A NEW FINITE ELEMENT FORMULATION FOR THE PLATE BENDING PROBLEM

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1 Introduction

In this paper we will introduce a new finite element formulation for the Reissner-Mindlin plate model. Our approach can be seen as a simplification of a stabilized mixed formulation by Hughes and Franca [10] or as a generalization of a low order method given by Pitkäranta [13]. The formulation is shown to be consistent and stable for any combination of finite element spaces chosen for the deflection and rotation. The order of the polynomials used is, however, dictated by the consistency error; the approximation for deflection should be one order higher than that for the rotation. In engineering terminology we have a formulation which does not suffer from a "locking" which is the problem with many finite element methods for this problem.

For a quite extensive survey of elements which are used for Reissner-Mindlin plates we refer to the recent book [4]. We remark, however, that for many methods used in practice there has not yet been presented any error analysis.

The organization of the paper is as follows. In the next section we introduce our notation. Our method is given in Section 3 where we also perform an error analysis in which we show the optimal order of convergence of the method. In the final section we point out the connection with the method of Hughes and Franca.

2 The Reissner-Mindlin plate model

For simplicity we consider the case of a clamped plate for which the variational form is: find the deflection $w \in H_0^1(\Omega)$ and the rotation $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that

$$Gt^3 a(\boldsymbol{\beta}, \boldsymbol{\eta}) + G\kappa t(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (f, v) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.1}$$

Here G denotes the shear modulus, t is the thickness of the plate and κ is the shear correction factor. The bilinear form a is defined by

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{1}{6}[(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + (\frac{\nu}{1-\nu})(\operatorname{div} \boldsymbol{\beta}, \operatorname{div} \boldsymbol{\eta})], \tag{2.2}$$

where ν is the Poisson ratio and $\boldsymbol{\varepsilon}(\cdot)$ denotes the linear strain operator. As usual, we denote L^2-inner products and norms by $(\cdot,\cdot)_D$ and $\|\cdot\|_{0,D}$, respectively. The subscript D is dropped when $D = \Omega$. We assume that the two dimensional region Ω has a polygonal boundary.

For the mathematical analysis we scale the problem by assuming the loading to be given by

$$f = Gt^3 g, \tag{2.3}$$

with g independent of t. We also let $\kappa = 1$. Hence, the problem is to find $w \in H_0^1(\Omega)$ and $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (g, v) \quad \forall (v,\boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.4}$$

Let us also introduce the (scaled) shear force

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}). \tag{2.5}$$

With this as an independent unknown the problem takes the form

$$\begin{aligned} a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) &= (g,v) \quad \forall (v,\boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2, \\ t^2(\boldsymbol{q},\boldsymbol{s}) + (\boldsymbol{\beta} - \nabla w, \boldsymbol{s}) &= 0 \qquad\qquad \forall \boldsymbol{s} \in [L^2(\Omega)]^2. \end{aligned} \tag{2.6}$$

The strong form of this variational problem is the following:

$$\begin{aligned} \boldsymbol{L}\,\boldsymbol{\beta} + \boldsymbol{q} &= \boldsymbol{0} \quad \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{q} &= g \quad \text{in } \Omega, \\ -t^2\boldsymbol{q} + \nabla w - \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega, \\ \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

with the abreviation $\boldsymbol{L}$ for the differential operator

$$\boldsymbol{L}\,\boldsymbol{\gamma} = \frac{1}{6}\,\mathbf{div}\,\{\boldsymbol{\varepsilon}(\boldsymbol{\gamma}) + (\frac{\nu}{1-\nu})\operatorname{div}\boldsymbol{\gamma}\,\boldsymbol{I}\}. \tag{2.8}$$

Here **div** denotes the divergence operator acting on second order tensors and $\boldsymbol{I}$ is the unit tensor.

If not otherwise stated we will use standard finite element notation, cf. [7, 9].

3 The finite element method

The stabilized method we propose is defined as follows. We choose two finite element subspaces $W_h \subset H_0^1(\Omega)$ and $\boldsymbol{V}_h \subset [H_0^1(\Omega)]^2$ for the deflection and rotation, respectively. The finite element approximation $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ is then defined as the solution to the problem

$$\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (g, v) \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{3.1}$$

with the bilinear form defined as

$$\begin{aligned}\mathcal{B}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta}) &= a(\boldsymbol{\phi},\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{L}\,\boldsymbol{\phi},\boldsymbol{L}\,\boldsymbol{\eta})_K \\ &+ \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}(\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K.\end{aligned} \tag{3.2}$$

From the solution $(w_h, \boldsymbol{\beta}_h)$ we then calculate the approximation for the shear by

$$\boldsymbol{q}_{h|K} = (t^2+\alpha h_K^2)^{-1}(\nabla w_h - \boldsymbol{\beta}_h - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}_h)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{3.3}$$

Here $\mathcal{C}_h$ stands for the finite element partitioning of $\bar{\Omega}$ into triangles or quadrilaterals and h_K denotes the diameter of $K \in \mathcal{C}_h$. As usual we denote $h = \max_{K\in\mathcal{C}_h} h_K$. The partitioning is assumed to satisfy the usual regularity and compatibility conditions. α is a positive parameter lying in a fixed range which will be specified below.

Note that (2.5) and the first equation of (2.7) give

$$\boldsymbol{q}_{|K} = (t^2+\alpha h_K^2)^{-1}(\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\beta})_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{3.4}$$

Hence, we see that the approximation (3.3) is consistent with the exact shear.

Let us first show that the formulation is both consistent and stable independent of the choice of finite element spaces.

Lemma 3.1 *The solution $(w, \boldsymbol{\beta})$ to (2.7) satisfies the equation*

$$\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) = (g,v) \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Recalling the first equation in (2.7), the expression (3.4), and the variational form (2.6), we get

$$\begin{aligned}&\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{L}\,\boldsymbol{\beta},\boldsymbol{L}\,\boldsymbol{\eta})_K \\ &\qquad + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}(\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) + \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{q},\boldsymbol{L}\,\boldsymbol{\eta})_K \\ &\qquad + \sum_{K\in\mathcal{C}_h} (\boldsymbol{q}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) + \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{q},\boldsymbol{L}\,\boldsymbol{\eta})_K + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{q},\boldsymbol{L}\,\boldsymbol{\eta})_K \\ &= a(\boldsymbol{\beta},\boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) \\ &= (g,v). \qquad \blacksquare\end{aligned}$$

For the stability we need the following inverse estimate which is valid for any $\boldsymbol{V}_h$ consisting of piecewise polynomials (cf. e.g. [7, 9]).

Lemma 3.2 *There is a positive constant C_I such that*

$$C_I \sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,\boldsymbol{\phi}\|_{0,K}^2 \le a(\boldsymbol{\phi},\boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h. \qquad \blacksquare$$

The stability and error estimates will be formulated using the following seminorm and norm.

$$|(v,\boldsymbol{\eta})|_h^2 = \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2, \tag{3.5}$$

$$|||(v,\boldsymbol{\eta})|||_h = \|v\|_1 + \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h. \tag{3.6}$$

We note that the following equivalence of norms holds.

Lemma 3.3 *There is a positive constant C such that*

$$C|||(v,\boldsymbol{\eta})|||_h \le \|\boldsymbol{\eta}\|_1 + |(v,\boldsymbol{\eta})|_h \le |||(v,\boldsymbol{\eta})|||_h \quad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Since $(t^2+\alpha h_K^2) \le C$, we get (using the Poincaré inequality)

$$\begin{aligned}
\|v\|_1^2 &\le C\|\nabla v\|_0^2 \le C(\|\nabla v - \boldsymbol{\eta}\|_0^2 + \|\boldsymbol{\eta}\|_0^2) \\
&\le C(\|\nabla v - \boldsymbol{\eta}\|_0^2 + \|\boldsymbol{\eta}\|_1^2) \\
&\le C(\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 + \|\boldsymbol{\eta}\|_1^2),
\end{aligned}$$

which proves the claim. $\blacksquare$

Now, we can formulate the stability condition.

Lemma 3.4 *For $0 < \alpha < C_I$ it holds*

$$\mathcal{B}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta}) \ge C|||(v,\boldsymbol{\eta})|||_h^2 \qquad \forall (v,\boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

Proof: Using the inverse estimate of Lemma 3.2 and Korn's inequality we get

$$\begin{aligned}
&\mathcal{B}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta}) \\
&\quad = a(\boldsymbol{\eta},\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 \\
&\quad \ge (1-\alpha C_I^{-1})\, a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2 \\
&\quad \ge C(a(\boldsymbol{\eta},\boldsymbol{\eta}) + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2) \qquad (3.7) \\
&\quad \ge C(\|\boldsymbol{\eta}\|_1^2 + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2).
\end{aligned}$$

Using the triangle inequality, the same inverse estimate and the boundedness of the bilinear form a we obtain

$$\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\|\nabla v - \boldsymbol{\eta}\|_{0,K}^2$$

$$
\begin{aligned}
&\leq 2\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)\\
&\leq 2\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+\alpha\sum_{K\in\mathcal{C}_h}h_K^2\|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+a(\boldsymbol{\eta},\boldsymbol{\eta})\Big) \qquad (3.8)\\
&\leq C\Big(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\|\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2+\|\boldsymbol{\eta}\|_1^2\Big).
\end{aligned}
$$

Combining (3.7) and (3.8) gives

$$\mathcal{B}_h(v,\boldsymbol{\eta};v,\boldsymbol{\eta})\geq C(\|\boldsymbol{\eta}\|_1^2+|(v,\boldsymbol{\eta})|_h^2),$$

and the claim follows from Lemma 3.3. ■

When the consistency and the stability is always valid, the choice of the finite element spaces is solely dictated by the proper balance between their interpolation properties with respect to the norms used. With the notation

$$R_m(K)=\begin{cases}P_m(K) & \text{if } K \text{ is a triangle,}\\ Q_m(K) & \text{if } K \text{ is a quadrilateral,}\end{cases} \qquad (3.9)$$

where for each positive integer m, $P_m(K)$ and $Q_m(K)$ are the usual triangular and and quadrilateral finite element subspaces [7], we are then led to the following choice

$$
\begin{aligned}
W_h &= \{\ v\in H_0^1(\Omega)\mid v_{|K}\in R_{k+1}(K)\ \forall K\in\mathcal{C}_h\ \}, &(3.10)\\
\boldsymbol{V}_h &= \{\ \boldsymbol{\eta}\in[H_0^1(\Omega)]^2\mid \boldsymbol{\eta}_{|K}\in[R_k(K)]^2\ \forall K\in\mathcal{C}_h\ \}, &(3.11)
\end{aligned}
$$

with the index $k\geq 1$. We are here allowing a mixing of triangles and quadrilaterals in the mesh.

Remark 3.1 For quadrilateral elements the spaces could be reduced to the "serendipity" elements (cf. e.g. [7]) of the same polynomial degrees. ■

Remark 3.2 For triangular elements with $k=1$ it holds

$$\boldsymbol{L}\,\boldsymbol{\phi}_{|K}=\boldsymbol{0},\quad \forall K\in\mathcal{C}_h,\quad \forall\boldsymbol{\phi}\in\boldsymbol{V}_h,$$

and hence the bilinear form $\mathcal{B}_h$ reduces to

$$\mathcal{B}_h(w,\boldsymbol{\beta};v,\boldsymbol{\eta})=a(\boldsymbol{\beta},\boldsymbol{\eta})+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\nabla w-\boldsymbol{\beta},\nabla v-\boldsymbol{\eta})_K,$$

and we obtain a formulation proposed by Pitkäranta [13]. Now, the method is stable for all positive values of α. It is easily seen that the above bilinear form also can be used in the quadrilateral case for $k=1$ without a decrease in accuracy. ■

Remark 3.3 For triangular elements and $k \geq 4$ it is shown in [3, 6] that the method is stable with $\alpha = 0$. However, it may be advantageous to choose $\alpha > 0$ also in that case since it gives a more well conditioned stiffness matrix, cf. [12] where the formulation by Hughes and Franca [10] is used for multigrid algorithms. ■

Remark 3.4 It is also possible to obtain methods with equal order approximations for the deflection and the rotation if one combines the above stabilization technique with the covariant interpolation of the shear strain used in the MITC elements, cf. [6] and also [11] where numerical results are presented. ■

We next derive error estimates for the method. Here $\|\cdot\|_{-1}$ denotes the norm in the dual space to $[H_0^1(\Omega)]^2$. We recall that the mesh is quasiuniform if $h_K \geq Ch$ for all elements $K \in \mathcal{C}_h$.

Theorem 3.1 *Suppose that the solution* $(w, \boldsymbol{\beta})$ *of (2.7) satisfies* $w \in H^{k+2}(\Omega)$ *and* $\boldsymbol{\beta} \in [H^{k+1}(\Omega)]^2$. *For* $0 < \alpha < C_I$ *we then have*

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_{-1} + t\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \leq Ch^k(|w|_{k+2} + |\boldsymbol{\beta}|_{k+1}), \tag{3.12}$$

When the mesh is quasiuniform we additionally have

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_0 \leq Ch^{k-1}(|w|_{k+2} + |\boldsymbol{\beta}|_{k+1}). \tag{3.13}$$

Proof: Let $\tilde{w} \in W_h$ and $\tilde{\boldsymbol{\beta}} \in \boldsymbol{V}_h$ be the usual Lagrange interpolants to w and $\boldsymbol{\beta}$, respectively. From Lemmas 3.4 and 3.1 there is $(v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h$ such that

$$|||(v, \boldsymbol{\eta})|||_h \leq C, \tag{3.14}$$

and

$$|||(w_h - \tilde{w}, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}})|||_h \leq \mathcal{B}_h(w_h - \tilde{w}, \boldsymbol{\beta}_h - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) = \mathcal{B}_h(w - \tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}). \tag{3.15}$$

Next, we write out the right hand side of (3.15)

$$\begin{aligned} \mathcal{B}_h(w - \tilde{w}, \boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}, \boldsymbol{\eta}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\,(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \boldsymbol{L}\,\boldsymbol{\eta})_K \qquad (3.16) \\ &+ \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla(w - \tilde{w}) - (\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}) - \alpha h_K^2 \boldsymbol{L}\,(\boldsymbol{\beta} - \tilde{\boldsymbol{\beta}}), \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K. \end{aligned}$$

Due to the inverse inequality of Lemma 3.2 we have

$$\Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2} \leq C|||(v, \boldsymbol{\eta})|||_h$$

and

$$\Big(\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta}\|_{0,K}^2\Big)^{1/2} \leq C|||(v, \boldsymbol{\eta})|||_h.$$

Using these and (3.14), (3.16) gives

$$\mathcal{B}_h(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta}) \tag{3.17}$$
$$\leq C(|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_1+(\sum_{K\in\mathcal{C}_h} h_K^{-2}|w-\tilde{w}|_{1,K}^2+h_K^{-2}\|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}\|_{0,K}^2+h_K^2|\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}}|_{2,K}^2)^{1/2}).$$

From the standard interpolation estimates we then get

$$\mathcal{B}_h(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}};v,\boldsymbol{\eta}) \leq Ch^k(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}), \tag{3.18}$$

and the estimate for $|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h$ then follows from (3.15) and (3.18), using the triangle inequality and the estimate

$$|||(w-\tilde{w},\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})|||_h \leq Ch^k(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}). \tag{3.19}$$

We also note that the proved estimate, an inverse estimate and a standard interpolation estimate imply

$$(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2))^{1/2} \leq Ch^k(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}). \tag{3.20}$$

Next, let us derive the estimates for the shear. Recalling (3.3) and (3.4) we get

$$(\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,K}^2)^{1/2} \leq |(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|_h+(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2))^{1/2}. \tag{3.21}$$

and two of the asserted estimates for the shear follow from (3.19) and (3.20).

To prove the estimate for $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1}$, let $\boldsymbol{\eta}\in[H_0^1(\Omega)]^2$ be arbitrary and let $\tilde{\boldsymbol{\eta}}^c\in\boldsymbol{V}_h$ be its Clément interpolant which satisfies (cf. [8] and [9, pages 109-111])

$$|\tilde{\boldsymbol{\eta}}^c|_1\leq C|\boldsymbol{\eta}|_1 \quad \text{and} \quad (\sum_{K\in\mathcal{C}_h} h_K^{-2}|\boldsymbol{\eta}-\tilde{\boldsymbol{\eta}}^c|_{0,K}^2)^{1/2}\leq C|\boldsymbol{\eta}|_1. \tag{3.22}$$

We thus have

$$\begin{aligned}(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}) &= (\boldsymbol{q}-\boldsymbol{q}_h,\tilde{\boldsymbol{\eta}}^c)+(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta}-\tilde{\boldsymbol{\eta}}^c)\\ &\leq (\boldsymbol{q}-\boldsymbol{q}_h,\tilde{\boldsymbol{\eta}}^c)+(\sum_{K\in\mathcal{C}_h} h_K^2\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,K}^2)^{1/2}\,|\boldsymbol{\eta}|_1.\end{aligned} \tag{3.23}$$

Next, using (3.1), Lemma 3.1, (3.22) and Lemma 3.2, we get

$$\begin{aligned}&(\boldsymbol{q}-\boldsymbol{q}_h,\tilde{\boldsymbol{\eta}}^c)\\ &= \sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h)-\alpha h_K^2\boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\tilde{\boldsymbol{\eta}}^c)_K\\ &= -a(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\tilde{\boldsymbol{\eta}}^c)\\ &\quad+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h)-\alpha h_K^2\boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h),\alpha h_K^2\boldsymbol{L}\,\tilde{\boldsymbol{\eta}}^c)_K\\ &\leq C(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h+(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2})(|\tilde{\boldsymbol{\eta}}^c|_1+(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{L}\,\tilde{\boldsymbol{\eta}}^c|_{0,K}^2)^{1/2})\\ &\leq C(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h+(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2})|\tilde{\boldsymbol{\eta}}^c|_1 \qquad (3.24)\\ &\leq C(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h+(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2})|\boldsymbol{\eta}|_1.\end{aligned}$$

Combining (3.23) and (3.24) gives

$$\begin{aligned}\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1} &= \sup_{\boldsymbol{\eta}\in[H_0^1(\Omega)]^2}\frac{(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{\eta})}{|\boldsymbol{\eta}|_1} \\ &\le C(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h+(\sum_{K\in\mathcal{C}_h}h_K^2(|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2+\|\boldsymbol{q}-\boldsymbol{q}_h\|_{0,K}^2))^{1/2})\end{aligned}\tag{3.25}$$

and asserted estimate then follows from (3.12), (3.20) and (3.21). ∎

Theorem 3.2 *Suppose that the solution of (2.7) satisfies* $(w,\boldsymbol{\beta})\in H^{k+2}(\Omega)\times[H^{k+1}(\Omega)]^2$ *and that the region* Ω *is convex. For* $0<\alpha<C_I$ *it then holds*

$$\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0\le Ch^{k+1}(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}),\tag{3.26}$$

and

$$\|w-w_h\|_1\le C(t+h)h^k(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}).\tag{3.27}$$

Proof: Let $(z,\boldsymbol{\phi},\boldsymbol{r})\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2\times[L^2(\Omega)]^2$ be the solution to

$$\begin{aligned}a(\boldsymbol{\phi},\boldsymbol{\eta})+(\boldsymbol{r},\nabla v-\boldsymbol{\eta}) &= (\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta}) \quad \forall(v,\boldsymbol{\eta})\in H_0^1(\Omega)\times[H_0^1(\Omega)]^2,\\ t^2(\boldsymbol{r},\boldsymbol{s})+(\boldsymbol{\phi}-\nabla z,\boldsymbol{s}) &= 0 \qquad\qquad\qquad \forall\boldsymbol{s}\in[L^2(\Omega)]^2.\end{aligned}\tag{3.28}$$

The strong form of this problem is

$$\begin{aligned}\boldsymbol{L}\,\boldsymbol{\phi}+\boldsymbol{r} &= \boldsymbol{\beta}-\boldsymbol{\beta}_h && \text{in }\Omega,\\ \operatorname{div}\boldsymbol{r} &= 0 && \text{in }\Omega,\\ -t^2\boldsymbol{r}+\nabla z-\boldsymbol{\phi} &= \boldsymbol{0} && \text{in }\Omega,\\ z=0,\quad \boldsymbol{\phi} &= \boldsymbol{0} && \text{on }\partial\Omega,\end{aligned}\tag{3.29}$$

From [1, Appendix] we can deduce the following regularity estimate

$$\|z\|_3+\|\boldsymbol{\phi}\|_2\le C\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0.\tag{3.30}$$

Using (3.28) and (3.29) we now get

$$\begin{aligned}&\mathcal{B}_h(z,\boldsymbol{\phi};v,\boldsymbol{\eta})\\ &= a(\boldsymbol{\phi},\boldsymbol{\eta})-\alpha\sum_{K\in\mathcal{C}_h}h_K^2(\boldsymbol{L}\,\boldsymbol{\phi},\boldsymbol{L}\,\boldsymbol{\eta})_K\\ &\quad+\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}(\nabla z-\boldsymbol{\phi}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\phi},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\ &= a(\boldsymbol{\phi},\boldsymbol{\eta})+\alpha\sum_{K\in\mathcal{C}_h}h_K^2(\boldsymbol{r},\boldsymbol{L}\,\boldsymbol{\eta})_K-\alpha\sum_{K\in\mathcal{C}_h}h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{L}\,\boldsymbol{\eta})_K\\ &\quad+\sum_{K\in\mathcal{C}_h}(\boldsymbol{r},\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\\ &\quad-\sum_{K\in\mathcal{C}_h}(t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K\end{aligned}$$

$$
\begin{aligned}
&= a(\boldsymbol{\phi},\boldsymbol{\eta}) + (\boldsymbol{r},\nabla v-\boldsymbol{\eta}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\quad - \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \nabla v-\boldsymbol{\eta}-\alpha h_K^2\boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= (\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\quad - \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \nabla v-\boldsymbol{\eta})_K.
\end{aligned}
$$

Hence, chosing $\boldsymbol{\eta}=\boldsymbol{\beta}-\boldsymbol{\beta}_h,\ v=w-w_h$, using Lemma 3.1, and the symmetry of the bilinear form we get

$$
\begin{aligned}
\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0^2 &= \mathcal{B}_h(z-\tilde{z},\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}; w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h) \\
&\quad + \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h))_K \qquad (3.31) \\
&\quad + \sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h))_K.
\end{aligned}
$$

Here $\tilde{z}$ and $\tilde{\boldsymbol{\phi}}$ are the Lagrange interpolants to z and $\boldsymbol{\phi}$, respectively. Next, we estimate the terms on the right hand side above. Using normal interpolation results and the regularity estimate (3.30) we get

$$
\begin{aligned}
&\mathcal{B}_h(z-\tilde{z},\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}; w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h) \\
&\leq Ch(|z|_3+|\boldsymbol{\phi}|_2)(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + (\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2}) \qquad (3.32) \\
&\leq Ch\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + (\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2}).
\end{aligned}
$$

For the second term we directly obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2(t^2+\alpha h_K^2)^{-1}(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h))_K \\
&\leq \sum_{K\in\mathcal{C}_h} \alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \boldsymbol{L}\,(\boldsymbol{\beta}-\boldsymbol{\beta}_h))_K \qquad (3.33) \\
&\leq Ch\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0(\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2}
\end{aligned}
$$

For the third term Schwartz inequality yields

$$
\begin{aligned}
&\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha h_K^2(\boldsymbol{\beta}-\boldsymbol{\beta}_h, \nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h))_K \\
&\leq (\sum_{K\in\mathcal{C}_h} (t^2+\alpha h_K^2)^{-1}\alpha^2 h_K^4\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_{0,K}^2)^{1/2}|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h \qquad (3.34) \\
&\leq Ch\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h.
\end{aligned}
$$

From (3.31) - (3.34) we get

$$
\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0 \leq Ch(|||(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|||_h + (\sum_{K\in\mathcal{C}_h} h_K^2|\boldsymbol{\beta}-\boldsymbol{\beta}_h|_{2,K}^2)^{1/2}) \qquad (3.35)
$$

and the estimate (3.26) for the rotation follows from (3.12) and (3.20).

Using the triangle inequality and recalling the definition of the seminorm $|\cdot|_h$ we get

$$\begin{aligned}\|\nabla(w-w_h)\|_0 &\le \|\nabla(w-w_h)-(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\|_0+\|\boldsymbol{\beta}-\boldsymbol{\beta}_h\|_0\\ &\le C(t+h)|(w-w_h,\boldsymbol{\beta}-\boldsymbol{\beta}_h)|_h+\|(\boldsymbol{\beta}-\boldsymbol{\beta}_h)\|_0,\end{aligned}$$

and hence the estimate for the deflection follows from (3.26) and (3.12). $\blacksquare$

Next, we will state som error estimates in the case when $t \le Ch_K,\ \forall K \in \mathcal{C}_h$. We then say that the plate is *numerically thin.* Following [2] we define the mesh dependent norm

$$\|u\|_{2,h}^2=\sum_{K\in\mathcal{C}_h}\|u\|_{2,K}^2+\sum_{T\in\Gamma_h}h_T^{-1}\int_T|[\![\frac{\partial u}{\partial n_T}]\!]|^2\,ds,$$

where Γ_h denotes the collection of edges in $\mathcal{C}_h$ and h_T is the length of the edge T. $[\![\partial u/\partial n_T]\!]$ denotes the jump in the normal derivative of u along the edge T. If T is on the boundary $\partial\Omega$ it denotes the value of the normal derivative in Ω. In [13, Lemma 3.1] the following inequality is proved

$$\|u\|_{2,h}\le C|||(u,\boldsymbol{\phi})|||_h \quad \forall(u,\boldsymbol{\phi})\in W_h\times\boldsymbol{V}_h.$$

With this inequality Theorem 3.1 gives the following result.

Corollary 3.1 *Suppose that the plate is numerically thin and that* $(w,\boldsymbol{\beta})\in H^{k+2}(\Omega)\times[H^{k+1}(\Omega)]^2$. *For* $0<\alpha<C_I$ *it then holds*

$$\|w-w_h\|_{2,h}\le Ch^k(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}). \tag{3.36}$$

If the region Ω *is convex we additionally have*

$$\|w-w_h\|_1\le Ch^{k+1}(|w|_{k+2}+|\boldsymbol{\beta}|_{k+1}). \qquad \blacksquare \tag{3.37}$$

We remark that in practice the assumption of a numerically thin plate seems to be quite natural. If the lengths of the elements are substantially smaller than the thickness of the plate, the error in the plate model is probably bigger that error in the finite element discretization. Hence, one should then turn to a direct three dimensional analysis of the plate.

4 The connection with the method of Hughes and Franca

The starting point for the Hughes-Franca method was a mixed formulation based on the variational form (2.6). As is well known, a direct finite element discretization based on that is difficult to design. Only the stability of the rotation is automatically ensured. To have stability for the deflection and shear, the method has to satisfy the "Z-ellipticity" and "inf-sup" conditions, cf. [5]. To avoid this Hughes and Franca [10] introduced a stabilized mixed method, which we present in a slightly altered form given by Peisker [12]:

$$\mathcal{M}_h(w_h, \boldsymbol{\beta}_h, \boldsymbol{q}_h; v, \boldsymbol{\eta}, \boldsymbol{s}) = (g, v) \qquad \forall (v, \boldsymbol{\eta}, \boldsymbol{s}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h, \tag{4.1}$$

with the bilinear form defined as

$$\begin{aligned} &\mathcal{M}_h(z, \boldsymbol{\phi}, \boldsymbol{r}; v, \boldsymbol{\eta}, \boldsymbol{s}) \\ &= a(\boldsymbol{\phi}, \boldsymbol{\eta}) + (\boldsymbol{r}, \nabla v - \boldsymbol{\eta}) + (\boldsymbol{s}, \nabla z - \boldsymbol{\phi}) - t^2(\boldsymbol{r}, \boldsymbol{s}) \\ &\quad -\alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\, \boldsymbol{\phi} + \boldsymbol{r}, \boldsymbol{L}\, \boldsymbol{\eta} + \boldsymbol{s})_K \\ &\quad +\gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} (\nabla z - \boldsymbol{\phi} - t^2 \boldsymbol{r}, \nabla v - \boldsymbol{\eta} - t^2 \boldsymbol{s})_K. \end{aligned} \tag{4.2}$$

The bilinear form consists of the bilinear form of the continous problem (2.6) with the addition of two terms which are locally weighted squares of two of the differential equations in (2.7). Hence, the formulation is consistent. Using the inverse inequality of Lemma 3.2 one now obtains

$$\begin{aligned} &\mathcal{M}_h(v, \boldsymbol{\eta}, \boldsymbol{s}; v, \boldsymbol{\eta}, -\boldsymbol{s}) \\ &= a(\boldsymbol{\eta}, \boldsymbol{\eta}) + t^2 \|\boldsymbol{s}\|_0^2 - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\, \boldsymbol{\eta}\|_{0,K}^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{s}\|_{0,K}^2 \\ &\quad +\gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 - \gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} t^4 \|\boldsymbol{s}\|_{0,K}^2 \\ &\geq C(\|\boldsymbol{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2 + \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2) \|\boldsymbol{s}\|_{0,K}^2) \\ &\qquad \forall (v, \boldsymbol{\eta}, \boldsymbol{s}) \in W_h \times \boldsymbol{V}_h \times \boldsymbol{S}_h, \end{aligned} \tag{4.3}$$

when $0 < \alpha < C_I$ and $0 < \gamma < 1$. Hence, the addition of the two new terms gives a stable method for every choice of the finite element spaces.

With discontinuous piecewise polynomials for the shear space $\boldsymbol{S}_h$ the shear can be eliminated on the element level. Let $\boldsymbol{\Pi}_K : [L^2(K)]^2 \to \boldsymbol{S}_{h|K}$ be the L^2-projection and denote

$$a_K = \frac{(1-\gamma) t^2 + \alpha h_K^2}{t^2 + \alpha h_K^2}\ . \tag{4.4}$$

The elimination of the shear then gives the following method:

$$\mathcal{B}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = (g, v) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{4.5}$$

with the bilinear form defined as

$$\begin{aligned} \mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\, \boldsymbol{\phi}, \boldsymbol{L}\, \boldsymbol{\eta})_K \\ &\quad +\gamma \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} (\nabla z - \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta})_K \\ &\quad + \sum_{K \in \mathcal{C}_h} (a_K t^2 + \alpha h_K^2)^{-1} (\boldsymbol{\Pi}_K [a_K (\nabla z - \boldsymbol{\phi}) - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\phi}], \boldsymbol{\Pi}_K [a_K (\nabla v - \boldsymbol{\eta}) - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\eta}])_K. \end{aligned} \tag{4.6}$$

The shear is given by

$$\boldsymbol{q}_{h|K} = (a_K t^2 + \alpha h_K^2)^{-1} \boldsymbol{\Pi}_K (a_K(\nabla w_h - \boldsymbol{\beta}_h) - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\beta}_h)_{|K} \quad \forall K \in \mathcal{C}_h. \tag{4.7}$$

Now, it is quite easily seen (cf. [13, Theorem 3.1]) that if we choose the finite element spaces such that $\nabla W_h \subset \boldsymbol{S}_h$ the method (4.1) is stable without the second stabilizing term in (4.2). With this we get the bilinear form from above with $\gamma = 0$ and $a_K = 1$, i.e.

$$\begin{aligned}\mathcal{B}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) &= a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\boldsymbol{L}\, \boldsymbol{\phi}, \boldsymbol{L}\, \boldsymbol{\eta})_K \\ &+ \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{\Pi}_K[\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\phi}], \boldsymbol{\Pi}_K[\nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\eta}])_K.\end{aligned} \tag{4.8}$$

Next, we chose the subspace for the shear big enough so that

$$\boldsymbol{\Pi}_K(\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\phi}) = \nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\phi} \quad \forall K \in \mathcal{C}_h, \quad \forall (z, \boldsymbol{\phi}) \in W_h \times \boldsymbol{V}_h. \tag{4.9}$$

With this we are lead back to our formulation (3.1) - (3.3). Note that the space for the shear now dissappears from the formulation and also from the error analysis as it is done in Section 3 above.

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