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A SIMPLE LINEAR NONCONFORMING SHELL ELEMENT

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Abstract. *A new low order facet type triangular shell finite element is presented. The element is combined from the Arnold-Falk plate bending element and the membrane element based on the noncorming element of Crouzeix-Raviart with conforming drilling rotation. Formulation of the membrane element is based on the variational principle of Hughes and Brezzi employing independent rotation field. Interpolation for the in-plane translational components is nonconforming and continuous only at the midpoints of the element edges, while conforming linear interpolation is used for the drilling rotation. It will be shown that the utilization of the drilling rotation supresses the mechanism which is present in the Crouzeix-Raviart element, applied to plane stress or plain strain problems, for certain boundary conditions. In this element all three rotation components appear at corner nodes and translational components at the midpoints of the element edges resulting in 18 degrees of freedom.*

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1 Introduction

In-plane rotational degrees of freedom, "drilling degrees of freedom", are particularly convenient in the analysis of shells. Typical shell elements have three translational and two rotational degrees of freedom at a node. This results in many difficulties of model construction and of programming and numerical ill-conditioning for certain types of element assemblages. Thus the presence of all three rotations at a node is advantageous from practical point of view.

Early attempts to construct membrane elements with drill rotations were unsuccessful. Allman [1] and Bergan and Felippa [5] formulated successful membrane elements with drill rotations by using special interpolation patterns. Allman derived the interpolation scheme by choosing the normal and tangential displacement components to be quadratic and linear functions, respectively. The only deficiences of this remarkably simple element are the existing zero energy mode and that the vertex rotations are not true rotations. These shortcomings are removed in a later development by Allman [2], at the expense of having a cubic displacement field. Bergan and Felippa used the so-called free formulation of the finite element method to derive their element.

Hughes and Brezzi [12] presented a simple variational formulation, which employs independent rotation field and is also stable in the discrete case. They also proved that elements based on their formulation are convergent for all standard interpolations including equal order interpolation for displacements and rotation. Numerical experiments reported by Hughes *et al.* [13] confirm the *a priori* theoretical convergence estimates. They compared linear and quadratic triangular and quadrilateral elements and also bilinear element with incompatible modes. Incompatible modes significantly improve the coarse mesh accuracy of the bilinear element. On the other hand, the static condensation needed to eliminate the nodeless generalized displacements is awkward, especially in nonlinear problems.

Ibrahimbegović *et al.* [15] amended the displacement interpolation of a four node quadrilateral element by the Allman type quadratic modes in order to improve the coarse mesh accuracy of the element. This considerably improves the bending behaviour of the element. They also added a hierarchical bubble interpolation mode to the displacement field.

In ref. [16] a membrane element with drilling rotation was introduced which is complementary to the present element with respect to interpolation scheme. This element has nonconfrming interpolation for all rotational quantities, thus resulting in an element with three degrees of freedom at each of its six node. Translational displacements are located at vertex nodes while rotations appear at the midside nodes.

In the present study, a new facet type triangular shell element which incorporates only linear interpolation is introduced. The membrane part of the element is based on Hughes-Brezzi type formulation with conforming linear interpolation for the drilling rotation, while the translational components are interpolated by nonconforming linear functions, continuous only at the midpoints of the element edges. Thus, translational and rotational components appear at different nodes.

2 Element formulation

2.1 Membrane element

Crouzeix and Raviart [7] showed than an optimally convergent method for the Stokes equations is obtained by using a triangular mesh with a piecewise constant approximation for the pressure combined with linear nonconforming approximations for both components of velocity. The corresponding plain strain element, with the pressure locally eliminated, would give a displacement method with nonconforming approximations for both components of displacements. However, it is easy to see (cf. [11], pp. 250–251) that spurious mechanisms can occur with this method and hence it is not of practical use.

In this paper a linear plain stress element using nonconforming elements for both displacement components. Stability of the method is obtained by adding conforming drill rotation. This is done within the framework of the Hughes-Brezzi formulation. The element is specially designed to be used in a shell element formulation containing all three rotation components.

The variational equation of the Hughes-Brezzi formulation in the case of Dirichlet boundary conditions has the form

$$\int_\Omega \delta\boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} d\Omega + \gamma \int_\Omega (\mathrm{skew}\nabla\delta\boldsymbol{u} - \delta\boldsymbol{\psi})(\mathrm{skew}\nabla\boldsymbol{u} - \boldsymbol{\psi}) d\Omega = \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{f} d\Omega \tag{1}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor, i.e. the symmetric part of the displacement gradient, $\boldsymbol{\varepsilon} = \mathrm{symm}\,\nabla\boldsymbol{u}$ and $\boldsymbol{\psi}$ is a skew symmetric tensor representing the in-plane rotation. The fourth order tensor $\boldsymbol{C}$ contains the material parameters and the body force vector is denoted by $\boldsymbol{f}$.

An appropriate value of the regularizing penalty parameter γ is chosen in accordance with the ellipticity condition. For the isotropic case the value $\gamma = \mu$ (shear modulus) seems to balance the terms in the estimate and thus seems reasonable [14]. However, the method is insensitive to the choise of the penalty parameter in the region $0 < \gamma \le \mu$ and recent results for conforming elements suggest to use values $10^{-3} \le \gamma/\mu \le 10^{-2}$, which minimizes the condition number of the global stiffness matrix.

The interpolation scheme for the membrane element can be written as

$$u = \sum_{i=n+1}^{2n} N^{\mathrm{nc}}_{i-n} u_i, \quad v = \sum_{i=n+1}^{2n} N^{\mathrm{nc}}_{i-n} v_i, \quad \psi = \sum_{i=1}^{n} N_i \psi_i, \tag{2}$$

where n is the number of vertex nodes ($n = 3$) and N_i's are the standard linear interpolation functions, which can be expressed simply by using the area-coordinates L_i as $N_i = L_i$.

For triangular element the nonconforming interpolation at edge i (connecting nodes i and $i+$) has the form

$$N^{\mathrm{nc}}_i = L_i + L_{i+} - L_{i-}. \tag{3}$$

To improve the coarse mesh accuracy of the conventional elements with conforming interpolation, the Allman-type in-plane qudratic displacement interpolation is linked to the drill-rotation. This can be done also within the present formulation, however, the effect of such addition has found to be insignificant.

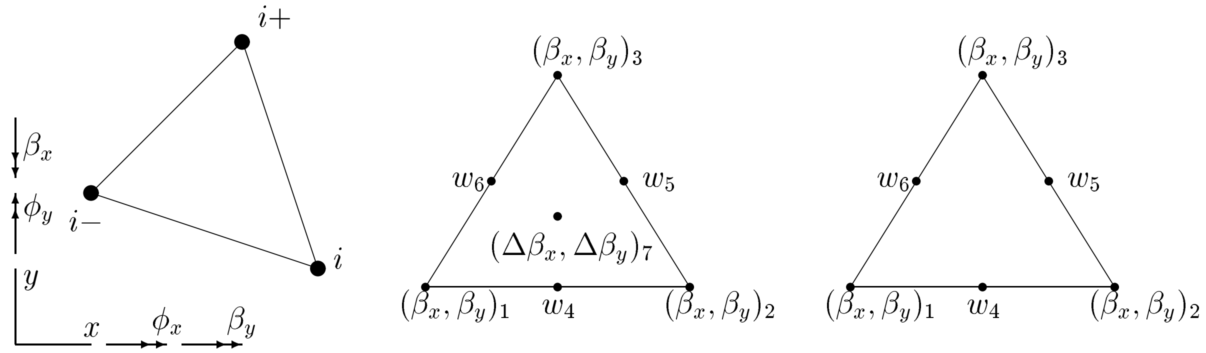


Figure 1: Arnold-Falk element and its modification.

2.2 Plate bending element

Perhaps the simplest finite element for the Reissner-Mindlin plate model for which the optimal order of convergence has rigorously been proven, is a method by Arnold and Falk [4]. A modification of this plate-bending element, which is even simpler to implement and computationally more advantageous, is described in refs. [8, 10]. Identical error estimates to those of the original method has been proven in these papers.

In the Arnold Falk element the linear nonconforming interpolation for the deflection is used (see fig. 1), i.e.

$$w = N_1^{\mathrm{nc}} w_4 + N_2^{\mathrm{nc}} w_5 + N_3^{\mathrm{nc}} w_6. \tag{4}$$

For the rotations linear conforming interpolation is used. In the original element the cubic bubble is augmented to the rotations,

$$\beta_i = L_1\beta_{i1} + L_2\beta_{i2} + L_3\beta_{i3} + 27L_1L_2L_3\Delta\beta_{i7}, \tag{5}$$

where $i = x$ or y. Shear strains are computed as averages

$$\gamma_{iz} = w_{,i} - \bar{\beta}_i, \tag{6}$$

where

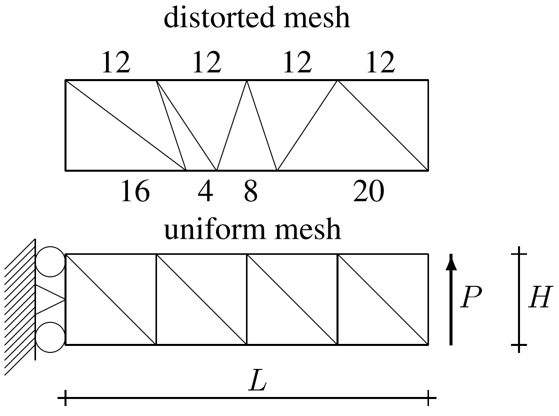
$$\bar{\beta}_i = \frac{1}{\mathrm{area}(\Omega^{(e)})}\int_{\Omega^{(e)}} \beta_i d\Omega.$$

In the modified element, the bubble function is not needed, but the shear forces are computed from

$$Q_i = \frac{k\mu t}{1+\alpha(h/t)^2}(w_{,i} - \bar{\beta}_i), \tag{7}$$

where α is a positive stabilization factor, t is the thickness of the plate and h is a characteristic length of an element, i.e. the length of the longest edge and k is the shear correction factor.

Combining the nonconforming membrane element described in the previous section with a nonconforming Arnold-Falk plate bending element, results in a facet shell element where the translational degrees of freedom are located at the midside nodes and the rotational degrees of freedom at the vertex nodes. This is just the opposite to the shell element introduced in ref. [16].



$E = 30000$, ν = 0.25, P = 40, L = 48, H = 12

Figure 2: Short cantilever beam.

3 Examples

3.1 A cantilever beam

A shear loaded cantilever beam is a common test case for membrane elements. The problem definition is shown in fig. 2. Computed tip displacements are tabulated in Table 1 for some triangular elements. Only a minimum number of restraints is imposed, hence no drilling degrees of freedom are restrained. The penalty parameter γ is chosen to be equal to the shear modulus in calculations shown in Table 1. The reference solution for the tip displacement with the properties shown in fig.2 is defined in ref. [15] as

$$v = \frac{PL^3}{3EI} + \frac{(4+5\nu)PL}{2EH} = 0.3553. \tag{8}$$

Behaviour of the nonconforming elements are comparable to the corresponding conforming elements and is similar to those reported by Hughes *et al.* [14]. As it can be seen from the figures in Table 1, the present nonconforming membrane element is the most flexible of all of the tested triangular elements. This can also be seen from fig. 3, where the normalized tip deflection is shown as a function of the regularizing γ-parameter.

The spectral condition number is also shown as a function of the penalty parameter in fig. 4. It seems that the range $10^{-4} < \gamma/\mu < 10^2$ can be used without any danger of deterioration of numerical conditioning of the equation system. However, the present nonconforming element will be too flexible in the range $\gamma/\mu < 10^{-2}$, thus, a value between the range $10^{-2} < \gamma/\mu < 10^2$ for the regularizing γ-parameter is recommended.

3.2 A pinched cylinder

The pinched cylinder with rigid end diaphragms is frequently analyzed test case to shell elements, see fig. 5. One octant of the shell is modelled and the normalized displacement under

Table 1: Short cantilever beam, tip deflections.

	uniform meshes				distorted
element	4 x 1	4 x 2	8 x 2	16 x 4	4 x 1
T3	0.0934	-	0.1977	0.2982	0.0900
T3D3	0.0851	-	0.1872	0.2882	0.0819
T3D3-A	0.2104	-	0.3170	0.3465	0.2020
T3ND3	0.0855	-	0.1842	0.2850	0.0822
NT3D3	-	0.4222	0.4649	0.3780	

Notations:

T3 = standard conforming linear triangle

T3D3 = conforming linear triangle with drilling rot

T3D3-A = T3D3 amended with Allman type displ. field

T3ND3 = element from ref. [16]

NT3D3 = present element

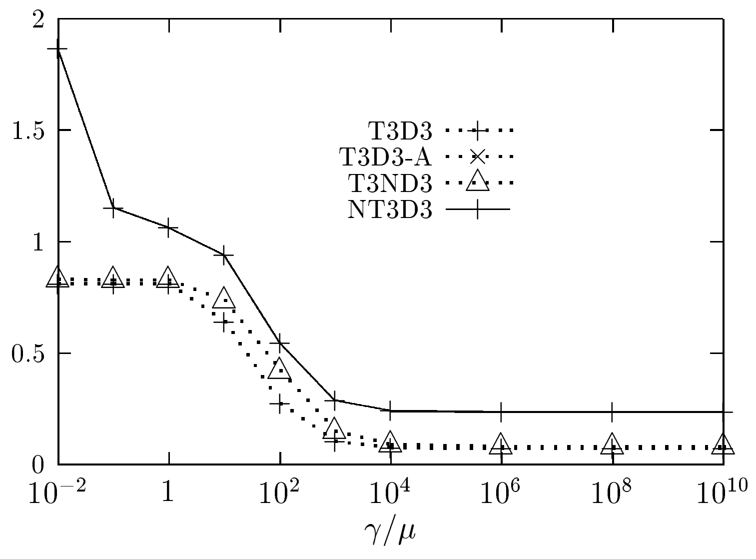


Figure 3: Normalized tip deflection as a function of γ. Uniform 16×4 mesh.

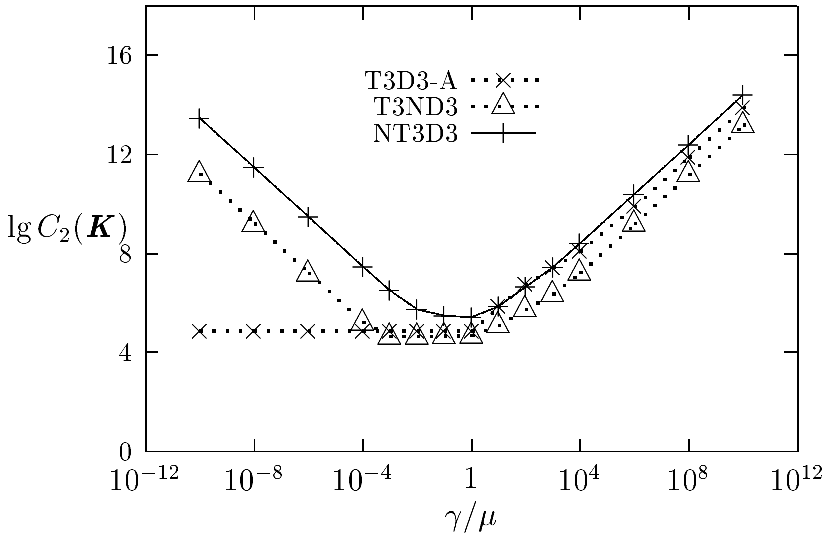


Figure 4: Spectral condition number of the stiffness matrix as a function of γ. Uniform 16×4 mesh.

the point loads is recorded in Table 2, where also some results from refs.[3], [6], [9] are shown. The reference displacement is [9]:

$$w_c = 164.24 \frac{P}{Et} = 1.82489 \times 10^{-5}. \tag{9}$$

The stabilization parameter used, is selected according to the results of ref.[17] and it is shown in Table 2. It can be seen that the displacement value is influenced of the choise of the parameter. However, one should not concentrate too much on the accuracy of displacements, which usually is good enough. As shown numerically by Lyly *et al.* [17] the moments are practically unchanged when the stabilization parameter is varied, while the shear forces become better when a larger value is chosen.

For the present nonconforming element the mesh used in computations, shown in Table 2, is not optimal since at crossing point of the symmetry lines, the corresponding node has no translational degrees of freedom. The tabulated results are computed by using the element's nonconforming interpolation functions.

4 Concluding remarks

A simple facet type linear shell element is introduced. Preliminary numerical tests show that the element performs reasonable well in comparison to other existing triangular shell elements. Due to the presence of the midside nodes, the number of unknowns will be doubled in comparison to conforming element with similar element division. However, further studies has to be done to asses the performance of the proposed element.

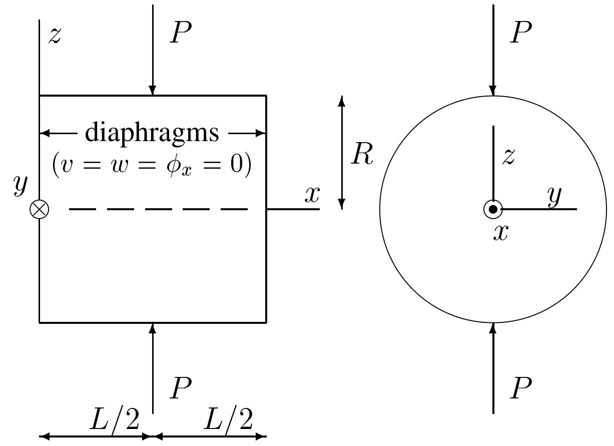


Figure 5: Pinched cylinder. ($L = 600, R = 300, t = 3, E = 3 \times 10^6, \nu = 0.3, P = 1$)

Table 2: Pinched cylinder, normalized radial displacement under loads.

		uniform meshes			
element	α	4×4	8×8	16×16	32×32
QD	0.4	0.6810	1.0416	1.1203	1.0584
QD	0.1	0.4863	0.8445	0.9807	1.0064
TD	0.4	0.6880	1.0384	1.1042	1.0528
TND	0.4	0.6591	1.4347	1.4489	1.2221
TND	0	0.6069	1.2310	1.2881	1.1438
NTD	0.1	0.6538	0.7627	0.8983	0.9665
NTD	0.4	1.1769	0.9912	1.0059	1.0114
[6]	-	0.535	0.857	0.980	-
[3]	-	0.599	0.909	0.973	-
		uniform meshes			
		5×5	10×10	20×20	
[9]	-	0.51	0.83	0.96	-

Notations:

TD	linear triangle with drill rot. + stab. MITC3
QD	bilinear quad with drill rot. + stab. MITC4
TND	triangular element from ref. [16]
QND	quadrilateral element from ref. [16]
NTD	present element

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