } \Omega, \\ w = 0,\ \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{on } \Gamma_C, \\ \boldsymbol{m} \cdot \boldsymbol{n} = \boldsymbol{0},\ \boldsymbol{q} \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma_F. \end{aligned} \tag{2.10}$$

Here the differential operator $\boldsymbol{L}$ is defined from

$$\boldsymbol{L}\boldsymbol{\eta} = \frac{1}{6}\mathbf{div}\left\{\boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \frac{\nu}{1-\nu}\operatorname{div} \boldsymbol{\eta} \boldsymbol{I}\right\} \tag{2.11}$$

and $\boldsymbol{m}$ is the moment tensor

$$\boldsymbol{m} = \frac{t^3}{6}\left\{\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \frac{\nu}{1-\nu}\operatorname{div} \boldsymbol{\beta} \boldsymbol{I}\right\}. \tag{2.12}$$

It holds

$$\boldsymbol{L\beta} = \mathbf{div}\, \boldsymbol{m}, \tag{2.13}$$

where we used the notation **div** for the divergence opertor applied to a second order tensor:

$$\mathbf{div}\, \boldsymbol{r} = (\frac{\partial r_{xx}}{\partial x} + \frac{\partial r_{xy}}{\partial y}, \frac{\partial r_{yx}}{\partial x} + \frac{\partial r_{yy}}{\partial y}). \tag{2.14}$$

The first two equations in (2.10) above are the local equilibrium equations between the moment, shear force and load. The third equation represents the constitutive relation between the shear strain and shear force.

In the limit $t \to 0$ the solution $(w, \boldsymbol{\beta}) = (w_t, \boldsymbol{\beta}_t)$ of the Reissner–Mindlin equations converges to the Kirchhoff solution with

$$\boldsymbol{\beta}_0 = \nabla w_0. \tag{2.15}$$

The limit solution w_0 satisfies the biharmonic equation in the domain Ω and only two boundary conditions on each part of the boundary, cf. [1, 3, 4]. This singularity gives rise to the boundary layers in the solution which complicates the convergence analysis of the methods.

In the analysis of our methods we will in the last step, the consistency estimate, assume that the plate is clamped, i.e. that $\Gamma_C = \partial\Omega$. This we do in order to get a estimate which shows hove the convergence depends on the load. We empasize, however, that the formulations of the methods and, in particular, the uniform stability with respect to the thickness do not depend on the choice of boundary conditions. In our consistency estimate we will use the following regularity estimate the proof of which is postponed to an appendix.

Theorem 2.1 *Let Ω be a convex polygonal domain and let Ω_i be a domain compactly embedded in Ω. Denote by $(w, \boldsymbol{\beta}, \boldsymbol{q})$ the Reissner–Mindlin solution for the clamped plate and let $w = w_0 + w_r$, where w_0 is the deflection obtained from the Kirchhoff model. With $g \in H^{s-2}(\Omega)$ and $tg \in H^{s-1}(\Omega)$, $s \geq 1$, it then holds*

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\boldsymbol{\beta}\|_2 + \|\boldsymbol{q}\|_0 + t\|\boldsymbol{q}\|_1 \leq C(\|g\|_{-1} + t\|g\|_0) \tag{2.16}$$

and

$$\begin{aligned}\|w_0\|_{s+2,\Omega_i} + t^{-1}\|w_r\|_{s+1,\Omega_i} + \|\boldsymbol{\beta}\|_{s+1,\Omega_i} + \|\boldsymbol{q}\|_{s-1,\Omega_i} + t\|\boldsymbol{q}\|_{s,\Omega_i} \\ \leq C(\|g\|_{s-2} + t\|g\|_{s-1}). \quad \square\end{aligned} \tag{2.17}$$

2.2 Finite element subspaces

For simplicity we will assume that the domain Ω is polygonal. We let $\mathcal{C}_h$ be the partitioning of $\bar{\Omega}$ into triangles or convex quadrilaterals satisfying the usual compatibility conditions, i.e. the intersection of two elements is either empty, an edge or a vertex. A mesh consisting of both triangles and quadrilaterals is allowed. The elements $K \in \mathcal{C}_h$ are the images of the reference element $\hat{K}$ (i.e. the unit triangle or unit square) under the affine or bilinear mapping $\boldsymbol{F}_K : \hat{K} \to K$. By $\boldsymbol{J}_K$ we denote the Jacobian matrix of $\boldsymbol{F}_K$. The elements $K \in \mathcal{C}_h$ are assumed to be regular in the usual sense [17]. For the quadrilaterals we in addition need the assumption that the maximum ratio between any pair of sides is bounded above independent of K.

On the reference element $\hat{K}$, $P_m(\hat{K})$ will stand for the space of polynomials of total degree $\leq m$. The space of polynomials of degree $\leq m$ in each variable will be denoted by $Q_m(\hat{K})$. We also use the notation

$$P_{m,n}(\hat{K}) = \{\hat{v} \mid \hat{v} = \sum_{0=1}^{m}\sum_{0=1}^{n} a_{ij}\xi^i\eta^j,\ a_{ij} \in \mathbb{R}\}, \tag{2.18}$$

$$Q'_m(\hat{K}) = P_m(\hat{K}) \oplus \operatorname{span}\{\xi^m\eta,\ \xi\eta^m\}, \tag{2.19}$$

where ξ and η are the coordinates of $\hat{K}$ (the natural coordinates of K). For a polynomial space $H(\hat{K})$ defined on the reference element $\hat{K}$ we define the corresponding space on K in the usual way, viz.

$$H(K) = \{v = \hat{v} \circ \boldsymbol{F}_K^{-1} \mid \hat{v} \in H(\hat{K})\}. \tag{2.20}$$

As usual h_K denotes the diameter of $K \in \mathcal{C}_h$ and h stands for the global mesh parameter

$$h = \max_{K \in \mathcal{C}_h} h_K. \tag{2.21}$$

To measure the size off the elements in the interior region Ω_i and the the boundary region $\Omega_b = \Omega \setminus \Omega_i$ we introduce the parameters

$$h_i = \max_{K \subset \Omega_i} h_K, \quad h_b = \max_{K \not\subset \Omega_i} h_K. \tag{2.22}$$

The finite element subspaces for the deflection, rotation vector and shear force we define with the index $k \geq 1$ as

$$W_h = \{v \in W \mid v_{|K} \in W_k(K),\ \forall K \in \mathcal{C}_h\}, \tag{2.23}$$

$$\boldsymbol{V}_h = \{\boldsymbol{\eta} \in \boldsymbol{V} \mid \boldsymbol{\eta}_{|K} \in \boldsymbol{V}_k(K),\ \forall K \in \mathcal{C}_h\}, \tag{2.24}$$

and

$$\boldsymbol{S}_h = \{\boldsymbol{s} \in \boldsymbol{S} \mid \boldsymbol{s}_{|K} \in \boldsymbol{S}_k(K),\ \forall K \in \mathcal{C}_h\}, \tag{2.25}$$

respectively.

The above finite elements are defined differently in the various methods to be considered below.

It will be convenient to introduce the notation

$$a_K(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{1}{6}\Big\{(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta}))_K + \frac{\nu}{1-\nu}(\operatorname{div}\boldsymbol{\beta}, \operatorname{div}\boldsymbol{\eta})_K\Big\}. \tag{2.26}$$

By C we will denote various positive constants which do not depend on the plate thickness t or the mesh $\mathcal{C}_h$.

3 Families with unequal interpolation

3.1 The method of Hughes and Franca

The starting point for the Hughes-Franca method was a mixed formulation based on the variational form (2.9). As is well known, a direct finite element discretization based on that is difficult to design. Only the stability of the rotation is automatically ensured. To have stability for the deflection and shear, the method has to satisfy the "Z-ellipticity" and "inf-sup" conditions, cf. [12]. To avoid this Hughes and Franca [24] introduced a stabilized mixed method, which we here present in a slightly altered form given by Peisker [31].

Method 3.1 (Hughes & Franca [24]) *Find* $(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h$ *such that*

$$\mathcal{M}_h(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h; v, \boldsymbol{\eta}, \boldsymbol{s}) = (g, v) \qquad \forall (v, \boldsymbol{\eta}, \boldsymbol{s}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h, \tag{3.1}$$

with the bilinear form defined as

$$\begin{aligned} \mathcal{M}_h(z, \boldsymbol{\phi}, \boldsymbol{r}; v, \boldsymbol{\eta}, \boldsymbol{s}) \;=\;& a(\boldsymbol{\phi}, \boldsymbol{\eta}) + (\boldsymbol{r}, \nabla v - \boldsymbol{\eta}) + (\boldsymbol{s}, \nabla z - \boldsymbol{\phi}) - t^2(\boldsymbol{r}, \boldsymbol{s}) \\ & -\alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\,\boldsymbol{\phi} + \boldsymbol{r}, \boldsymbol{L}\,\boldsymbol{\eta} + \boldsymbol{s})_K \\ & +\gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} (\nabla z - \boldsymbol{\phi} - t^2 \boldsymbol{r}, \nabla v - \boldsymbol{\eta} - t^2 \boldsymbol{s})_K. \qquad \square \end{aligned} \tag{3.2}$$

The bilinear form consists of the bilinear form of the continous problem (2.9) with the addition of two terms which are locally weighted squares of two of the differential equations in (2.10). Hence, the formulation is consistent:

Theorem 3.1 *The solution* $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in W \times \boldsymbol{V} \times \boldsymbol{S}$ *to* (2.9) *satisfies the equation*

$$\mathcal{M}_h(w, \boldsymbol{\beta}, \boldsymbol{q}; v, \boldsymbol{\eta}, \boldsymbol{s}) = (g, v) \qquad \forall (v, \boldsymbol{\eta}, \boldsymbol{s}) \in W \times \boldsymbol{V} \times \boldsymbol{S}. \qquad \square \tag{3.3}$$

The stability and error estimates will be formulated using the following seminorm and norm.

Definition 3.1 *For* $(v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}$ *we define*

$$|(v, \boldsymbol{\eta})|_h^2 = \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2, \tag{3.4}$$

$$|||(v, \boldsymbol{\eta})|||_h = \|v\|_1 + \|\boldsymbol{\eta}\|_1 + |(v, \boldsymbol{\eta})|_h, \tag{3.5}$$

and for $\boldsymbol{q} \in \boldsymbol{S}$

$$\|\boldsymbol{q}\|_{-1,h} = \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{q}\|_{0,K}^2 \Big)^{1/2}. \qquad \square \tag{3.6}$$

We note that the following equivalence of norms holds.

Lemma 3.1 *There is a positive constant* C *such that*

$$C |||(v, \boldsymbol{\eta})|||_h \le \|\boldsymbol{\eta}\|_1 + |(v, \boldsymbol{\eta})|_h \le |||(v, \boldsymbol{\eta})|||_h \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the Poincaré inequality and the inequality $(t^2 + h_K^2) \leq C$, we get

$$\begin{aligned}
\|v\|_1^2 &\leq C\|\nabla v\|_0^2 \leq C(\|\nabla v - \boldsymbol{\eta}\|_0^2 + \|\boldsymbol{\eta}\|_0^2) \\
&\leq C(\|\nabla v - \boldsymbol{\eta}\|_0^2 + \|\boldsymbol{\eta}\|_1^2) \\
&\leq C(\sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2),
\end{aligned}$$

which proves the claim. □

For the stability we need the following inverse estimate which is valid for any polynomial space $\boldsymbol{V}_k(K)$ consisting of polynomials (cf. e.g. [17, 22]).

Lemma 3.2 *There is a positive constant C_I such that*

$$C_I h_K^2 \|\boldsymbol{L}\,\boldsymbol{\phi}\|_{0,K}^2 \leq a_K(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_k(K). \qquad \square$$

Remark 3.1 The constant C_I above will play an important role, not only in the analysis of the methods, but also in numerical computations, cf. Section 5 below. □

Theorem 3.2 *Suppose that $0 < \alpha < C_I$ and $0 < \gamma < 1$. Then the Method 3.1 is stable, i.e. there is a positive constant C such that, for all $(v, \boldsymbol{\eta}, \boldsymbol{s}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h$ there exists $(z, \boldsymbol{\xi}, \boldsymbol{r}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h$ with*

$$\mathcal{M}_h(v, \boldsymbol{\eta}, \boldsymbol{s}; z, \boldsymbol{\xi}, \boldsymbol{r}) \geq C\ (|||(v, \boldsymbol{\eta})|||_h^2 + \|\boldsymbol{q}\|_{-1,h}^2 + t^2\|\boldsymbol{q}\|_0^2)$$

and

$$|||(z, \boldsymbol{\xi})|||_h + \|\boldsymbol{r}\|_{-1,h} + t\|\boldsymbol{r}\|_0 \leq |||(v, \boldsymbol{\eta})|||_h + \|\boldsymbol{q}\|_{-1,h} + t\|\boldsymbol{q}\|_0.$$

Proof: By simply choosing $(z, \boldsymbol{\xi}, \boldsymbol{r}) = (v, \boldsymbol{\eta}, -\boldsymbol{s})$ one obtains

$$\begin{aligned}
&\mathcal{M}_h(v, \boldsymbol{\eta}, \boldsymbol{s}; v, \boldsymbol{\eta}, -\boldsymbol{s}) \\
&\quad = \ a(\boldsymbol{\eta}, \boldsymbol{\eta}) + t^2\|\boldsymbol{s}\|_0^2 - \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{s}\|_{0,K}^2 \\
&\qquad\quad + \gamma \sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 - \gamma \sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)^{-1} t^4 \|\boldsymbol{s}\|_{0,K}^2 \qquad (3.7) \\
&\quad \geq \ C(\|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)\|\boldsymbol{s}\|_{0,K}^2) \\
&\qquad \forall (v, \boldsymbol{\eta}, \boldsymbol{s}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h,
\end{aligned}$$

when $0 < \alpha < C_I$ and $0 < \gamma < 1$. □

Hence, the addition of the two new terms gives a stable method for every choice of the finite element spaces.

When in addition the consistency is valid, the choice of the finite element spaces is solely dictated by the proper balance between their interpolation properties with respect to the norms used. With the notation

$$R_m(K) = \begin{cases} P_m(K) \ \text{ when } K \text{ is a triangle,} \\ Q_m(K) \text{ or } \ Q'_m(K) \text{ when } K \text{ is a quadrilateral,} \end{cases} \qquad (3.8)$$

where for each positive integer m, $P_m(K)$ and $Q_m(K)$ are the usual triangular and and quadrilateral finite element subspaces [17], we are then led to the following choice

$$W_k(K) = R_{k+1}(K), \quad \boldsymbol{V}_k(K) = [R_k(K)]^2, \quad \text{and} \quad \boldsymbol{S}_k(K) = [R_{k-1}(K)]^2, \tag{3.9}$$

with the index $k \geq 1$. Note, that we are here allowing a mixing of triangles and quadrilaterals in the mesh. From the stability and consistency an error estimate now follows in the standard manner, cf. the original work of Hughes and Franca [24]. Such a straightforward analysis, however, gives an estimate which assumes a regularity of the solution wich is more realistic for the Kirchhoff model than for the Reissner-Mindlin counterpart. As a consequence, the method has often been disregarded as a "Kirchhoff" element. By doing a more detailed consistency analysis using Theorem 2.1 it is possible to derive an estimate which only requires the Reissner-Mindlin regularity and shows how the regularity depends on the thickness t. As this estimation of the consistency error is completely equivalent to the one to be performed for the method in next section we postpone it and only give the final result:

Theorem 3.3 *Suppose that* $0 < \alpha < C_I$ *and* $0 < \gamma < 1$. *For the solution* $(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h)$ *of* (3.1) *it then holds*

$$\begin{aligned} &|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + t\,\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \\ &\qquad \leq C\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \qquad \square \end{aligned}$$

Let us now show how a new formulation emerges by studying this formulation. First, we point out that we have chosen the shear to be discontinuous. This we have done in order that that the method should be competitive in structural mechanics as it makes it possible to locally eliminate the shear leading to a final linear system which is symmetric and positive definite in the displacement variables. More precisely, let $\boldsymbol{\pi}_K : [L^2(K)]^2 \to \boldsymbol{S}_k(K)$ be the element L^2-projection. With the notation

$$\tau_K = \frac{(1-\gamma)t^2 + \alpha h_K^2}{t^2 + \alpha h_K^2} \tag{3.10}$$

the elimination of the shear then gives the following method:

$$\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{3.11}$$

with the bilinear form defined as

$$\begin{aligned} \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\,\boldsymbol{\phi}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &+ \gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} (\nabla z - \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta})_K \\ &+ \sum_{K \in \mathcal{C}_h} (\tau_K t^2 + \alpha h_K^2)^{-1} (\boldsymbol{\pi}_K[\tau_K(\nabla z - \boldsymbol{\phi}) - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\phi}], \boldsymbol{\pi}_K[\tau_K(\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}])_K. \end{aligned} \tag{3.12}$$

The shear is given by

$$\boldsymbol{q}_{h|K} = (\tau_K t^2 + \alpha h_K^2)^{-1} \boldsymbol{\pi}_K (\tau_K(\nabla w_h - \boldsymbol{\beta}_h) - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}_h)_{|K} \quad \forall K \in \mathcal{C}_h. \tag{3.13}$$

Now, it is quite easily seen (cf. [33, Theorem 3.1]) that if we choose the finite element spaces such that $\nabla W_h \subset \boldsymbol{S}_h$ the method (3.1) is stable without the second stabilizing term in (3.2). With this we get the bilinear form from above with $\gamma = 0$ and $\tau_K = 1$, i.e.

$$\begin{aligned}\mathcal{B}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta}) = a(\boldsymbol{\phi},\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{L}\,\boldsymbol{\phi},\boldsymbol{L}\,\boldsymbol{\eta})_K \qquad (3.14)\\ + \sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\boldsymbol{\pi}_K[\nabla z - \boldsymbol{\phi} - \alpha h_K^2\boldsymbol{L}\,\boldsymbol{\phi}], \boldsymbol{\pi}_K[\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}])_K.\end{aligned}$$

Next, we chose the subspace for the shear big enough so that

$$\boldsymbol{\pi}_K(\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\phi}) = \nabla z - \boldsymbol{\phi} - \alpha h_K^2\boldsymbol{L}\,\boldsymbol{\phi} \quad \forall K\in\mathcal{C}_h, \quad \forall (z,\boldsymbol{\phi}) \in W_h\times \boldsymbol{V}_h. \qquad (3.15)$$

This leads to the following formulation introduced in [36].

3.2 A displacement formulation

By the arguments presented above we arrive in a formulation in diplacement variables, i.e. two finite element subspaces $W_h \subset W$ and $\boldsymbol{V}_h \subset \boldsymbol{V}$ for the deflection and rotation, respectively. The finite element approximation $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ is then defined as

Method 3.2 *Find $(w_h,\boldsymbol{\beta}_h) \in W_h\times\boldsymbol{V}_h$ such that*

$$\mathcal{B}_h(w_h,\boldsymbol{\beta}_h;v,\boldsymbol{\eta}) = (g,v) \quad \forall (v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h, \qquad (3.16)$$

with the bilinear form defined as

$$\begin{aligned}\mathcal{B}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta}) \;=\; & a(\boldsymbol{\phi},\boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h}\alpha_K h_K^2(\boldsymbol{L}\,\boldsymbol{\phi},\boldsymbol{L}\,\boldsymbol{\eta})_K \qquad (3.17)\\ & + \sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}(\nabla z - \boldsymbol{\phi} - \alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K.\end{aligned}$$

From the solution $(w_h,\boldsymbol{\beta}_h)$ we then calculate the approximation for the shear by

$$\boldsymbol{q}_{h|K} = (t^2+\alpha_K h_K^2)^{-1}(\nabla w_h - \boldsymbol{\beta}_h - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}_h)_{|K} \qquad \forall K\in\mathcal{C}_h. \qquad (3.18)$$

Here α_K is a positive parameter lying in a fixed range which will be specified in Theorem 3.6 below. □

The formulation above is slightly more general as we allow the parameter α_K to depend on the element K. Note that (2.8) and the first equation of (2.10) give

$$\boldsymbol{q}_{|K} = (t^2+\alpha_K h_K^2)^{-1}(\nabla w - \boldsymbol{\beta} - \alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\beta})_{|K} \qquad \forall K\in\mathcal{C}_h. \qquad (3.19)$$

Hence, we see that the approximation (3.18) is consistent with the exact shear.

From the way we arrived at the formulation it is clear that it should is both stable and consistent. This and the convergence can however, proved directly.

Theorem 3.4 *The solution* $(w, \boldsymbol{\beta})$ *to* (2.10) *satisfies the equation*

$$\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}.$$

Proof: Recalling the first equation in (2.10), the expression (3.19), and the variational form (2.9), we get

$$\begin{aligned}
&\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{L}\,\boldsymbol{\beta}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\nabla w - \boldsymbol{\beta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}, \nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{q}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (\boldsymbol{q}, \nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{q}, \boldsymbol{L}\,\boldsymbol{\eta})_K + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{q}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) \\
&= (g, v). \qquad \square
\end{aligned}$$

Next, we formulate the stability condition.

Theorem 3.5 *Suppose that there is a positive constant* α *such that* $0 < \alpha < \alpha_K < C_I$ *for all* $K \in \mathcal{C}_h$. *Then there is a positive constant* C *such that*

$$\mathcal{B}_h(v, \boldsymbol{\eta}; v, \boldsymbol{\eta}) \geq C |||(v, \boldsymbol{\eta})|||_h^2 \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the inverse estimate of Lemma 3.2 and Korn's inequality we get

$$\begin{aligned}
&\mathcal{B}_h(v, \boldsymbol{\eta}; v, \boldsymbol{\eta}) \\
&= a(\boldsymbol{\eta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 \|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 \\
&\geq \sum_{K \in \mathcal{C}_h} (1 - \alpha_K C_I^{-1})\, a_K(\boldsymbol{\eta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 \\
&\geq C\Big(a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big) \qquad (3.20) \\
&\geq C\Big(\|\boldsymbol{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big).
\end{aligned}$$

Using the triangle inequality, the same inverse estimate and the boundedness of the bilinear form a we obtain

$$\begin{aligned}
&\sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 \\
&\leq 2\Big(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} \|\alpha_K h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq 2\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+\sum_{K\in\mathcal{C}_h}\alpha_K h_K^2\|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big) \qquad (3.21)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+a(\boldsymbol{\eta},\boldsymbol{\eta})\Big)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+\|\boldsymbol{\eta}\|_1^2\Big).
\end{aligned}$$

Combining (3.20) and (3.21) gives

$$\mathcal{B}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta})\geq C(\|\boldsymbol{\eta}\|_1^2+|(v,\boldsymbol{\eta})|_h^2),$$

and the claim follows from Lemma 3.1. □

Remark 3.2 For triangular elements with $k=1$ it holds

$$\boldsymbol{L}\,\boldsymbol{\phi}_{|K}=\mathbf{0},\quad \forall K\in\mathcal{C}_h,\quad \forall\boldsymbol{\phi}\in\boldsymbol{V}_h,$$

and hence the bilinear form $\mathcal{B}_h$ reduces to

$$\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta})=a(\boldsymbol{\beta},\boldsymbol{\eta})+\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}(\nabla w-\boldsymbol{\beta},\nabla v-\boldsymbol{\eta})_K,$$

and we obtain a formulation proposed by Pitkäranta [33]. Now, the method is stable for all positive values of α_K. It is easily seen that the above bilinear form also can be used in the quadrilateral case for $k=1$ without a decrease in accuracy. □

Remark 3.3 For triangular elements and $k\geq 4$ it is shown in [8, 13] that the method is stable with $\alpha_K=0$. However, it may be advantageous to choose $\alpha_K>0$ also in that case since it gives a more well conditioned stiffness matrix, cf. [31] where the formulation by Hughes and Franca [24] is used for multigrid algorithms. □

We next derive error estimates for the methods.

Theorem 3.6 *Suppose that $0<\alpha<\alpha_K<C_I$ for all $K\in\mathcal{C}_h$. For the solution $(w_h,\boldsymbol{\beta}_h,\boldsymbol{q}_h)$ of (3.16) it then holds*

$$\begin{aligned}
&|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1,h}+t\|\boldsymbol{q}-\boldsymbol{q}_h\|_0\\
&\qquad\leq C\Big\{h_i^k(\|g\|_{k-2}+t\|g\|_{k-1})+h_b(\|g\|_{-1}+t\|g\|_0)\Big\}.
\end{aligned}$$

Proof: Let $\tilde{w}\in W_h$ and $\tilde{\boldsymbol{\beta}}\in\boldsymbol{V}_h$ be the usual Lagrange interpolants to w and $\boldsymbol{\beta}$, respectively. From Lemmas 3.5 and 3.4 there is $(v,\boldsymbol{\eta})\in W_h\times\boldsymbol{V}_h$ such that

$$|||(v,\boldsymbol{\eta})|||_h\leq C, \qquad (3.22)$$

and

$$|||(w_h-\tilde{w},\boldsymbol{\beta}_h-\tilde{\boldsymbol{\beta}})|||_h\leq\mathcal{B}_h(w_h-\tilde{w},\boldsymbol{\beta}_h-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta})=\mathcal{B}_h(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta}). \qquad (3.23)$$

Next, we write out the right hand side of (3.23)

$$\begin{aligned}\mathcal{B}_h(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta}) &= a(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}},\boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h}\alpha_K h_K^2(\boldsymbol{L}\,(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}),\boldsymbol{L}\,\boldsymbol{\eta})_K \qquad (3.24)\\ &+\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}(\nabla(w-\tilde{w})-(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})-\alpha_K h_K^2\boldsymbol{L}\,(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}),\nabla v-\boldsymbol{\eta}-\alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K.\end{aligned}$$

Due to the inverse inequality of Lemma 3.2 we have

$$\Big(\sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\le C|||(v,\boldsymbol{\eta})|||_h$$

and

$$\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha_K h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha_K h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2}\le C|||(v,\boldsymbol{\eta})|||_h.$$

Using these and (3.22), (3.24) gives

$$\begin{aligned}&\mathcal{B}_h(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta}) \qquad (3.25)\\ &\le\ C\Big(|||(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})|||_h+\Big(\sum_{K\in\mathcal{C}_h}h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\Big)^{1/2}\Big).\end{aligned}$$

From (3.23) and (3.25) we thus get

$$|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h\le C\Big(|||(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})|||_h+\Big(\sum_{K\in\mathcal{C}_h}h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\Big)^{1/2}\Big). \qquad (3.26)$$

Writing out the right hand side above we obtain

$$\begin{aligned}&|||(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})|||_h+\Big(\sum_{K\in\mathcal{C}_h}h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\Big)^{1/2} \qquad (3.27)\\ &\le\ C\Big\{\sum_{K\in\mathcal{C}_h}\Big(|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{1,K}^2+(t^2+h_K^2)^{-1}(|w-\tilde{w}|_{1,K}^2+\|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}\|_{0,K}^2)+h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\Big)\Big\}^{1/2}.\end{aligned}$$

Let us next estimate the different terms. To this end let us measure the smoothness of the solution in the element K by the parameter s_K, $1\le s_K\le k$, by assuming that

$$w_{0|K}\in H^{s_K+2}(K),\quad w_{r|K}\in H^{s_K+1}(K)\ \text{ and }\boldsymbol{\beta}_{|K}\in[H^{s_K+1}(K)]^2.$$

Standard interpolation estimates then gives

$$\begin{aligned}&|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{1,K}^2+(t^2+h_K^2)^{-1}\|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}\|_{0,K}^2+h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\\ &\le|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{1,K}^2+h_K^{-2}\|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}\|_{0,K}^2+h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2\\ &\le h_K^{2s_K}\|\boldsymbol{\beta}\|_{s_K+1,K}^2 \qquad (3.28)\end{aligned}$$

and

$$\begin{aligned}&(t^2+h_K^2)^{-1}|w-\tilde{w}|_{1,K}^2=(t^2+h_K^2)^{-1}|w_0-\tilde{w}_0+w_r-\tilde{w}_r|_{1,K}^2\\ &\le 2(t^2+h_K^2)^{-1}|w_0-\tilde{w}_0|_{1,K}^2+2(t^2+h_K^2)^{-1}|w_r-\tilde{w}_r|_{1,K}^2\end{aligned}$$

$$\begin{aligned} &\leq 2h_K^{-2}|w_0 - \tilde{w}_0|_{1,K}^2 + 2t^{-2}|w_r - \tilde{w}_r|_{1,K}^2 \\ &\leq C(h_K^{2s_K}\|w_0\|_{s_K+2,K}^2 + t^{-2}h_K^{2s_K}\|w_r\|_{s_K+1,K}^2). \end{aligned} \tag{3.29}$$

Hence, we have

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \leq C\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2} \tag{3.30}$$

with

$$E_K^2 = h_K^{2s_K}\|w_0\|_{s_K+2,K}^2 + t^{-2}h_K^{2s_K}\|w_r\|_{s_K+1,K}^2 + h_K^{2s_K}\|\boldsymbol{\beta}\|_{s_K+1,K}^2. \tag{3.31}$$

Now, chosing $s_K = k$ when $K \subset \Omega_i$ and $s_K = 1$ for $K \subset \Omega_b$ this yields

$$\begin{aligned} |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \ \leq\ & Ch_i^k\Big(\|w_0\|_{k+2,\Omega_i} + t^{-1}\|w_r\|_{k+1,\Omega_i} + \|\boldsymbol{\beta}\|_{k+1,\Omega_i}\Big) \\ & + Ch_b\Big(\|w_0\|_{3,\Omega_b} + t^{-1}\|w_r\|_{2,\Omega_b} + \|\boldsymbol{\beta}\|_{2,\Omega_b}\Big). \end{aligned} \tag{3.32}$$

The asserted estimate for the displacement variables $(w, \boldsymbol{\beta})$ now follows from Theorem 2.1.

The estimate for the shear force follows from the above estimates since (3.18) and (3.19) yield

$$\Big(\sum_{K\in\mathcal{C}_h}(t^2 + \alpha_K h_K^2)\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,K}^2\Big)^{1/2} \leq |(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|_h + \Big(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta} - \boldsymbol{\beta}_h|_{2,K}^2)\Big)^{1/2}. \tag{3.33}$$

□

Remark 3.4 By the "Pitkäranta–Verfürth trick (cf. [32, 42]) it is possible to derive the estimate for the shear in the "right norm", i.e. the norm $\|\cdot\|_{-1}$ dual to energy norm in $\boldsymbol{V}$, cf. [36]:

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1} \leq C\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \tag{3.34}$$

□

By the usual duality arguments it is possible to derive error estimates in lower order norms. Here we only state the results and refer to our previous paper for the derivations.

Theorem 3.7 *Suppose that $0 < \alpha < \alpha_K < C_I$ for all $K \in \mathcal{C}_h$ and that the region Ω is convex. Then it holds*

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 \leq Ch\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}, \tag{3.35}$$

and

$$\|w - w_h\|_1 \leq C(t + h)\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \quad \square \tag{3.36}$$

We will close this section by giving some error estimates in the case when $t \leq Ch_K$ for all elements $K \in \mathcal{C}_h$. We then say that the plate is *numerically thin.* Following [7] we define the mesh dependent norm

$$\|u\|_{2,h}^2 = \sum_{K\in\mathcal{C}_h} \|u\|_{2,K}^2 + \sum_{T\in\Gamma_h} h_T^{-1} \int_T |[\![\frac{\partial u}{\partial n_T}]\!]|^2 \, ds, \tag{3.37}$$

where Γ_h denotes the collection of edges in $\mathcal{C}_h$ and h_T is the length of the edge T. $[\![\partial u/\partial n_T]\!]$ denotes the jump in the normal derivative of u along the edge T. If T is on the boundary $\partial\Omega$ it denotes the value of the normal derivative in Ω. In [33, Lemma 3.1] the following result is proved.

Lemma 3.3 *There is a constant $C > 0$ such that*

$$\|u\|_{2,h} \leq C|||(u, \boldsymbol{\phi})|||_h \quad \forall (u, \boldsymbol{\phi}) \in W_h \times \boldsymbol{V}_h. \qquad \square$$

We then have the following estimates.

Corollary 3.1 *Suppose that $0 < \alpha < \alpha_K < C_I$ for all $K \in \mathcal{C}_h$ and that the plate is numerically thin. Then it holds*

$$\|w - w_h\|_{2,h} \leq C\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \tag{3.38}$$

If the region Ω is convex we additionally have

$$\|w - w_h\|_1 \leq Ch\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \qquad \square \tag{3.39}$$

We remark that in practice the assumption of a numerically thin plate seems to be quite natural. If the lengths of the elements are substantially smaller than the thickness of the plate, the error in the plate model is probably bigger that error in the finite element discretization. Hence, one should then turn to a direct three dimensional analysis of the plate.

4 Families with equal interpolation

In these families of methods identical basis funtions will be used for both the deflection and the rotation vector. Hence, for the index $k \geq 1$ they are defined as

$$W_k(K) = R_k(K) \quad \text{and} \quad \boldsymbol{V}_k(K) = [R_k(K)]^2. \tag{4.1}$$

We recall that the local space is defined with the usual mapping technique from the corresponding space in the reference element

$$R_k(K) = \{v = \hat{v} \circ \boldsymbol{F}_K^{-1} \mid \hat{v} \in R_k(\hat{K})\}. \tag{4.2}$$

For the discrete shear strain will be modified by the MITC interpolation techniques and the above convention is not used for the local strain space. Instead, it is defined through

$$\boldsymbol{S}_k(K) = \{\boldsymbol{\eta} = \boldsymbol{J}_K^{-T}\hat{\boldsymbol{\eta}} \circ \boldsymbol{F}_K^{-1} \mid \hat{\boldsymbol{\eta}} \in \boldsymbol{S}_k(\hat{K})\}, \tag{4.3}$$

where $\boldsymbol{J}_K^{-T}$ is the transpose of $\boldsymbol{J}_K^{-1}$. With $\boldsymbol{R}_K : [H^1(K)]^2 \to \boldsymbol{S}_k(K)$ we denote the the local MITC reduction operator, which is defined from the operator on the reference element by a covariant transformation through the equation

$$\boldsymbol{R}_K\boldsymbol{\eta} = \boldsymbol{J}_K^{-T}\boldsymbol{R}_{\hat{K}}\boldsymbol{J}_K^T\boldsymbol{\eta}. \tag{4.4}$$

We will first define a general formulation, refered to as the "Method", and the particular choices, to be specified by the local finite element spaces and reduction operators, will be refered to as "Elements".

Method 4.1 *Find* $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ *such that*

$$\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{4.5}$$

with

$$\begin{aligned}\mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = {} & a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{R}_K \boldsymbol{L}\boldsymbol{\phi}, \boldsymbol{R}_K \boldsymbol{L}\boldsymbol{\eta})_K \\ & + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K(\nabla z - \boldsymbol{\phi}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\phi}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K.\end{aligned} \tag{4.6}$$

The approximate shear is computed from

$$\boldsymbol{q}_{h|K} = (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K(\nabla w_h - \boldsymbol{\beta}_h) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\beta}_h)_{|K}, \ \forall K \in \mathcal{C}_h. \quad \square \tag{4.7}$$

Next, we will give the three different alternatives. We point out that all three elements are compatible in the sense that they can be used simulataneously for different triangles and quadrilaterals (which can be mixed) in the mesh.

Element 4.1 *For a triangle* K *we choose* $R_k(\hat{K}) = P_k(\hat{K})$ *and*

$$\boldsymbol{S}_k(\hat{K}) = [P_{k-1}(\hat{K})]^2 \oplus (\eta, -\xi)\widetilde{P}_{k-1}(\hat{K}), \tag{4.8}$$

where $\widetilde{P}_{k-1}(\hat{K})$ *is space of homogeneous polynomials of degree* $k-1$ *on* $\hat{K}$*. This is the rotated Raviart-Thomas space* [35]*. The reduction operator* $\boldsymbol{R}_{\hat{K}}$ *is defined through the conditions*

$$\int_{\hat{E}} [(\boldsymbol{R}_{\hat{K}}\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{\tau}}]\hat{v} \, d\hat{s} = 0, \ \forall \hat{v} \in P_{k-1}(\hat{E}), \ \textit{for every edge } \hat{E} \text{ of } \hat{K}, \tag{4.9}$$

and

$$\int_{\hat{K}} (\boldsymbol{R}_{\hat{K}}\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{s}} \, d\xi d\eta = 0, \ \forall \hat{\boldsymbol{s}} \in [P_{k-2}(\hat{K})]^2. \tag{4.10}$$

Here $\hat{\boldsymbol{\tau}}$ *is the unit tangent to the edge* $\hat{E}$.

Remark 4.1 When $k = 1$, it holds

$$\boldsymbol{L}\boldsymbol{\eta}_{|K} = \boldsymbol{0}, \quad \forall K \in \mathcal{C}_h, \ \forall \boldsymbol{\eta} \in \boldsymbol{V}_h, \tag{4.11}$$

and the bilinear form $\mathcal{B}_h$ reduces to

$$\mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K(\nabla z - \boldsymbol{\phi}), \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}))_K. \tag{4.12}$$

This gives the linear element introduced and analyzed in [13]. In [28] it has been shown that this element is essentially equivalent to an element introduced by Hughes and Tessler [40], and later rediscovered in [43, 6, 39]. □

For quadrilaterals we have two possibilities.

Element 4.2 *For a quadrilateral K, let $R_k(\hat{K}) = Q_k(\hat{K})$ and*

$$\boldsymbol{S}_k(\hat{K}) = P_{k-1,k}(\hat{K}) \times P_{k,k-1}(\hat{K}), \tag{4.13}$$

which is the rotated rectangular Raviart-Thomas space [35]. *The reduction operator is defined through the conditions*

$$\int_{\hat{E}} [(\boldsymbol{R}_{\hat{K}} \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{\tau}}] \hat{v} \, d\hat{s} = 0, \ \forall \hat{v} \in P_{k-1}(\hat{E}), \ \textit{for every edge } \hat{E} \textit{ of } \hat{K}, \tag{4.14}$$

and

$$\int_{\hat{K}} (\boldsymbol{R}_{\hat{K}} \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{s}} \, d\xi d\eta = 0, \ \forall \hat{\boldsymbol{s}} \in P_{k-1,k-2}(\hat{K}) \times P_{k-2,k-1}(\hat{K}). \tag{4.15}$$

Element 4.3 *A quadrilateral K with $R_k(\hat{K}) = Q_k^r(\hat{K}) = Q_k(\hat{K}) \cap P_{k+1}(\hat{K})$ and*

$$\boldsymbol{S}_k(\hat{K}) = [P_k(\hat{K})]^2 \backslash \mathrm{span}\{(\xi^k, 0), (0, \eta^k)\}, \tag{4.16}$$

which is the rotated rectangular Brezzi-Douglas-Fortin-Marini (BDFM) space [14]. *The reduction operator is defined from*

$$\int_{\hat{E}} [(\boldsymbol{R}_{\hat{K}} \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{\tau}}] \hat{v} \, d\hat{s} = 0, \ \forall \hat{v} \in P_{k-1}(\hat{E}), \ \textit{for every edge } \hat{E} \textit{ of } \hat{K}, \tag{4.17}$$

and

$$\int_{\hat{K}} (\boldsymbol{R}_{\hat{K}} \hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\eta}}) \cdot \hat{\boldsymbol{s}} \, d\xi d\eta = 0, \ \forall \hat{\boldsymbol{s}} \in [P_{k-2}(\hat{K})]^2. \tag{4.18}$$

Remark 4.2 For the Elements 4.2 and 4.3 it is also possible to use the reduced bilinear form (4.12) when $k = 1$. This gives the stabilized MITC4 element introduced in [13, 30]. □

Remark 4.3 It is easily seen that in each case we have $\nabla R_k(K) \subset \boldsymbol{S}_k(K)$ and hence it holds

$$\boldsymbol{R}_K \nabla v = \nabla v, \ \forall v \in R_k(K). \tag{4.19}$$

□

Under some (minor) restrictive assumptions on the mesh (see [41]) we have the following result which states that the operator $\boldsymbol{R}_K$ has optimal interpolation properties.

Lemma 4.1 [41] *There exist a positive constant C such that for $\boldsymbol{\eta} \in [H^m(K)]^2$, with $1 \leq m \leq k$, it holds*

$$\|\boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta}\|_{0,K} \leq Ch_K^m \|\boldsymbol{\eta}\|_{m,K}, \ \forall K \in \mathcal{C}_h. \quad \square$$

For the stability and error analysis of these methods it is necessary to alter the mesh dependent norm as follows. Due to the presence of the reduction operator we only define them in the subspaces.

Definition 4.1 *For $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$ we define the seminorms*

$$|(v,\boldsymbol{\eta})|_K = \Big((t^2 + h_K^2)^{-1} \|\boldsymbol{R}_K(\nabla v - \boldsymbol{\eta})\|_{0,K}^2\Big)^{1/2}, \tag{4.20}$$

$$|(v,\boldsymbol{\eta})|_h = \Big(\sum_{K\in\mathcal{C}_h} |(v,\boldsymbol{\eta})|_K^2\Big)^{1/2}, \tag{4.21}$$

and the norm

$$|||(v,\boldsymbol{\eta})|||_h = \|v\|_1 + \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h. \tag{4.22}$$

For the shear space $\boldsymbol{S}$ we use the norm defined in (3.6).

We again have an equivalence in norms.

Lemma 4.2 *There exists a positive constant C such that*

$$C|||(v,\boldsymbol{\eta})|||_h \leq \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h \leq |||(v,\boldsymbol{\eta})|||_h \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the results (4.19) and the estimate (cf. Lemma 4.1)

$$\|\boldsymbol{R}_K\boldsymbol{\eta}\|_{0,K} \leq \|\boldsymbol{\eta}\|_{1,K}$$

this result follows as in the proof of Lemma 3.1. $\square$

The proof of the stability of the formulation is identical as that of Theorem 3.5.

Theorem 4.1 *There exists a constant $C > 0$ such that for $0 < \alpha < \alpha_K < C_I$ it holds*

$$\mathcal{B}_h(v,\boldsymbol{\eta}; v,\boldsymbol{\eta}) \geq C|||(v,\boldsymbol{\eta})|||_h^2 \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \quad \square$$

Next, we note that in the bilinear form $\mathcal{B}_h$ the shear energy term has been modified so that the methods are not consistent with the exact shear. In order to characterize the consistency error produced by the action of the reduction operator $\boldsymbol{R}_K$, we define

$$\begin{aligned}\mathcal{E}_K(\boldsymbol{s}; v,\boldsymbol{\eta}) = {} & (\boldsymbol{s}, (\boldsymbol{R}_K - \boldsymbol{I})(\nabla v - \boldsymbol{\eta}))_K \\ & + t^2(t^2 + \alpha_K h_K^2)^{-1}(\boldsymbol{R}_K\boldsymbol{s} - \boldsymbol{s}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K\end{aligned} \tag{4.23}$$

and

$$\mathcal{E}_h(\boldsymbol{s}; v,\boldsymbol{\eta}) = \sum_{K\in\mathcal{C}_h} \mathcal{E}_K(\boldsymbol{s}; v,\boldsymbol{\eta}). \tag{4.24}$$

We then have

Theorem 4.2 *The solution $(w, \boldsymbol{\beta})$ to (2.7) satisfies*

$$\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = (g, v) + \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}.$$

Proof: Using the constitutive relation (2.8) and the moment equilibrium equation in (2.10), we get

$$\begin{aligned}
\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K(\nabla w - \boldsymbol{\beta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\beta}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{q}, \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (t^2 \boldsymbol{R}_K \boldsymbol{q} + \alpha_K h_K^2 \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha_K h_K^2 (\boldsymbol{q}, \boldsymbol{L}\boldsymbol{\eta})_K + \sum_{K \in \mathcal{C}_h} (\boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&\quad + t^2 \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K \boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} (\boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}))_K \\
&\quad + t^2 \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K \boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) + (\boldsymbol{q}, (\boldsymbol{R}_K - \boldsymbol{I})(\nabla v - \boldsymbol{\eta})) \\
&\quad + t^2 \sum_{K \in \mathcal{C}_h} (t^2 + \alpha_K h_K^2)^{-1} (\boldsymbol{R}_K \boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta}) - \alpha_K h_K^2 \boldsymbol{L}\boldsymbol{\eta})_K \\
&= (g, v) + \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}). \quad \square
\end{aligned}$$

The following auxiliary result is needed in estimating the consistency error.

Lemma 4.3 *For $\boldsymbol{s} \in [H^{m-1}(\Omega)]^2$, with $1 \le m \le k$, it holds*

$$|(\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_K \boldsymbol{\eta})_K| \le C h_K^m \|\boldsymbol{s}\|_{m-1,K} \|\boldsymbol{\eta}\|_{1,K} \quad \forall \boldsymbol{\eta} \in [H^1(K)]^2.$$

Proof: If $m = 1$ the result follow directly from the Schwarz inequality and Lemma 4.1.

For $2 \le m \le k$, we let $\boldsymbol{P}_K : [L^2(\hat{K})]^2 \to [L^2(K)]^2$ be the Piola transformation defined through [12, page 97]

$$\boldsymbol{P}_K \hat{\boldsymbol{s}} = |\boldsymbol{J}_K|^{-1} \boldsymbol{J}_K \hat{\boldsymbol{s}}, \quad \hat{\boldsymbol{s}} \in [L^2(\hat{K})]^2, \tag{4.25}$$

and define the space $\boldsymbol{A}(\hat{K})$ by

$$\boldsymbol{A}(\hat{K}) = \begin{cases} [P_{k-2}(\hat{K})]^2 & \text{for Elements 4.1 and 4.3,} \\ P_{k-1,k-2}(\hat{K}) \times P_{k-2,k-1}(\hat{K}) & \text{for Element 4.2,} \end{cases} \tag{4.26}$$

Using the definition of the operator $\boldsymbol{R}_K$ and the properties (4.10), (4.15) and (4.18) we then get

$$\begin{aligned}
&(\boldsymbol{P}_K\hat{\boldsymbol{s}}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K \\
&= \int_K (\boldsymbol{P}_K\hat{\boldsymbol{s}})(x,y)\cdot(\boldsymbol{\eta}(x,y) - \boldsymbol{R}_K\boldsymbol{\eta}(x,y))dxdy \\
&= \int_{\hat{K}} |\boldsymbol{J}_K^{-1}|^{-1}\boldsymbol{J}_K\hat{\boldsymbol{s}}(\xi,\eta)\cdot(\boldsymbol{\eta}(\xi,\eta) - \boldsymbol{J}_K^{-T}\boldsymbol{R}_{\hat{K}}\boldsymbol{J}_K^T\boldsymbol{\eta}(\xi,\eta))|\boldsymbol{J}_K|d\xi d\eta \qquad (4.27)\\
&= \int_{\hat{K}} \hat{\boldsymbol{s}}(\xi,\eta)\cdot(\boldsymbol{J}_K^T\boldsymbol{\eta}(\xi,\eta) - \boldsymbol{R}_{\hat{K}}\boldsymbol{J}_K^T\boldsymbol{\eta}(\xi,\eta))d\xi d\eta \\
&= 0, \qquad \forall \hat{\boldsymbol{s}} \in \boldsymbol{A}(\hat{K}).
\end{aligned}$$

Next, we let $\boldsymbol{\Pi}_{\hat{K}} : [L_2(\hat{K})]^2 \to \boldsymbol{A}(\hat{K})$ be the L_2-projection and define the mapping $\boldsymbol{\Pi}_K$ through

$$\boldsymbol{\Pi}_K = \boldsymbol{P}_K\boldsymbol{\Pi}_{\hat{K}}\boldsymbol{P}_K^{-1}. \qquad (4.28)$$

From (4.27) it then follows

$$(\boldsymbol{\Pi}_K\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K = 0. \qquad (4.29)$$

By known interpolation estimates [41] we have

$$\|\boldsymbol{s} - \boldsymbol{\Pi}_K\boldsymbol{s}\|_{0,K} \le Ch_K^{m-1}\|\boldsymbol{s}\|_{m-1,K}. \qquad (4.30)$$

We then have

$$\begin{aligned}
(\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K &= (\boldsymbol{s} - \boldsymbol{\Pi}_K\boldsymbol{s}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K \qquad (4.31)\\
&\le \|\boldsymbol{s} - \boldsymbol{\Pi}_K\boldsymbol{s}\|_{0,K}\|\boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta}\|_{0,K} \le Ch_K^m\|\boldsymbol{s}\|_{m-1,K}\|\boldsymbol{\eta}\|_{1,K}. \qquad \Box
\end{aligned}$$

For the consistency error we now have the following result.

Lemma 4.4 *Suppose that the exact shear force satisfies $\boldsymbol{q} \in [H^{m-1}(K)]^2$ and $t\boldsymbol{q} \in [H^m(K)]^2$, with $1 \le m \le k$. Then it holds*

$$|\mathcal{E}_K(\boldsymbol{q}; v, \boldsymbol{\eta})| \le Ch_K^m(\|\boldsymbol{q}\|_{m-1,K} + t\|\boldsymbol{q}\|_{m,K})(|(v,\boldsymbol{\eta})|_K^2 + \|\boldsymbol{\eta}\|_{1,K}^2)^{1/2}, \ \forall (v,\boldsymbol{\eta}) \in W_k(K)\times\boldsymbol{V}_k(K).$$

Proof: We first note that the boundedness of the bilinear form a_K together with Lemmas 3.2 and 4.1 imply

$$\begin{aligned}
&((t^2+\alpha_K h_K^2)^{-1}\|\boldsymbol{R}_K(\nabla v - \boldsymbol{\eta})_K - \alpha_K h_K^2\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2)^{1/2} \qquad (4.32)\\
&\le C((t^2+\alpha_K h_K^2)^{-1}\|\boldsymbol{R}_K(\nabla v - \boldsymbol{\eta})\|_{0,K}^2 + \alpha_K \sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\boldsymbol{\eta}\|_{0,K}^2)^{1/2} \\
&\le C(|(v,\boldsymbol{\eta})|_K^2 + \alpha_K C_I^{-1}a_K(\boldsymbol{\eta},\boldsymbol{\eta}))^{1/2} \le C(|(v,\boldsymbol{\eta})|_K^2 + \|\boldsymbol{\eta}\|_{1,K}^2)^{1/2}.
\end{aligned}$$

Hence, Lemmas 4.3 and 4.1 give

$$\begin{aligned}
\mathcal{E}_K(\boldsymbol{q}; v, \boldsymbol{\eta}) &= (\boldsymbol{q}, (\boldsymbol{R}_K - \boldsymbol{I})(\nabla v - \boldsymbol{\eta}))_K \qquad (4.33)\\
&\quad + t^2(t^2+\alpha_K h_K^2)^{-1}(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta})_K - \alpha_K h_K^2\boldsymbol{L}\boldsymbol{\eta})_K \\
&= (\boldsymbol{q}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K + t^2(t^2+\alpha_K h_K^2)^{-1}(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla v - \boldsymbol{\eta})_K - \alpha_K h_K^2\boldsymbol{L}\boldsymbol{\eta})_K \\
&\le (\boldsymbol{q}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K + Ct^2((t^2+\alpha_K h_K^2)^{-1}\|\boldsymbol{q} - \boldsymbol{R}_K\boldsymbol{q}\|_{0,K}^2)^{1/2}(|(v,\boldsymbol{\eta})|_K^2 + \|\boldsymbol{\eta}\|_{1,K}^2)^{1/2} \\
&\le (\boldsymbol{q}, \boldsymbol{\eta} - \boldsymbol{R}_K\boldsymbol{\eta})_K + Ct\|\boldsymbol{q} - \boldsymbol{R}_K\boldsymbol{q}\|_{0,K}(|(v,\boldsymbol{\eta})|_K^2 + \|\boldsymbol{\eta}\|_{1,K}^2)^{1/2} \\
&\le Ch^m(\|\boldsymbol{q}\|_{m-1,K} + t\|\boldsymbol{q}\|_{m,K})(|(v,\boldsymbol{\eta})|_K^2 + \|\boldsymbol{\eta}\|_{1,K}^2)^{1/2}. \qquad \Box
\end{aligned}$$

For the rest of the error analysis we will next define a special interpolation operator.

Definition 4.2 *We define $I_h : H_0^1(\Omega) \to W_h$ through the following three conditions for $I_{h|K} = I_K$:*

$$((v - I_K v) \circ \boldsymbol{F}_K)(\hat{p}) = 0, \ \forall \text{ vertices } \hat{p} \text{ of } \hat{K}, \tag{4.34}$$

$$\int_{\hat{E}} ((v - I_K v) \circ \boldsymbol{F}_K)\hat{r} \ d\hat{s} = 0, \ \forall \hat{r} \in P_{k-2}(\hat{E}), \ \forall \text{ edges } \hat{E} \text{ of } \hat{K}, \tag{4.35}$$

and

$$\int_{\hat{K}} ((v - I_K v) \circ \boldsymbol{F}_K)\hat{s} \ d\xi d\eta = 0, \ \begin{cases} \forall \hat{s} \in P_{k-3}(\hat{K}), & \textit{for the Elements 4.1 and 4.3,} \\ \forall \hat{s} \in Q_{k-2}(\hat{K}), & \textit{for the Element 4.2,} \end{cases} \tag{4.36}$$

for every element $K \in \mathcal{C}_h$.

The operator I_h has optimal interpolation properties (cf. [22, Lemma A.3. p. 100]):

Lemma 4.5 *There exists a positive constant C such that for $v \in H^m(K)$, with $1 < m \leq k+1$, it holds*

$$\|v - I_K v\|_{s,K} \leq C h_K^{m-s} \|v\|_{m,K}, \quad s = 0, 1. \quad \square$$

The reason for introducing the operator I_h is the following technical result.

Lemma 4.6 *For $v \in H^s(K)$, $s > 1$ it holds*

$$\boldsymbol{R}_K \nabla (v - I_K v) = \boldsymbol{0}.$$

Proof: Let $\hat{E}$ be an edge of $\hat{K}$. The conditions (4.34) and (4.35) then give (where $\int_{\partial \hat{E}}$ denotes substituting at the two endpoints of $\hat{E}$)

$$\begin{aligned} \int_{\hat{E}} \hat{\nabla}((v - I_K v) \circ \boldsymbol{F}_K) \cdot \hat{\boldsymbol{\tau}} \ \hat{r} \ d\hat{s} &= \int_{\hat{E}} \frac{\partial}{\partial \hat{s}}((v - I_K v) \circ \boldsymbol{F}_K) \ \hat{r} \ d\hat{s} \\ &= \int_{\partial \hat{E}} ((v - I_K v) \circ \boldsymbol{F}_K)\hat{r} - \int_{\hat{E}} ((v - I_K v) \circ \boldsymbol{F}_K) \frac{\partial \hat{r}}{\partial \hat{s}} \ d\hat{s} = 0. \end{aligned} \tag{4.37}$$

for $\hat{r} \in P_{k-1}(\hat{E})$. Next, let $\hat{\boldsymbol{s}} \in \boldsymbol{A}(\hat{K})$, with $\boldsymbol{A}(\hat{K})$ defined in (4.26). Then it holds

$$\widehat{\mathrm{div}} \hat{\boldsymbol{s}} \in \begin{cases} P_{k-3}(\hat{K}) \text{ for Elements 4.1 and 4.3,} \\ Q_{k-2}(\hat{K}) \text{ for Element 4.2.} \end{cases} \tag{4.38}$$

Hence, (4.35) and (4.36) give

$$\begin{aligned} \int_{\hat{K}} \hat{\nabla}((v - I_K v) \circ \boldsymbol{F}_K) \cdot \hat{\boldsymbol{s}} \ d\xi d\eta &= \int_{\partial \hat{K}} ((v - I_K v) \circ \boldsymbol{F}_K)\hat{\boldsymbol{s}} \cdot \hat{\boldsymbol{n}} \ d\hat{s} \\ &- \int_{\hat{K}} ((v - I_K v) \circ \boldsymbol{F}_K) \ \widehat{\mathrm{div}} \hat{\boldsymbol{s}} \ d\xi d\eta = 0. \end{aligned} \tag{4.39}$$

From(4.37), (4.39) and the definitions of the operators $\boldsymbol{R}_{\hat{K}}$, we get

$$\boldsymbol{R}_{\hat{K}} \hat{\nabla}((v - I_K v) \circ \boldsymbol{F}_K) = \boldsymbol{0}. \tag{4.40}$$

Since it holds $\boldsymbol{J}_K^T(\nabla v \circ \boldsymbol{F}_K) = \hat{\nabla}(v \circ \boldsymbol{F}_K)$, we conclude that

$$\begin{aligned} \boldsymbol{R}_K \nabla (v - I_K v) &= \boldsymbol{J}_K^{-T} \boldsymbol{R}_{\hat{K}} \boldsymbol{J}_K^T \nabla((v - I_K v) \circ \boldsymbol{F}_K) \\ &= \boldsymbol{J}_K^{-T} \boldsymbol{R}_{\hat{K}} \hat{\nabla}((v - I_K v) \circ \boldsymbol{F}_K) = \boldsymbol{0}. \quad \square \end{aligned} \tag{4.41}$$

We will next state our main result.

Theorem 4.3 *Suppose that $0 < \alpha < \alpha_K < C_I$ for all $K \in \mathcal{C}_h$. Then it holds*

$$\begin{aligned} \|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1,h} + t\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \\ \leq C\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \end{aligned}$$

Proof: Let $\tilde{\boldsymbol{\beta}} \in \boldsymbol{V}_h$ be the usual Lagrange interpolant to $\boldsymbol{\beta}$ and let $I_h w \in W_h$ the interpolant defined through (4.34)–(4.36). From Theorems 4.1 and 4.2, there exists a pair $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$ such that

$$|||(v, \boldsymbol{\eta})|||_h \leq C, \tag{4.42}$$

and

$$\begin{aligned} |||(w_h - I_h w, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h &\leq \mathcal{B}_h(w_h - I_h w, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) \\ &= \mathcal{B}_h(w - I_h w, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) - \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}). \end{aligned} \tag{4.43}$$

Now, using Lemma 4.6 we have

$$\mathcal{B}_h(w - I_h w, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) = \mathcal{B}_h(0, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}), \tag{4.44}$$

(which is the property that enables us to use equal order interpolation) and we obtain

$$|||(w_h - I_h w, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h \leq \mathcal{B}_h(0, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) - \mathcal{E}_h(\boldsymbol{q}; v, \boldsymbol{\eta}). \tag{4.45}$$

The proof now follows the lines of that of Theorem 3.6. Estimating as in (3.27) the term for the deflection falls away, but we additionally obtain a term for the consistency error, for which we have an estimate in Lemma 4.4, and the term

$$(t^2 + h_K^2)^{-1}\|\boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K}^2. \tag{4.46}$$

This we estimate as follows

$$(t^2 + h_K^2)^{-1}\|\boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K}^2 \leq h_K^{-2}\|\boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K}^2 \tag{4.47}$$

and

$$\begin{aligned} \|\boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K} &= \|(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) - (\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) + \boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K} \\ &\leq \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{0,K} + \|(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) - \boldsymbol{R}_K(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}})\|_{0,K} \\ &\leq \|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{0,K} + Ch_K\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{1,K}. \end{aligned} \tag{4.48}$$

We again measure the smoothness of the solution in the element K by the parameter s_K, $1 \leq s_K \leq k$, by assuming that

$$\begin{aligned} &w_{|K} \in H^{s_K+1}(K), \quad \boldsymbol{\beta}_{|K} \in [H^{s_K+1}(K)]^2, \\ &\boldsymbol{q}_{|K} \in [H^{s_K-1}(K)]^2 \ \text{ and } \ t\boldsymbol{q}_{|K} \in [H^{s_K}(K)]^2. \end{aligned} \tag{4.49}$$

Using Lemmas 4.5 and 4.4 we then get

$$|||(w_h - I_h w, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h \le C\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2}, \tag{4.50}$$

with

$$E_K^2 = h_K^{2s_K}\|w\|_{s_K+1,K}^2 + h_K^{2s_K}\|\boldsymbol{\beta}\|_{s_K+1,K}^2 + +h_K^{2s_K}\|\boldsymbol{q}\|_{s_K-1,K}^2 + t^2 h_K^{2s_K}\|\boldsymbol{q}\|_{s_K,K}^2. \tag{4.51}$$

Since

$$\|w_h - I_h w\|_1 + \|\boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}}\|_1 \le C|||(w_h - I_h w, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h, \tag{4.52}$$

the triangle inequality and the interpolation estimates now yield

$$\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 \le C\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2}. \tag{4.53}$$

Now, we again choose $s_K = k$ when $K \subset \Omega_i$ and $s_K = 1$ for $K \subset \Omega_b$ which gives

$$\begin{aligned}\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2} \;\le\;& Ch_i^k\Big(\|w\|_{k+1,\Omega_i} + \|\boldsymbol{\beta}\|_{k+1,\Omega_i} + \|\boldsymbol{q}\|_{k-1,\Omega_i} + t\|\boldsymbol{q}\|_{k,\Omega_i}\Big)\\ &+Ch_b\Big(\|w\|_{2,\Omega_b} + \|\boldsymbol{\beta}\|_{2,\Omega_b} + \|\boldsymbol{q}\|_{0,\Omega_b} + t\|\boldsymbol{q}\|_{1,\Omega_b}\Big).\end{aligned} \tag{4.54}$$

The estimate for $(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)$ then follows from Theorem 2.1.

In the sequel we will also need the estimate

$$\|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{-1,h} \le C\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2}. \tag{4.55}$$

This estimate is easily verified from the previous estimates.

Next, let us derive the asserted estimates for the shear. Recalling the definition (4.7) of $\boldsymbol{q}_h$ and the relation (3.19) for $\boldsymbol{q}$ we have

$$\begin{aligned}(\boldsymbol{q} - \boldsymbol{q}_h)_{|K} \;=\;& (t^2 + \alpha_K h_K^2)^{-1}(\nabla w - \boldsymbol{\beta} - (\boldsymbol{R}_K(\nabla w_h - \boldsymbol{\beta}_h))_{|K}\\ &-\alpha_K h_K^2 \boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)_{|K}.\end{aligned} \tag{4.56}$$

Now, using (2.8) and Lemma 4.6 we write

$$\begin{aligned}&(\nabla w - \boldsymbol{\beta})_{|K} - \boldsymbol{R}_K(\nabla w_h - \boldsymbol{\beta}_h)_{|K}\\ &= (\nabla w - \boldsymbol{\beta})_{|K} - \boldsymbol{R}_K(\nabla w - \boldsymbol{\beta})_{|K} + \boldsymbol{R}_K(\nabla w - \boldsymbol{\beta})_{|K} - \boldsymbol{R}_K(\nabla w_h - \boldsymbol{\beta}_h)_{|K}\\ &= t^2(\boldsymbol{q}_{|K} - \boldsymbol{R}_K\boldsymbol{q}_{|K}) + \boldsymbol{R}_K(\nabla I_h w - \boldsymbol{\beta})_{|K} - \boldsymbol{R}_K(\nabla w_h - \boldsymbol{\beta}_h)_{|K}\\ &= t^2(\boldsymbol{q}_{|K} - \boldsymbol{R}_K\boldsymbol{q}_{|K}) + \boldsymbol{R}_K\nabla(I_h w - w_h)_{|K} - \boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_h)_{|K} + \boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})_{|K}.\end{aligned} \tag{4.57}$$

Hence, we get

$$\begin{aligned}&(t^2 + \alpha_K h_K^2)^{1/2}\|\boldsymbol{q} - \boldsymbol{q}_h\|_{0,K}\\ &\quad\le C(t\|\boldsymbol{q} - \boldsymbol{R}_K\boldsymbol{q}\|_{0,K} + |(\tilde{w} - w_h, \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_h)|_K + h_K^{-1}\|\boldsymbol{R}_K(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})\|_{0,K} + h_K|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)|_{0,K}).\end{aligned} \tag{4.58}$$

The estimate for the shear thus follows from the estimates derived above. □

Remark 4.4 We the estimate for $w_h - I_h w$ obtained from(4.50) and (4.54) is a superconvergence result. $\square$

Let us finally prove the L_2-estimates for the deflection. For this we need quadratic or higher order elements.

Theorem 4.4 *Suppose that $0 < \alpha < \alpha_K < C_I$ for all $K \in \mathcal{C}_h$ and that the region Ω is convex. For $k \geq 2$ it then holds*

$$\|w - w_h\|_0 \leq Ch\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \tag{4.59}$$

Proof: Let $(z, \boldsymbol{\phi}, \boldsymbol{r}) \in W \times \boldsymbol{V} \times \boldsymbol{S}$ be the solution to the following auxiliary problem

$$\begin{aligned} a(\boldsymbol{\phi}, \boldsymbol{\eta}) + (\boldsymbol{r}, \nabla v - \boldsymbol{\eta}) &= (I_h w - w_h, v), \quad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}, & (4.60)\\ (\nabla z - \boldsymbol{\phi}, \boldsymbol{s}) - t^2(\boldsymbol{r}, \boldsymbol{s}) &= 0, \qquad \forall \boldsymbol{s} \in \boldsymbol{S}. & (4.61)\end{aligned}$$

From the consistency estimate of Theorem 4.2 we get

$$\mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = (I_h w - w_h, v) + \mathcal{E}_h(\boldsymbol{r}; v, \boldsymbol{\eta}). \tag{4.62}$$

Now, we chose $v = I_h w - w_h$ and $\boldsymbol{\eta} = \boldsymbol{\beta} - \boldsymbol{\beta}_h$ which gives gives

$$\|I_h w - w_h\|_0^2 = \mathcal{B}_h(z, \boldsymbol{\phi}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) - \mathcal{E}_h(\boldsymbol{r}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h). \tag{4.63}$$

Next, let us recall Lemma 4.6, viz.

$$\boldsymbol{R}_K \nabla w = \boldsymbol{R}_K \nabla I_K w, \qquad \forall K \in \mathcal{C}_h,$$

which implies that

$$\mathcal{B}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = \mathcal{B}_h(I_h w, \boldsymbol{\beta}; v, \boldsymbol{\eta}), \qquad \forall (v, \boldsymbol{\eta}) \in W \times \boldsymbol{V}.$$

Let now $\tilde{\boldsymbol{\phi}} \in \boldsymbol{V}_h$ be the Lagrange interpolant to $\boldsymbol{\phi}$ and $I_h z I_h z \in W_h$ the interpolant to z. Hence, the above equation and the consistency estimate of Lemma 4.2 yield

$$\mathcal{B}_h(I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h; I_h z, \tilde{\boldsymbol{\phi}}) = \mathcal{E}_h(\boldsymbol{q}; I_h z, \tilde{\boldsymbol{\phi}}).$$

Since the bilinear form $\mathcal{B}_h$ is symmetric we then get

$$\begin{aligned} \|I_h w - w_h\|_0^2 &= \mathcal{B}_h(z - I_h z, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ &\quad - \mathcal{E}_h(\boldsymbol{r}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) - \mathcal{E}_h(\boldsymbol{q}; I_h z, \tilde{\boldsymbol{\phi}}). \end{aligned} \tag{4.64}$$

From Theorem 2.1 we have

$$\|\boldsymbol{\phi}\|_2 + t\|\boldsymbol{r}\|_1 + \|\boldsymbol{r}\|_0 \leq C(\|I_h w - w_h\|_0. \tag{4.65}$$

Now, using (4.65) and similar arguments as in the proof of Theorem 4.3,

$$\begin{aligned} &\mathcal{B}_h(z - I_h z, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ &= \mathcal{B}_h(0, \boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ &\leq Ch\|\boldsymbol{\phi}\|_2 |||(I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \\ &\leq Ch\|I_h w - w_h\|_0 \, |||(I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h. \end{aligned} \tag{4.66}$$

For the first consistency error term in (4.64) we directly obtain

$$
\begin{aligned}
&|\mathcal{E}_h(\boldsymbol{r}; I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)| \\
&\quad \leq Ch(\|\boldsymbol{r}\|_0 \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + t\|\boldsymbol{r}\|_1(|I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h|_h + \|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{-1,h})) \\
&\quad \leq Ch\|I_h w - w_h\|_0 \left(|||I_h w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h|||_h + \|\boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{-1,h}\right).
\end{aligned}
\tag{4.67}
$$

The second consistency error term is more complicated. The local contribution is

$$
\begin{aligned}
\mathcal{E}_K(\boldsymbol{q}; I_h z, \tilde{\boldsymbol{\phi}}) &= (\boldsymbol{q}, (\boldsymbol{R}_K - \boldsymbol{I})(\nabla I_h z - \tilde{\boldsymbol{\phi}}))_K \\
&\quad + t^2(t^2 + \alpha_K h_K^2)^{-1}(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla I_h z - \tilde{\boldsymbol{\phi}}) - \alpha_K h_K^2 \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K
\end{aligned}
\tag{4.68}
$$

Since it is supposed that $k \geq 2$, (4.19), (4.27) and (4.30) give

$$
\begin{aligned}
(\boldsymbol{q}, (\boldsymbol{R}_K - \boldsymbol{I})(\nabla I_h z - \tilde{\boldsymbol{\phi}}))_K &= (\boldsymbol{q}, \boldsymbol{R}_K\tilde{\boldsymbol{\phi}} - \tilde{\boldsymbol{\phi}})_K = (\boldsymbol{q} - \boldsymbol{\Pi}_K\boldsymbol{q}, \boldsymbol{R}_K\tilde{\boldsymbol{\phi}} - \tilde{\boldsymbol{\phi}})_K \\
&= (\boldsymbol{q} - \boldsymbol{\Pi}_K\boldsymbol{q}, (\boldsymbol{\phi} - \boldsymbol{R}_K\boldsymbol{\phi}) - (\boldsymbol{I} - \boldsymbol{R}_K)(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}))_K, \\
&\leq \|\boldsymbol{q} - \boldsymbol{\Pi}_K\boldsymbol{q}\|_{0,K}\Big(\|\boldsymbol{\phi} - \boldsymbol{R}_K\boldsymbol{\phi}\|_{0,K} + \|(\boldsymbol{I} - \boldsymbol{R}_K)(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})\|_{0,K}\Big) \qquad (4.69) \\
&\leq Ch_K^{s_K-1}\|\boldsymbol{q}\|_{s_K-1,K}\Big(h_K^2\|\boldsymbol{\phi}\|_{2,K} + h_K\|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\|_{1,K}\Big) \\
&\leq Ch_K^{s_K+1}\|\boldsymbol{q}\|_{s_K-1,K}\|\boldsymbol{\phi}\|_{2,K}.
\end{aligned}
$$

For the second term in $\mathcal{E}_K$ we again use Lemma 4.6

$$
(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla I_h z - \tilde{\boldsymbol{\phi}}) - \alpha_K h_K^2 \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K = (\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla z - \tilde{\boldsymbol{\phi}}) - \alpha_K h_K^2 \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K. \tag{4.70}
$$

From (4.60) we have

$$
\nabla z = \boldsymbol{\phi} - t^2\boldsymbol{r}
$$

and this gives

$$
\begin{aligned}
&(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\nabla z - \tilde{\boldsymbol{\phi}}) - \alpha_K h_K^2 \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K \\
&= (\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\boldsymbol{\phi} - t^2\boldsymbol{r} - \tilde{\boldsymbol{\phi}}) - \alpha_K h_K^2 \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K \\
&= (\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}))_K - t^2(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K\boldsymbol{r})_K - \alpha_K h_K^2(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K.
\end{aligned}
\tag{4.71}
$$

Using Lemmas 4.1 and 4.3 we get

$$
\begin{aligned}
|(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}))_K| &\leq \|\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}\|_{0,K}\|\boldsymbol{R}_K(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})\|_{0,K} \\
&\leq \|\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}\|_{0,K}\Big(\|(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})\|_{0,K} + \|(\boldsymbol{I} - \boldsymbol{R}_K)(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}})\|_{0,K}\Big) \qquad (4.72) \\
&\leq h_K^{s_K+1}\|\boldsymbol{q}\|_{s_K-1,K}\|\boldsymbol{\phi}\|_{2,K},
\end{aligned}
$$

$$
\begin{aligned}
|t^2(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{R}_K\boldsymbol{r})_K| &= t^2|(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{r} - \boldsymbol{r} + \boldsymbol{R}_K\boldsymbol{r})_K| \\
&= t^2|(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{r}) + (\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{r} - \boldsymbol{R}_K\boldsymbol{r})_K| \qquad (4.73) \\
&\leq Cth_K^{s_K+1}\|\boldsymbol{q}\|_{s_K,K} \cdot t\|\boldsymbol{r}\|_{1,K}
\end{aligned}
$$

and

$$
|h_K^2(\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}, \boldsymbol{L}\tilde{\boldsymbol{\phi}})_K| \leq h_K^2\|\boldsymbol{R}_K\boldsymbol{q} - \boldsymbol{q}\|_{0,K}\|\boldsymbol{L}\tilde{\boldsymbol{\phi}}\| \leq h_K^{s_K+1}\|\boldsymbol{q}\|_{s_K-1,K}\|\boldsymbol{\phi}\|_{2,K}. \tag{4.74}
$$

Collecting (4.68) to (4.74) and using the fact that $t^2(t^2+\alpha_K h_K^2)^{-1} \leq 1$ gives

$$|\mathcal{E}_K(\boldsymbol{q}; I_h z, \tilde{\boldsymbol{\phi}})| \leq C h_K^{s_K+1}\Big(\|\boldsymbol{q}\|_{s_K-1,K} + t\|\boldsymbol{q}\|_{s_K,K}\Big)\Big(\|\boldsymbol{\phi}\|_{2,K} + t\|\boldsymbol{r}\|_{1,K}\Big) \tag{4.75}$$

and hence it holds

$$\begin{aligned} |\mathcal{E}_h(\boldsymbol{q}; I_h z, \tilde{\boldsymbol{\phi}})| &\leq Ch\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2}(\|\boldsymbol{\phi}\|_2 + t\|\boldsymbol{r}\|_1) \\ &\leq Ch\Big(\sum_{K\in\mathcal{C}_h} E_K^2\Big)^{1/2}\|I_h w - w_h\|_0. \end{aligned} \tag{4.76}$$

The asserted estimate now follows by combining (4.66), (4.67) and (4.76), and estimates derived in Theorem 4.3

Remark 4.5 For the rotation the duality argument also yields the estimate

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 \leq Ch\Big\{h_i^k(\|g\|_{k-2} + t\|g\|_{k-1}) + h_b(\|g\|_{-1} + t\|g\|_0)\Big\}. \tag{4.77}$$

□

5 Numerical results

The numerical results of this section will be given for a clamped square plate occupying the region $\Omega = [-1,1]\times[-1,1]$. We let $G = 1$, $\nu = 0.3$, $t = 0$, and define the load such that

$$w = \cos^2\left(\frac{\pi x}{2}\right)\cos^2\left(\frac{\pi y}{2}\right).$$

That is, we take $f = \Delta\Delta w$. Due to the symmetry of the deflection, only the positive quadrant $[0,1]\times[0,1]$ of the midsurface will be discretized.

The results will be given for the unequally interpolated elements of Method 3.2 with

$$R_k(K) = \begin{cases} P_k(K) & \text{when } K \text{ is a triangle,} \\ Q_k(K) & \text{when } K \text{ is quadrilateral,} \end{cases}$$

and for the equally interpolated Elements 4.1 and 4.2 of Method 4.1 with $k = 1,2,3$. For quadrilateral elements the computational domain is divided uniformly into $N\times N$ elements according to Figure 1. For triangular elements, the finite element meshes are generated by a Delanay triangulation algorithm, see Figure 2.

For the linear elements with $k = 1$, we choose the value of the stabilization factor as $\alpha_K = 0.1$ when K is a quadrilateral and $\alpha_K = 0.2$ when K is a triangle. When $k \geq 2$ the procedure is more complicated. In order to respect the stability condition $0 < \alpha < \alpha_K < C_I$, where C_I is the constant of the inverse inequality of Lemma 3.2, we let $\alpha_K = \gamma/\lambda_K$, where $0 < \gamma < 1$ is a numerical parameter, and

$$\lambda_K = \max_{\substack{\boldsymbol{\eta}\in \boldsymbol{V}_k(K) \\ a_K(\boldsymbol{\eta},\boldsymbol{\eta})\neq 0}} \frac{h_K^2||L_K\boldsymbol{\eta}||_{0,K}^2}{a_K(\boldsymbol{\eta},\boldsymbol{\eta})}.$$

Here $\boldsymbol{V}_k(K) = [R_k(K)]^2$ is the local rotation space as before. Choosing $\gamma = 0.2$ (by numerical experimentation), we get $\alpha_K = 0.0648$ and $\alpha_K = 0.0164$ for the quadrilateral elements with $k = 2$ and $k = 3$, respectively. For the triangular elements the corresponding values vary in the ranges $0.0089 \leq \alpha_K \leq 0.1143$ and $0.0090 \leq \alpha_K \leq 0.0223$. The average values of the weighting factors are $\bar{\alpha}_K = 0.0805$ and $\bar{\alpha}_K = 0.0166$ for $k = 2$ and $k = 3$, respectively.

In the numerical calculations we approximate the operator $\boldsymbol{L}$ elementwise by

$$\boldsymbol{L}_K \boldsymbol{\eta} = \frac{1}{6} \mathrm{Div}\ \pi_K \left\{ \boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \frac{\nu}{1+\nu} (\mathrm{div}\ \boldsymbol{\eta}) \boldsymbol{I} \right\}$$

where π_K is the usual Lagrange interpolation operator in $[R_k(K)]^{2\times 2}$. In other words, we first compute from $\boldsymbol{\eta}$ the moment tensor $\boldsymbol{m}(\boldsymbol{\eta}) = \boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \frac{\nu}{1+\nu}(\mathrm{div}\ \boldsymbol{\eta})\boldsymbol{I}$ in the node points, interpolate the node values using polynomials of degree k, and finally compute the divergence of the interpolant to get $\boldsymbol{L}_K \boldsymbol{\eta}$. The direct implementation of $\boldsymbol{L}$ is in practice too complicated.

In what follows, we compute the values of the norms $|| \cdot ||_0$ numerically on each element and then sum over the mesh. We use the 5×5 point Gauss rule for quadrilateral elements, and the usual 7th-degree accurate 13-point rule for triangles.

The results from the calculations are shown in Figures 3-8. The figures show the relative L_2-errors in deflection, bending moment and shear force on a log-log scale with respect to the mesh parameter h. The number r in the figures is the approximate convergence rate exponent, obtained by fitting a function of the form Ch^r (the dashed extrapolation lines in the figures) to the results in a least squares sense.

According to Figure 3, the convergence rate of the deflection of the unequally interpolated elements is $r \approx k + 2$ for $k = 2, 3$, and $r \approx 2$ for $k = 1$. For the equally interpolated elements the corresponding convergence rates are $r \approx k + 1$ for $k = 1, 2, 3$, as can be seen from Figure 4. The results are in complete accordance with the theory.

From Figures 5 and 6 we see that the bending moment of all elements converges with the rate $r \approx k$, again as predicted by the theory.

According to Figures 7 and 8, the shear force of the linear and bilinear elements with $k = 1$ do not converge. The error remains, however, on a bounded level independently of the value of the mesh parameter h. For $k \geq 2$ the convergence rate of the shear force is $r \approx k - 1$, which again is in good agreement with the theoretically obtained results.

6 Appendix

Below we will prove Theorem 2.1 the statement of which we for convinience repeat.

Theorem 6.1 *Let Ω be a convex polygonal domain and let Ω_i be a domain compactly embedded in Ω. Denote by $(w, \boldsymbol{\beta}, \boldsymbol{q})$ the Reissner–Mindlin solution for the clamped plate and let $w = w_0 + w_r$, where w_0 is the deflection obtained from the Kirchhoff model. With $g \in H^{s-2}(\Omega)$ and $tg \in H^{s-1}(\Omega)$, $s \geq 1$, it then holds*

$$||w_0||_3 + t^{-1}||w_r||_2 + ||\boldsymbol{\beta}||_2 + ||\boldsymbol{q}||_0 + t||\boldsymbol{q}||_1 \leq C(||g||_{-1} + t||g||_0) \tag{6.1}$$

and

$$\|w_0\|_{s+2,\Omega_i} + t^{-1}\|w_r\|_{s+1,\Omega_i} + \|\boldsymbol{\beta}\|_{s+1,\Omega_i} + \|\boldsymbol{q}\|_{s-1,\Omega_i} + t\|\boldsymbol{q}\|_{s,\Omega_i} \leq C(\|g\|_{s-2} + t\|g\|_{s-1}). \tag{6.2}$$

Proof: As w_0 is the Kirchhoff solution it is clear that

$$\|w_0\|_3 \leq C\|g\|_{-1} \quad \text{and} \quad \|w_0\|_{s+2,\Omega_i} \leq C\|g\|_{s-2}. \tag{6.3}$$

To obtain the other estimates we rely on results proved by Arnold and Liu [5]. First we use the Helmholtz theorem (cf. [12]) to orthogonally decompose the shear $\boldsymbol{q} \in \boldsymbol{S} = [L^2(\Omega)]^2$ as

$$\boldsymbol{q} = \nabla\psi + \mathbf{rot}\, p, \tag{6.4}$$

with $\psi \in W = H_0^1(\Omega)$ and $p \in H^1(\Omega) \cap L_0^2(\Omega)$. Here we denote

$$\mathbf{rot}\, p = (\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x}). \tag{6.5}$$

With the same decomposition for the test function, the problem (3.22) decouples into a sequence of problems. First a Poisson problem: find $\psi \in H_0^1(\Omega)$ such that

$$(\nabla\psi, \nabla\xi) = (g, \xi), \qquad \xi \in H_0^1(\Omega). \tag{6.6}$$

The solution to this is the data for the second problem: find $(\boldsymbol{\beta}, p) \in [H_0^1(\Omega)]^2 \times H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\, p, \boldsymbol{\eta}) &= (\nabla\psi, \boldsymbol{\eta}), & \boldsymbol{\eta} &\in [H_0^1(\Omega)]^2, \\ t^2(\mathbf{rot}\, p, \mathbf{rot}\, q) + (\boldsymbol{\beta}, \mathbf{rot}\, q) &= 0, & q &\in H^1(\Omega) \cap L_0^2(\Omega). \end{aligned} \tag{6.7}$$

The deflection $w \in H_0^1(\Omega)$ is then determined from $\boldsymbol{\beta}$ and ψ from the Poisson problem

$$(\nabla w, \nabla v) = (\boldsymbol{\beta}, \nabla v) + t^2(\nabla\psi, \nabla v), \quad v \in H_0^1(\Omega). \tag{6.8}$$

Note that by (6.6) this is equivalent with

$$(\nabla w, \nabla v) = (\boldsymbol{\beta}, \nabla v) + t^2(g, v), \quad v \in H_0^1(\Omega). \tag{6.9}$$

Let now $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, (with $\subset\subset$ denoting a compact imbedding) be arbitrary. For the solution of the first Poisson problem above we have

$$\|\psi\|_l \leq C\|g\|_{l-2}, \text{ for } l = 1, 2, \quad \text{and} \quad \|\psi\|_{s,\Omega''} \leq C\|g\|_{s-2}. \tag{6.10}$$

In [2, 5] the following general problem was studied: find

$$\begin{aligned} a(\boldsymbol{\Phi}, \boldsymbol{\eta}) - (\mathbf{rot}\, P, \boldsymbol{\eta}) &= (\boldsymbol{F}, \boldsymbol{\eta}), & \boldsymbol{\eta} &\in [H_0^1(\Omega)]^2, \\ t^2(\mathbf{rot}\, P, \mathbf{rot}\, q) + (\boldsymbol{\Phi}, \mathbf{rot}\, q) &= (K, q), & q &\in H^1(\Omega) \cap L_0^2(\Omega). \end{aligned} \tag{6.11}$$

For this problem the following estimates are in essence proved in [2, 5]:

$$\|\boldsymbol{\Phi}\|_2 + \|P\|_1 + t\|P\|_2 + t^2\|P\|_{3,\Omega''} \leq C(\|\boldsymbol{F}\|_0 + \|K\|_1) \tag{6.12}$$

and

$$\|\mathbf{\Phi}\|_{l+2,\Omega'} + \|P\|_{l+1,\Omega'} + t\|P\|_{l+2,\Omega'} + t^2\|P\|_{l+3,\Omega'} \leq C(\|\boldsymbol{F}\|_{l,\Omega''} + \|K\|_{l+1,\Omega''}), \quad (6.13)$$

with $l \geq 0$. Applying these result with $\mathbf{\Phi} = \boldsymbol{\beta}$, $P = p$, $\boldsymbol{F} = \nabla\psi$, $K = 0$, $\Omega' = \Omega_i$, and $l = s - 1$, gives

$$\|\boldsymbol{\beta}\|_2 + \|p\|_1 + t\|p\|_2 \leq C\|\nabla\psi\|_0 \quad (6.14)$$

and

$$\|\boldsymbol{\beta}\|_{s+1,\Omega_i} + \|p\|_{s,\Omega_i} + t\|p\|_{s+1,\Omega_i} \leq C\|\nabla\psi\|_{s-1,\Omega''}. \quad (6.15)$$

Using the decomposition (6.4) these estimates combined with the estimates (6.10) gives all of the asserted estimates except the one for w_r. To this end, let $(\boldsymbol{\beta}_0, p_0)$ be the solution of (6.7) of with $t = 0$. From the results in [2, 5] one obtains

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_1 \leq Ct\|\nabla\psi\|_0. \quad (6.16)$$

The difference $w_r = w - w_0$ satisfies

$$(\nabla w_r, \nabla v) = (\boldsymbol{\beta} - \boldsymbol{\beta}_0, \nabla v) + t^2(g, v), \quad v \in H_0^1(\Omega), \quad (6.17)$$

and hence the H^2-regularity for the Poisson problem gives

$$\|w_r\|_2 \leq C(\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_1 + t^2\|g\|_0). \quad (6.18)$$

Combining this with (6.16) and (6.10) gives the remaining part of the global regularity estimate (6.1). Next, let us derive the remaining local estimates. From (6.7) we obtain

$$a(\boldsymbol{\beta} - \boldsymbol{\beta}_0, \boldsymbol{\eta}) - (\mathbf{rot}\,(p - p_0), \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \quad (6.19)$$

$$(\boldsymbol{\beta} - \boldsymbol{\beta}_0, \mathbf{rot}\, q) = -t^2(\mathbf{rot}\, p, \mathbf{rot}\, q), \qquad q \in H^1(\Omega) \cap L_0^2(\Omega). \quad (6.20)$$

From (6.7) follows that p satisfies the natural boundary condition $\partial p/\partial n = 0$ on $\partial\Omega$, the right hand side of (6.20) above is equal to $-t^2(\Delta p, q)$. Hence, applying the estimate (6.15) with $t = 0$, $\mathbf{\Phi} = \boldsymbol{\beta} - \boldsymbol{\beta}_0$, $P = p - p_0$, $\boldsymbol{F} = \mathbf{0}$, $K = -t^2\Delta p$ and $l = s - 2$, gives

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{s,\Omega'} \leq Ct^2\|p\|_{s+1,\Omega''}. \quad (6.21)$$

Exactly in the same way as before we now get

$$t\|p\|_{s+1,\Omega''} \leq C\|g\|_{s-2} \quad (6.22)$$

and hence

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{s,\Omega'} \leq Ct\|g\|_{s-2}. \quad (6.23)$$

The equation (6.17) and interior elliptic regularity for the Poisson problem the gives

$$\|w_r\|_{s+1,\Omega_i} \leq C(\|\boldsymbol{\beta} - \boldsymbol{\beta}_0\|_{s,\Omega'} + t^2\|g\|_{s-1}) \leq Ct(\|g\|_{s-2} + t\|g\|_{s-1}). \quad (6.24)$$

which concludes the proof. □

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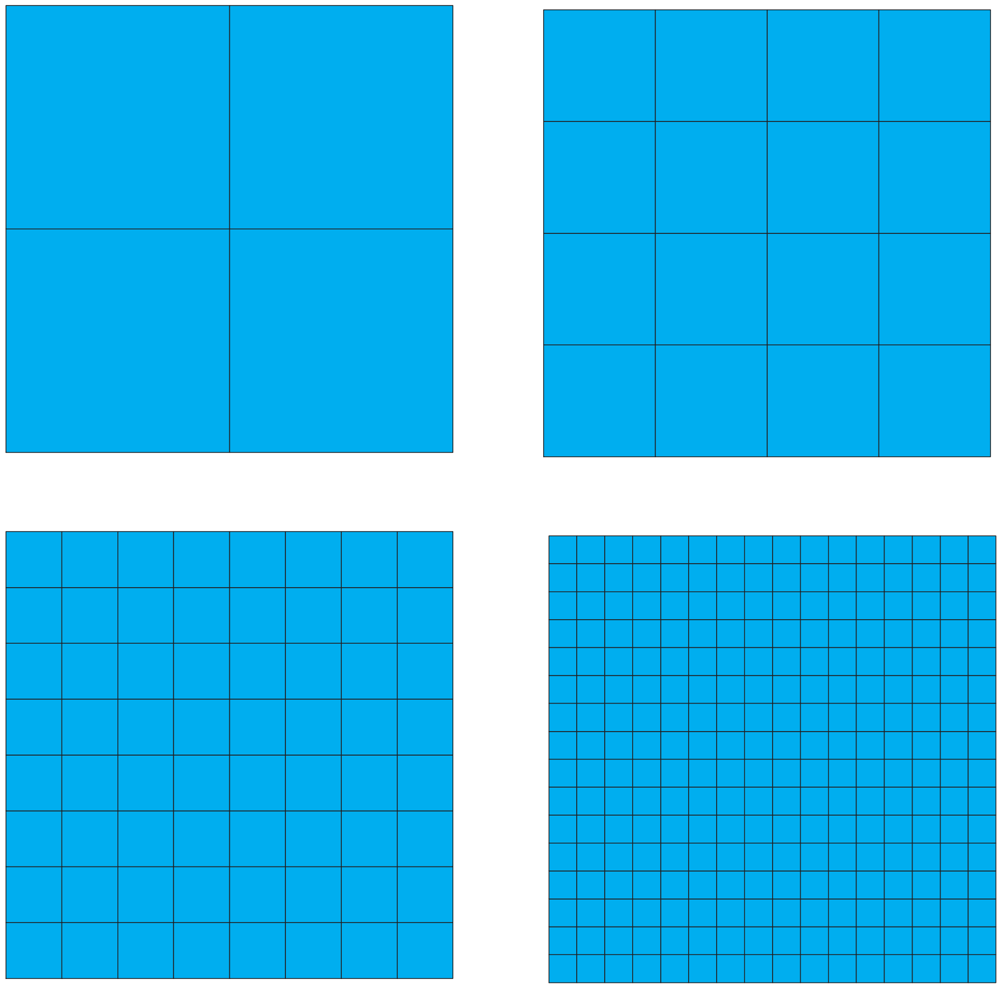

Figure 1. The quadriulateral meshes for the square plate test problem.

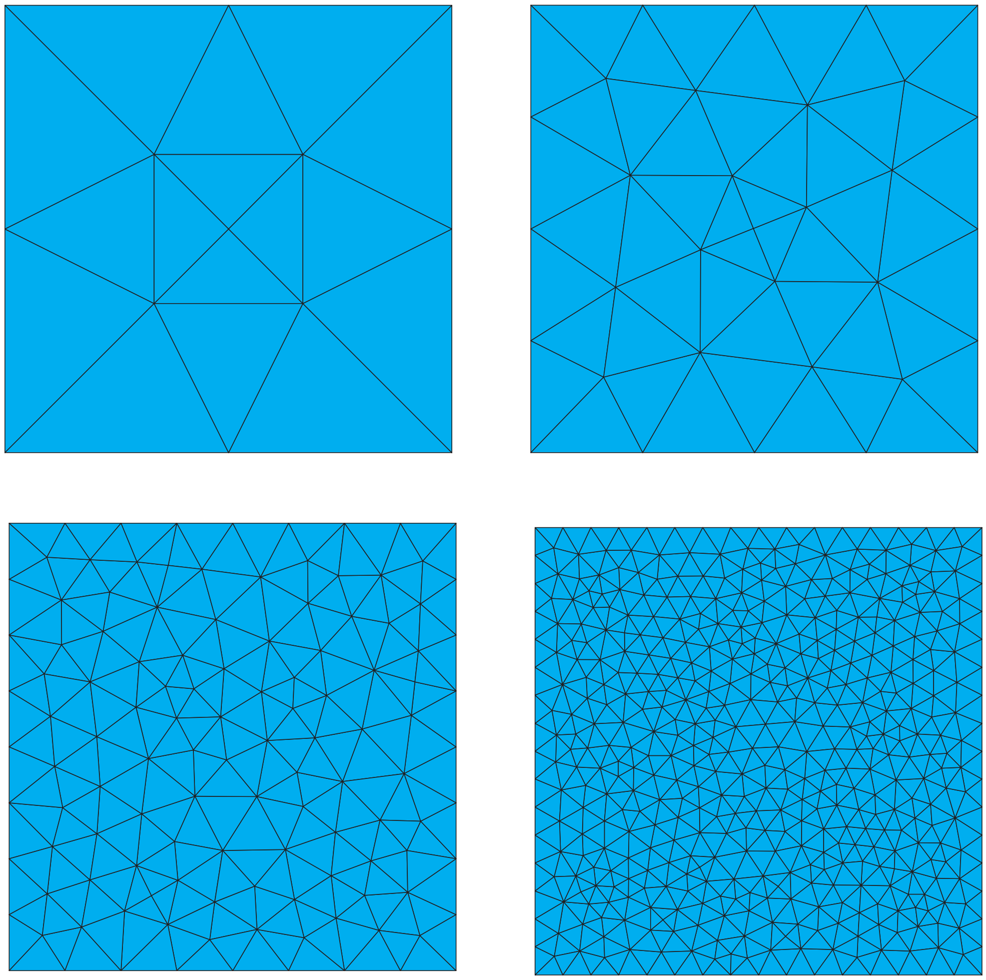

Figure 2. The triangular meshes for the square plate test problem.

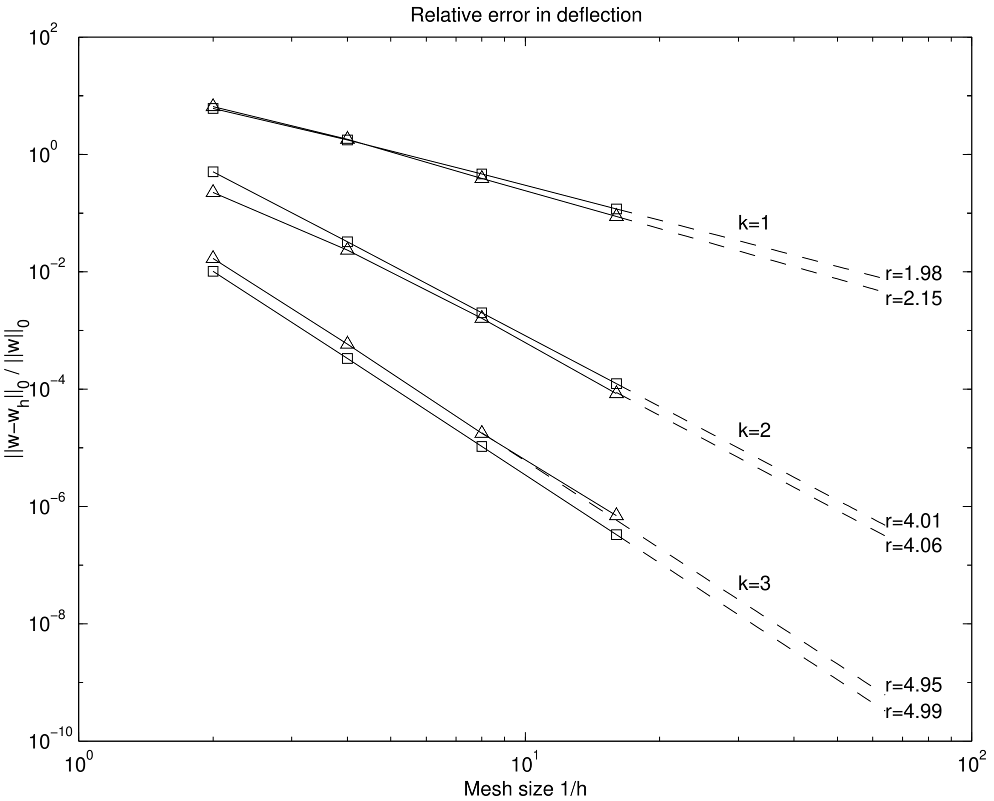

Figure 3. The relative error in deflection for the unequally interpolated elements with $k = 1, 2, 3$.

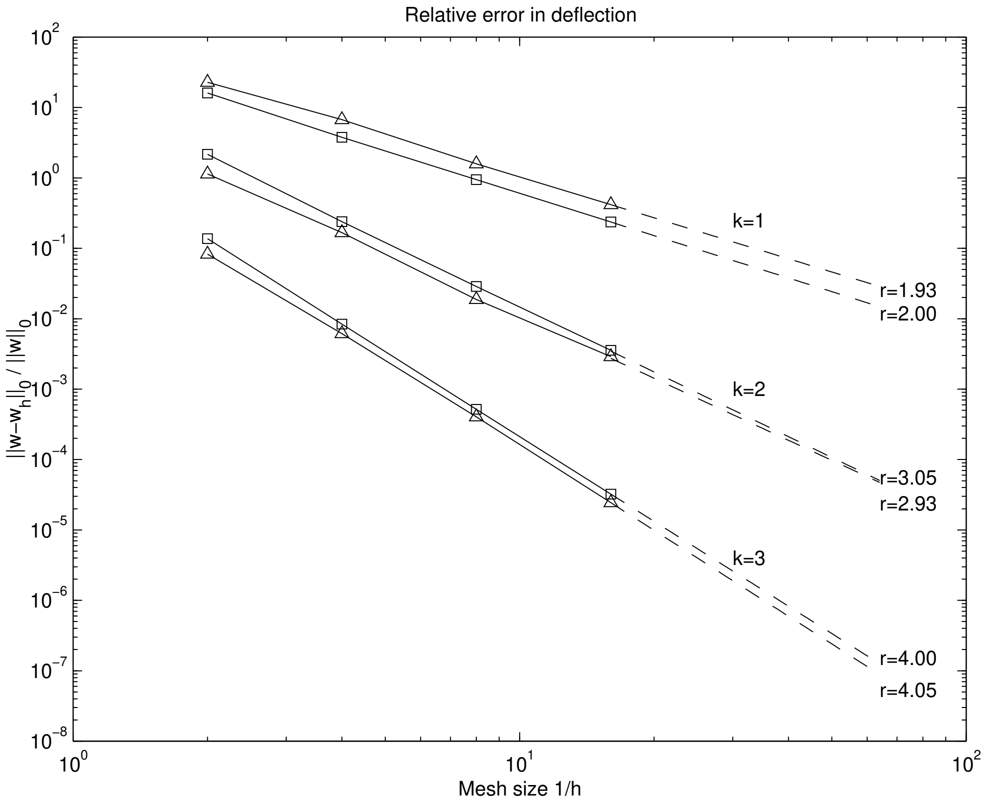

Figure 4. The relative error in deflection for the equally interpolated elements with $k = 1, 2, 3$.

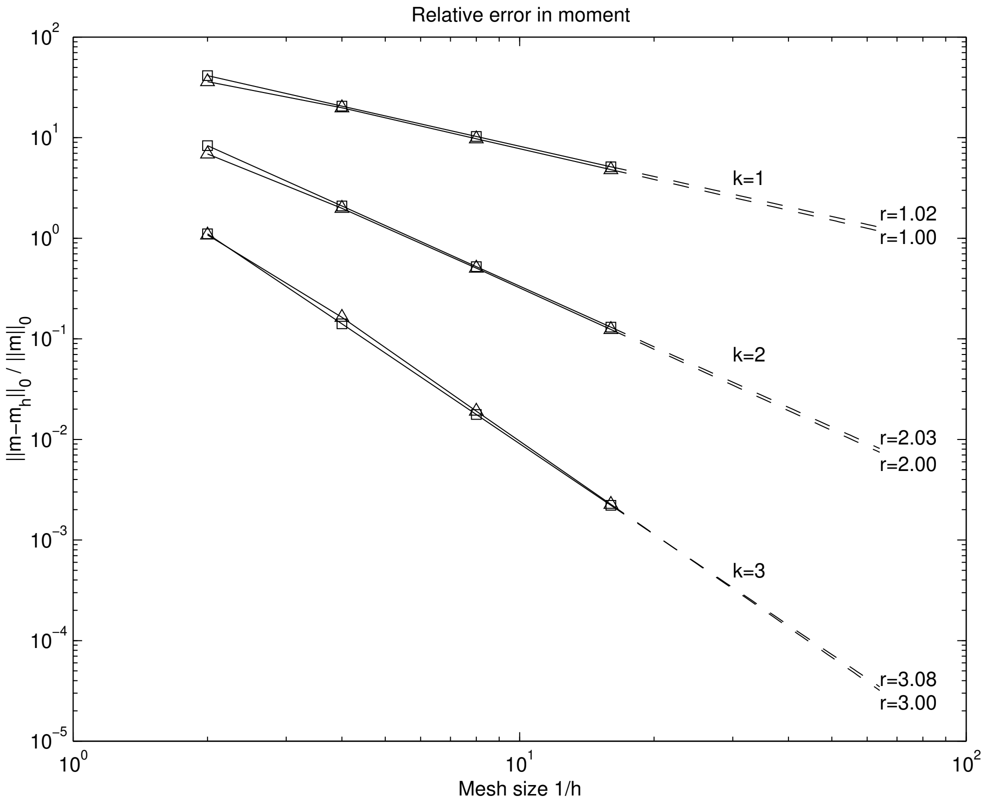

Figure 5. The relative error in bending moment for the unequally interpolated elements with $k = 1, 2, 3$.

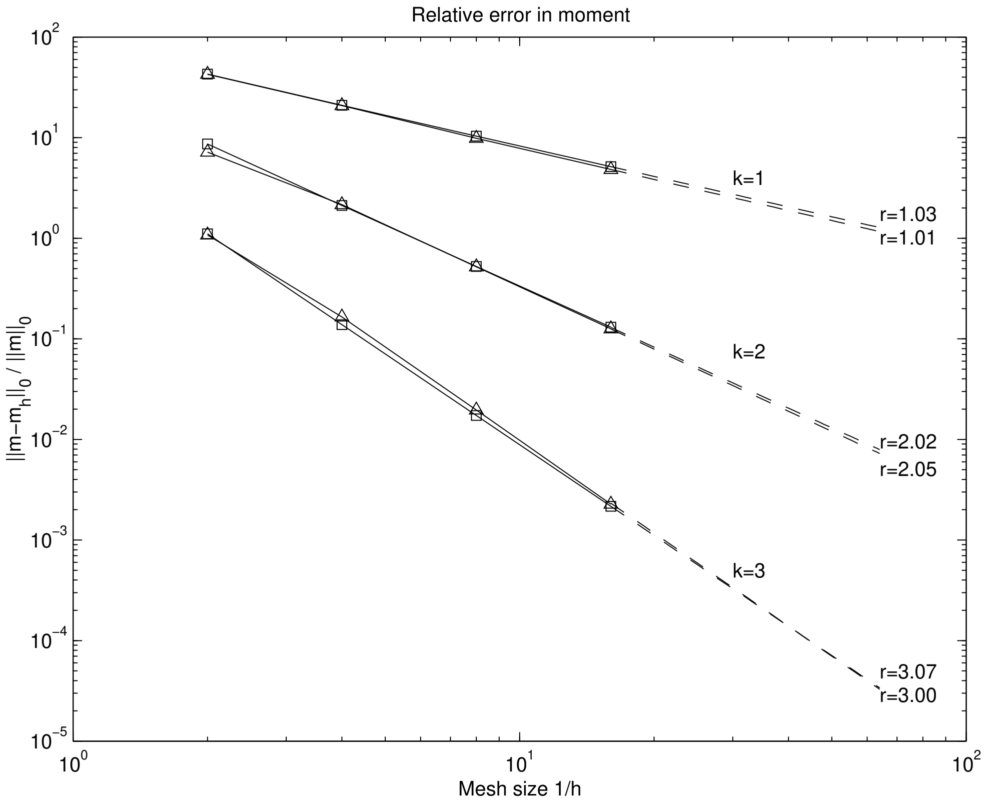

Figure 6. The relative error in bending moment for the equally interpolated elements with $k = 1, 2, 3$.

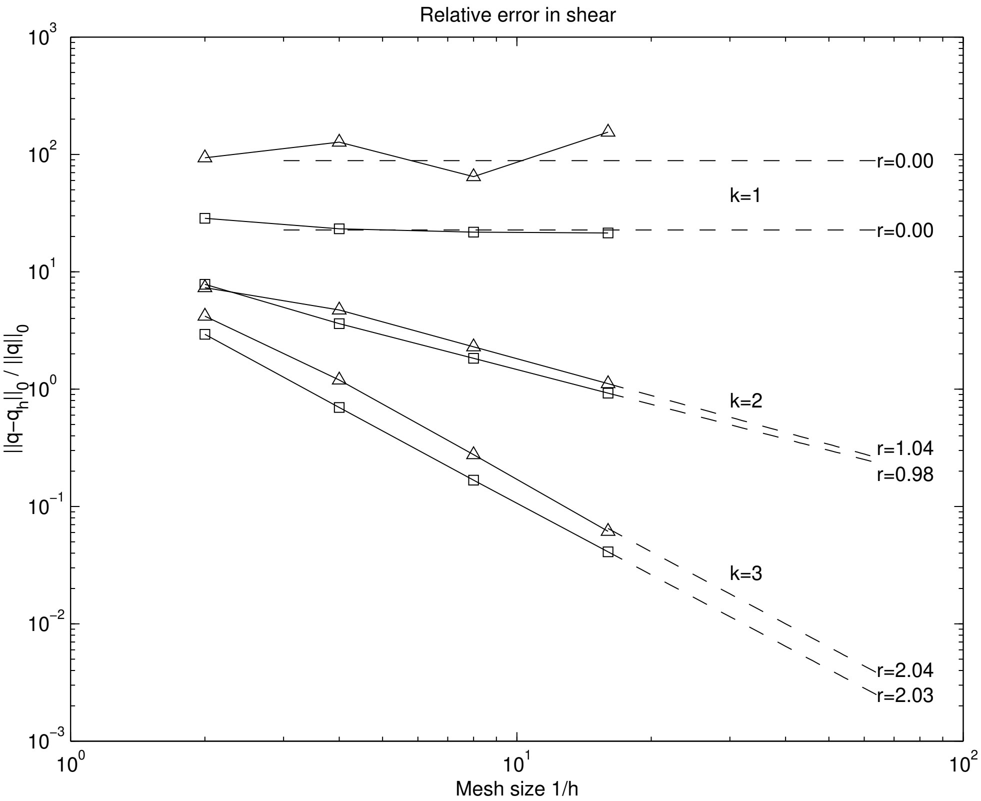

Figure 7. The relative error in shear force for the unequally interpolated elements with $k = 1, 2, 3$.

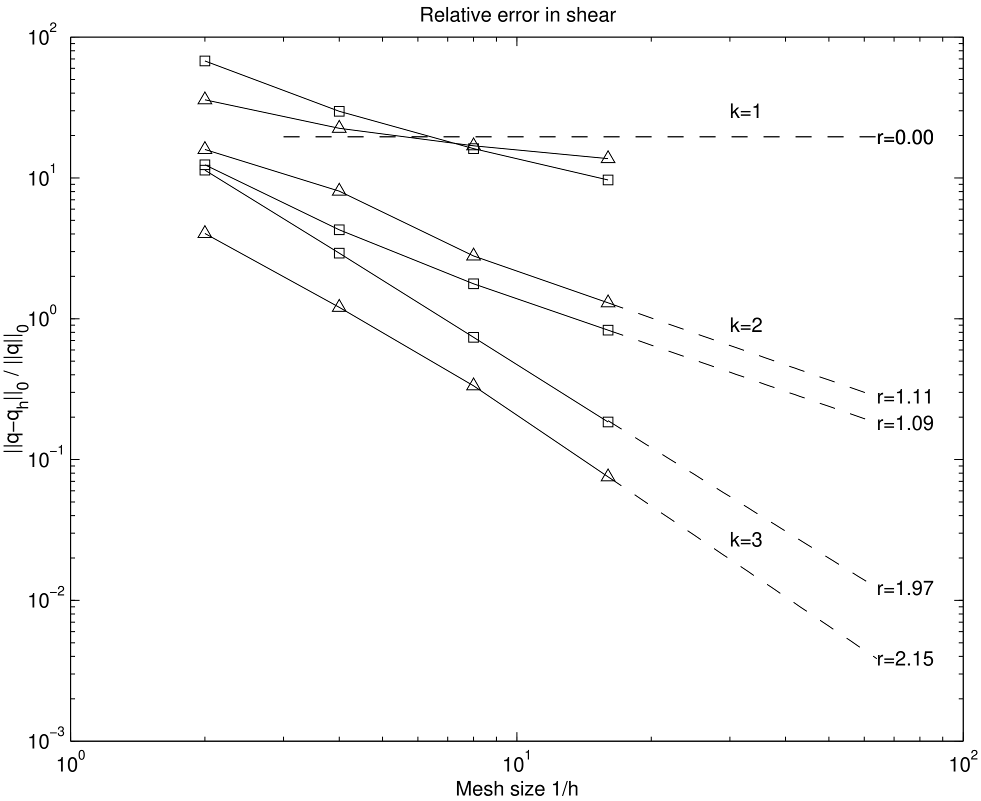

Figure 8. The relative error in shear force for the equally interpolated elements with $k = 1, 2, 3$.