An optimal low-order locking-free finite element method for Reissner-Mindlin plates

D. Chapelle*
Laboratoire Central des Ponts et Chaussées
2 allée Kepler, 77420 Champs/Marne, France

R. Stenberg
Universität Innsbruck - Institut für Mathematik und Geometrie
Technikerstrasse 13, A-6020 Innsbruck, Austria

Abstract

We propose a simple modification of a recently introduced locking-free finite element method for the Reissner-Mindlin plate model. By this modification, we are able to obtain optimal convergence rates on numerical benchmarks. These results are substantiated by a complete mathematical analysis which provides optimal a priori error estimates.

1 Introduction

The problem of shear locking which affects standard finite element methods used to approximate the Reissner-Mindlin plate model has been—and still remains—the subject of an extensive literature. Its causes are by now well understood, and ad hoc finite element schemes, based in general on mixed formulations where the shear force is used as an auxiliary unknown, have been designed to overcome this difficulty. However, one common drawback of these methods is that, unless special stabilization techniques are used (see [12, 13]), they do not apply to low-order finite elements.

A new mixed method was recently introduced and analysed in both [1] and [7], and numerically tested in [9]. This method uses relatively low-order finite elements, namely quadratic for the displacement and essentially linear for the rotations. The mathematical analyses performed in [1, 7] establish error estimates independent of the thickness parameter, thereby proving that locking is avoided. The estimates obtained in H^1-norm, however, are linear for the displacement as well as for the rotations, and therefore sub-optimal considering the quadratic displacement shape

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functions. The purpose of this paper is to show that, by a slight modification of this mixed method, one obtains improved convergence rates, both in theory and in numerical practice, that are optimal for the rotations and the displacement.

In the next section, we recall the Reissner-Mindlin variational formulation with the discretization introduced in [1, 7]. We also describe the modification we propose in order to improve error estimates. We proceed to present, in Section 3, numerical results that have been computed on two different benchmarks. In Section 4, we perform the mathematical analysis of the proposed finite element method, by which we derive optimal a priori error estimates for the rotation and the transverse displacement. Eventually, in Section 5, we carry out an analysis of the original scheme of [1, 7]. The conclusion of this analysis is that, whereas the approximate displacement is under-optimal, this procedure provides a better accuracy for the shear force.

2 Reissner-Mindlin formulation and discretization scheme

The standard Reissner-Mindlin variational formulation reads

$$Gt^3 a(\underline{\beta}, \underline{\eta}) + Gkt(\underline{\nabla} w - \underline{\beta}, \underline{\nabla} v - \underline{\eta}) = (f, v), \quad \forall (v, \underline{\eta}) \in W \times V, \tag{1}$$

with

$$a(\underline{\beta}, \underline{\eta}) \stackrel{\text{def}}{=} \frac{1}{6}\left[(\varepsilon(\underline{\beta}), \varepsilon(\underline{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div}\underline{\beta}, \operatorname{div}\underline{\eta})\right].$$

Here $\underline{\beta}$ and w respectively represent the unknown rotation vector and transverse displacement of the plate, whereas $\underline{\eta}$ and v are the corresponding trial functions. We assume that the definition of W and V incorporates the relevant boundary conditions. f is the transverse load, t the thickness, G the shear modulus, k the shear correction factor and ν stands for Poisson's ratio. The ε symbol denotes the linear strain operator, and $(\cdot, \cdot)$ the L^2-inner-product.

It is well known that, when the plate is very thin (t very small compared to the overall size of the plate), the load must be of the order of t^3 for the solution of (1) to remain bounded and non-vanishing (see e.g. [6]). Therefore we will assume that f is of the form

$$f = Gt^3 g, \tag{2}$$

so that we can rewrite (1) as follows

$$a(\underline{\beta}, \underline{\eta}) + \frac{k}{t^2}(\underline{\nabla} w - \underline{\beta}, \underline{\nabla} v - \underline{\eta}) = (g, v), \quad \forall (v, \underline{\eta}) \in W \times V. \tag{3}$$

The finite element scheme introduced in [1, 7] amounts to solving

$$\begin{aligned} a(\underline{\beta}_h, \underline{\eta}) + k\alpha(\underline{\nabla} w_h - \underline{\beta}_h, \underline{\nabla} v - \underline{\eta}) + k(\frac{1}{t^2} - \alpha)(\Pi_h\left[\underline{\nabla} w_h - \underline{\beta}_h\right], \Pi_h\left[\underline{\nabla} v - \underline{\eta}\right]) \quad (4) \\ = (g, v), \quad \forall (v, \underline{\eta}) \in W_h \times V_h, \end{aligned}$$

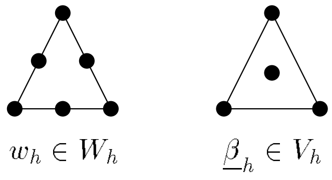

Figure 1: Degrees of freedom

where α is a positive parameter chosen so that, for the values of t of practical interest

$$\frac{1}{t^2} - \alpha > 0,$$

and Π_h is the operator that substitutes to a function its mean value over each element. The displacement w_h is sought in the space of piecewise-quadratic functions (P_2), whereas the rotation components are discretized in the space of piecewise-linear functions with an additional "bubble" degree of freedom within each element ($P_1 + B_3$). This bubble is necessary to avoid locking, but it can be condensed out at the element level, so that no significant computational cost is added compared to a simple piecewise-linear discretization. These finite element spaces are visualized in Figure 1.

The numerical scheme we propose corresponds to a modification of (4) by which we allow α to depend on the mesh parameter h and the thickness t by a simple rule (inspired from stabilization techniques, see [15, 16]) of the form

$$\alpha = \frac{\mu}{t^2 + h^2}, \tag{5}$$

where μ is a constant independent of t and h. It is to be noted that several modifications of (4) were already proposed in [9], namely $\alpha = O(h^{-2})$ and $\alpha = O(h^{-1})$, and reported to provide very good numerical results. The second of these strategies was mathematically analysed in [5], although for discretization spaces different from here, and improved—albeit still sub-optimal—error estimates were obtained.

3 Numerical results

In order to study the influence of α on actual numerical solutions, and assess the relevance of a rule such as (5), we used two benchmarks to compute approximation errors in Sobolev norms for different values of α. In both cases the system analysed was a square plate of unit side length. In the actual computations we used $\nu = 0.3$ and $k = 1$.

In the first example, clamped boundary conditions are imposed on the whole boundary ($w = 0$, $\underline{\beta} = \underline{0}$), and the load is determined so that the exact limit solution when t tends to zero (Kirchhoff solution) is exactly

$$w = x^2(x-1)^2 y^2 (y-1)^2, \ (x, y) \in [0, 1]^2. \tag{6}$$

We set $t = 0.001$ in order to be able to use this limit solution as a reference for approximations computed with the Reissner-Mindlin model.

In the second case, the plate is simply-supported ($w = 0$, $\underline{\beta} \cdot \underline{t} = 0$, where $\underline{t}$ is the tangent vector to the boundary). Note that we enforce "hard" simply-supported conditions, with the tangential component of the rotation set to zero. This is done in order to minimize the boundary layer effect (see [3]). Thus the solution computed for small values of t can again be compared to the exact limit Kirchhoff solution to derive approximation errors with very good accuracy. The numerical results presented here correspond to $t = 0.001$. The load is such that the Kirchhoff solution is now

$$w = \sin(\pi x)\sin(\pi y),\ (x, y) \in [0, 1]^2. \tag{7}$$

In both cases, for symmetry reasons, only one fourth of the square was actually computed, and we considered regular meshes obtained by splitting each side of the (reduced) square into N parts, $N = 10, 20, 40$ (see the mesh for $N = 10$ in Figure 2). We computed the H^1-norm of the difference between the finite element solutions and the Kirchhoff solutions, separately for the displacement and the rotation vector. We also computed the L^2-norm of the same difference for the shear vector, the approximation of which was provided as a by-product of the finite element procedure (see Remark 1) by

$$\underline{q}_h = \frac{k}{t^2}\Pi_h\left[\underline{\nabla} w_h - \underline{\beta}_h\right], \tag{8}$$

to be compared with

$$\underline{q} = \frac{k}{t^2}(\underline{\nabla} w - \underline{\beta}) \tag{9}$$

for the exact solution.

The numerical results obtained for the two different benchmarks turned out to be very similar, particularly with regard to the influence of α. Therefore, while the actual numbers are listed in separate tables, the comments that follow pertain to both cases.

Tables 1 and 2 feature the interpolation errors. Then, for each mesh, we determined the approximate value of α for which the relative error was optimal for the displacement error. These values are displayed in Tables 3 and 4 with the corresponding errors. These optimal values do not seem to follow a rule in h^{-2}, as one might have hoped (note that here $t^2 \ll h^2$). However, we also observed that the sensitivity of errors with respect to α is rather low. In fact, by setting $\alpha = \mu h^{-2}$, one still obtains an excellent convergence behaviour, as shown in Tables 5 and 6, quadratic for the displacements and linear for the rotations. By contrast, setting α at a fixed value, one obtains the results shown in Table 7 and 8, where the convergence of the displacement is now suboptimal, even though it remains within the (linear) bound guaranteed by theory.

These numerical results, and in particular those of Tables 5 and 6 clearly support the conjecture that setting α according to a rule like (5) leads to improved convergence rates, optimal for both the rotation and the displacement. It remains

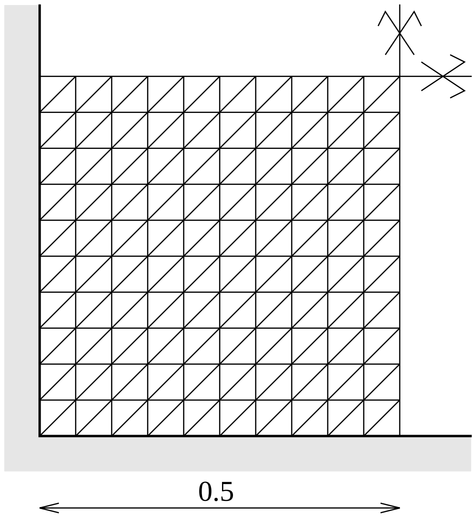

Figure 2: Mesh used for $N = 10$

N	$\frac{\lvert w-w_i\rvert_1}{\lvert w\rvert_1}$	$\frac{\lvert\underline{\beta}-\underline{\beta}_i\rvert_1}{\lvert\underline{\beta}\rvert_1}$	$\frac{\lVert\underline{q}-\underline{q}_i\rVert_0}{\lVert\underline{q}\rVert_0}$
10	$6.14\,10^{-3}$	0.184	$6.50\,10^{-2}$
20	$1.54\,10^{-3}$	$9.26\,10^{-2}$	$3.26\,10^{-2}$
40	$3.85\,10^{-4}$	$4.64\,10^{-2}$	$1.63\,10^{-2}$

Table 1: Clamped case - Interpolation errors

N	$\frac{\lvert w-w_i\rvert_1}{\lvert w\rvert_1}$	$\frac{\lvert\underline{\beta}-\underline{\beta}_i\rvert_1}{\lvert\underline{\beta}\rvert_1}$	$\frac{\lVert\underline{q}-\underline{q}_i\rVert_0}{\lVert\underline{q}\rVert_0}$
10	$2.82\,10^{-3}$	0.135	$4.67\,10^{-2}$
20	$7.06\,10^{-4}$	$6.75\,10^{-2}$	$2.34\,10^{-2}$
40	$1.77\,10^{-4}$	$3.38\,10^{-2}$	$1.17\,10^{-2}$

Table 2: Simply-supported case - Interpolation errors

N	α	$\frac{\lvert w-w_h\rvert_1}{\lvert w\rvert_1}$	$\frac{\lvert\underline{\beta}-\underline{\beta}_h\rvert_1}{\lvert\underline{\beta}\rvert_1}$	$\frac{\lVert\underline{q}-\underline{q}_h\rVert_0}{\lVert\underline{q}\rVert_0}$
10	200	$1.21\,10^{-2}$	0.125	0.227
20	800	$2.93\,10^{-3}$	$6.19\,10^{-2}$	0.156
40	1600	$7.07\,10^{-4}$	$3.08\,10^{-2}$	$9.52\,10^{-2}$

Table 3: Clamped case - Optimal values of α

N	α	$\frac{\vert w-w_h\vert_1}{\vert w\vert_1}$	$\frac{\vert \underline{\beta}-\underline{\beta}_h\vert_1}{\vert \underline{\beta}\vert_1}$	$\frac{\Vert \underline{q}-\underline{q}_h\Vert_0}{\Vert \underline{q}\Vert_0}$
10	200	$5.35\,10^{-3}$	$8.77\,10^{-2}$	0.193
20	400	$1.30\,10^{-3}$	$4.38\,10^{-2}$	0.130
40	1100	$3.16\,10^{-4}$	$2.19\,10^{-2}$	$8.69\,10^{-2}$

Table 4: Simply-supported case - Optimal values of α

N	α	$\frac{\vert w-w_h\vert_1}{\vert w\vert_1}$	$\frac{\vert \underline{\beta}-\underline{\beta}_h\vert_1}{\vert \underline{\beta}\vert_1}$	$\frac{\Vert \underline{q}-\underline{q}_h\Vert_0}{\Vert \underline{q}\Vert_0}$
10	200	$1.21\,10^{-2}$	0.125	0.227
20	800	$2.93\,10^{-3}$	$6.19\,10^{-2}$	0.156
40	3200	$7.15\,10^{-4}$	$3.08\,10^{-2}$	0.103
Conv. rate		2.03	1.01	0.60

Table 5: Clamped case - Errors for $\alpha = \mu h^{-2}$

N	α	$\frac{\vert w-w_h\vert_1}{\vert w\vert_1}$	$\frac{\vert \underline{\beta}-\underline{\beta}_h\vert_1}{\vert \underline{\beta}\vert_1}$	$\frac{\Vert \underline{q}-\underline{q}_h\Vert_0}{\Vert \underline{q}\Vert_0}$
10	200	$5.35\,10^{-3}$	$8.77\,10^{-2}$	0.193
20	800	$1.32\,10^{-3}$	$4.37\,10^{-2}$	0.140
40	3200	$3.24\,10^{-4}$	$2.18\,10^{-2}$	$9.69\,10^{-2}$
Conv. rate		2.03	1.00	0.53

Table 6: Simply-supported case - Errors for $\alpha = \mu h^{-2}$

N	α	$\frac{\vert w-w_h\vert_1}{\vert w\vert_1}$	$\frac{\vert \underline{\beta}-\underline{\beta}_h\vert_1}{\vert \underline{\beta}\vert_1}$	$\frac{\Vert \underline{q}-\underline{q}_h\Vert_0}{\Vert \underline{q}\Vert_0}$
10	200	$1.21\,10^{-2}$	0.125	0.227
20	200	$3.13\,10^{-3}$	$6.20\,10^{-2}$	0.138
40	200	$9.49\,10^{-4}$	$3.09\,10^{-2}$	$8.70\,10^{-2}$
Conv. rate		1.72	1.00	0.67

Table 7: Clamped case - Errors for α fixed

N	α	$\frac{\lvert w-w_h\rvert_1}{\lvert w\rvert_1}$	$\frac{\lvert\underline{\beta}-\underline{\beta}_h\rvert_1}{\lvert\underline{\beta}\rvert_1}$	$\frac{\lVert\underline{q}-\underline{q}_h\rVert_0}{\lVert\underline{q}\rVert_0}$
10	200	$5.35\,10^{-3}$	$8.77\,10^{-2}$	0.193
20	200	$1.32\,10^{-3}$	$4.39\,10^{-2}$	0.124
40	200	$3.63\,10^{-4}$	$2.19\,10^{-2}$	$8.18\,10^{-2}$
Conv. rate		1.86	1.00	0.60

Table 8: Simply-supported case - Errors for α fixed

to substantiate this conjecture by a mathematical analysis, which is done in the next section.

4 Mathematical analysis of the approximation scheme

In this section, the symbol C will be used to denote a generic, strictly positive constant, which is allowed to take different values at different occurrences.

We start by recasting the variational problem in the appropriate mathematical framework. For the sake of simplicity we will focus on the clamped case. Denoting by Ω the planar region occupied by the plate, we define

$$W \stackrel{\text{def}}{=} H_0^1(\Omega), \qquad V \stackrel{\text{def}}{=} \left[H_0^1(\Omega)\right]^2.$$

The variational problem can be stated as:
$(\mathcal{P}_t)$: *Find* $(w, \underline{\beta}) \in W \times V$ *such that (3) holds for any* $(v, \underline{\eta}) \in W \times V$.
For any given value of t, $(\mathcal{P}_t)$ is a standard elliptic problem with a unique solution. Moreover, even in the limit $t \to 0$, this problem remains well-posed, in that its solution admits a bounded (and non-vanishing) limit [6].

It is a standard procedure, in such a formulation where numerical locking is of particular concern, to define an equivalent mixed problem by introducing an auxiliary unknown. Here we use the scaled shear force

$$\underline{q} \stackrel{\text{def}}{=} \frac{k}{t^2}(\underline{\nabla} w - \underline{\beta}). \tag{10}$$

This new unknown lies in $Q \stackrel{\text{def}}{=} [L^2(\Omega)]^2$, and we can rewrite this definition in a variational form which, combined with $(\mathcal{P}_t)$, yields the following mixed problem:
$(\mathcal{P}'_t)$: *Find* $(w, \underline{\beta}, \underline{q}) \in W \times V \times Q$ *such that*

$$B(w, \underline{\beta}, \underline{q}; v, \underline{\eta}, \underline{r}) = (g, v), \tag{11}$$

for any $(v, \underline{\eta}, \underline{r}) \in W \times V \times Q$.
The bilinear form B is defined by

$$B(w, \underline{\beta}, \underline{q}; v, \underline{\eta}, \underline{r}) \stackrel{\text{def}}{=} a(\underline{\beta}, \underline{\eta}) + (\underline{q}, \underline{\nabla} v - \underline{\eta}) + (\underline{r}, \underline{\nabla} w - \underline{\beta}) - \frac{t^2}{k}(\underline{q}, \underline{r}).$$

It is $(\mathcal{P}'_t)$, rather than $(\mathcal{P}_t)$, that will be the starting point of the finite element procedure which we now introduce.

Defining $\mathcal{C}_h$ as a partitioning of Ω into triangles, assumed to be regular in the sense of [10], we denote by h_K the diameter of each triangle K and set

$$h \stackrel{\text{def}}{=} \max_{K \in \mathcal{C}_h} h_K.$$

The finite element spaces used respectively for the displacement and rotation are (see Figure 1)

$$W_h \stackrel{\text{def}}{=} \{v \in W \ \mid \ v_{|K} \in P_2(K),\ \forall K \in \mathcal{C}_h\},$$

$$V_h \stackrel{\text{def}}{=} \{\underline{\eta} \in V \mid \underline{\eta}_{|K} \in [P_1(K) + B_3(K)]^2,\ \forall K \in \mathcal{C}_h\},$$

where we recall that, in general, the space $B_k(K)$ $(k \geq 3)$ is defined by

$$B_k(K) \stackrel{\text{def}}{=} \{v \in P_k(K),\ v_{|\partial K} = 0\}.$$

In addition we define the shear discretization space as

$$Q_h \stackrel{\text{def}}{=} \{\underline{r} \in Q \ \mid \ \underline{r}_{|K} \in [P_0(K)]^2,\ \forall K \in \mathcal{C}_h\}.$$

The finite element mixed problem is:
$(\mathcal{P}'_{t,h})$: *Find* $(w_h, \underline{\beta}_h, \underline{q}_h) \in W_h \times V_h \times Q_h$ *such that*

$$B_h(w_h, \underline{\beta}_h, \underline{q}_h; v, \underline{\eta}, \underline{r}) = (g, v), \tag{12}$$

for any $(v, \underline{\eta}, \underline{r}) \in W_h \times V_h \times Q_h$,
with

$$\begin{aligned} B_h(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) &\stackrel{\text{def}}{=} B(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) \\ &\quad + \sum_{K \in \mathcal{C}_h} \frac{\mu k}{t^2 + h_K^2} (\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k} \underline{s}, \underline{\nabla} v - \underline{\eta} - \frac{t^2}{k} \underline{r})_K, \end{aligned}$$

where μ is a positive constant.

Remark 1: From (12), q_h can be expressed in terms of the original unknowns as

$$q_h = \frac{k}{t^2} \Pi_h [\underline{\nabla} w_h - \underline{\beta}_h], \tag{13}$$

where Π_h is the L^2-projection operator onto Q_h. Taking into account the definition of Q_h, Π_h appears to be the already-mentioned operator that substitutes to a function its mean value on each triangle. Then, using (13) in (12), one obtains a reduced finite element problem in $(w_h, \underline{\beta}_h)$ which reads

$$\begin{aligned} a(\underline{\beta}_h, \underline{\eta}) + \sum_{K \in \mathcal{C}_h} k \Bigg\{ & \frac{\mu}{t^2 + h_K^2} (\underline{\nabla} w_h - \underline{\beta}_h, \underline{\nabla} v - \underline{\eta})_K \\ & + (\frac{1}{t^2} - \frac{\mu}{t^2 + h_K^2})(\Pi_h [\underline{\nabla} w_h - \underline{\beta}_h], \Pi_h [\underline{\nabla} v - \underline{\eta}])_K \Bigg\} \\ & = (g, v), \quad \forall (v, \underline{\eta}) \in W_h \times V_h. \end{aligned} \tag{14}$$

The only difference between this and (4), assuming $\alpha = \mu/(t^2 + h^2)$, is that here we allow "α" to be set locally according to the size of each element. (e.g. in case the mesh is non-uniform). ■

Remark 2: The reduced problem (14) deduced from the mixed finite element method we propose is of the same type as that obtained by the approach of [1, 7]. There is, however, a difference in the mixed formulation itself: here we directly approximate the shear, instead of an artificial unknown that coincides with it in the limit $t \to 0$. ■

We proceed to establish the convergence of our method by a stability-consistency argument. The consistency is satisfied "by construction" from (10).

Proposition 1 *The mixed finite element scheme is consistent, i.e. the solution $(w, \underline{\beta}, \underline{q})$ of $(\mathcal{P}_t')$ satisfies*

$$B_h(w, \underline{\beta}, \underline{q}; v, \underline{\eta}, \underline{r}) = (g, v), \tag{15}$$

for any $(v, \underline{\eta}, \underline{r}) \in W \times V \times Q$.

The stability property that we want to prove involves specific norms that we now define.

$$|||v, \underline{\eta}|||_V \stackrel{\text{def}}{=} \left\{ \|v\|_1^2 + \|\underline{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} \frac{1}{t^2 + h_K^2} \|\underline{\nabla} v - \underline{\eta}\|_{0,K}^2 \right\}^{\frac{1}{2}},$$

$$|||\underline{r}|||_Q \stackrel{\text{def}}{=} \left\{ t^2 \|\underline{r}\|_0^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{r}\|_{0,K}^2 \right\}^{\frac{1}{2}},$$

$$|||v, \underline{\eta}, \underline{r}||| \stackrel{\text{def}}{=} \left\{ |||v, \underline{\eta}|||_V^2 + |||\underline{r}|||_Q^2 \right\}^{\frac{1}{2}}.$$

We note for reference that $||| \cdot |||_V$ satisfies the following equivalence property (detailed in [16]):

$$c|||v, \underline{\eta}|||_V^2 \leq \|\underline{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} \frac{1}{t^2 + h_K^2} \|\underline{\nabla} v - \underline{\eta}\|_{0,K}^2 \leq C|||v, \underline{\eta}|||_V^2. \tag{16}$$

Furthermore these norms are tailored to provide the following continuity property.

Proposition 2 *B_h is continuous for the norm $||| \cdot |||$, i.e. for any $(z, \underline{\phi}, \underline{s})$ and $(v, \underline{\eta}, \underline{r})$ in $W \times V \times Q$, it holds that*

$$B_h(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) \leq C|||z, \underline{\phi}, \underline{s}||| \times |||v, \underline{\eta}, \underline{r}|||. \tag{17}$$

Proof: For the cross terms, we have

$$\begin{aligned}(\underline{s}, \underline{\nabla} v - \underline{\eta}) &= \sum_{K\in\mathcal{C}_h} (\underline{s}, \underline{\nabla} v - \underline{\eta})_K \\ &\le \left[\sum_{K\in\mathcal{C}_h} (t^2 + h_K^2)\|\underline{s}\|_{0,K}^2\right]^{\frac{1}{2}} \left[\sum_{K\in\mathcal{C}_h} \frac{1}{t^2 + h_K^2}\|\underline{\nabla} v - \underline{\eta}\|_{0,K}^2\right]^{\frac{1}{2}} \\ &\le |||\underline{s}|||_Q |||v, \underline{\eta}|||_V,\end{aligned}$$

and a similar bound for $(\underline{r}, \underline{\nabla} z - \underline{\phi})$. Furthermore,

$$\begin{aligned}&\sum_{K\in\mathcal{C}_h} \frac{\mu k}{t^2 + h_K^2}(\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k}\underline{s}, \underline{\nabla} v - \underline{\eta} - \frac{t^2}{k}\underline{r})_K \\ &\qquad \le C\left[\sum_{K\in\mathcal{C}_h} \frac{1}{t^2 + h_K^2}\|\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k}\underline{s}\|_{0,K}^2\right]^{\frac{1}{2}} \\ &\qquad\quad \times \left[\sum_{K\in\mathcal{C}_h} \frac{1}{t^2 + h_K^2}\|\underline{\nabla} v - \underline{\eta} - \frac{t^2}{k}\underline{r}\|_{0,K}^2\right]^{\frac{1}{2}} \\ &\qquad \le C\left[\sum_{K\in\mathcal{C}_h} \frac{\|\underline{\nabla} z - \underline{\phi}\|_{0,K}^2}{t^2 + h_K^2} + t^2\|\underline{s}\|_{0,K}^2\right]^{\frac{1}{2}} \\ &\qquad\quad \times \left[\sum_{K\in\mathcal{C}_h} \frac{\|\underline{\nabla} v - \underline{\eta}\|_{0,K}^2}{t^2 + h_K^2} + t^2\|\underline{r}\|_{0,K}^2\right]^{\frac{1}{2}} \\ &\qquad \le C|||z, \underline{\phi}, \underline{s}|||\;|||v, \underline{\eta}, \underline{r}|||.\end{aligned}$$

The remaining terms are obviously bounded, so that (17) is proved. ■

We now establish the stability result.

Proposition 3 *Assuming* $0 < \mu < 1$, *formulation* $(\mathcal{P}'_{t,h})$ *is stable for the norm* $|||\cdot|||$, *i.e. for any* $(z, \underline{\phi}, \underline{s}) \in W_h \times V_h \times Q_h$, *there exists* $(v, \underline{\eta}, \underline{r}) \in W_h \times V_h \times Q_h$ *such that*

$$|||v, \underline{\eta}, \underline{r}||| \le C|||z, \underline{\phi}, \underline{s}|||, \tag{18}$$

and

$$B_h(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) \ge C|||z, \underline{\phi}, \underline{s}|||^2. \tag{19}$$

Proof: In two steps.
i) Set $v_1 = z$, $\underline{\eta}_1 = \underline{\phi}$ and $\underline{r}_1 = -\underline{s}$. Then

$$|||v_1, \underline{\eta}_1, \underline{r}_1||| = |||z, \underline{\phi}, \underline{s}||| \tag{20}$$

and

$$B_h(z, \underline{\phi}, \underline{s}; v_1, \underline{\eta}_1, \underline{r}_1)$$

$$
\begin{aligned}
&= a(\underline{\phi},\underline{\phi}) + \frac{t^2}{k}\|\underline{s}\|_0^2 + \sum_{K\in\mathcal{C}_h} \frac{\mu k}{t^2+h_K^2}\left[\|\underline{\nabla} z - \underline{\phi}\|_{0,K}^2 - \frac{t^4}{k^2}\|\underline{s}\|_{0,K}^2\right] \\
&\geq a(\underline{\phi},\underline{\phi}) + \frac{1-\mu}{k}t^2\|\underline{s}\|_0^2 + \mu k \sum_{K\in\mathcal{C}_h} \frac{1}{t^2+h_K^2}\|\underline{\nabla} z - \underline{\phi}\|_{0,K}^2 \\
&\geq C\left\{\|\underline{\phi}\|_1^2 + \sum_{K\in\mathcal{C}_h} \frac{\|\underline{\nabla} z - \underline{\phi}\|_{0,K}^2}{t^2+h_K^2} + t^2\|\underline{s}\|_0^2\right\} \\
&\geq C\left\{|||z,\underline{\phi}|||_V^2 + t^2\|\underline{s}\|_0^2\right\}, \qquad (21)
\end{aligned}
$$

using (16).
ii) To obtain the second part of the shear norm, we now set $v_2 = 0$, $\underline{r}_2 = \underline{0}$ and $\underline{\eta}_2 = -h_K^2 b_h \underline{s}$, where b_h is the bubble function within each element. An inverse inequality yields

$$
\begin{aligned}
|||v_2,\underline{\eta}_2,\underline{r}_2|||^2 &= \|\underline{\eta}_2\|_1^2 + \sum_{K\in\mathcal{C}_h} \frac{1}{t^2+h_K^2}\|\underline{\eta}_2\|_{0,K}^2 \\
&\leq C \sum_{K\in\mathcal{C}_h} h_K^{-2}\|\underline{\eta}_2\|_{0,K}^2 \\
&\leq C \sum_{K\in\mathcal{C}_h} h_K^2\|\underline{s}\|_{0,K}^2 \leq C|||\underline{s}|||_Q^2. \qquad (22)
\end{aligned}
$$

Moreover,

$$
B_h(z,\underline{\phi},\underline{s};v_2,\underline{\eta}_2,\underline{r}_2) = a(\underline{\phi},\underline{\eta}_2) - (\underline{s},\underline{\eta}_2) - \sum_{K\in\mathcal{C}_h} \frac{\mu k}{t^2+h_K^2}(\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k}\underline{s},\underline{\eta}_2)_K.
$$

By a simple scaling argument, we get

$$
-(\underline{s},\underline{\eta}_2) = \sum_{K\in\mathcal{C}_h} h_K^2(\underline{s},b_h\underline{s})_K \geq C \sum_{K\in\mathcal{C}_h} h_K^2\|\underline{s}\|_{0,K}^2,
$$

and we bound the two other terms like in the proof of Proposition 2. Hence,

$$
\begin{aligned}
B_h(z,\underline{\phi},\underline{s};v_2,\underline{\eta}_2,\underline{r}_2) &\geq C \sum_{K\in\mathcal{C}_h} h_K^2\|\underline{s}\|_{0,K}^2 - C\left[|||z,\underline{\phi}|||_V^2 + t^2\|\underline{s}\|_0^2\right]^{\frac{1}{2}} |||0,\underline{\eta}_2|||_V \\
&\geq C\left\{\sum_{K\in\mathcal{C}_h} h_K^2\|\underline{s}\|_{0,K}^2 - \left[|||z,\underline{\phi}|||_V^2 + t^2\|\underline{s}\|_0^2\right]\right\}, \qquad (23)
\end{aligned}
$$

using the bound on $|||0,\underline{\eta}_2|||_V$ obtained in (22). To conclude the proof, it suffices to consider convex combinations of the form

$$
(v,\underline{\eta},\underline{r}) = \lambda(v_1,\underline{\eta}_1,\underline{r}_1) + (1-\lambda)(v_2,\underline{\eta}_2,\underline{r}_2),
$$

where $0 < \lambda < 1$. Then (20) and (22) imply (18), and combining (21) with (23), it is always possible to choose λ big enough to get (19). ■

Since the continuity, consistency and stability properties are satisfied, the approximation bound automatically follows by a classical argument. We therefore state the result and omit the proof.

Proposition 4 *Assuming $0 < \mu < 1$, problem $(\mathcal{P}'_{t,h})$ has a unique solution $(w_h, \underline{\beta}_h, \underline{q}_h)$. This solution satisfies*

$$|||w - w_h, \underline{\beta} - \underline{\beta}_h, \underline{q} - \underline{q}_h||| \leq C \inf_{(v,\underline{\eta},\underline{r}) \in W_h \times V_h \times Q_h} |||w - v, \underline{\beta} - \underline{\eta}, \underline{q} - \underline{r}|||. \quad (24)$$

Remark 3: The extension of this result to higher order schemes is very straightforward. Considering indeed, for $k \geq 1$ the following discretization spaces

$$W_h = \{v \in W \mid v_{|K} \in P_{k+1}(K), \ \forall K \in \mathcal{C}_h\},$$

$$V_h = \{\underline{\eta} \in V \mid \underline{\eta}_{|K} \in [P_k(K) + B_{k+2}(K)]^2, \forall K \in \mathcal{C}_h\},$$

$$Q_h = \{\underline{r} \in Q \mid \underline{r}_{|K} \in [P_{k-1}(K)]^2, \ \forall K \in \mathcal{C}_h\},$$

not a single word need be changed in the consistency, continuity and stability arguments (note in particular that the choice $\underline{\eta}_2 = -h_K^2 b_h \underline{s}$ at step ii) of the stability proof is still allowed). ■

In the following two lemmas, we state some regularity results that are needed to derive error estimates.

Lemma 1 *Assume Ω is convex. Given $\underline{\delta} \in [L^2(\Omega)]^2$, there is a unique $(z, \underline{\phi}, \underline{s}) \in W \times V \times Q$ such that*

$$B(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) = (\underline{\delta}, \underline{\eta}), \quad (25)$$

for any $(v, \underline{\eta}, \underline{r}) \in W \times V \times Q$, and it satisfies

$$\|z\|_3 + \|\underline{\phi}\|_2 + \|\underline{s}\|_0 + t\|\underline{s}\|_1 \leq C\|\underline{\delta}\|_0. \quad (26)$$

Proof: Directly inferred from [2, Appendix]. ■

In the next lemma, we consider the decomposition of the solution of problem $(\mathcal{P}'_t)$, $(w, \underline{\beta}, \underline{q})$, into the Kirchhoff-Love solution $(w_0, \underline{\beta}_0)$, defined as the limit of $(w, \underline{\beta})$ when the parameter t tends to zero, and a residual term $(w_r, \underline{\beta}_r)$, viz.

$$(w, \underline{\beta}) = (w_0 + w_r, \underline{\beta}_0 + \underline{\beta}_r). \quad (27)$$

This decomposition was introduced in a general framework in [4], and already used for plates in [17].

Lemma 2 *Assume that Ω is convex and that $g \in L^2(\Omega)$. Then we have*

$$\|w_0\|_3 + \|\underline{\beta}\|_2 + \|\underline{q}\|_0 + t\|\underline{q}\|_1 \leq C(\|g\|_{-1} + t\|g\|_0), \quad (28)$$

$$\|w_r\|_2 \leq Ct(\|g\|_{-1} + t\|g\|_0). \quad (29)$$

Proof: Estimate (28) is directly obtained from [2, Appendix], where we also get

$$\|\underline{\beta}_r\|_1 \leq Ct\|g\|_{-1}. \tag{30}$$

Then recall

$$\underline{\nabla} w_0 = \underline{\beta}_0, \tag{31}$$

$$\underline{\nabla} w = \underline{\beta} + \frac{t^2}{k}\underline{q}. \tag{32}$$

Hence,

$$\underline{\nabla} w_r = \underline{\beta}_r + \frac{t^2}{k}\underline{q}, \tag{33}$$

and since w_r is in $H_0^1(\Omega)$ we therefore have, using (28) and (30),

$$\|w_r\|_2 \leq C(\|\underline{\beta}_r\|_1 + t^2\|\underline{q}\|_1) \leq Ct(\|g\|_{-1} + t\|g\|_0). \tag{34}$$

■

We now state and prove the final result of this section.

Proposition 5 *Assume* $0 < \mu < 1$, Ω *convex, and* $g \in L^2(\Omega)$. *Then it holds that*

$$|||w - w_h, \underline{\beta} - \underline{\beta}_h, \underline{q} - \underline{q}_h||| \leq Ch(\|g\|_{-1} + t\|g\|_0) \tag{35}$$

Moreover, we have the improved estimates

$$\|\underline{\beta} - \underline{\beta}_h\|_0 \leq Ch^2(\|g\|_{-1} + t\|g\|_0), \tag{36}$$

$$\|w - w_h\|_1 \leq Ch(t+h)(\|g\|_{-1} + t\|g\|_0). \tag{37}$$

Proof: We decompose the proof into three steps.
i) Let $\tilde{w}_0$, $\tilde{w}_r$ and $\tilde{\underline{\beta}}$ be the Lagrange interpolants of w_0, w_r, $\underline{\beta}$, respectively, and $\tilde{\underline{q}} \stackrel{\text{def}}{=} \Pi_h(\underline{q})$. Defining

$$\tilde{w} \stackrel{\text{def}}{=} \tilde{w}_0 + \tilde{w}_r,$$

we have

$$|||w - \tilde{w}, \underline{\beta} - \tilde{\underline{\beta}}, \underline{q} - \tilde{\underline{q}}||| \leq I_1 + I_2 + I_3 + I_4 + I_5 \tag{38}$$

with

$$I_1 \stackrel{\text{def}}{=} \|w - \tilde{w}\|_1,$$

$$I_2 \stackrel{\text{def}}{=} \|\underline{\beta} - \tilde{\underline{\beta}}\|_1,$$

$$I_3 \stackrel{\text{def}}{=} \left\{ \sum_{K \in \mathcal{C}_h} \frac{1}{t^2 + h_K^2} |w - \tilde{w}|_{1,K}^2 \right\}^{\frac{1}{2}},$$

$$I_4 \stackrel{\text{def}}{=} \left\{ \sum_{K \in \mathcal{C}_h} \frac{1}{t^2 + h_K^2} \|\underline{\beta} - \tilde{\underline{\beta}}\|_{0,K}^2 \right\}^{\frac{1}{2}},$$

$$I_5 \stackrel{\text{def}}{=} |||\underline{q} - \tilde{\underline{q}}|||_Q.$$

We now analyse these five terms separately, using the regularity estimates (28) and (29). In order to bound I_1 and I_3, we use the above decomposition. We get

$$\begin{aligned} I_1 &\leq \|w_0-\tilde{w}_0\|_1+\|w_r-\tilde{w}_r\|_1 \\ &\leq C(h^2\|w_0\|_3+h\|w_r\|_2) \\ &\leq Ch(h+t)(\|g\|_{-1}+t\|g\|_0), \end{aligned} \tag{39}$$

and

$$\begin{aligned} I_3 &\leq \left\{\sum_{K\in\mathcal{C}_h}\frac{1}{t^2+h_K^2}|w_0-\tilde{w}_0|^2_{1,K}\right\}^{\frac12}+\left\{\sum_{K\in\mathcal{C}_h}\frac{1}{t^2+h_K^2}|w_r-\tilde{w}_r|^2_{1,K}\right\}^{\frac12} \\ &\leq C(h\|w_0\|_3+\frac{h}{t}\|w_r\|_2) \\ &\leq Ch(\|g\|_{-1}+t\|g\|_0). \end{aligned} \tag{40}$$

Bounding the three remaining terms is more straightforward. We have

$$I_2 \leq Ch\|\underline{\beta}\|_2 \leq Ch(\|g\|_{-1}+t\|g\|_0), \tag{41}$$

$$I_4 \leq Ch\|\underline{\beta}\|_2 \leq Ch(\|g\|_{-1}+t\|g\|_0), \tag{42}$$

and

$$I_5 \leq Ch(t\|\underline{q}\|_1+\|\underline{q}\|_0) \leq Ch(\|g\|_{-1}+t\|g\|_0). \tag{43}$$

Combining Eqs. (38)–(43) and using Proposition 4, we get (35).

ii) To prove (36), we use a duality argument. Invoking Lemma 1 with $\underline{\delta}=\underline{\beta}-\underline{\beta}_h$, $(z,\underline{\phi},\underline{s})$ satisfies

$$B(z,\underline{\phi},\underline{s};v,\underline{\eta},\underline{r})=(\underline{\beta}-\underline{\beta}_h,\underline{\eta}), \tag{44}$$

for any $(v,\underline{\eta},\underline{r})$. Note that we also have

$$B_h(z,\underline{\phi},\underline{s};v,\underline{\eta},\underline{r})=(\underline{\beta}-\underline{\beta}_h,\underline{\eta}), \tag{45}$$

since (44) implies

$$\underline{\nabla}z-\underline{\phi}=\frac{t^2}{k}\underline{s}.$$

Letting $(v,\underline{\eta},\underline{r})=(w-w_h,\underline{\beta}-\underline{\beta}_h,\underline{q}-\underline{q}_h)$ in (45), we get

$$\begin{aligned} \|\underline{\beta}-\underline{\beta}_h\|_0^2 &= B_h(z,\underline{\phi},\underline{s};w-w_h,\underline{\beta}-\underline{\beta}_h,\underline{q}-\underline{q}_h) \\ &= B_h(z-\tilde{z},\underline{\phi}-\tilde{\underline{\phi}},\underline{s}-\tilde{\underline{s}};w-w_h,\underline{\beta}-\underline{\beta}_h,\underline{q}-\underline{q}_h) \end{aligned} \tag{46}$$

for any $(\tilde{z},\tilde{\underline{\phi}},\tilde{\underline{s}})\in W_h\times V_h\times Q_h$ by the consistency property (Proposition 1). We choose for $\tilde{z}$ and $\tilde{\underline{\phi}}$ the Lagrange interpolants of z and $\underline{\phi}$, and let $\tilde{\underline{s}}=\Pi_h(\underline{s})$. Then,

$$\begin{aligned} |||z-\tilde{z},\underline{\phi}-\tilde{\underline{\phi}},\underline{s}-\tilde{\underline{s}}||| &\leq Ch\left\{\|z\|_3+\|\underline{\phi}\|_2+\|\underline{s}\|_0+t\|\underline{s}\|_1\right\} \\ &\leq Ch\|\underline{\beta}-\underline{\beta}_h\|_0, \end{aligned} \tag{47}$$

using Lemma 1. Thus, applying Proposition 2 in (46), and using (47) and (35), we have

$$\begin{aligned}
\|\underline{\beta} - \underline{\beta}_h\|_0^2 &\leq C|||z - \tilde{z}, \underline{\phi} - \tilde{\underline{\phi}}, \underline{s} - \tilde{\underline{s}}||| \times |||w - w_h, \underline{\beta} - \underline{\beta}_h, \underline{q} - \underline{q}_h||| \\
&\leq Ch\|\underline{\beta} - \underline{\beta}_h\|_0 \times h(\|g\|_{-1} + t\|g\|_0),
\end{aligned}$$

so that (36) is proved.
iii) Finally, to obtain the last estimate, we write

$$\begin{aligned}
\|w - w_h\|_1 &\leq C\|\underline{\nabla}(w - w_h)\|_0 \\
&\leq C\left\{\|\underline{\nabla}(w - w_h) - (\underline{\beta} - \underline{\beta}_h)\|_0 + \|\underline{\beta} - \underline{\beta}_h\|_0\right\} \\
&\leq C\left\{(t + h)|||w - w_h, \underline{\beta} - \underline{\beta}_h|||_V + \|\underline{\beta} - \underline{\beta}_h\|_0\right\} \\
&\leq C(t + h)h(\|g\|_{-1} + t\|g\|_0),
\end{aligned}$$

from (35) and (36). This concludes the proof. ■

Remark 4: In estimate (37), we considered t and h as independent parameters. In practice, however, unless a plate is very thick (and then a three-dimensional analysis may well be preferable), one will seldom perform actual computations on cases where h is smaller than t, so that

$$t \leq h \tag{48}$$

seems a very reasonable assumption. When this holds (it is then sometimes said that the plate is "numerically thin", cf. [15]), estimate (37) becomes

$$\|w - w_h\|_1 \leq Ch^2. \tag{49}$$

Therefore, unless exceedingly fine meshes that violate (48) are used, the transverse displacement approximation behaves quadratically in the H^1-norm. Thus, although the discretization spaces used are unbalanced (P_2 for the displacement and P_1 for the rotations), all estimates are optimal, which is what we observed in the numerical examples. ■

Remark 5: Problem $(\mathcal{P}'_t)$ can also be considered as a penalized form of the limit problem (namely the Kirchhoff model):
$(\mathcal{P}'_0)$: *Find* $(w_0, \underline{\beta}_0, \underline{q}_0) \in W \times V \times Q$ *such that*

$$a(\underline{\beta}_0, \underline{\eta}) + (\underline{q}_0, \underline{\nabla} v - \underline{\eta}) + (\underline{r}, \underline{\nabla} w_0 - \underline{\beta}_0) = (g, v), \tag{50}$$

for any $(v, \underline{\eta}, \underline{r}) \in W \times V \times Q$.
Thus, we can use the scheme analysed above as an approximation of the solution of $(\mathcal{P}'_0)$. Then the term $t\|g\|_0$ appearing in (35)–(36) vanishes from the error estimates (since we consider the limit problem), but a consistency error is present. Namely, assuming $0 < \mu < 1$ and Ω convex, we obtain

$$|||w_0 - w_h, \underline{\beta}_0 - \underline{\beta}_h, \underline{q}_0 - \underline{q}_h||| \leq C(t + h)\|g\|_{-1}, \tag{51}$$

$$\|\underline{\beta}_0 - \underline{\beta}_h\|_0 + \|w_0 - w_h\|_1 \leq C(t + h)^2\|g\|_{-1}. \tag{52}$$

Therefore, if we now *set* $t = h$, the convergence of the scheme is again optimal for both the rotation and the deflection. Note further that the numerical results of Section 1, where we used the Kirchhoff solution as the reference, are rigorously justified by these error estimates. ■

5 Compared shear error estimates for the original scheme

In the previous section, we have proved that the proposed finite element scheme, compared with the original scheme introduced in [1, 7], provides improved error estimates for the transverse displacement with unchanged estimates for the rotation. Furthermore, as shown in Section 3, these theoretical results are fully confirmed by the convergence behaviours observed in actual computations. The purpose of the present section is to compare error estimates for the shear force. Those derived in [1, 7] are indeed expressed with abstract norms for which any direct comparison with our results is impossible. We will therefore carry out an analysis of the original scheme involving discrete norms of the type previously introduced.

We suppose that the values of interest for t lie in an interval $[0, t_0]$. We rewrite the original scheme as

$(\tilde{\mathcal{P}}'_{t,h})$: *Find* $(w_h, \underline{\beta}_h, \underline{q}_h) \in W_h \times V_h \times Q_h$ *such that*

$$\tilde{B}_h(w_h, \underline{\beta}_h, \underline{q}_h; v, \underline{\eta}, \underline{r}) = (g, v), \tag{53}$$

for any $(v, \underline{\eta}, \underline{r}) \in W_h \times V_h \times Q_h$,
with

$$\tilde{B}_h(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) \stackrel{\text{def}}{=} B(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) + \frac{\rho k}{t_0^2}(\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k}\underline{s}, \underline{\nabla} v - \underline{\eta} - \frac{t^2}{k}\underline{r}),$$

where ρ is a positive constant. Again, this discretization scheme is obviously consistent with the continuous problem.

Proposition 6 *Problem* $(\tilde{\mathcal{P}}'_{t,h})$ *is consistent, i.e. the solution* $(w, \underline{\beta}, \underline{q})$ *of* $(\mathcal{P}'_t)$ *satisfies*

$$\tilde{B}_h(w, \underline{\beta}, \underline{q}; v, \underline{\eta}, \underline{r}) = (g, v), \tag{54}$$

for any $(v, \underline{\eta}, \underline{r}) \in W \times V \times Q$.

Denoting by Γ_h the collection of all the edges in the mesh that are not on $\partial\Omega$, we define the new discrete norm for the shear for any $\underline{r} \in Q_h$ as follows

$$|||\underline{r}|||_{\tilde{Q}} \stackrel{\text{def}}{=} \left\{ t^2 \|\underline{r}\|_0^2 + \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{r}\|_{0,K}^2 + \sum_{E \in \Gamma_h} h_E \|[[\underline{r} \cdot \underline{n}]]\|_{0,E}^2 \right\}^{\frac{1}{2}},$$

where h_E denotes the length of each edge E, and $[[\cdot]]$ the jump of the quantity across the edge. For the rotation and displacement we use the H^1-norm, so that the new global norm is

$$|||v, \underline{\eta}, \underline{r}|||_O \stackrel{\text{def}}{=} \left\{ \|v, \underline{\eta}\|_1^2 + |||\underline{r}|||_Q^2 \right\}^{\frac{1}{2}}.$$

Using this norm, we directly establish the stability of $(\tilde{\mathcal{P}}'_{t,h})$.

Proposition 7 *Assuming $0 < \rho < 1$, formulation $\tilde{B}_h$ is stable for the norm $|||\cdot|||_O$, i.e. for any $(z, \underline{\phi}, \underline{s}) \in W_h \times V_h \times Q_h$, there exists $(v, \underline{\eta}, \underline{r}) \in W_h \times V_h \times Q_h$ such that*

$$|||v, \underline{\eta}, \underline{r}|||_O \le C|||z, \underline{\phi}, \underline{s}|||_O, \tag{55}$$

and

$$\tilde{B}_h(z, \underline{\phi}, \underline{s}; v, \underline{\eta}, \underline{r}) \ge C|||z, \underline{\phi}, \underline{s}|||_O^2. \tag{56}$$

Proof: In three steps, out of which the first two closely resemble those of the proof of Proposition 3.
i) Set $v_1 = z$, $\underline{\eta}_1 = \underline{\phi}$ and $\underline{r}_1 = -\underline{s}$. Then

$$|||v_1, \underline{\eta}_1, \underline{r}_1|||_O = |||z, \underline{\phi}, \underline{s}|||_O, \tag{57}$$

and

$$\begin{aligned} \tilde{B}_h(z, \underline{\phi}, \underline{s}; v_1, \underline{\eta}_1, \underline{r}_1) &= a(\underline{\phi}, \underline{\phi}) + \frac{\rho k}{t_0^2}\|\underline{\nabla} z - \underline{\phi}\|_0^2 + \frac{t^2}{k}(1 - \rho\frac{t^2}{t_0^2})\|\underline{s}\|_0^2 \\ &\ge C\left\{\|z, \underline{\phi}\|_1^2 + t^2\|\underline{s}\|_0^2\right\}, \end{aligned} \tag{58}$$

since $0 < \rho < 1$ and $0 < t \le t_0$.
ii) Next, we define $v_2 = 0$, $\underline{r}_2 = \underline{0}$ and $\underline{\eta}_2 = -h_K^2 b_h \underline{s}$. Like in the proof of Proposition 3 we have

$$|||v_2, \underline{\eta}_2, \underline{r}_2|||_O^2 = \|\underline{\eta}_2\|_1^2 \le C \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{s}\|_{0,K}^2 \le C|||z, \underline{\phi}, \underline{s}|||_0. \tag{59}$$

Furthermore,

$$\tilde{B}_h(z, \underline{\phi}, \underline{s}; v_2, \underline{\eta}_2, \underline{r}_2) = a(\underline{\phi}, \underline{\eta}_2) - (\underline{s}, \underline{\eta}_2) - \frac{\rho k}{t_0^2}(\underline{\nabla} z - \underline{\phi} - \frac{t^2}{k}\underline{s}, \underline{\eta}_2).$$

Again, by a scaling argument we obtain

$$-(\underline{s}, \underline{\eta}_2) = \sum_{K \in \mathcal{C}_h} h_K^2 (\underline{s}, b_h \underline{s})_K \ge C \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{s}\|_{0,K}^2,$$

and the other two terms are easily bounded, so that we infer

$$\begin{aligned} \tilde{B}_h(z, \underline{\phi}, \underline{s}; v_2, \underline{\eta}_2, \underline{r}_2) &\ge C \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{s}\|_{0,K}^2 - C\left[\|z, \underline{\phi}\|_1^2 + t^2\|\underline{s}\|_0^2\right]^{\frac{1}{2}} \|\underline{\eta}_2\|_1 \\ &\ge C\left\{ \sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{s}\|_{0,K}^2 - \left[\|z, \underline{\phi}\|_1^2 + t^2\|\underline{s}\|_0^2\right] \right\}, \end{aligned} \tag{60}$$

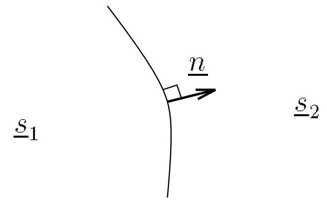

Figure 3: $[[\underline{s}\cdot\underline{n}]] \stackrel{\rm def}{=} (\underline{s}_2-\underline{s}_1)\cdot\underline{n}$

using the intermediate bound on $\|\underline{\eta}_2\|_1$ featured in (59).
iii) To obtain the jump term of the shear norm, we finally choose $\underline{\eta}_3=\underline{0}$, $\underline{r}_3=\underline{0}$ and set the degrees of freedom of v_3 so that $v_3=0$ at all vertices and $v_3=-h_E[[\underline{s}\cdot\underline{n}]]$ at the midpoint of each edge E in Γ_h (see Figure 3 for the sign of the jump). Thus,

$$v_3\,|_E = -h_E[[\underline{s}\cdot\underline{n}]]b_E,$$

where b_E is the quadratic "1D-bubble function" along the edge. By an inverse inequality and scaling arguments, we infer

$$\begin{aligned}
|||v_3,\underline{\eta}_3,\underline{r}_3|||_O^2 = \|v_3\|_1^2 \quad &\le C\sum_{K\in\mathcal{C}_h} h_K^{-2}\|v_3\|_{0,K}^2\\
&\le C\sum_{E\in\Gamma_h} h_E^{-1}\|v_3\|_{0,E}^2\\
&\le C\sum_{E\in\Gamma_h} h_E\|[[\underline{s}\cdot\underline{n}]]\|_{0,E}^2 &(61)\\
&\le C|||z,\underline{\phi},\underline{s}|||_0. &(62)
\end{aligned}$$

On the other hand,

$$\tilde{B}_h(z,\underline{\phi},\underline{s};v_3,\underline{\eta}_3,\underline{r}_3) = (\underline{s},\underline{\nabla}v_3)+\frac{\rho k}{t_0^2}(\underline{\nabla}z-\underline{\phi}-\frac{t^2}{k}\underline{s},\underline{\nabla}v_3),$$

and an integration by part within each element yields

$$(\underline{s},\underline{\nabla}v_3) = -\sum_{E\in\Gamma_h}\int_E[[\underline{s}\cdot\underline{n}]]v_3d\tau = \sum_{E\in\Gamma_h}h_E[[\underline{s}\cdot\underline{n}]]^2\int_E b_E d\tau = \frac{2}{3}\sum_{E\in\Gamma_h}h_E\|[[\underline{s}\cdot\underline{n}]]\|_{0,E}^2.$$

Using this equation and (61), we obtain

$$\begin{aligned}
\tilde{B}_h(z,\underline{\phi},\underline{s};v_3,\underline{\eta}_3,\underline{r}_3) \quad &\ge C\sum_{E\in\Gamma_h}h_E\|[[\underline{s}\cdot\underline{n}]]\|_{0,E}^2-\left[\|z,\underline{\phi}\|_1^2+t^2\|\underline{s}\|_0^2\right]^{\frac{1}{2}}\|v_3\|_1\\
&\ge C\left\{\sum_{E\in\Gamma_h}h_E\|[[\underline{s}\cdot\underline{n}]]\|_{0,E}^2-\left[\|z,\underline{\phi}\|_1^2+t^2\|\underline{s}\|_0^2\right]\right\}. &(63)
\end{aligned}$$

To conclude the proof, it suffices to use appropriate convex combinations of the sets $(v_i,\underline{\eta}_i,\underline{r}_i)$, $i=1,2,3$.
■

We proceed to derive a priori error estimates in the norms that we defined.

Proposition 8 *Assume $0 < \rho < 1$, Ω convex, and $g \in L^2(\Omega)$. Then problem $(\tilde{\mathcal{P}}'_{t,h})$ has a unique solution $(w_h, \underline{\beta}_h, \underline{q}_h)$. Moreover this solution satisfies the following error estimate*

$$|||w - w_h, \underline{\beta} - \underline{\beta}_h, \underline{q} - \underline{q}_h|||_O \leq Ch\|g\|_0. \tag{64}$$

Proof: The existence and uniqueness follow from the stability property. Regarding the regularity of the exact solution, we will use an estimate deduced from [2, Appendix] and slightly stronger than (28)-(29), namely

$$\|w\|_2 + \|\underline{\beta}\|_2 + \|\underline{q}\|_{H(\text{div})} + t\|\underline{q}\|_1 \leq C\|g\|_0. \tag{65}$$

Note that, since $\underline{q} \in H(\text{div};\Omega)$, the jump terms in $|||\underline{q}|||_{\tilde{Q}}$ are well-defined and equal zero (cf. e.g. [6]).

We start by choosing interpolants for the exact solutions. Set $\tilde{w}$ and $\tilde{\underline{\beta}}$ as the P_1-Lagrange interpolants of w and $\underline{\beta}$ respectively. We have the standard interpolation estimates

$$\|w - \tilde{w}\|_1 + \|\underline{\beta} - \tilde{\underline{\beta}}\|_1 \leq Ch\|g\|_0, \tag{66}$$

$$\left\{\sum_{K\in\mathcal{C}_h} h_K^{-2}\left(\|w - \tilde{w}\|_{0,K}^2 + \|\underline{\beta} - \tilde{\underline{\beta}}\|_{0,K}^2\right)\right\}^{\frac{1}{2}} \leq Ch\|g\|_0. \tag{67}$$

For $\underline{q}$, we need to consider several non-standard norms. In addition to the above-mentioned discrete norm, we will use the $H^{-1}(\text{div})$-norm, for which one possible definition is (see [6])

$$\|\underline{s}\|_{H^{-1}(\text{div})} = \sup_{(v,\underline{\eta})\in H_0^1(\Omega)\times\left[H_0^1(\Omega)\right]^2} \frac{(\underline{s}, \underline{\nabla} v - \underline{\eta})}{\|v, \underline{\eta}\|_1}.$$

Note that this expression is well-defined for instance if $\underline{s} \in L^2(\Omega)$, which will be the case for our purposes. We now state and prove the required interpolation result for the shear in a separate lemma.

Lemma 3 *Suppose that $\underline{s} \in \left[H^1(\Omega)\right]^2$. Then, there exists $\tilde{\underline{s}} \in Q_h$ such that*

$$\|\underline{s} - \tilde{\underline{s}}\|_{H^{-1}(\text{div})} \leq Ch\|\underline{s}\|_{H(\text{div})}, \tag{68}$$

$$\left\{\sum_{K\in\mathcal{C}_h} h_K^2\|\underline{s} - \tilde{\underline{s}}\|_{0,K}^2 + \sum_{E\in\Gamma_h} h_E\|[[(\underline{s} - \tilde{\underline{s}})\cdot\underline{n}]]\|_{0,E}^2\right\}^{\frac{1}{2}} \leq Ch\|\underline{s}\|_{H(\text{div})}, \tag{69}$$

$$\|\underline{s} - \tilde{\underline{s}}\|_0 \leq Ch\|\underline{s}\|_1. \tag{70}$$

Proof: We define the interpolant exactly like in [7, Prop. 4.3]. Using the Helmholtz decomposition theorem (see e.g. [6]), we can find $\psi_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\psi_2 \in H^2(\Omega)$ such that

$$\underline{s} = \underline{\nabla}\psi_1 + (\underline{\nabla}\psi_2)^\perp, \tag{71}$$

$$\|\psi_1\|_2 + \|\psi_2\|_1 \leq C\|\underline{s}\|_{H(\text{div})}, \tag{72}$$

$$\|\psi_2\|_2 \leq C\|\underline{s}\|_1. \tag{73}$$

We set $\tilde{\psi}_1$ as the standard P_1-Lagrange interpolant of ψ_1. For ψ_2, we use the Clément operator to construct $\tilde{\psi}_2$ as a P_1-interpolant which satisfies [11]

$$|\psi_2 - \tilde{\psi}_2|_k \leq Ch^{l-k}|\psi_2|_l \qquad l = 1, 2,\ 0 \leq k \leq l. \tag{74}$$

For interpolating $\underline{s}$, we now set

$$\underline{\tilde{s}} \stackrel{\text{def}}{=} \underline{\nabla}\tilde{\psi}_1 + (\underline{\nabla}\tilde{\psi}_2)^\perp.$$

Then $\underline{\tilde{s}}$ is in Q_h. Estimate (70) follows from standard interpolation properties. (68) is proved in [7, Prop. 4.3]. Furthermore, using standard interpolation estimates for ψ_1, (74) for ψ_2 and (72), we have

$$\begin{aligned}
\sum_{K \in \mathcal{C}_h} h_K^2 \|\underline{s} - \underline{\tilde{s}}\|_{0,K}^2 &\leq Ch^2 \left(|\psi_1 - \tilde{\psi}_1|_1^2 + |\psi_2 - \tilde{\psi}_2|_1^2\right) \\
&\leq Ch^2 \left(h^2|\psi_1|_2^2 + |\psi_2|_1^2\right) \\
&\leq Ch^2 \|s\|_{H(\mathrm{div})}^2,
\end{aligned}$$

It remains to study the jump terms. On any edge E, we have

$$[[(\underline{s} - \underline{\tilde{s}}) \cdot \underline{n}]] = [[(\underline{\nabla}\psi_1 - \underline{\nabla}\tilde{\psi}_1) \cdot \underline{n}]] + [[((\underline{\nabla}\psi_2)^\perp - (\underline{\nabla}\tilde{\psi}_2)^\perp) \cdot \underline{n}]],$$

wherein $[[(\underline{\nabla}\psi_2)^\perp \cdot \underline{n}]] = 0$ because $\psi_2 \in H^2(\Omega)$, and

$$[[(\underline{\nabla}\tilde{\psi}_2)^\perp \cdot \underline{n}]] = [[\frac{\partial \tilde{\psi}_2}{\partial \tau}]] = 0,$$

since $\tilde{\psi}_2$ is uniquely defined along the edge. Finally, from a scaling argument we deduce

$$\begin{aligned}
\sum_{E \in \Gamma_h} h_E \|[[(\underline{\nabla}\psi_1 - \underline{\nabla}\tilde{\psi}_1) \cdot \underline{n}]]\|_{0,E}^2 &\leq C \sum_{K \in \mathcal{C}_h} \left(h_K^2 |\psi_1|_{2,K}^2 + |\psi_1 - \tilde{\psi}_1|_{1,K}^2\right) \\
&\leq Ch^2 \|\psi_1\|_2^2 \\
&\leq Ch^2 \|\underline{s}\|_{H(\mathrm{div})}^2
\end{aligned}$$

and (69) follows. ∎

Proof of Proposition 8 (continued): We define $\underline{\tilde{q}}$ as an interpolant of $\underline{q}$ of the type provided by Lemma 3. It therefore satisfies

$$|||\underline{q} - \underline{\tilde{q}}|||_{\tilde{Q}} + \|\underline{q} - \underline{\tilde{q}}\|_{H^{-1}(\mathrm{div})} \leq Ch \left(t\|\underline{q}\|_1 + \|\underline{q}\|_{H(\mathrm{div})}\right) \leq Ch\|g\|_0, \tag{75}$$

taking (65) into account.

From the stability property, we know that we can find $(v, \underline{\eta}, \underline{r}) \in W_h \times V_h \times Q_h$ such that

$$|||v, \underline{\eta}, \underline{r}|||_O \leq C|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}|||_O, \tag{76}$$

$$\tilde{B}_h(w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}_h; v, \underline{\eta}, \underline{r}) \geq C|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}|||_O^2. \tag{77}$$

Combining (77) and the consistency property, we infer

$$|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}_h|||_O^2 \leq C\tilde{B}_h(w - \tilde{w}, \underline{\beta} - \underline{\tilde{\beta}}, \underline{q} - \underline{\tilde{q}}; v, \underline{\eta}, \underline{r}). \tag{78}$$

All the terms that the right hand side decomposes into are easily bounded by the Schwarz inequality, except the mixed ones. For $(\underline{q} - \underline{\tilde{q}}, \underline{\nabla} v - \underline{\eta})$, we use the definition of the $H^{-1}(\mathrm{div})$-norm and Eqs. (75)-(76). We get

$$(\underline{q}-\underline{\tilde{q}}, \underline{\nabla} v-\underline{\eta}) \leq \|\underline{q}-\underline{\tilde{q}}\|_{H^{-1}(\mathrm{div})}\|v, \underline{\eta}\|_1 \leq Ch\|g\|_0|||w_h-\tilde{w}, \underline{\beta}_h-\underline{\tilde{\beta}}, \underline{q}_h-\underline{\tilde{q}}|||_O. \tag{79}$$

We split the other mixed term $(\underline{r}, \underline{\nabla}(w - \tilde{w}) - (\underline{\beta} - \underline{\tilde{\beta}}))$ in two as follows

$$\begin{aligned}|(\underline{r}, \underline{\beta} - \underline{\tilde{\beta}})| &\leq C\left(\sum_{K\in\mathcal{C}_h} h_K^2\|\underline{r}\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{C}_h} h_K^{-2}\|\underline{\beta} - \underline{\tilde{\beta}}\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\leq Ch\|g\|_0|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}|||_O,\end{aligned} \tag{80}$$

from (67) and (76). The remaining part is treated by an integration by parts (over each element) and scaling arguments. We get

$$\begin{aligned}&|(\underline{r}, \underline{\nabla}(w - \tilde{w}))| \\ &\quad= |\sum_{E\in\Gamma_h}\int_E [[\underline{r}\cdot\underline{n}]](w - \tilde{w})d\tau| \\ &\quad\leq \left(\sum_{E\in\Gamma_h} h_E\|[[\underline{r}\cdot\underline{n}]]\|_{0,E}^2\right)^{\frac{1}{2}}\left(\sum_{E\in\Gamma_h} h_E^{-1}\|w - \tilde{w}\|_{0,E}^2\right)^{\frac{1}{2}} \\ &\quad\leq C\left(\sum_{E\in\Gamma_h} h_E\|[[\underline{r}\cdot\underline{n}]]\|_{0,E}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{C}_h}\left[|w - \tilde{w}|_{1,K}^2 + h_K^{-2}\|w - \tilde{w}\|_{0,K}^2\right]\right)^{\frac{1}{2}} \\ &\quad\leq Ch\|g\|_0|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}|||_O,\end{aligned} \tag{81}$$

from (66), (67) and (76). Finally, we infer from (78) that

$$|||w_h - \tilde{w}, \underline{\beta}_h - \underline{\tilde{\beta}}, \underline{q}_h - \underline{\tilde{q}}_h|||_O \leq Ch\|g\|_0, \tag{82}$$

and a triangle inequality concludes the proof. ∎

Remark 6: This analysis of the scheme originally proposed in [1, 7] shows that, although it is detrimental to the convergence of the transverse displacement which is under-optimal in theory *and* in practice, it appears to provide stronger error estimates for the shear with an additional control on the jumps between elements. This is consistent with the numerical results that showed slightly better convergence behaviours for the shear in L^2-norm when "α was fixed", although we did not obtain theoretical error estimates in this norm. Whether or not it is worth relinquishing some significant accuracy on the displacement in order to gain on the shear jumps is probably a question of engineering judgement. ∎

Remark 7: We can use the estimates of Proposition 8 to obtain an error estimate for the shear force in the $H^{-1}(\text{div}\,)$ norm by an argument inspired from the "Pitkäranta-Verfürth trick" (see [14, 18], and also [8] for an application in a similar case). We thus get

$$\|\underline{q} - \underline{q}_h\|_{H^{-1}(\text{div})} \leq Ch\|g\|_0, \tag{83}$$

which confirms the estimate obtained in [7]. ■

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