SIAM J. NUMER. ANAL.
Vol. 34, No. 2, pp. 544–568, April 1997
© 1997 Society for Industrial and Applied Mathematics
008

AN *hp* ERROR ANALYSIS OF MITC PLATE ELEMENTS*

ROLF STENBERG† AND MANIL SURI‡

Abstract. We give an *hp* error analysis of several rectangular families of finite elements for the Reissner–Mindlin plate bending equations. We consider both the original MITC families [K. J. Bathe, F. Brezzi, and M. Fortin, *Internat. J. Numer. Methods Engrg.*, 28 (1989), pp. 1787–1801] and some new ones introduced in this paper. For the deflection and rotation we give error estimates which are optimal with respect to the mesh size h and optimal up to $\mathcal{O}(k^{\varepsilon})$, $\varepsilon > 0$ arbitrary, with respect to the polynomial degree k. We also obtain estimates for the error in the shear force, calculated via two different methods. Our analysis utilizes some recent results of ours for the mixed method for the Stokes problem, as well as *hp* interpolation estimates for mixed methods for second-order elliptic equations. In this regard, we derive new *hp* results in this paper for the Brezzi–Douglas–Fortin–Marini spaces and improve upon previous estimates for the Brezzi–Douglas–Marini spaces.

Key words. *hp*, mixed method, Reissner–Mindlin plates

AMS subject classification. 65N30

PII. S0036142994278486

1. Introduction.

The Reissner–Mindlin (RM) plate model is often used as a two-dimensional model for three-dimensional plates that are "moderately thick." This is in contrast to "thin" plates, for which the Kirchhoff (KI) plate model is sometimes used instead. The latter is the asymptotic limit of the three-dimensional equations as the thickness $t \to 0$. For plates where the thickness t is small but perhaps not in the asymptotic range, the RM model may provide a more accurate approximation to the three-dimensional equations, with the solution incorporating features such as boundary layers, which do not appear in the asymptotic KI solution.

In terms of numerical approximation by the finite element (FE) method, the second-order RM system offers an advantage over the fourth-order KI one: whereas $C^{(1)}$ continuity is required of the FE spaces in the KI case, the RM model can be discretized using functions that are only $C^{(0)}$ continuous. Unfortunately, for values of t close to zero, this freedom breaks down, since the RM equations tend to the KI ones. As a result, the phenomenon of *shear locking* may be observed, due to the imposition of the *Kirchhoff constraint* on the FE subspaces, which effectively limits the approximability that may be achieved.

Two methods are commonly used to reduce or overcome the effects of locking. The first idea is to modify the variational form so that the limiting Kirchhoff constraint is only enforced in a weaker sense. This can usually be implemented by reformulating the problem as a *mixed method* that contains the shear force as an independent unknown. Such *reduced constraint* methods have been proposed and analyzed in, e.g., [2], [7], [12], [18] (see also the references in [11]). These methods are based on the h version, where the mesh size h is decreased, and a fixed (low) polynomial degree k is used.

The second idea is to use sufficiently *high-order* FEs, so that the Kirchhoff constraint can be enforced *exactly*, without loss in the approximability. The p and *hp*

*Received by the editors December 9, 1994; accepted for publication (in revised form) May 30, 1995. This research was supported in part by Air Force Office of Scientific Research, Air Force Systems Command, USAF, grant F49620-92-J-0100.

http://www.siam.org/journals/sinum/34-2/27848.html

†Institut für Mathematik und Geometrie, Universität Innsbruck, Technikerstrasse 13, A-6020 Innsbruck, Austria (stenberg@mat1.uibk.ac.at).

‡Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21250 (suri@umbc2.umbc.edu).

versions of the FE method, where the polynomial degree k can be increased [3], are obvious choices for this type of *exact constraint* approach. In this case, the standard variational form is used without modification. In [24], it is shown that the p version is free of locking when the error is measured in the energy norm (the same is also shown for certain hp and high-order h version FE methods). Computational results illustrating these conclusions are presented in [21].

The idea of this paper is to analyze the *combination* of the above two approaches, i.e., the hp mixed method. Such a method could be interpreted in two ways—an implementation of an h version family of mixed elements, where elements of different degrees can be chosen as needed from the family, or a p (or hp) implementation, where instead of the standard variational form, a modified one is used to reduce the effect of the Kirchhoff constraint. For our analysis here, we consider quadrilateral MITC-type elements, both the original ones from [5, 7], as well as new families introduced in section 3. Elements of this type have been shown to be locking-free with respect to the h version (see [5, 7, 12, 18]) and perform excellently from a computational standpoint (see [6, 8, 15]). However, the error analysis so far has only investigated the effect of letting $h \to 0$.

Here, we give error estimates for the hp version, i.e., in terms of both h and k. We prove that the estimate for the H^1 error in the deflection and rotation is optimal with respect to h and optimal up to $\mathcal{O}(k^\varepsilon)$, $\varepsilon > 0$ arbitrary, with respect to the polynomial degree k. Our estimates therefore give the expected asymptotic rate of convergence when accuracy is attained by using successively higher-degree elements from such families. They also indicate the advantage of using the pure h version with a high-degree element versus a low-degree element from the same family (see Remark 4.2).

We also give error estimates for the shear force, calculated both directly from the constitutive equation and by a simple postprocessing from the moment equilibrium equations. We obtain estimates which are *uniform* in t, in contrast to what is the case for the pure p version. These estimates may not be optimal—we comment on this in Remark 4.1. Our estimates show how a mixed hp implementation then provides the additional capability of mesh refinement to further increase accuracy in the calculated shear.

Let us remark that commercial p and hp FE codes increasingly allow the user the option of either h refinement or p refinement, with low-order polynomials on refined meshes recommended in the vicinity of corners. The use of a reduced constraint scheme as the underlying method would then ensure that there is no locking in such areas. In this context, our results establish that replacing the exact constraint method in a p or hp implementation by an MITC reduced constraint scheme will not adversely affect the p convergence rate either (in fact, it will guarantee a uniform rate for the shears).

The outline of our paper is as follows. In section 2, we state our problem and reformulate it as a system of two Poisson equations and a Stokes system, using a Helmholtz decomposition (this idea was first used in [10]). Section 3 contains the description of several MITC-type elements, including new families introduced here. The error estimates are derived in section 4. Section 5 contains some new hp estimates, needed in section 4, for the Brezzi–Douglas–Fortin–Marini (BDFM) spaces [13]. These results are interesting in their own right, since they establish optimal (up to $\mathcal{O}(k^\varepsilon)$, $\varepsilon > 0$ arbitrary) error estimates for the mixed hp formulation of second-order elliptic problems using these elements. An hp estimate for an interpolation operator related

to the Brezzi–Douglas–Marini (BDM) spaces [14] can also be improved using our technique (see Remark 5.2).

2. The RM equations. For notational simplicity we will assume that the plate is clamped along the whole boundary of the domain $\Omega \subset \mathbb{R}^2$, which is assumed to be simply connected and polygonal. The RM plate model is then as follows: find $w \in H_0^1(\Omega)$ and $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that

$$Gt^3 a(\boldsymbol{\beta}, \boldsymbol{\eta}) + G\kappa t(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (f, v) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.1}$$

Here $w, \boldsymbol{\beta}$ are the deflection and rotation, respectively, (of the midplane), G denotes the shear modulus, t is the thickness of the plate, and κ is the shear correction factor. The bilinear form a is defined by

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{1}{6}\left[(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \left(\frac{\nu}{1-\nu} \right) (\operatorname{div} \boldsymbol{\beta}, \operatorname{div} \boldsymbol{\eta}) \right], \tag{2.2}$$

where ν is the Poisson ratio and $\boldsymbol{\varepsilon}(\cdot)$ denotes the linear strain operator.

For the mathematical analysis we scale the problem by assuming the loading to be given by

$$f = Gt^3 g, \tag{2.3}$$

with g independent of t. We also let $\kappa = 1$. Hence, the problem is to find $w \in H_0^1(\Omega)$ and $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.4}$$

We now take the scaled shear force

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}), \tag{2.5}$$

as an independent unknown, to get the variational problem

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2, \tag{2.6}$$

$$t^2(\boldsymbol{q}, \boldsymbol{r}) - (\nabla w - \boldsymbol{\beta}, \boldsymbol{r}) = 0 \qquad \forall \boldsymbol{r} \in [L^2(\Omega)]^2. \tag{2.7}$$

We recall that expression (2.5) represents the constitutive equation. The (scaled) shear force also satisfies the equilibrium equation

$$\boldsymbol{L}\boldsymbol{\beta} + \boldsymbol{q} = \boldsymbol{0}, \tag{2.8}$$

obtained by putting $v = 0$ in (2.6) and integrating by parts. Here we have denoted

$$\boldsymbol{L}\boldsymbol{\eta} = \frac{1}{6} \operatorname{\mathbf{div}} \left\{ \boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \left(\frac{\nu}{1-\nu} \right) \operatorname{div} \boldsymbol{\eta}\, \boldsymbol{I} \right\}, \tag{2.9}$$

where $\mathbf{div}$ is the divergence operator acting on second-order tensors and $\boldsymbol{I}$ is the unit tensor.

By the Helmholtz theorem [11] every $\boldsymbol{q} \in [L^2(\Omega)]^2$ can be uniquely written as

$$\boldsymbol{q} = \nabla \psi + \mathbf{rot}\, p, \tag{2.10}$$

with $\psi \in H_0^1(\Omega)$ and $p \in H^1(\Omega) \cap L_0^2(\Omega)$. Here we denote

$$\mathbf{rot}\, p = \left(\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x} \right). \tag{2.11}$$

By using this decomposition the problem can be written as a system of two Poisson equations and a singularly perturbed Stokes problem: find $w \in H_0^1(\Omega)$, $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$, $\psi \in H_0^1(\Omega)$, and $p \in H^1(\Omega) \cap L_0^2(\Omega)$ such that

$$(\nabla\psi, \nabla v) = (f, v), \qquad v \in H_0^1(\Omega), \tag{2.12}$$

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\, p, \boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.13}$$

$$t^2(\mathbf{rot}\, p, \mathbf{rot}\, q) + (\boldsymbol{\beta}, \mathbf{rot}\, q) = 0, \qquad q \in H^1(\Omega) \cap L_0^2(\Omega), \tag{2.14}$$

$$(\nabla w, \nabla\xi) - (\boldsymbol{\beta}, \nabla\xi) - t^2(\nabla\psi, \nabla\xi) = 0, \qquad \xi \in H_0^1(\Omega). \tag{2.15}$$

The basic stability estimate for this problem is

$$\|w\|_1 + \|\boldsymbol{\beta}\|_1 + \|\psi\|_1 + \|p\|_0 + t\|p\|_1 \le C\|f\|_{-1}. \tag{2.16}$$

For deriving L^2 estimates we also need the following regularity result.

THEOREM 2.1 (see [10, 2]). *Let Ω be a convex polygonal domain. For $f \in L^2(\Omega)$ and $\boldsymbol{g} \in [L^2(\Omega)]^2$ the solution to the problem*

$$(\nabla\psi, \nabla v) = (f, v), \qquad v \in H_0^1(\Omega), \tag{2.17}$$

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\, p, \boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = (\boldsymbol{g}, \boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.18}$$

$$t^2(\mathbf{rot}\, p, \mathbf{rot}\, q) + (\boldsymbol{\beta}, \mathbf{rot}\, q) = 0, \qquad q \in H^1(\Omega) \cap L_0^2(\Omega), \tag{2.19}$$

$$(\nabla w, \nabla\xi) - (\boldsymbol{\beta}, \nabla\xi) - t^2(\nabla\psi, \nabla\xi) = 0, \qquad \xi \in H_0^1(\Omega), \tag{2.20}$$

satisfies

$$\|w\|_2 + \|\boldsymbol{\beta}\|_2 + \|\psi\|_2 + \|p\|_1 + t\|p\|_2 \le C(\|f\|_0 + \|\boldsymbol{g}\|_0).$$

For the analysis of the MITC methods we need one more step. It consists of writing the problem by introducing $\mathbf{rot}\, p$ as an independent unknown $\boldsymbol{\alpha}$. The natural boundary condition for p will now be imposed on the new variable $\boldsymbol{\alpha}$ which will be in the space

$$\boldsymbol{H}_0(\mathrm{rot}\,;\Omega) = \{\ \boldsymbol{q} \in [L^2(\Omega)]^2 \mid \mathrm{rot}\, \boldsymbol{q} \in L^2(\Omega), \quad \boldsymbol{q} \cdot \boldsymbol{t} = 0 \text{ on } \partial\Omega\ \}. \tag{2.21}$$

Here $\boldsymbol{t}$ denotes the tangent to $\partial\Omega$, and

$$\mathrm{rot}\, \boldsymbol{q} = \mathrm{rot}\, (q_x, q_y) = \frac{\partial q_y}{\partial x} - \frac{\partial q_x}{\partial y}\ . \tag{2.22}$$

We then have the following.

THEOREM 2.2. *The solution $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$ of (2.6)–(2.7) can be found by solving the following problem: find $(w, \boldsymbol{\beta}, \psi, p, \boldsymbol{\alpha}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times H_0^1(\Omega) \times L_0^2(\Omega) \times \boldsymbol{H}_0(\mathrm{rot}\,;\Omega)$ such that*

$$(\nabla\psi, \nabla v) = (f, v), \quad v \in H_0^1(\Omega), \tag{2.23}$$

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\operatorname{rot}\boldsymbol{\eta}, p) - (\nabla\psi, \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.24}$$

$$t^2(\operatorname{rot}\boldsymbol{\alpha}, q) + (\operatorname{rot}\boldsymbol{\beta}, q) = 0, \qquad q \in L_0^2(\Omega), \tag{2.25}$$

$$(\boldsymbol{\alpha}, \boldsymbol{\delta}) - (\operatorname{rot}\boldsymbol{\delta}, p) = 0, \qquad \boldsymbol{\delta} \in \boldsymbol{H}_0(\operatorname{rot}; \Omega), \tag{2.26}$$

$$(\nabla w, \nabla\xi) - (\boldsymbol{\beta}, \nabla\xi) - t^2(\nabla\psi, \nabla\xi) = 0, \qquad \xi \in H_0^1(\Omega), \tag{2.27}$$

and setting $\boldsymbol{q} = \nabla\psi + \boldsymbol{\alpha}$.

3. The FE methods. We first recall the procedure for defining mixed interpolated FEs given by Bathe, Brezzi, and Fortin in [7]. It consists of the following steps:

1. We start from a space $\boldsymbol{B}_n$ for the rotation which together with an auxiliary space R_n is known to work for the (rotated) Stokes problem. More precisely, we require the spaces to satisfy a stability inequality

$$\sup_{\mathbf{0}\neq\eta\in\boldsymbol{B}_n} \frac{(\operatorname{rot}\boldsymbol{\eta}, q)}{\|\boldsymbol{\eta}\|_1} \geq C(h,k)\|q\|_0 \quad \forall q \in R_n, \tag{3.1}$$

   with a positive constant $C(h,k)$ depending as little as possible on the mesh length h and the polynomial degree k. We have studied this problem in our earlier paper [20] in which we introduced and analyzed several combinations for which $C(h,k) \geq Ck^{-1/2}$.
2. With R_n now fixed, we have to find another auxiliary space $\boldsymbol{\Gamma}_n$ and an operator $\boldsymbol{\Pi}_n : [H^1(\Omega)]^2 \to \boldsymbol{\Gamma}_n$ such that

$$(\operatorname{rot}(\boldsymbol{\sigma} - \boldsymbol{\Pi}_n\boldsymbol{\sigma}), q) = 0 \quad \forall q \in R_n. \tag{3.2}$$

   Pairs of spaces $(\boldsymbol{\Gamma}_n, R_n)$ of this type are well known in the literature (cf. [11]) as they are used for mixed FE approximations of second-order elliptic problems. They include, for instance, the Raviart–Thomas (RT), BDM, and BDFM spaces.
3. The final step is to choose the space W_n for the deflection such that it holds

$$\nabla W_n = \{\ \boldsymbol{\sigma} \in \boldsymbol{\Gamma}_n \mid \operatorname{rot}\boldsymbol{\sigma} = 0\ \}. \tag{3.3}$$

The methods are then defined in the following way: find $w_n \in W_n \subset H_0^1(\Omega)$ and $\boldsymbol{\beta}_n \in \boldsymbol{B}_n \subset [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\beta}_n, \boldsymbol{\eta}) + t^{-2}(\boldsymbol{\Pi}_n\boldsymbol{\beta}_n - \nabla w_n, \boldsymbol{\Pi}_n\boldsymbol{\eta} - \nabla v) = (f, v) \quad \forall (v, \boldsymbol{\eta}) \in W_n \times \boldsymbol{B}_n. \tag{3.4}$$

An approximation for the shear can be calculated in two ways. The first alternative is to use the approximate constitutive relation, viz.,

$$\boldsymbol{q}_n = t^{-2}(\nabla w_n - \boldsymbol{\Pi}_n\boldsymbol{\beta}_n). \tag{3.5}$$

The second alternative would be to use the equilibrium equations separately on each element; i.e., we calculate the approximation $\boldsymbol{q}_n^e$ through

$$\boldsymbol{q}_{n|K}^e = -\boldsymbol{L}\boldsymbol{\beta}_{n|K} \quad \forall K \in \mathcal{C}_h. \tag{3.6}$$

We will analyze both alternatives.

Next let us turn to the detailed definition of the methods. For completeness, we will define the methods for general curved quadrilaterals. Our analysis will, however, be limited to the case of parallelograms. This is because in [20], our results for the Stokes problem were only established for this case.

We let $\mathcal{C}_h$ be the FE partitioning of $\bar{\Omega}$. Each element $K \in \mathcal{C}_h$ is the image of the reference square $\hat{K} = (-1,1)^2$ under a bijective mapping $\boldsymbol{F}_K$, viz., $K = \boldsymbol{F}_K(\hat{K})$. The partitioning is assumed to satisfy the usual compatibility assumptions; i.e., the intersection of two elements of $\mathcal{C}_h$ is either empty, a vertex, or a common edge. Furthermore, it is assumed to be regular; there is a constant C such that

$$h_K \le C\rho_K \quad \forall K \in \mathcal{C}_h, \tag{3.7}$$

where h_K is the diameter of K and ρ_K is the diameter of the largest circle inscribed in K. As usual we denote $h = \max_{K\in\mathcal{C}_h} h_K$. The coordinates of a point in Ω will be denoted by (x,y) and those of a point in the reference square $\hat{K}$ will be denoted by (ξ,η).

The spaces W_n, $\boldsymbol{B}_n$, and R_n are all defined in the usual way:

$$W_n = \{\ v \in H_0^1(\Omega) \mid v_{|K} \in W_k(K)\ \forall K \in \mathcal{C}_h\ \}, \tag{3.8}$$

$$\boldsymbol{B}_n = \{\ \boldsymbol{\beta} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\beta}_{|K} \in \boldsymbol{B}_k(K)\ \forall K \in \mathcal{C}_h\ \}, \tag{3.9}$$

$$R_n = \{\ p \in L_0^2(\Omega) \mid p_{|K} \in R_k(K)\ \forall K \in \mathcal{C}_h\ \}, \tag{3.10}$$

with the aid of the spaces $\boldsymbol{B}_k(\hat{K}), W_k(\hat{K}), R_k(\hat{K})$ on the reference square:

$$\boldsymbol{B}_k(K) = \{\boldsymbol{\beta} = \hat{\boldsymbol{\beta}} \circ \boldsymbol{F}_K^{-1} \mid \hat{\boldsymbol{\beta}} \in \boldsymbol{B}_k(\hat{K})\ \}, \tag{3.11}$$

$$W_k(K) = \{v = \hat{v} \circ \boldsymbol{F}_K^{-1} \mid \hat{v} \in W_k(\hat{K})\ \}, \tag{3.12}$$

$$R_k(K) = \{p = \hat{p} \circ \boldsymbol{F}_K^{-1} \mid \hat{p} \in R_k(\hat{K})\ \}. \tag{3.13}$$

The space $\boldsymbol{\Gamma}_n$ is defined differently. We let

$$\boldsymbol{\Gamma}_n = \{\ \boldsymbol{q} \in \boldsymbol{H}_0(\mathrm{rot}\,;\Omega) \mid \boldsymbol{q}_{|K} \in \boldsymbol{\Gamma}_k(K)\ \forall\ K \in \mathcal{C}_h\ \}, \tag{3.14}$$

where the space $\boldsymbol{\Gamma}_k(K)$ is now defined from the space $\boldsymbol{\Gamma}_k(\hat{K})$ on the reference square through a covariant transformation:

$$\boldsymbol{\Gamma}_k(K) = \{\ \boldsymbol{\sigma} \mid \boldsymbol{\sigma} = \boldsymbol{J}_K^{-T}\hat{\boldsymbol{\sigma}} \circ \boldsymbol{F}_K^{-1},\ \hat{\boldsymbol{\sigma}} \in \boldsymbol{\Gamma}_k(\hat{K})\ \}. \tag{3.15}$$

Here $\boldsymbol{J}_K^{-T}$ is the transpose of the inverse of the Jacobian matrix $\boldsymbol{J}_K$ of the mapping $\boldsymbol{F}_K$. We remark that the assumption $\boldsymbol{r} \in \boldsymbol{H}_0(\mathrm{rot}\,;\Omega)$, for $\boldsymbol{r} \in \boldsymbol{\Gamma}_n$, implies that the tangential component of $\boldsymbol{r}$ is continuous along interelement boundaries and vanishes on $\partial\Omega$.

The reduction operator $\boldsymbol{\Pi}_n$ is also defined locally on each element from the reduction operator $\widehat{\boldsymbol{\Pi}}_k$ defined on the reference element with the same covariant interpolation:

$$\widehat{\boldsymbol{\Pi}_n\boldsymbol{\sigma}}_{|K} = \widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}, \tag{3.16}$$

or, more explicitly stated,

$$\boldsymbol{\Pi}_n\boldsymbol{\sigma}_{|K} = \boldsymbol{J}_K^{-T}\widehat{\boldsymbol{\Pi}}_k\boldsymbol{J}_K^T\boldsymbol{\sigma}_{|K}. \tag{3.17}$$

Let us now specify the methods we are going to analyze. Here $P_l(\hat{K})$ and $P_l(\hat{E})$ denote the polynomials of total degree $\leq l$ on $\hat{K}$ and on an edge $\hat{E}$ of $\hat{K}$, respectively. $Q_l(\hat{K})$ denotes the polynomials of degree $\leq l$ in each variable, while $Q'_l(\hat{K})$ denotes the serendipity space, which is the span of polynomials of total degree $\leq l$ together with $\xi^l\eta, \xi\eta^l$. $Q_{l,m}(\hat{K})$ is the space of polynomials of degree $\leq l$ in the first variable and degree $\leq m$ in the second. In all the methods below, k will be an integer ≥ 2; this excludes the possibility of piecewise constants being taken for the space of "pressures" R_n.

Method 1. We first consider the "minimal" Stokes combination considered in [20]. In this, the pressure space is chosen as

$$R_k(\hat{K}) = P_{k-1}(\hat{K}) \tag{3.18}$$

and it was shown that with the choice

$$\boldsymbol{B}_k(\hat{K}) = [Q_k(\hat{K}) \cap P_{k+2}(\hat{K})]^2, \tag{3.19}$$

there exists a constant C such that the stability inequality (3.1) is valid with $C(h,k) = Ck^{-1/2}$ for any $k \geq 2$.

We then choose $\boldsymbol{\Gamma}_k(\hat{K})$ as the BDFM space [13, 11]:

$$\boldsymbol{\Gamma}_k(\hat{K}) = \{\ \hat{\boldsymbol{\sigma}} \mid \hat{\boldsymbol{\sigma}} \in [P_k(\hat{K}) \setminus \{\xi^k\}] \times [P_k(\hat{K}) \setminus \{\eta^k\}]\ \ \}, \tag{3.20}$$

and the reduction operator is defined by

$$\int_{\hat{E}} [(\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}}]\, \hat{v} = 0, \quad \hat{v} \in P_{k-1}(\hat{E}) \ \text{ for every edge } \hat{E} \text{ of } \hat{K}, \tag{3.21}$$

$$\int_{\hat{K}} (\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{r}} = 0, \quad \hat{\boldsymbol{r}} \in [P_{k-2}(\hat{K})]^2. \tag{3.22}$$

Here $\hat{\boldsymbol{t}}$ is the tangent to $\hat{E}$.

The pair $(\boldsymbol{\Gamma}_n, R_n)$ satisfies condition (3.2).

Finally, condition (3.3) leads to the choice

$$W_k(\hat{K}) = Q_k(\hat{K}) \cap P_{k+1}(\hat{K}). \tag{3.23}$$

Method 2. This is the original MITC family in which W_n, R_n, and $\boldsymbol{\Gamma}_n$ are as in Method 1 above, but the space for the rotation is taken as

$$\boldsymbol{B}_k(\hat{K}) = [Q_k(\hat{K})]^2. \tag{3.24}$$

We point out that compared with the first method this choice will lead to $\mathcal{O}(k^2)$ more degrees of freedom. Since the two methods have the same order of convergence, the first appears to be preferable.

Method 3. In this choice the space for the rotation and the auxiliary pressure are chosen as in Method 1, viz.,

$$\boldsymbol{B}_k(\hat{K}) = [Q_k(\hat{K}) \cap P_{k+2}(\hat{K})]^2, \quad R_k(\hat{K}) = P_{k-1}(\hat{K}). \tag{3.25}$$

In order to get a stable pair $(\boldsymbol{\Gamma}_n, R_n)$ we now choose the second known choice, i.e., the BDM space [14, 11]:

$$\boldsymbol{\Gamma}_k(\hat{K}) = \{\ \hat{\boldsymbol{\sigma}} \mid \hat{\boldsymbol{\sigma}} \in [P_k(\hat{K})]^2 \oplus \nabla(\xi\eta^{k+1}) \oplus \nabla(\xi^{k+1}\eta)\ \} \tag{3.26}$$

with the reduction operator defined by

$$\int_{\hat{E}} [(\widehat{\boldsymbol{\Pi}}_k \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}}]\, \hat{v} = 0, \quad \hat{v} \in P_k(\hat{E}) \ \text{ for every edge } \hat{E} \text{ of } \hat{K}, \tag{3.27}$$

$$\int_{\hat{K}} (\widehat{\boldsymbol{\Pi}}_k \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{r}} = 0, \quad \hat{\boldsymbol{r}} \in [P_{k-2}(\hat{K})]^2. \tag{3.28}$$

The condition (3.3) then gives

$$W_k(\hat{K}) = Q'_{k+1}(\hat{K}). \tag{3.29}$$

Remark 3.1. This method could also be modified by choosing $\boldsymbol{B}_k(\hat{K}) = [Q_k(\hat{K})]^2$, but again it does not seem to lead to any advantage.

Method 4. Here, we choose the pressure space as

$$R_k(\hat{K}) = Q_{k-1}(\hat{K}). \tag{3.30}$$

With the pressure space now increased, we have to include additional internal degrees of freedom ("bubbles") into the rotation space. In [20] we showed that the choice

$$\boldsymbol{B}_k(\hat{K}) = \{\, \boldsymbol{\beta} \in [Q_{k+1}(\hat{K})]^2 \,|\, \boldsymbol{\beta}_{|\hat{E}} \in [P_k(\hat{E})]^2 \text{ for every edge } \hat{E} \text{ of } \hat{K} \,\} \tag{3.31}$$

gives a method with a stability constant $Ck^{-1/2}$.

The corresponding space for the shear is then the RT space [19, 11]:

$$\boldsymbol{\Gamma}_k(\hat{K}) = \{\ \hat{\boldsymbol{\sigma}} \mid \hat{\boldsymbol{\sigma}} \in Q_{k-1,k}(\hat{K}) \times Q_{k,k-1}(\hat{K})\ \} \tag{3.32}$$

with the reduction operator defined by

$$\int_{\hat{E}} [(\widehat{\boldsymbol{\Pi}}_k \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{t}}]\, \hat{v} = 0, \quad \hat{v} \in P_{k-1}(\hat{E}) \ \text{ for every edge } \hat{E} \text{ of } \hat{K}, \tag{3.33}$$

$$\int_{\hat{K}} (\widehat{\boldsymbol{\Pi}}_k \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}) \cdot \hat{\boldsymbol{r}} = 0, \quad \hat{\boldsymbol{r}} \in Q_{k-1,k-2}(\hat{K}) \times Q_{k-2,k-1}(\hat{K}). \tag{3.34}$$

The space for the deflection is now

$$W_k(\hat{K}) = Q_k(\hat{K}). \tag{3.35}$$

Remark 3.2. Again, the rotation space could be increased and for p-dominated extension it seems simplest to let

$$\boldsymbol{B}_k(\hat{K}) = [Q_{k+1}(\hat{K})]^2. \tag{3.36}$$

For completeness, we would like to point out that the mapping between $\boldsymbol{\Gamma}_k(K)$ and $\boldsymbol{\Gamma}_k(\hat{K})$ is a kind of "Piola transformation" for the "rot" operator. (The Piola transformation is defined for the divergence operator in, for example [11, pp. 97–98].) For an edge $\hat{E}$ of $\hat{K}$ we let $E = \boldsymbol{F}_K(\hat{E})$ be the corresponding edge of K. The unit tangents to E and $\hat{E}$ we denote by $\boldsymbol{t}$ and $\hat{\boldsymbol{t}}$, respectively. The determinant of the Jacobian $\boldsymbol{J}_K$ is denoted by $|\boldsymbol{J}_K|$. We further let

$$\widehat{\operatorname{rot}}\, \hat{\boldsymbol{\sigma}} = \widehat{\operatorname{rot}}\, (\sigma_\xi, \sigma_\eta) = \frac{\partial \sigma_\eta}{\partial \xi} - \frac{\partial \sigma_\xi}{\partial \eta}\ . \tag{3.37}$$

Then we have the following results which are easily established.

LEMMA 3.1. *Let* $\hat{v} \in H^1(\hat{K})$ *and* $\hat{\boldsymbol{\sigma}} \in [H^1(\hat{K})]^2$, *and define* $v \in H^1(K)$ *and* $\boldsymbol{\sigma} \in [H^1(K)]^2$ *through*

$$v = \hat{v} \circ \boldsymbol{F}_K^{-1}, \qquad \boldsymbol{\sigma} = \boldsymbol{J}_K^{-T} \hat{\boldsymbol{\sigma}} \circ \boldsymbol{F}_K^{-1}. \tag{3.38}$$

Then it holds

$$\operatorname{rot} \boldsymbol{\sigma} = |\boldsymbol{J}_K|^{-1} \widehat{\operatorname{rot}} \hat{\boldsymbol{\sigma}}, \tag{3.39}$$

$$\int_K \operatorname{rot} \boldsymbol{\sigma}\, v = \int_{\hat{K}} \widehat{\operatorname{rot}} \hat{\boldsymbol{\sigma}}\, \hat{v}\,, \tag{3.40}$$

$$\int_E \boldsymbol{\sigma} \cdot \boldsymbol{t}\, v = \int_{\hat{E}} \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{t}}\, \hat{v}. \tag{3.41}$$

4. Error analysis. In our error analysis we will follow the framework set up in [12]. The main extension to the earlier analysis is that we now have to analyze the method also with respect to the polynomial degree in which the methods are not uniformly stable.

In our analysis we concentrate on analyzing the stability of the method and therefore we give the error estimates assuming that the solution is sufficiently smooth. As is well known, this is not the case in general due to the corner singularities and, in particular, due to the boundary layer in the solution; cf. [1].

We start by collecting some results on the reduction operators and the spaces used for the shear.

LEMMA 4.1. *Suppose that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^2$ *with* $r > 1/2$. *Then for every* $\delta > 0$ *there is a positive constant* C_δ *such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_n \boldsymbol{\sigma}\|_0 \le C_\delta h^l k^{1/2+\delta-r} \|\boldsymbol{\sigma}\|_r, \tag{4.1}$$

with $l = \min\{k, r\}$ *for the RT and BDFM spaces and* $l = \min\{k+1, r\}$ *for the BDM space.*

Proof. For the RT space this result is proved in [23, 17]. The analysis for the BDFM space is given in section 5 below. The result for the BDM space follows from the results proved in [23] as indicated in Remark 5.2 below. ∎

In what follows we will assume that the elements are parallelograms. Then the following relations hold:

$$\operatorname{rot} \boldsymbol{\Gamma}_n \subset R_n \tag{4.2}$$

and

$$\operatorname{rot} \boldsymbol{\Pi}_n \boldsymbol{\eta} = \Pi_n \operatorname{rot} \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{4.3}$$

where Π_n is the L^2 projection onto R_n. From (3.22), (3.28), and (3.34) it also follows that

$$(\boldsymbol{z}, \boldsymbol{\eta} - \boldsymbol{\Pi}_n \boldsymbol{\eta})_K = 0 \quad \forall \boldsymbol{z} \in \boldsymbol{K}_n, \ \ \forall \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{4.4}$$

with

$$\boldsymbol{K}_n = \{\boldsymbol{z} \in [L^2(\Omega)]^2 \mid \boldsymbol{z}_{|K} \in [P_{k-2}(K)]^2 \qquad \forall K \in \mathcal{C}_h \ \}. \tag{4.5}$$

The analysis will be based on the following discrete Helmholtz decomposition introduced in [12]. For completeness we will repeat the short proof.

LEMMA 4.2. *For every* $\boldsymbol{q} \in \boldsymbol{\Gamma}_n$ *there exist unique* $\psi \in W_n$, $p \in \boldsymbol{B}_n$, *and* $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}_n$ *such that*

$$\boldsymbol{q} = \nabla\psi + \boldsymbol{\alpha}, \tag{4.6}$$

$$(\boldsymbol{\alpha}, \boldsymbol{\sigma}) = (\operatorname{rot}\boldsymbol{\sigma}, p) \qquad \forall \boldsymbol{\sigma} \in \boldsymbol{\Gamma}_n, \tag{4.7}$$

and

$$(\boldsymbol{\alpha}, \nabla z) = 0 \qquad \forall z \in W_n. \tag{4.8}$$

Proof. Since the pair $(\boldsymbol{\Gamma}_n, R_n)$ gives a stable mixed method for second-order elliptic equations, the following FE discretization has a unique solution $(\boldsymbol{\alpha}, p) \in \boldsymbol{\Gamma}_n \times R_n$:

$$(\boldsymbol{\alpha}, \boldsymbol{\sigma}) - (\operatorname{rot}\boldsymbol{\sigma}, p) = 0 \qquad \forall \boldsymbol{\sigma} \in \boldsymbol{\Gamma}_n, \tag{4.9}$$

$$(\operatorname{rot}\boldsymbol{\alpha}, q) = (\operatorname{rot}\boldsymbol{q}, q) \quad \forall q \in R_n. \tag{4.10}$$

Due to the inclusion (4.2) we conclude that the second equation above gives

$$\operatorname{rot}(\boldsymbol{q} - \boldsymbol{\alpha}) = 0. \tag{4.11}$$

The condition (3.3) then implies that there is $\psi \in W_n$ such that

$$\boldsymbol{q} - \boldsymbol{\alpha} = \nabla\psi. \tag{4.12}$$

The uniqueness of ψ follows from the uniqueness of $\boldsymbol{\alpha}$. The orthogonality condition (4.8) then follows from (4.9). ∎

Using this decomposition we get the following discrete analogue to Theorem 2.2.

THEOREM 4.1. *The solution* $(w_n, \boldsymbol{\beta}_n, \boldsymbol{q}_n) \in W_n \times \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n$ *of* (3.4) *and* (3.5) *can be found by solving the following problem: find* $(w_n, \boldsymbol{\beta}_n, \psi_n, p_n, \boldsymbol{\alpha}_n) \in W_n \times \boldsymbol{B}_n \times W_n \times R_n \times \boldsymbol{\Gamma}_n$ *such that*

$$(\nabla\psi_n, \nabla v) = (f, v), \quad v \in W_n, \tag{4.13}$$

$$a(\boldsymbol{\beta}_n, \boldsymbol{\eta}) - (\operatorname{rot}\boldsymbol{\eta}, p_n) - (\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in \boldsymbol{B}_n, \tag{4.14}$$

$$t^2(\operatorname{rot}\boldsymbol{\alpha}_n, q) + (\operatorname{rot}\boldsymbol{\beta}_n, q) = 0, \qquad q \in R_n, \tag{4.15}$$

$$(\boldsymbol{\alpha}_n, \boldsymbol{\delta}) - (\operatorname{rot}\boldsymbol{\delta}, p_n) = 0, \qquad \boldsymbol{\delta} \in \boldsymbol{\Gamma}_n, \tag{4.16}$$

$$(\nabla w_n, \nabla\xi) - (\boldsymbol{\Pi}_n\boldsymbol{\beta}_n, \nabla\xi) - t^2(\nabla\psi_n, \nabla\xi) = 0, \qquad \xi \in W_n, \tag{4.17}$$

and setting $\boldsymbol{q}_n = \nabla\psi_n + \boldsymbol{\alpha}_n$.

Before giving the stability and error analysis of the method, let us state the stability of the Stokes system explicitly as a lemma.

LEMMA 4.3 (see [20, Theorem 5.1]). *There is a positive constant* C *such that (with* $k \geq 2$*)*

$$\sup_{\mathbf{0} \neq \eta \in \boldsymbol{B}_n} \frac{(\operatorname{rot}\boldsymbol{\eta}, q)}{\|\boldsymbol{\eta}\|_1} \geq Ck^{-1/2}\|q\|_0 \quad \forall q \in R_n, \tag{4.18}$$

holds for the pairs $(\boldsymbol{B}_n, R_n)$ *of Methods* 1–4.

The fact that the Stokes system is not stable uniformly in the polynomial degree has several consequences in the stability analysis, which we give in detail below. First, we define the bilinear form

$$\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\eta},\boldsymbol{\delta},q) = a(\boldsymbol{\beta},\boldsymbol{\eta}) - (\operatorname{rot}\boldsymbol{\eta},p) - (\operatorname{rot}\boldsymbol{\beta},q) + t^2((\boldsymbol{\alpha},\boldsymbol{\delta}) - (\operatorname{rot}\boldsymbol{\alpha},q) - (\operatorname{rot}\boldsymbol{\delta},p)). \tag{4.19}$$

With this, the continuous problem (2.24)–(2.26) can be written as follows: find $(\boldsymbol{\beta},\boldsymbol{\alpha},p) \in [H_0^1(\Omega)]^2 \times \boldsymbol{H}_0(\operatorname{rot};\Omega) \times L_0^2(\Omega)$ such that

$$\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\eta},\boldsymbol{\delta},q) = (\nabla\psi,\boldsymbol{\eta}) \quad \forall(\boldsymbol{\eta},\boldsymbol{\delta},q) \in [H_0^1(\Omega)]^2 \times \boldsymbol{H}_0(\operatorname{rot};\Omega) \times L_0^2(\Omega). \tag{4.20}$$

Its discretization (4.14)–(4.16) can be written as follows: find $(\boldsymbol{\beta}_n,\boldsymbol{\alpha}_n,p_n) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n$ such that

$$\mathcal{B}(\boldsymbol{\beta}_n,\boldsymbol{\alpha}_n,p_n;\boldsymbol{\eta},\boldsymbol{\delta},q) = (\nabla\psi_n,\boldsymbol{\Pi}_n\boldsymbol{\eta}) \quad \forall(\boldsymbol{\eta},\boldsymbol{\delta},q) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n. \tag{4.21}$$

Stability is established in terms of the following norm:

$$|||(\boldsymbol{\beta},\boldsymbol{\alpha})|||^2 = \|\boldsymbol{\beta}\|_1^2 + t^2\|\boldsymbol{\alpha}\|_0^2 + t^4\|\operatorname{rot}\boldsymbol{\alpha}\|_0^2. \tag{4.22}$$

LEMMA 4.4. *There is a positive constant C such that the following condition holds: for every $(\boldsymbol{\beta},\boldsymbol{\alpha},p) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n$ there is a $(\boldsymbol{\eta},\boldsymbol{\delta},q) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n$ such that*

$$\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\eta},\boldsymbol{\delta},q) \geq |||(\boldsymbol{\beta},\boldsymbol{\alpha})||| + k^{-1/2}\|p\|_0 \tag{4.23}$$

and

$$|||(\boldsymbol{\eta},\boldsymbol{\delta})||| + k^{-1/2}\|q\|_0 \leq C. \tag{4.24}$$

Proof. Let $(\boldsymbol{\beta},\boldsymbol{\alpha},p) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n$ be given. First, we note that

$$\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\beta},\boldsymbol{\alpha},-p) = a(\boldsymbol{\beta},\boldsymbol{\beta}) + t^2(\boldsymbol{\alpha},\boldsymbol{\alpha}) \geq C_1(\|\boldsymbol{\beta}\|_1^2 + t^2\|\boldsymbol{\alpha}\|_0^2). \tag{4.25}$$

By Lemma 4.3 there exists $\boldsymbol{\omega} \in \boldsymbol{B}_n$ such that

$$(\operatorname{rot}\boldsymbol{\omega},p) \geq C_2k^{-1}\|p\|_0^2 \ \text{ and } \ \|\boldsymbol{\omega}\|_1 \leq C_3k^{-1/2}\|p\|_0. \tag{4.26}$$

We then define $r \in R_n$ by $r = t^2\operatorname{rot}\boldsymbol{\alpha}$ (which is possible due to (4.2)).

Now let $(\boldsymbol{\eta},\boldsymbol{\delta},q) = (\boldsymbol{\beta}-\kappa\boldsymbol{\omega},\boldsymbol{\alpha},-p-\kappa r)$ with $\kappa > 0$. Using the Schwarz inequality and the arithmetic–geometric-mean inequality we then get

$$\begin{aligned}
&\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\eta},\boldsymbol{\delta},q)\\
&= \mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\beta},\boldsymbol{\alpha},-p) - \kappa\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{\omega},\boldsymbol{0},0) - \kappa\mathcal{B}(\boldsymbol{\beta},\boldsymbol{\alpha},p;\boldsymbol{0},\boldsymbol{0},r)\\
&\geq C_1(\|\boldsymbol{\beta}\|_1^2 + t^2\|\boldsymbol{\alpha}\|_0^2) - \kappa a(\boldsymbol{\beta},\boldsymbol{\omega}) + \kappa(\operatorname{rot}\boldsymbol{\omega},p)\\
&\quad + \kappa(\operatorname{rot}\boldsymbol{\beta},r) + \kappa t^2(\operatorname{rot}\boldsymbol{\alpha},r)\\
&\geq C_1(\|\boldsymbol{\beta}\|_1^2 + t^2\|\boldsymbol{\alpha}\|_0^2) - \kappa C_4\|\boldsymbol{\beta}\|_1\|\boldsymbol{\omega}\|_1 + \kappa C_2k^{-1}\|p\|_0^2
\end{aligned} \tag{4.27}$$

$$
\begin{aligned}
&- \kappa t^2 C_5 \|\boldsymbol{\beta}\|_1 \|\mathrm{rot}\,\boldsymbol{\alpha}\|_0 + \kappa t^4 \|\mathrm{rot}\,\boldsymbol{\alpha}\|_0^2 \\
&\geq (C_1 - \frac{\kappa(C_4+C_5)}{2\epsilon})\|\boldsymbol{\beta}\|_1^2 + \kappa k^{-1}(C_2 - \frac{\epsilon C_4 C_3^2}{2})\|p\|_0^2 \\
&\quad + \kappa t^4 (1 - \frac{\epsilon C_5}{2})\|\mathrm{rot}\,\boldsymbol{\alpha}\|_0^2 + C_1 t^2 \|\boldsymbol{\alpha}\|_0^2 \\
&\geq C(|||(\boldsymbol{\beta},\boldsymbol{\alpha})|||^2 + k^{-1}\|p\|_0^2),
\end{aligned}
$$

when first choosing $0 < \epsilon < \min\{2C_2 C_4^{-1} C_3^{-2},\, 2C_5^{-1}\}$ and then $0 < \kappa < 2\epsilon C_1 (C_4 + C_5)^{-1}$.

On the other hand we also have

$$
|||(\boldsymbol{\eta},\boldsymbol{\delta})||| + k^{-1/2}\|q\|_0 \leq C(|||(\boldsymbol{\beta},\boldsymbol{\alpha})||| + k^{-1/2}\|p\|_0), \tag{4.28}
$$

which proves the assertion. □

THEOREM 4.2. *Suppose that for the solution of* (2.12)–(2.15) *we have* $w \in H^m(\Omega)$, $\boldsymbol{\beta} \in [H^m(\Omega)]^2$, $\psi \in H^m(\Omega)$, *and* $p \in H^m(\Omega)$. *For every* $\epsilon > 0$ *there is a positive constant* C_ϵ *such that*

$$
\|w - w_n\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1 \leq C_\epsilon h^l k^{-m+1+\epsilon}(\|w\|_m + \|\boldsymbol{\beta}\|_m + \|\psi\|_m + \|p\|_{m-1} + t\|p\|_m),
$$

with $l = \min\{k, m-1\}$.

Proof. Throughout the proof we let $l = \min\{k, m-1\}$. For the solutions to (2.23) and (4.13) it holds [4]

$$
\|\psi - \psi_n\|_1 \leq C h^l k^{-m+1} \|\psi\|_m. \tag{4.29}
$$

Next, let us estimate the distance between the solution to the continuous (2.24)–(2.26) and discrete (4.14)–(4.16) Stokes systems. We let $\boldsymbol{I}_n\boldsymbol{\beta} \in \boldsymbol{B}_n$ be the H^1 projection of $\boldsymbol{\beta}$. For this it holds [4]

$$
\|\boldsymbol{\beta} - \boldsymbol{I}_n\boldsymbol{\beta}\|_1 \leq C h^l k^{-m+1} \|\boldsymbol{\beta}\|_m. \tag{4.30}
$$

We let $\Pi_n p \in R_n$ be the L^2 projection of p. It then holds

$$
\|p - \Pi_n p\|_0 \leq C h^l k^{-m+1} \|p\|_{m-1}. \tag{4.31}
$$

By the stability estimate of Lemma 4.4 there is $(\boldsymbol{\eta},\boldsymbol{\delta},q) \in \boldsymbol{B}_n \times \boldsymbol{\Gamma}_n \times R_n$ such that

$$
\begin{aligned}
&|||(\boldsymbol{\beta}_n - \boldsymbol{I}_n\boldsymbol{\beta}, \boldsymbol{\alpha}_n - \boldsymbol{\Pi}_n\boldsymbol{\alpha})||| + k^{-1/2}\|p_n - \Pi_n p\|_0 \\
&\qquad \leq \mathcal{B}(\boldsymbol{\beta}_n - \boldsymbol{I}_n\boldsymbol{\beta}, \boldsymbol{\alpha}_n - \boldsymbol{\Pi}_n\boldsymbol{\alpha}, p_n - \Pi_n p; \boldsymbol{\eta},\boldsymbol{\delta},q)
\end{aligned} \tag{4.32}
$$

and

$$
|||(\boldsymbol{\eta},\boldsymbol{\delta})||| + k^{-1/2}\|q\|_0 \leq C. \tag{4.33}
$$

Moreover, from (4.20) and (4.21) we get

$$
\begin{aligned}
&\mathcal{B}(\boldsymbol{\beta}_n - \boldsymbol{I}_n\boldsymbol{\beta}, \boldsymbol{\alpha}_n - \boldsymbol{\Pi}_n\boldsymbol{\alpha}, p_n - \Pi_n p; \boldsymbol{\eta},\boldsymbol{\delta},q) \\
&\qquad = \mathcal{B}(\boldsymbol{\beta} - \boldsymbol{I}_n\boldsymbol{\beta}, \boldsymbol{\alpha} - \boldsymbol{\Pi}_n\boldsymbol{\alpha}, p - \Pi_n p; \boldsymbol{\eta},\boldsymbol{\delta},q) + (\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}).
\end{aligned} \tag{4.34}
$$

Let us now estimate the different terms arising on the right-hand side of (4.34). First, we will consider the terms in the bilinear form (defined by (4.19)). From (4.30) and (4.33) we directly get

$$|a(\boldsymbol{\beta}-\boldsymbol{I}_n\boldsymbol{\beta},\boldsymbol{\eta})| \le C\|\boldsymbol{\beta}-\boldsymbol{I}_n\boldsymbol{\beta}\|_1\|\boldsymbol{\eta}\|_1 \le Ch^l k^{-m+1}\|\boldsymbol{\beta}\|_m. \tag{4.35}$$

For the next term (4.31) and (4.33) give

$$|(\operatorname{rot}\boldsymbol{\eta}, p-\Pi_n p)| \le C\|\boldsymbol{\eta}\|_1\|p-\Pi_n p\|_0 \le Ch^l k^{-m+1}\|p\|_{m-1}. \tag{4.36}$$

Since it holds that $\|q\|_0 \le Ck^{1/2}$, the third term is estimated as follows:

$$\begin{aligned}|(\operatorname{rot}(\boldsymbol{\beta}-\boldsymbol{I}_n\boldsymbol{\beta}), q)| &\le C\|\boldsymbol{\beta}-\boldsymbol{I}_n\boldsymbol{\beta}\|_1\|q\|_0 \le Ck^{1/2}\|\boldsymbol{\beta}-\boldsymbol{I}_n\boldsymbol{\beta}\|_1\\ &\le Ch^l k^{-m+3/2}\|\boldsymbol{\beta}\|_m.\end{aligned} \tag{4.37}$$

Next, using (4.33) with Lemma 4.1 gives, for arbitrary $\delta > 0$,

$$\begin{aligned}|t^2(\boldsymbol{\alpha}-\boldsymbol{\Pi}_n\boldsymbol{\alpha},\boldsymbol{\delta})| &\le t^2\|\boldsymbol{\alpha}-\boldsymbol{\Pi}_n\boldsymbol{\alpha}\|_0\|\boldsymbol{\delta}\|_0 \le Ct\|\boldsymbol{\alpha}-\boldsymbol{\Pi}_n\boldsymbol{\alpha}\|_0\\ &\le C_\delta h^l k^{-m+3/2+\delta}t\|\boldsymbol{\alpha}\|_{m-1} \le C_\delta h^l k^{-m+3/2+\delta}t\|p\|_m.\end{aligned} \tag{4.38}$$

By (3.2), (4.2), and the definition of the operators $\boldsymbol{\Pi}_n, L_0^2(\Omega)$, we have for the last two terms of the bilinear form

$$(\operatorname{rot}(\boldsymbol{\alpha}-\boldsymbol{\Pi}_n\boldsymbol{\alpha}), q) = 0 \quad\text{and}\quad (\operatorname{rot}\boldsymbol{\delta}, p-\Pi_n p) = 0. \tag{4.39}$$

Let $\boldsymbol{\Sigma}_n$ be the L^2 projection onto the space $\boldsymbol{K}_n$ defined in (4.5). Using the orthogonality property (4.4) of the reduction operator we can now estimate the final two terms in (4.34) as follows:

$$\begin{aligned}|(\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{\eta}) - (\nabla\psi,\boldsymbol{\eta})| &= |(\nabla(\psi_n-\psi), \boldsymbol{\Pi}_n\boldsymbol{\eta}) + (\nabla\psi, \boldsymbol{\Pi}_n\boldsymbol{\eta}-\boldsymbol{\eta})|\\ &= |(\nabla(\psi_n-\psi), \boldsymbol{\Pi}_n\boldsymbol{\eta}) + (\nabla\psi - \boldsymbol{\Sigma}_n\nabla\psi, \boldsymbol{\Pi}_n\boldsymbol{\eta}-\boldsymbol{\eta})|\\ &\le \|\psi_n-\psi\|_1\|\boldsymbol{\Pi}_n\boldsymbol{\eta}\|_0 + \|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|\boldsymbol{\Pi}_n\boldsymbol{\eta}-\boldsymbol{\eta}\|_0\\ &\le C\|\psi_n-\psi\|_1\|\boldsymbol{\eta}\|_1 + C_\delta hk^{-1/2+\delta}\|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|\boldsymbol{\eta}\|_1,\\ &\le (Ch^l k^{-m+1} + C_\delta h^l k^{-m+1/2+\delta})\|\psi\|_m\|\boldsymbol{\eta}\|_1\\ &\le Ch^l k^{-m+1}\|\psi\|_m.\end{aligned} \tag{4.40}$$

Collecting the above estimates then gives

$$\begin{aligned}&|||(\boldsymbol{\beta}_n-\boldsymbol{I}_n\boldsymbol{\beta}, \boldsymbol{\alpha}_n-\boldsymbol{\Pi}_n\boldsymbol{\alpha})||| + k^{-1/2}\|p_n-\Pi_n p\|_0\\ &\le C_\delta h^l k^{-m+3/2+\delta}(\|\boldsymbol{\beta}\|_m + \|\psi\|_m + t\|p\|_m + \|p\|_{m-1}).\end{aligned} \tag{4.41}$$

We now recall (cf. (2.25)) that $t^2\operatorname{rot}\boldsymbol{\alpha} = -\operatorname{rot}\boldsymbol{\beta}$. Hence, we can estimate as follows:

$$\begin{aligned}t^2\|\operatorname{rot}(\boldsymbol{\alpha}-\boldsymbol{\Pi}_n\boldsymbol{\alpha})\|_0 &= t^2\|\operatorname{rot}\boldsymbol{\alpha} - \Pi_n\operatorname{rot}\boldsymbol{\alpha}\|_0\\ &= \|\operatorname{rot}\boldsymbol{\beta} - \Pi_n\operatorname{rot}\boldsymbol{\beta}\|_0 \le Ch^l k^{-m+1}\|\boldsymbol{\beta}\|_m.\end{aligned} \tag{4.42}$$

Using the triangle inequality together with (4.41), (4.42), Lemma 4.1, (4.30), and (4.31) then gives

$$\begin{aligned}&|||(\boldsymbol{\beta}-\boldsymbol{\beta}_n, \boldsymbol{\alpha}-\boldsymbol{\alpha}_n)||| + k^{-1/2}\|p-p_n\|_0\\ &\le C_\delta h^l k^{-m+3/2+\delta}(\|\boldsymbol{\beta}\|_m + \|\psi\|_m + t\|p\|_m + \|p\|_{m-1}).\end{aligned} \tag{4.43}$$

From the stability inequality of Lemma 4.4 and (4.21) it also follows that

$$
\begin{aligned}
(4.44)\qquad |||(\boldsymbol{\beta}_n, \boldsymbol{\alpha}_n)||| + k^{-1/2}\|p_n\|_0 &\le C(\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{\eta}) \\
&\le\ C\|\psi\|_1\|\boldsymbol{\Pi}_n\boldsymbol{\eta}\|_0 \le C\|\psi\|_1\|\boldsymbol{\eta}\|_1 \le C\|\psi\|_1,
\end{aligned}
$$

and hence the triangle inequality gives

$$
(4.45)\qquad |||(\boldsymbol{\beta}-\boldsymbol{\beta}_n, \boldsymbol{\alpha}-\boldsymbol{\alpha}_n)||| + k^{-1/2}\|p-p_n\|_0 \le C(\|\boldsymbol{\beta}\|_1 + \|\psi\|_1 + t\|p\|_1 + \|p\|_0).
$$

As in [20, Lemma 5.2 and Theorem 5.2], we then interpolate between (4.45) and (4.43) using the K-method [16], which gives

$$
\begin{aligned}
(4.46)\qquad &|||(\boldsymbol{\beta}-\boldsymbol{\beta}_n, \boldsymbol{\alpha}-\boldsymbol{\alpha}_n)||| + k^{-1/2}\|p-p_n\|_0 \\
&\le\ C_\epsilon h^l k^{-m+1+\epsilon}(\|\boldsymbol{\beta}\|_m + \|\psi\|_m + t\|p\|_m + \|p\|_{m-1}),
\end{aligned}
$$

with $\epsilon > 0$ and $l = \min\{k, m-1\}$. The asserted estimate for the rotation is then proved.

Next, we let $I_n w \in W_n$ be the H^1 projection of w for which we have [4]

$$
(4.47)\qquad \|w - I_n w\|_1 \le C h^{\tilde{l}} k^{-m+1}\|w\|_m,
$$

where $\tilde{l} = l$ for Methods 1, 2, and 4, $\tilde{l} = \min\{k+1, m-1\}$ for Method 3. Since $(\nabla\psi_n, \nabla v) = (\nabla\psi, \nabla v) = (f, v)$, (2.27) and (4.17) give

$$
\begin{aligned}
\|\nabla(I_n w - w_n)\|_0^2 &= (\nabla(I_n w - w_n), \nabla(I_n w - w_n)) \\
(4.48)\qquad &= (\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}_n, \nabla(I_n w - w_n)) \\
&\le \|\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}_n\|_0\|\nabla(I_n w - w_n)\|_0.
\end{aligned}
$$

Hence,

$$
(4.49)\qquad \|\nabla(I_n w - w_n)\|_0 \le \|\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}_n\|_0.
$$

From Lemma 4.1 we have

$$
(4.50)\qquad \|(\boldsymbol{I} - \boldsymbol{\Pi}_n)\boldsymbol{\eta}\|_0 \le C_\delta h k^{-1/2+\delta}\|\boldsymbol{\eta}\|_1, \quad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2,
$$

and hence we get

$$
\begin{aligned}
\|\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}_n\|_0 &= \|(\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}) + (\boldsymbol{\beta} - \boldsymbol{\beta}_n) - (\boldsymbol{I} - \boldsymbol{\Pi}_n)(\boldsymbol{\beta} - \boldsymbol{\beta}_n)\|_0 \\
(4.51)\qquad &\le\ \|\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_0 + \|(\boldsymbol{I} - \boldsymbol{\Pi}_n)(\boldsymbol{\beta} - \boldsymbol{\beta}_n)\|_0 \\
&\le\ \|\boldsymbol{\beta} - \boldsymbol{\Pi}_n\boldsymbol{\beta}\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_0 + C_\delta h k^{-1/2+\delta}\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1.
\end{aligned}
$$

Using the triangle inequality together with (4.47), (4.49), (4.51), Lemma 4.1, and (4.46) then gives the asserted estimate

$$
(4.52)\qquad \|w - w_n\|_1 \le C_\epsilon h^l k^{-m+1+\epsilon}(\|w\|_m + \|\boldsymbol{\beta}\|_m + \|\psi\|_m + t\|\boldsymbol{\alpha}\|_{m-1} + \|p\|_{m-1}),
$$

with $\epsilon > 0$ and $l = \min\{k, m-1\}$. □

THEOREM 4.3. *Suppose that Ω is convex and that we have $w \in H^m(\Omega)$, $\boldsymbol{\beta} \in [H^m(\Omega)]^2$, $\psi \in H^m(\Omega)$, and $p \in H^m(\Omega)$. For every $\epsilon > 0$ there is a positive constant C_ϵ such that*

$$\|w-w_n\|_0+\|\boldsymbol{\beta}-\boldsymbol{\beta}_n\|_0 \le C_\epsilon h^{l+1}k^{-m+1/2+\epsilon}(\|w\|_m+\|\boldsymbol{\beta}\|_m+\|\psi\|_m+\|p\|_{m-1}+t\|p\|_m),$$

with $l = \min\{k, m-1\}$. If $w \in H^{m+1}(\Omega)$, for the solution with Method 3, we additionally have

$$\|w - w_n\|_1 \le C_\epsilon h^{l+1}k^{-m+1/2+\epsilon}(\|w\|_{m+1} + \|\boldsymbol{\beta}\|_m + \|\psi\|_m + \|p\|_{m-1} + t\|p\|_m).$$

Proof. Since Ω is convex, it holds that

$$\|\psi - \psi_n\|_0 \le Ch^{l+1}k^{-m}\|\psi\|_m. \tag{4.53}$$

Next, we let $(\boldsymbol{\theta}, \boldsymbol{\gamma}, r) \in [H_0^1(\Omega)]^2 \times \boldsymbol{H}_0(\mathrm{rot}\,;\Omega) \times L_0^2(\Omega)$ be the solution to the problem

$$\mathcal{B}(\boldsymbol{\theta}, \boldsymbol{\gamma}, r; \boldsymbol{\eta}, \boldsymbol{\delta}, q) = (\boldsymbol{\beta}-\boldsymbol{\beta}_n, \boldsymbol{\eta}) \quad \forall (\boldsymbol{\eta}, \boldsymbol{\delta}, q) \in [H_0^1(\Omega)]^2 \times \boldsymbol{H}_0(\mathrm{rot}\,;\Omega) \times L_0^2(\Omega). \tag{4.54}$$

With $\boldsymbol{\eta} = \boldsymbol{\beta} - \boldsymbol{\beta}_n$, $\boldsymbol{\delta} = \boldsymbol{\alpha} - \boldsymbol{\alpha}_n$, and $q = p - p_n$, we get

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_0^2 = \mathcal{B}(\boldsymbol{\theta}, \boldsymbol{\gamma}, r; \boldsymbol{\beta} - \boldsymbol{\beta}_n, \boldsymbol{\alpha} - \boldsymbol{\alpha}_n, p - p_n). \tag{4.55}$$

We let $\Pi_n r \in R_n$ be the L^2 projection of r and let $\boldsymbol{I}_n\boldsymbol{\theta} \in \boldsymbol{B}_n$ be the H^1 projection of $\boldsymbol{\theta}$. From (4.20) and (4.21) it follows

$$\mathcal{B}(\boldsymbol{\beta} - \boldsymbol{\beta}_n, \boldsymbol{\alpha} - \boldsymbol{\alpha}_n, p - p_n; \boldsymbol{I}_n\boldsymbol{\theta}, \boldsymbol{\Pi}_n\boldsymbol{\gamma}, \Pi_n r) = (\nabla\psi, \boldsymbol{I}_n\boldsymbol{\theta}) - (\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{I}_n\boldsymbol{\theta}). \tag{4.56}$$

Using the symmetry of the bilinear form $\mathcal{B}$ we then get

$$\begin{aligned}\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_0^2 = {}& \mathcal{B}(\boldsymbol{\theta} - \boldsymbol{I}_n\boldsymbol{\theta}, \boldsymbol{\gamma} - \boldsymbol{\Pi}_n\boldsymbol{\gamma}, r - \Pi_n r; \boldsymbol{\beta} - \boldsymbol{\beta}_n, \boldsymbol{\alpha} - \boldsymbol{\alpha}_n, p - p_n) \\ & + (\nabla\psi, \boldsymbol{I}_n\boldsymbol{\theta}) - (\nabla\psi_n, \boldsymbol{\Pi}_n\boldsymbol{I}_n\boldsymbol{\theta}).\end{aligned} \tag{4.57}$$

For the different terms in the bilinear form we obtain

$$\begin{aligned}|a(\boldsymbol{\theta} - \boldsymbol{I}_n\boldsymbol{\theta}, \boldsymbol{\beta} - \boldsymbol{\beta}_n)| &\le C\|\boldsymbol{\theta} - \boldsymbol{I}_n\boldsymbol{\theta}\|_1\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1 \\ &\le Chk^{-1}\|\boldsymbol{\theta}\|_2\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1,\end{aligned} \tag{4.58}$$

$$\begin{aligned}|(\mathrm{rot}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_n), r - \Pi_n r)| &\le C\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1\|r - \Pi_n r\|_0 \\ &\le Chk^{-1}\|r\|_1\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_1,\end{aligned} \tag{4.59}$$

$$\begin{aligned}|(\mathrm{rot}\,(\boldsymbol{\theta} - \boldsymbol{I}_n\boldsymbol{\theta}), p - p_n)| &\le C\|\boldsymbol{\theta} - \boldsymbol{I}_n\boldsymbol{\theta}\|_1\|p - p_n\|_0 \\ &\le Chk^{-1}\|\boldsymbol{\theta}\|_2\|p - p_n\|_0,\end{aligned} \tag{4.60}$$

$$\begin{aligned}|t^2(\boldsymbol{\gamma} - \boldsymbol{\Pi}_n\boldsymbol{\gamma}, \boldsymbol{\alpha} - \boldsymbol{\alpha}_n)| &\le t\|\boldsymbol{\gamma} - \boldsymbol{\Pi}_n\boldsymbol{\gamma}\|_0 \cdot t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_n\|_0 \\ &\le C_\delta hk^{-1/2+\delta}\, t\|\boldsymbol{\gamma}\|_1 \cdot t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_n\|_0,\end{aligned} \tag{4.61}$$

$$|t^2(\mathrm{rot}\,(\boldsymbol{\gamma}-\boldsymbol{\Pi}_n\boldsymbol{\gamma}),p-p_n)| = |((\mathrm{rot}\,\boldsymbol{\theta}-\Pi_n\mathrm{rot}\,\boldsymbol{\theta}),p-p_n)| \le Chk^{-1}\|\boldsymbol{\theta}\|_2\|p-p_n\|_0, \tag{4.62}$$

and

$$|t^2(\mathrm{rot}\,(\boldsymbol{\alpha}-\boldsymbol{\alpha}_n),r-\Pi_n r)| \le t^2\|\mathrm{rot}\,(\boldsymbol{\alpha}-\boldsymbol{\alpha}_n)\|_0\|r-\Pi_n r\|_0 \le Chk^{-1}\|r\|_1\cdot t^2\|\mathrm{rot}\,(\boldsymbol{\alpha}-\boldsymbol{\alpha}_n)\|_0. \tag{4.63}$$

The second term in (4.57) we first write as

$$(\nabla\psi,\boldsymbol{I}_n\boldsymbol{\theta})-(\nabla\psi_n,\boldsymbol{\Pi}_n\boldsymbol{I}_n\boldsymbol{\theta}) = (\nabla(\psi-\psi_n),\boldsymbol{I}_n\boldsymbol{\theta})+(\nabla\psi_n,(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{I}_n\boldsymbol{\theta}). \tag{4.64}$$

The first term above is directly estimated by integration by parts:

$$(\nabla(\psi-\psi_n),\boldsymbol{I}_n\boldsymbol{\theta}) = (\psi_n-\psi,\mathrm{div}\,\boldsymbol{I}_n\boldsymbol{\theta}) \le \|\psi-\psi_n\|_0\|\boldsymbol{I}_n\boldsymbol{\theta}\|_1 \le C\|\psi-\psi_n\|_0\|\boldsymbol{\theta}\|_1. \tag{4.65}$$

The second term we write as the sum

$$(\nabla\psi_n,(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{I}_n\boldsymbol{\theta}) = (\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)(\boldsymbol{I}_n\boldsymbol{\theta}-\boldsymbol{\theta})) + (\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{\theta})+(\nabla(\psi_n-\psi),(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{I}_n\boldsymbol{\theta}). \tag{4.66}$$

Using the L^2 projection $\boldsymbol{\Sigma}_n$ onto the space $\boldsymbol{K}_n$ and the orthogonality property (4.4), we estimate the above terms as follows:

$$\begin{aligned}
&(\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)(\boldsymbol{I}_n\boldsymbol{\theta}-\boldsymbol{\theta}))\\
&\quad= (\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)(\boldsymbol{I}_n\boldsymbol{\theta}-\boldsymbol{\theta}))\\
&\quad\le \|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|(\boldsymbol{I}-\boldsymbol{\Pi}_n)(\boldsymbol{I}_n\boldsymbol{\theta}-\boldsymbol{\theta})\|_0\\
&\quad\le C_\delta hk^{-1/2+\delta}\|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|\boldsymbol{\theta}-\boldsymbol{I}_n\boldsymbol{\theta}\|_1\\
&\quad\le C_\delta h^2k^{-3/2+\delta}\|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|\boldsymbol{\theta}\|_2\\
&\quad\le C_\delta h^{l+1}k^{-m-1/2+\delta}\|\psi\|_m\|\boldsymbol{\theta}\|_2,
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
(\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{\theta}) &= (\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi,(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{\theta})\\
&\le \|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\|(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{\theta}\|_0\\
&\le C_\delta h^2k^{-3/2+\delta}\|\boldsymbol{\theta}\|_2\|\nabla\psi-\boldsymbol{\Sigma}_n\nabla\psi\|_0\\
&\le C_\delta h^{l+1}k^{-m-1/2+\delta}\|\psi\|_m\|\boldsymbol{\theta}\|_2,
\end{aligned} \tag{4.68}$$

$$(\nabla(\psi_n-\psi),(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{I}_n\boldsymbol{\theta}) \le \|\nabla(\psi_n-\psi)\|_0\|(\boldsymbol{I}-\boldsymbol{\Pi}_n)\boldsymbol{I}_n\boldsymbol{\theta}\|_0 \le C_\delta hk^{-1/2+\delta}\|\boldsymbol{I}_n\boldsymbol{\theta}\|_1\|\psi-\psi_n\|_1 \le C_\delta hk^{-1/2+\delta}\|\boldsymbol{\theta}\|_1\|\psi-\psi_n\|_1. \tag{4.69}$$

Collecting the estimates above we get

(4.70)

$$\begin{aligned}
&\|\boldsymbol{\beta}-\boldsymbol{\beta}_n\|_0^2\\
&\le C_\delta h k^{-1/2+\delta}(\|\boldsymbol{\theta}\|_2+\|r\|_1+t\|\boldsymbol{\gamma}\|_1)(|||(\boldsymbol{\beta}-\boldsymbol{\beta}_n,\boldsymbol{\alpha}-\boldsymbol{\alpha}_n)|||+k^{-1/2}\|p-p_n\|_0)\\
&\quad + C_\delta h^{l+1}k^{-m-1/2+\delta}\|\psi\|_m\|\boldsymbol{\theta}\|_2+\|\psi-\psi_n\|_0\|\boldsymbol{\theta}\|_1+C_\delta h k^{-1/2+\delta}\|\psi-\psi_n\|_1\|\boldsymbol{\theta}\|_1.
\end{aligned}$$

From Theorem 2.1 it holds that

$$\|\boldsymbol{\theta}\|_2+\|r\|_1+t\|\boldsymbol{\gamma}\|_1\le C\|\boldsymbol{\beta}-\boldsymbol{\beta}_n\|_0. \tag{4.71}$$

Hence, (4.46) and (4.53) give the asserted L^2 estimate for the rotation.

The estimate for $\|w-w_n\|_1$ for Method 3 now follows from (4.47), (4.49), (4.51), and Lemma 4.1.

Next, we will derive the L^2 estimate for the deflection. Let $z\in H_0^1(\Omega)$ be the solution to

$$(\nabla z,\nabla v)=(w-w_n,v),\quad v\in H_0^1(\Omega). \tag{4.72}$$

Using (2.23), (2.27), (4.13), and (4.17) we then have

$$\begin{aligned}
\|w-w_n\|_0^2 &= (\nabla(w-w_n),\nabla z)\\
&= (\nabla(w-w_n),\nabla(z-I_nz))+(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta}_n,\nabla I_nz)\\
&\le Chk^{-1}\|z\|_2\|\nabla(w-w_n)\|_0+(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta}_n,\nabla I_nz).
\end{aligned} \tag{4.73}$$

Again let $\boldsymbol{\Sigma}_n$ be the L^2 projection onto the space $\boldsymbol{K}_n$ as defined in (4.5). Using the orthogonality property (4.4) we then get

$$\begin{aligned}
&(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta}_n,\nabla I_nz)\\
&\quad=(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta},\nabla(I_nz-z))+(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta},\nabla z)+(\boldsymbol{\Pi}_n(\boldsymbol{\beta}-\boldsymbol{\beta}_n),\nabla I_nz)\\
&\quad=(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta},\nabla(I_nz-z))+(\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta},\nabla z-\boldsymbol{\Sigma}_n\nabla z)+(\boldsymbol{\Pi}_n(\boldsymbol{\beta}-\boldsymbol{\beta}_n),\nabla I_nz)\\
&\quad\le\|\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta}\|_0\|\nabla(I_nz-z)\|_0+\|\boldsymbol{\beta}-\boldsymbol{\Pi}_n\boldsymbol{\beta}\|_0\|\nabla z-\boldsymbol{\Sigma}_n\nabla z\|_0\\
&\qquad\qquad+\|\boldsymbol{\Pi}_n(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0\|\nabla I_nz\|_0\\
&\quad\le C_\delta h^{l+1}k^{-1/2+\delta-m}\|\boldsymbol{\beta}\|_m\|z\|_2+\|\boldsymbol{\Pi}_n(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0\|z\|_1.
\end{aligned}$$

We now estimate by the last term above

$$\begin{aligned}
\|\boldsymbol{\Pi}_n(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0&\le\|(\boldsymbol{I}-\boldsymbol{\Pi}_n)(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0+\|(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0\\
&\le Chk^{-1/2+\delta}\|(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_1+\|(\boldsymbol{\beta}-\boldsymbol{\beta}_n)\|_0.
\end{aligned} \tag{4.74}$$

The assertion now follows from the previous estimates and the regularity estimate $\|z\|_2\le C\|w-w_n\|_0$. □

Next, we will prove the error estimates for the approximations to the shear force. We have assumed that the mesh is regular and consists of parallelograms, and hence the mesh is *quasi uniform*; i.e., it holds that $h\le Ch_K\ \forall K\in\mathcal{C}_h$. This means that we

can use global inverse inequalities and with them we can state the estimates for the shear force in the L^2-norm.

THEOREM 4.4. *Suppose that we have* $w \in H^m(\Omega)$, $\boldsymbol{\beta} \in [H^m(\Omega)]^2$, $\psi \in H^m(\Omega)$, *and* $p \in H^m(\Omega)$. *For every* $\epsilon > 0$ *there is a positive constant* C_ϵ *such that*

$$\|\boldsymbol{q} - \boldsymbol{q}_n\|_0 \le C_\epsilon h^{l-1} k^{-m+7/2+\epsilon}(\|w\|_m + \|\boldsymbol{\beta}\|_m + \|\psi\|_m + \|p\|_{m-1} + t\|p\|_m),$$

and

$$\|\boldsymbol{q} - \boldsymbol{q}_n^e\|_0 \le C_\epsilon h^{l-1} k^{-m+3+\epsilon}(\|w\|_m + \|\boldsymbol{\beta}\|_m + \|\psi\|_m + \|p\|_{m-1} + t\|p\|_m),$$

with $l = \min\{k, m-1\}$.

Proof. Let us first consider the case when the shear is calculated through the constitutive relation. We recall that

$$\boldsymbol{q}_n = \boldsymbol{\alpha}_n + \nabla\psi_n \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{\alpha} + \nabla\psi. \tag{4.75}$$

Next, (2.26) and (4.16) give, for all $\boldsymbol{\delta} \in \boldsymbol{\Gamma}_n$,

$$(\boldsymbol{\alpha} - \boldsymbol{\alpha}_n, \boldsymbol{\delta}) = (p - p_n, \operatorname{rot} \boldsymbol{\delta}). \tag{4.76}$$

Let $\tilde{\boldsymbol{\alpha}} \in \boldsymbol{\Gamma}_n$ be the L^2 projection of $\boldsymbol{\alpha}$. Choosing $\boldsymbol{\delta} = \tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n$ in the relation above and using an inverse estimate [3] we then get

$$\begin{aligned} \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n\|_0^2 &= (\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n, \tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n) = (\boldsymbol{\alpha} - \boldsymbol{\alpha}_n, \tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n) \\ &= (p - p_n, \operatorname{rot}(\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n)) \le \|p - p_n\|_0 \|\operatorname{rot}(\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n)\|_0 \\ &\le Ch^{-1}k^2\|p - p_n\|_0\|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n\|_0, \end{aligned} \tag{4.77}$$

and hence

$$\|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_n\|_0 \le Ch^{-1}k^2\|p - p_n\|_0. \tag{4.78}$$

For the L^2 projection we have the estimate

$$\|\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}\|_0 \le Ch^{l-1}k^{-m+2}\|\boldsymbol{\alpha}\|_{m-2} \le Ch^{l-1}k^{-m+2}\|p\|_{m-1}. \tag{4.79}$$

The asserted estimate then follows from (4.46) and (4.29).

For the shear calculated from the equilibrium condition, we note that for any $K \in \mathcal{C}_h$, using an inverse estimate,

$$\begin{aligned} &\|\boldsymbol{L}\boldsymbol{\beta} - \boldsymbol{L}\boldsymbol{\beta}_n\|_{0,K} \\ &\quad\le C\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_{2,K} \le C(\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{2,K} + h^{-1}k^2\|\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_n\|_{1,K}) \\ &\quad\le C(\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{2,K} + h^{-1}k^2\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{1,K} + h^{-1}k^2\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_{1,K}). \end{aligned} \tag{4.80}$$

Let $\tilde{\boldsymbol{\beta}} \in \boldsymbol{B}_k(K)$ be the projection of $\boldsymbol{\beta}$ in the $[H^2(K)]^2$-norm, satisfying

$$\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{2,K} + h^{-1}k\|\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}\|_{1,K} \le Ch^{l-1}k^{-m+2}\|\boldsymbol{\beta}\|_{m,K}. \tag{4.81}$$

(Such a $\tilde{\boldsymbol{\beta}}$ may be found, using the results of [22], together with scaling.) Then

$$\|\boldsymbol{L}\boldsymbol{\beta} - \boldsymbol{L}\boldsymbol{\beta}_n\|_{0,K} \le Ch^{l-1}k^{-m+3}\|\boldsymbol{\beta}\|_{m,K} + Ch^{-1}k^2\|\boldsymbol{\beta} - \boldsymbol{\beta}_n\|_{1,K}. \tag{4.82}$$

Hence, using Theorem 4.2,

$$\|\boldsymbol{q}-\boldsymbol{q}_n^e\|_0 = \left(\sum_{K\in\mathcal{C}_h}\|\boldsymbol{L}\boldsymbol{\beta}-\boldsymbol{L}\boldsymbol{\beta}_n\|_{0,K}^2\right)^{1/2} \le C\left(\sum_{K\in\mathcal{C}_h}\|\boldsymbol{\beta}-\boldsymbol{\beta}_n\|_{2,K}^2\right)^{1/2} \tag{4.83}$$

$$\le C_\epsilon h^{l-1}k^{-m+3+\epsilon}(\|w\|_m+\|\boldsymbol{\beta}\|_m+\|\psi\|_m+\|p\|_{m-1}+t\|p\|_m). \qquad \square$$

Remark 4.1. We note that the estimates in Theorem 4.4 are of lower order than the asymptotic rate of best approximation, both in terms of h and k. In h, the above rates are the ones obtained in [12], and it is likely that these will be the actual rates observed. With respect to p, the loss is more severe, due to the fact that we lose powers when using the inverse inequality. We do not know if the rates in p obtained here can be improved.

Remark 4.2. The *hp* estimates derived in this section also show that for the h version, using a higher-degree element from the family can reduce the error due to a reduction in the "constant" present in h version error estimates.

5. Analysis of the BDFM space. In this final section, we will give the details of the error analysis of the BDFM interpolation operator. We first recall the definition of the space and the interpolation operator in coordinate form [13] on the reference element $\hat{K}$.

Let $\hat{\boldsymbol{\sigma}}=(\sigma_1,\sigma_2)$ and

$$\boldsymbol{\Gamma}_k(\hat{K}) = \{\ \hat{\boldsymbol{\sigma}} \mid \hat{\boldsymbol{\sigma}}\in[P_k(\hat{K})\setminus\{\xi^k\}]\times[P_k(\hat{K})\setminus\{\eta^k\}]\ \ \}. \tag{5.1}$$

Denoting $\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}=(\tilde{\sigma}_1,\tilde{\sigma}_2)$, the reduction operator is defined by

$$\int_{-1}^1\int_{-1}^1(\sigma_1-\tilde{\sigma}_1)\phi(\xi,\eta)\,d\xi d\eta=0 \quad \forall\phi\in P_{k-2}(\hat{K}), \tag{5.2}$$

$$\int_{-1}^1(\sigma_1-\tilde{\sigma}_1)(\xi,\pm1)\phi(\xi)\,d\xi=0 \quad \forall\phi\in P_{k-1}([-1,1]), \tag{5.3}$$

and

$$\int_{-1}^1\int_{-1}^1(\sigma_2-\tilde{\sigma}_2)\phi(\xi,\eta)\,d\xi d\eta=0 \quad \forall\phi\in P_{k-2}(\hat{K}), \tag{5.4}$$

$$\int_{-1}^1(\sigma_2-\tilde{\sigma}_1)(\pm1,\eta)\phi(\eta)\,d\eta=0 \quad \forall\phi\in P_{k-1}([-1,1]). \tag{5.5}$$

Let us now derive an estimate for $\hat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}$ over $\hat{K}$.

LEMMA 5.1. *For any $\hat{\boldsymbol{\sigma}}\in[H^r(\hat{K})]^2$, $r>1/2$, let $\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}=(\tilde{\sigma}_1,\tilde{\sigma}_2)$ be defined as in* (5.2)–(5.5). *Then for any $0<\epsilon<r-1/2$, there exists a constant C_ϵ, independent of k and $\hat{\boldsymbol{\sigma}}$, such that for all $\hat{\boldsymbol{\sigma}}\in[H^r(\hat{K})]^2$,*

$$\|\hat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}\|_{0,\hat{K}}\le C_\epsilon k^{1/2-r+\epsilon}\|\hat{\boldsymbol{\sigma}}\|_{r,\hat{K}}. \tag{5.6}$$

Proof. We expand σ_1 and $\tilde{\sigma}_1$ in terms of the Legendre polynomials L_i, viz.,

$$\tilde{\sigma}_1=\sum_{i,j=0}^{\infty}a_{ij}L_i(\xi)L_j(\eta) \tag{5.7}$$

and

$$\tilde{\sigma}_1 = \sum_{i=0}^{k-1} \sum_{j=0}^{k-i} b_{ij} L_i(\xi) L_j(\eta). \tag{5.8}$$

The Legendre polynomials we normalize in the usual way:

$$\int_{-1}^{1} L_i(\xi) L_j(\xi)\, d\xi = \gamma_i \delta_{ij}, \tag{5.9}$$

where δ_{ij} is the Kronecker symbol and

$$\gamma_i = \frac{2}{2i+1}\,. \tag{5.10}$$

Then

$$\|\sigma_1\|^2_{0,\hat{K}} = \sum_{i,j=0}^{\infty} \gamma_i \gamma_j a_{ij}^2 \tag{5.11}$$

and for $r \geq 0$ (cf. [23])

$$\|\sigma_1\|^2_{r,\hat{K}} \geq C \sum_{i,j=0}^{\infty} (1 + i^2 + j^2)^r \gamma_i \gamma_j a_{ij}^2. \tag{5.12}$$

Now condition (5.2) directly gives

$$b_{ij} = a_{ij} \quad \text{for} \quad 0 \leq i + j \leq k - 2. \tag{5.13}$$

Since $L_j(1) = 1$, using (5.7)–(5.8) with $\eta = 1$, we obtain

$$\int_{-1}^{1} \sigma_1(\xi, 1) L_l(\xi)\, d\xi = \gamma_l \sum_{j=0}^{\infty} a_{lj} \tag{5.14}$$

and

$$\int_{-1}^{1} \tilde{\sigma}_1(\xi, 1) L_l(\xi)\, d\xi = \gamma_l \sum_{j=0}^{k-l} b_{lj}\,. \tag{5.15}$$

Condition (5.3) then implies

$$\sum_{j=0}^{k-l} b_{lj} = \sum_{j=0}^{\infty} a_{lj} \tag{5.16}$$

for $0 \leq l \leq k - 1$. Using (5.13) this gives

$$b_{l,k-l-1} + b_{l,k-l} = \sum_{j=k-l-1}^{\infty} a_{lj}\,. \tag{5.17}$$

Since $L_j(-1) = (-1)^j$, we obtain in the same way

$$(-1)^{k-l-1} b_{l,k-l-1} + (-1)^{k-l} b_{l,k-l} = \sum_{j=k-l-1}^{\infty} (-1)^j a_{lj}\,, \tag{5.18}$$

which gives

$$b_{l,k-l-1} - b_{l,k-l} = \sum_{j=k-l-1}^{\infty} (-1)^{j-k+l+1} a_{lj} \text{ for } 0 \le l \le k-1. \tag{5.19}$$

From (5.17) and (5.19) we then get

$$b_{l,k-l-1} = \frac{1}{2} \sum_{j=k-l-1}^{\infty} (1 + (-1)^{j-k+l+1}) a_{lj} \quad \left(= \sum_{t=0}^{\infty} a_{l,k-l-1+2t} \right), \tag{5.20}$$

and

$$\begin{aligned} b_{l,k-l} &= \frac{1}{2} \sum_{j=k-l-1}^{\infty} (1 - (-1)^{j-k+l+1}) a_{lj} \\ &= \frac{1}{2} \sum_{j=k-l}^{\infty} (1 + (-1)^{j-k+l}) a_{lj} \qquad \left(= \sum_{t=0}^{\infty} a_{l,k-l+2t} \right) \end{aligned} \tag{5.21}$$

for $0 \le l \le k-1$.

The coefficients in the expansion (5.8) for the interpolant $\tilde{\sigma}_1$ are then explicitly given by (5.13), (5.20), and (5.21).

Using (5.11) we have (where $b_{ij} = 0$, when $i + j > k$ or $i = k$)

$$\begin{aligned} \|\sigma_1 - \tilde{\sigma}_1\|^2_{0,\hat{K}} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \gamma_i \gamma_j (a_{ij} - b_{ij})^2 \\ &= \left(\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-2} + \sum_{i=0}^{k-1} \sum_{j=k-i-1}^{k-i} + \sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} + \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \right) \gamma_i \gamma_j (a_{ij} - b_{ij})^2 \\ &= \sum_{i=0}^{k-1} \sum_{j=k-i-1}^{k-i} \gamma_i \gamma_j (a_{ij} - b_{ij})^2 + \left(\sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} + \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \right) \gamma_i \gamma_j a_{ij}^2 \\ &= \sum_{i=0}^{k-1} \gamma_i \gamma_{k-i-1} (a_{i,k-i-1} - b_{i,k-i-1})^2 + \sum_{i=0}^{k-1} \gamma_i \gamma_{k-i} (a_{i,k-i} - b_{i,k-i})^2 \\ &\quad + \left(\sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} + \sum_{i=k}^{\infty} \sum_{j=0}^{\infty} \right) \gamma_i \gamma_j a_{ij}^2 \\ &= S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{5.22}$$

Here we used the relation (5.13) to drop the first sum.

Now let us estimate the sums above. By (5.20) we obtain

$$(a_{i,k-i-1} - b_{i,k-i-1})^2 = \left(\frac{1}{2} \sum_{j=k-i+1}^{\infty} (1 + (-1)^{j-k+i+1}) a_{ij} \right)^2. \tag{5.23}$$

By the Schwarz inequality this gives for $r > 1$

$$(a_{i,k-i-1} - b_{i,k-i-1})^2 \le \left(\sum_{j=k-i+1}^{\infty} |a_{ij}| \right)^2$$

$$\le \left(\sum_{j=k-i+1}^{\infty} \gamma_j^{-1} (1 + i^2 + j^2)^{-r} \right) \left(\sum_{j=k-i+1}^{\infty} (1 + i^2 + j^2)^r \gamma_j a_{ij}^2 \right)$$

$$\le Ck^{2-2r} \left(\sum_{j=k-i+1}^{\infty} (1 + i^2 + j^2)^r \gamma_j a_{ij}^2 \right). \tag{5.24}$$

Above we used (5.10) and estimated as follows:

$$\sum_{j=k-i+1}^{\infty} (\gamma_j^{-1} (1 + i^2 + j^2)^{-r})$$

$$= \frac{1}{2} \sum_{j=k-i+1}^{\infty} (2j+1)(1 + i^2 + j^2)^{-r} \le C \sum_{j=k-i+1}^{\infty} j (1 + i^2 + j^2)^{-r}$$

$$\le C \int_{k-i}^{\infty} x (1 + i^2 + x^2)^{-r} \, dx \le C (1 + i^2 + (k-i)^2)^{1-r}$$

$$\le Ck^{2-2r}. \tag{5.25}$$

Using the fact that $\gamma_{k-i-1} \le C$, (5.24) gives

$$S_1 \le Ck^{2-2r} \sum_{i=0}^{k-1} \gamma_i \gamma_{k-i-1} \left(\sum_{j=k-i+1}^{\infty} (1 + i^2 + j^2)^r \gamma_j a_{ij}^2 \right)$$

$$\le Ck^{2-2r} \sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} (1 + i^2 + j^2)^r \gamma_i \gamma_j a_{ij}^2$$

$$\le Ck^{2-2r} \|\sigma_1\|_{r,\hat{K}}^2. \tag{5.26}$$

For the second sum we similarly get

$$S_2 \le Ck^{2-2r} \|\sigma_1\|_{r,\hat{K}}^2 . \tag{5.27}$$

In the sum S_3, we have $i + j > k$, and we thus get for any $r \ge 0$

$$S_3 = \sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} \gamma_i \gamma_j a_{ij}^2 = k^{-2r} \sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} k^{2r} \gamma_i \gamma_j a_{ij}^2$$

$$\le k^{-2r} \sum_{i=0}^{k-1} \sum_{j=k-i+1}^{\infty} (i+j)^{2r} \gamma_i \gamma_j a_{ij}^2$$

$$\leq Ck^{-2r}\sum_{i=0}^{k-1}\sum_{j=k-i+1}^{\infty}(1+i^2+j^2)^r\gamma_i\gamma_j a_{ij}^2$$

$$\leq Ck^{-2r}\|\sigma_1\|^2_{r,\hat{K}}. \tag{5.28}$$

In S_4, we have $i \geq k$, which gives for $r \geq 0$

$$S_4 = \sum_{i=k}^{\infty}\sum_{j=0}^{\infty}\gamma_i\gamma_j a_{ij}^2 = k^{-2r}\sum_{i=k}^{\infty}\sum_{j=0}^{\infty}k^{2r}\gamma_i\gamma_j a_{ij}^2$$

$$\leq k^{-2r}\sum_{i=k}^{\infty}\sum_{j=0}^{\infty}i^{2r}\gamma_i\gamma_j a_{ij}^2 \leq Ck^{-2r}\sum_{i=k}^{\infty}\sum_{j=0}^{\infty}(1+i^2+j^2)^r\gamma_i\gamma_j a_{ij}^2$$

$$\leq Ck^{-2r}\|\sigma_1\|^2_{r,\hat{K}}. \tag{5.29}$$

Combining (5.22) and (5.26)–(5.29), we obtain for $r > 1$

$$\|\sigma_1 - \widetilde{\sigma_1}\|_{0,\hat{K}} \leq Ck^{1-r}\|\sigma_1\|_{r,\hat{K}}. \tag{5.30}$$

Let us now obtain a different bound for $\|\sigma_1 - \tilde{\sigma}_1\|_{0,\hat{K}}$. We begin with S_1. Using (5.23), we have

$$(a_{i,k-i-1} - b_{i,k-i-1})^2 \leq C\left(\left(\sum_{j=k-i+1}^{\infty}a_{ij}\right)^2 + \left(\sum_{j=k-i+1}^{\infty}(-1)^{j-k+i+1}a_{ij}\right)^2\right)$$

$$\leq C\left(\left(\sum_{j=0}^{\infty}a_{ij}\right)^2 + \left(\sum_{j=0}^{k-i}a_{ij}\right)^2 + \left(\sum_{j=0}^{\infty}(-1)^j a_{ij}\right)^2 + \left(\sum_{j=0}^{k-i}(-1)^j a_{ij}\right)^2\right). \tag{5.31}$$

Since $\hat{\boldsymbol{\sigma}} \in [H^r(\hat{K})]^2$, $r > 1/2$, we see that $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{t}$, $\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}$, and hence σ_1 lie in $L_2(\partial\hat{K})$. Let $\ell_\pm$ denote the sides of $\partial\hat{K}$ where $\eta = \pm 1$. Then we have by the imbedding theorem and (5.7)

$$\|\sigma_1\|^2_{0,\ell\pm} = \sum_{i=0}^{\infty}\gamma_i\left(\sum_{j=0}^{\infty}(\pm 1)^j a_{ij}\right)^2 \leq \|\sigma_1\|^2_{1/2+\epsilon,\hat{K}} \tag{5.32}$$

for any $0 < \epsilon \leq r - 1/2$. Hence, using (5.32), since $\gamma_{k-i-1} \leq C$, we have

$$\sum_{i=0}^{k-1}\gamma_i\gamma_{k-i-1}\left(\sum_{j=0}^{\infty}(\pm 1)^j a_{ij}\right)^2 \leq C\sum_{i=0}^{\infty}\gamma_i\left(\sum_{j=0}^{\infty}(\pm 1)^j a_{ij}\right)^2 \leq C\|\sigma_1\|^2_{1/2+\epsilon,\hat{K}}. \tag{5.33}$$

Next,

$$\sum_{i=0}^{k-1}\gamma_i\gamma_{k-i-1}\left(\sum_{j=0}^{k-i}(\pm 1)^j a_{ij}\right)^2 \tag{5.34}$$

$$\leq \sum_{i=0}^{k-1}\gamma_i\gamma_{k-i-1}\left(\sum_{j=0}^{k-i}(1+i^2+j^2)^{1/2+\epsilon}\gamma_j a_{ij}^2\right)\left(\sum_{j=0}^{k-i}\gamma_j^{-1}(1+i^2+j^2)^{-1/2-\epsilon}\right).$$

Now since $j \le k-i$ in the second sum above, we have $\gamma_{k-i-1} \le C\gamma_j$ there. Hence,

$$\gamma_{k-i-1}\left(\sum_{j=0}^{k-i}\gamma_j^{-1}(1+i^2+j^2)^{-1/2-\epsilon}\right) \le C\sum_{j=0}^{k-i}(1+i^2+j^2)^{-1/2-\epsilon} \le C.$$

(Above, we have used the boundedness of the series $\sum_{j=0}^{\infty} j^{-1-2\epsilon}$.) Substituting this in (5.34), and using (5.12) with $r = 1/2+\epsilon$, we obtain

$$\sum_{i=0}^{k-1}\gamma_i\gamma_{k-i-1}\left(\sum_{j=0}^{k-1}(\pm 1)^j a_{ij}\right)^2 \le C\|\sigma_1\|^2_{1/2+\epsilon,\hat{K}}. \tag{5.35}$$

Now by (5.31), (5.33), (5.35), and the expression for S_1, we have

$$S_1 \le C\|\sigma_1\|^2_{1/2+\epsilon,\hat{K}}. \tag{5.36}$$

Similarly, it can be shown that S_2 satisfies the same bound (5.36). Hence, combining (5.36) for S_1, S_2 and (5.28)–(5.29) with $r = 1/2+\epsilon$ gives

$$\|\sigma_1 - \widetilde{\sigma_1}\|_{0,\hat{K}} \le C\|\sigma_1\|_{1/2+\epsilon,\hat{K}}\ . \tag{5.37}$$

Now we use a standard interpolation argument for the p version (Theorem 4.2 of [3]) on the operator $\widehat{\boldsymbol{\Pi}}_{k,1}$ (where $\widehat{\boldsymbol{\Pi}}_{k,1}(\sigma_1 - \tilde{\sigma}_1 ix)$). Let $s > 1/2+\epsilon$ and let r in (5.30) be such that $r - 1/2 - \epsilon > (1/2-\epsilon)(s-1/2-\epsilon)/\epsilon$. Then using the K-method of interpolation [9] and the fact that $[H^{1/2+\epsilon}(\hat{K}),\ H^r(\hat{K})]_\theta = H^s(\hat{K})$ for $\theta = (s-1/2-\epsilon)/(r-1/2-\epsilon)$, we obtain from (5.30), (5.37) the estimate

$$\|\sigma_1 - \tilde{\sigma}_1\|_{0,\hat{K}} \le Ck^{1/2-s+2\epsilon}\|\sigma_1\|_{s,\hat{K}}.$$

A similar estimate may be established for $\sigma_2 - \tilde{\sigma}_2$, proving (5.6).

Once Lemma 5.1 is established, the following theorem follows by using a scaling argument. The proof is identical to that for RT elements given in Theorem 3.1 of [23].

THEOREM 5.1. *For the BDFM interpolation operator $\boldsymbol{\Pi}_n$, given arbitrary $\epsilon > 0$, there exists a constant $C = C(\epsilon)$ independent of h, k, $\boldsymbol{\sigma}$ such that with $\mu = \min\{k, r\}$*

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_n\boldsymbol{\sigma}\|_0 \le Ch^\mu k^{1/2-r+\epsilon}\|\boldsymbol{\sigma}\|_r \tag{5.38}$$

for all $\boldsymbol{\sigma} \in [H^r(\Omega)]^2$, $r > 1/2+\epsilon$. Moreover, if $\operatorname{rot}\boldsymbol{\sigma} \in H^r(\Omega)$, then

$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}_n\boldsymbol{\sigma}\|_{\mathbf{H}_0(\operatorname{rot};\Omega)} \le Ch^\mu k^{1/2-r+\epsilon}(\|\boldsymbol{\sigma}\|_r + \|\operatorname{rot}\boldsymbol{\sigma}\|_r). \tag{5.39}$$

Remark 5.1. Using the arguments of [23], we can show that the BDFM spaces satisfy optimal stability and convergence results in k as well as h for the mixed formulation of second-order elliptic equations.

Remark 5.2. In [23], we also discussed the BDM spaces and proved that for the corresponding interpolation operator,

$$\|\hat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\Pi}}_k\hat{\boldsymbol{\sigma}}\|_0 \le Ck^{1-r}\|\hat{\boldsymbol{\sigma}}\|_{r,\hat{K}} \tag{5.40}$$

for $\hat{\boldsymbol{\sigma}} \in [H^r(\hat{K})]^2$, $r > 1$. Using the techniques of Lemma 5.1, (5.40) can be improved to (5.6), $(r > 1/2+\epsilon)$. This improves the analogues of (5.38), (5.39) in Theorem 3.2 of [23] for the BDM spaces as well (the *stability* of BDM spaces for the mixed formulation of second-order elliptic problems like the Poisson equation may be deduced as a consequence).

REFERENCES

[1] D. N. Arnold and R. S. Falk, *Edge effects in the Reissner-Mindlin plate theory*, in Analytical and Computational Models of Shells, A. K. Noor, T. Belytschko, and J. C. Simo, eds., ASME, New York, 1989, pp. 71–89.

[2] D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

[3] I. Babuška and M. Suri, *The p and h-p versions of the finite element method: Basic principles and properties*, SIAM Rev., 36 (1994), pp. 578–632.

[4] I. Babuška and M. Suri, *The h-p version of the finite element method with quasiuniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[5] K. J. Bathe and F. Brezzi, *A simpified analysis of two plate bending elements - the MITC4 and MITC9 elements*, in NUMETA 87, Vol. 1, Numerical Techniques for Engineering Analysis and Design, G. N. Pande and J. Middleton, eds., Martinus Nijhoff, Dordrecht, the Netherlands, 1987.

[6] K. J. Bathe, F. Brezzi, and S. W. Cho, *The MITC7 and MITC9 plate elements*, Comput. and Struct., 32 (1989), pp. 797–814.

[7] K. J. Bathe, F. Brezzi, and M. Fortin, *Mixed-interpolated elements for Reissner-Mindlin plates*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 1787–1801.

[8] K. J. Bathe, M. L. Bucalem, and F. Brezzi, *Displacement and stress convergence of our MITC plate bending elements*, Engrg. Comput., 7 (1990), pp. 291–302.

[9] I. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, New York, 1976.

[10] F. Brezzi and M. Fortin, *Numerical approximation of Mindlin-Reissner plates*, Math. Comp., 47 (1986), pp. 151–158.

[11] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, Berlin, New York, 1991.

[12] F. Brezzi, M. Fortin, and R. Stenberg, *Error analysis of mixed-interpolated elements for Reissner-Mindlin plates*, Math. Models Methods Appl. Sci., 1 (1991), pp. 125–151.

[13] F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 237–250.

[14] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[15] L. Delle Croce and T. Scapolla, *Some applications of hierarchic high order MITC finite elements for Reissner-Mindlin plates*, in Finite Element Methods. Fifty Years of the Courant Element, P. Neittaanmäki, M. Krizek, and R. Stenberg, eds., Marcel Dekker, New York, 1994, pp. 183–190.

[16] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications* I, Springer-Verlag, Berlin, New York, 1972.

[17] F. A. Milner and M. Suri, *Mixed finite element methods for quasilinear second order elliptic problems: the p-version*, RAIRO Modél. Math. Anal. Numér., 26 (1992), pp. 913–931.

[18] P. Peisker and D. Braess, *Uniform convergence of mixed interpolated elements for Reissner-Mindlin plates*, RAIRO Modél. Math. Anal. Numér., 26 (1992), pp. 557–574.

[19] P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Math. 606, Springer-Verlag, New York, 1977, pp. 292–315.

[20] R. Stenberg and M. Suri, *Mixed hp finite element methods for problems in elasticity and Stokes flow*, Numer. Math., 72 (1996), pp. 367–389.

[21] M. Suri, *Analytic and computational assessment of locking in the hp finite element method*, Comput. Methods Appl. Mech. Engrg., 133 (1996), pp. 347–371.

[22] M. Suri, *The p-version of the finite element method for elliptic equations of order* $2l$, RAIRO Modél. Math. Anal. Numér., 24 (1990), pp. 265–304.

[23] M. Suri, *On the stability and convergence of higher-order mixed finite element methods for second-order elliptic problems*, Math. Comp., 54 (1990), pp. 1–19.

[24] M. Suri, I. Babuška, and C. Schwab, *Locking effects in the finite element approximation of plate models*, Math. Comp., 64 (1995), pp. 461–482.