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On some techniques for approximating boundary conditions in the finite element method

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Abstract

We discuss the stabilization of finite element methods in which essential boundary conditions are approximated by Babuška's method of Lagrange multipliers and we show that there is a close connection with this technique and a classical method by Nitsche.

Keywords: FEM; Boundary conditions; Lagrange multipliers; Stabilization; Nitsche's method

1. Introduction

The foundations for the theory of mixed finite element methods were laid down in a paper by Babuška [1] in which he introduced the idea of approximating essential boundary conditions in the Dirichlet problem by using a Lagrange multiplier. It was shown that the method will converge optimally if the finite element spaces, i.e. the space for the unknown in the domain and the space for the multiplier on the boundary, satisfy an "inf–sup" condition. In the original paper [1] the actual question of how to construct subspaces satisfying the inf–sup condition was more or less left open. This problem was later studied in detail in a series of papers by Pitkäranta [19–21], and it was shown that there is not too much freedom in choosing the spaces if the optimal order of convergence of the method is desired.

Traditionally, the method of using Lagrange multipliers has perhaps not been considered as a method to be used in practice, but more as a model problem for studying the mixed finite element method. Lately, there has, however, been a renewed interest in the method and it has been proposed to be used in domain decomposition [8, 9, 16], in fictitious domain methods [11, 12] and

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for contact problems [13, 15, 17]. However, in view of the difficulties with satisfying the inf–sup condition the usefulness of the technique can be questioned.

During the last few years it has been shown that the problem with the stability of a mixed finite element formulation can to a large extent be avoided by adding well-chosen stabilizing terms to the bilinear form defining the mixed method (cf. [10, 14] and the references therein). In two recent papers by Barbosa and Hughes [4, 5] this stabilization technique was proposed for the original method of Babuška. Since the finite element spaces now do not have to satisfy an inf–sup condition, the method now seem to have a much greater potential in the applications mentioned above. Similar ideas was independently introduced by Verfürth [22] for the approximation of slip boundary conditions for the Navier–Stokes equations and by Baiocchi et al. [3] for domain decomposition.

The plan of this paper is the following. First, we recall the method of Babuška. Next, we propose and analyze a simplification of the stabilized formulation of Barbosa and Hughes. We then show that this method is closely related to a classical method by Nitsche [18].

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with a smooth boundary Γ. We consider the model Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{2.1}$$

$$u = g \quad \text{on } \Gamma. \tag{2.2}$$

This problem is chosen only for notational simplicity; our statements are also valid for other second-order elliptic problems such as, e.g., the equations of linear elasticity and Stokes problem.

The Sobolev spaces $H^s(S)$ for $S \subset \Omega$ or $S \subset \Gamma$, and $s \geqslant 0$, are defined in the standard way (cf. [1, 2, 7]. The norms are denoted $\|\cdot\|_{l,S}$ with the subscript S dropped when $S = \Omega$. We recall that the following trace inequality holds for $v \in H^s(\Omega)$, with $s > \frac{1}{2}$,

$$\|v\|_{s-1/2,\Gamma} \leqslant C\|v\|_s. \tag{2.3}$$

(In the paper C and C_j will denote positive constants which all are independent of the mesh parameter h.) We will also use the space $H^{-1/2}(\Gamma)$, i.e. the dual space of $H^{1/2}(\Omega)$, with the norm

$$\|\mu\|_{-1/2,\Gamma} = \sup_{z \in H^{1/2}(\Gamma)} \frac{\langle \mu, z\rangle}{\|z\|_{1/2,\Gamma}}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing.

For functions $v \in H^1(\Omega)$ with $\Delta v \in L^2(\Omega)$, it holds (cf. [1, 2]) $\partial v/\partial n \in H^{-1/2}(\Gamma)$ with

$$\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,\Gamma} \leqslant C(\|v\|_1 + \|\Delta v\|_0). \tag{2.5}$$

The problem is then given in the following variational formulation: given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma)$ such that

$$\mathscr{B}(u,\lambda;v,\mu) = (f,v) + \langle g,\mu\rangle \quad \forall (v,\mu) \in H^1(\Omega)\times H^{-1/2}(\Gamma), \tag{2.6}$$

with the bilinear form defined by

$$\mathscr{B}(u,\lambda;v,\mu) = (\nabla u, \nabla v) + \langle \lambda, v\rangle + \langle \mu, u\rangle. \tag{2.7}$$

Above $(\cdot\,,\cdot)$ denotes the inner product in $L^2(\Omega)$.

The problems has a unique solution (cf. [1]). By using Green's formula in (2.6) we get the relation

$$\lambda + \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma. \tag{2.8}$$

3. The finite element methods

When discussing the finite element methods we will, for notational simplicity, consider the case when simplicial meshes are used, but we emphasize already now that the big advantage of the stabilized methods is that a very big freedom can be allowed in chosing the finite element spaces since no stability conditions are needed.

Let now $\mathscr{R}_h$ be a partitioning of the whole of $\mathbb{R}^d$ into closed simplices (i.e. triangles and tetrahedrons, respectively) and assume that the partitioning satisfies the usual requirements that the intersection of two simplices is either empty, a vertex, an edge or a face. The partitioning of $\bar{\Omega}$ is now defined as

$$\mathscr{C}_h = \{K \mid K = S \cap \bar{\Omega} \text{ for some } S \in \mathscr{R}_h\}. \tag{3.1}$$

The finite element subspace for the field variable is then defined as

$$V_h = \{v \in H^1(\Omega) \mid v_{|K} \in P_k(K)\ \forall K \in \mathscr{C}_h\}, \tag{3.2}$$

where $P_k(K)$ denotes the polynomials of degree $k \geqslant 1$ on K.

We will assume that the elements of $\mathscr{C}_h$ verify the usual regularity condition

$$h_K \leqslant C\rho_K \quad \forall K \in \mathscr{C}_h, \tag{3.3}$$

where h_K and ρ_K are the diameter of K and the diameter of the biggest circle (sphere) inscribed in K, respectively. We have to remark here that this condition is not automatically valid for every mesh $\mathscr{R}_h$ satisfying the same condition.

To define the space Λ_h for the Lagrange multiplier on the boundary we proceed as follows. For $d = 2$ the finite element partitioning $\mathscr{E}_h$ of the boundary consists of segments and for $d = 3$ the elements are curved triangles. This partitioning is also assumed to satisfy the usual compatibility conditions, i.e. the intersection of two elements is assumed to be either empty, a point or a curved edge (for $d = 3$). We further assume that each element $E \in \mathscr{E}_h$ is the image of the reference element

$\hat{E}$ (i.e. the unit interval or unit triangle) under a smooth mapping F_E. We then define

$$\Lambda_h = \{\mu \in L^2(\Gamma) \mid \mu(x)_{|E} = \hat{\mu}(F_E^{-1}(x)) \text{ for some } \hat{\mu} \in P_l(\hat{E}),\ \forall E \in \mathscr{E}_h\}, \tag{3.4}$$

with $l \geqslant 0$.

The regularity assumptions for the boundary mesh we state as follows:

$$\|F_E\| \leqslant Ch_E, \qquad \|F_E^{-1}\| \leqslant Ch_E^{-1} \quad \forall E \in \mathscr{E}_h, \tag{3.5}$$

where $\|\cdot\|$ denotes the Euclidian norm and h_E is the diameter of $E \in \mathscr{E}_h$.

We will also make the natural assumption that there are constants C_1, C_2 such that

$$C_1 h_K \leqslant h_E \leqslant C_2 h_K \quad \text{for all } K \in \mathscr{C}_h \text{ and } E \in \mathscr{E}_h \text{ with } K \cap E \neq \emptyset. \tag{3.6}$$

As usual we will denote $h = \max_{K \in \mathscr{C}_h} h_K$, and from (3.6) it then follows that $h_E \leqslant Ch$ for all $E \in \mathscr{E}_h$.

First, we will consider the original method.

Method 1: *Babuška's method of Lagrange multipliers*

In this the variational formulation (2.6) is transferred to the finite element subspaces: find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\mathscr{B}(u_h, \lambda_h; v, \mu) = (f, v) + \langle g, \mu \rangle \quad \forall (v, \mu) \in V_h \times \Lambda_h. \tag{3.7}$$

The convergence of the method is given by the following classical result.

Theorem 1A (Babuška [1], Brezzi [6]). *Suppose that the finite element subspaces satisfy the conditions*

$$\sup_{v \in V_h \setminus \{0\}} \frac{\langle \mu, v \rangle}{\|v\|_1} \geqslant C \|\mu\|_{-1/2,\Gamma} \quad \forall \mu \in \Lambda_h, \tag{3.8}$$

and

$$|v|_1^2 \geqslant C \|v\|_1^2 \quad \forall v \in \{v \in V_h \mid \langle \mu, v \rangle = 0\ \forall \mu \in \Lambda_h\}. \tag{3.9}$$

For the solution (u_h, λ_h) *to the problem* (3.7) *it then holds*

$$\|u - u_h\|_1 + \|\lambda - \lambda_h\|_{-1/2,\Gamma} \leqslant C(h^k \|u\|_{k+1} + h^{l+3/2} \|\lambda\|_{l+1,\Gamma}), \tag{3.10}$$

when $u \in H^{k+1}(\Omega)$ *and* $\lambda \in H^{l+1}(\Gamma)$.

This method has been thoroughly studied by Pitkäranta. Among other things, he showed that the stability and error analysis is most easily performed using the following mesh-dependent norms (introduced in [20] with a different notation)

$$\|v\|_{1/2,h}^2 = \sum_{E \in \mathscr{E}_h} h_E^{-1} \|v\|_{0,E}^2 \quad \text{for } v \in H^1(\Omega) \tag{3.11}$$

and

$$\|z\|_{-1/2,h}^2 = \sum_{E \in \mathscr{E}_h} h_E \|z\|_{0,E}^2 \quad \text{for } z \in L^2(\Gamma). \tag{3.12}$$

For these norms it holds

$$\langle v,z\rangle \leqslant \|v\|_{1/2,h}\|z\|_{-1/2,h} \quad \forall (v,z)\in H^1(\Omega)\times L^2(\Gamma). \tag{3.13}$$

We also denote

$$\|v\|_{1,h} = \|v\|_1 + \|v\|_{1/2,h} \quad \forall v\in H^1(\Omega). \tag{3.14}$$

The interpolation estimates in these norms are proved by scaling (cf. [20]).

Lemma 1. *For $u\in H^{k+1}(\Omega)$ it holds*

$$\inf_{v\in V_h} \|u-v\|_{1,h} \leqslant Ch^k\|u\|_{k+1}. \tag{3.15}$$

Lemma 2. *For $\lambda\in H^{l+1}(\Gamma)$ it holds*

$$\inf_{\mu\in \Lambda_h} \|\lambda-\mu\|_{-1/2,h} \leqslant Ch^{l+3/2}\|\lambda\|_{l+1,\Gamma}. \tag{3.16}$$

In the sequel we will also need the following inequality, which is proved by scaling using the regularity conditions.

Lemma 3. *There exist a constant C_I such that*

$$C_I\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,h} \leqslant \|\nabla v\|_0 \quad \forall v\in V_h. \tag{3.17}$$

In [20] the following result is proved.

Theorem 1B (Pitkäranta [20]). *Suppose that the finite element subspaces satisfy the conditions*

$$\sup_{v\in V_h\setminus\{0\}} \frac{\langle \mu,v\rangle}{\|v\|_{1,h}} \geqslant C\|\mu\|_{-1/2,h} \quad \forall \mu\in \Lambda_h. \tag{3.18}$$

and

$$|v|_1^2 \geqslant C\|v\|_{1,h}^2 \quad \forall v\in \{v\in V_h \mid \langle \mu,v\rangle = 0\ \forall \mu\in\Lambda_h\}. \tag{3.19}$$

For the solution (u_h,λ_h) to the problem (3.7) *it then holds*

$$\|u-u_h\|_{1,h} + \|\lambda-\lambda_h\|_{-1/2,h} \leqslant C(h^k\|u\|_{k+1} + h^{l+3/2}\|\lambda\|_{l+1,\Gamma}), \tag{3.20}$$

when $u\in H^{k+1}(\Omega)$ and $\lambda\in H^{l+1}(\Gamma)$.

In the papers [19–21], it is shown that the spaces V_h and Λ_h should be designed quite carefully in order that the stability conditions would be valid. Hence, there are reason to be quite pessimistic with regards to the general usefulness of this approach in the applications for which the methods has been proposed.

Therefore, it is natural to try to modify the method with similar techniques as those that has been successfully used for the Stokes problem [14, 10]. This has also been done by Barbosa and Hughes

[4, 5]. Their methods did, however, contain terms that are not necessary for the stability. By dropping them we obtain the following method.

Method 2: *A simplification of the symmetric formulation of Barbosa and Hughes* [4, 5]
Find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\mathscr{B}_h(u_h, \lambda_h; v, \mu) = (f, v) + \langle g, \mu \rangle \quad \forall (v, \mu) \in V_h \times \Lambda_h, \tag{3.21}$$

with

$$\mathscr{B}_h(u, \lambda; v, \mu) = \mathscr{B}(u, \lambda; v, \mu) - \alpha \sum_{E \in \mathscr{E}_h} h_E \left\langle \lambda + \frac{\partial u}{\partial n}, \mu + \frac{\partial v}{\partial n} \right\rangle_E, \tag{3.22}$$

where $\mathscr{B}$ is the original bilinear form (2.7) and $\langle \cdot, \cdot \rangle_E$ denotes the L^2-inner product on E.
The first observation concerning this method is that it is consistent due to (2.8).

Lemma 4. *For the exact solution* (u, λ) *to* (2.6) *it holds*

$$\mathscr{B}_h(u, \lambda; v, \mu) = (f, v) + \langle g, \mu \rangle \quad \forall (v, \mu) \in V_h \times \Lambda_h, \tag{3.23}$$

provided that $\lambda \in L^2(\Gamma)$.

The next observation is that the modified bilinear form is bounded with respect to the mesh-dependent norms.

Lemma 5. *There is a positive constant C such that*

$$|\mathscr{B}_h(v, \mu; z, \eta)| \leqslant C(\|v\|_{1,h} + \|\mu\|_{-1/2,h})(\|z\|_{1,h} + \|\eta\|_{-1/2,h})$$
$$\forall (v, \mu) \in H^1(\Omega) \times L^2(\Gamma)\ \forall (z, \eta) \in H^1(\Omega) \times L^2(\Gamma). \tag{3.24}$$

Next, we will prove the stability and optimal order of convergence.

Lemma 6. *Suppose that* $0 < \alpha < C_I$. *Then it holds*

$$\sup_{(z,\eta) \in V_h \times \Lambda_h} \frac{\mathscr{B}_h(v, \mu; z, \eta)}{\|z\|_{1,h} + \|\eta\|_{-1/2,h}} \geqslant C(\|v\|_{1,h} + \|\mu\|_{-1/2,h}). \tag{3.25}$$

Proof. Let $(v, \mu) \in V_h \times \Lambda_h$ be arbitrary. We first note that the estimate of Lemma 3 gives

$$\begin{aligned} \mathscr{B}_h(v, \mu; v, -\mu) &= \|\nabla v\|_0^2 + \alpha \sum_{E \in \mathscr{E}_h} h_E \left(\|\mu\|_{0,E}^2 - \left\| \frac{\partial v}{\partial n} \right\|_{0,E}^2 \right) \\ &\geqslant (1 - \alpha C_I^{-1}) \|\nabla v\|_0^2 + \alpha \sum_{E \in \mathscr{E}_h} h_E \|\mu\|_{0,E}^2 \\ &\geqslant C_1(\|\nabla v\|_0^2 + \|\mu\|_{-1/2,h}^2), \end{aligned} \tag{3.26}$$

since it was assumed that $0 < \alpha < C_I$. Next, let $\Pi_h : L^2(\Gamma) \to \Lambda_h$ be the L^2-projection. Since the functions of Λ_h are discontinuous, we can define $\bar{\mu} \in \Lambda_h$ by $\bar{\mu}_{|E} = h_E^{-1} \Pi_h v_{|E}$ for all $E \in \mathscr{E}_h$. We then have

$$\|\bar{\mu}\|_{-1/2,h} = \|\Pi_h v\|_{1/2,h}. \tag{3.27}$$

By using (3.13), Lemma 3 and the Young inequality we then get

$$\begin{aligned}
\mathscr{B}_h(v,\mu;0,\bar{\mu}) &= \langle v,\bar{\mu}\rangle - \alpha \sum_{E\in\mathscr{E}_h} h_E \left\langle \mu + \frac{\partial v}{\partial n}, \bar{\mu} \right\rangle_E \\
&= \sum_{E\in\mathscr{E}_h} h_E^{-1} \|\Pi_h v\|_{0,E}^2 - \alpha \sum_{E\in\mathscr{E}_h} \left\langle \mu + \frac{\partial v}{\partial n}, \Pi_h v \right\rangle_E \\
&\geqslant \|\Pi_h v\|_{1/2,h}^2 - \left(\left\| \frac{\partial v}{\partial n} \right\|_{-1/2,h} + \|\mu\|_{-1/2,h} \right) \|\Pi_h v\|_{1/2,h} \\
&\geqslant \|\Pi_h v\|_{1/2,h}^2 - C_2 (\|\nabla v\|_0 + \|\mu\|_{-1/2,h}) \|\Pi_h v\|_{1/2,h} \\
&\geqslant \|\Pi_h v\|_{1/2,h}^2 - \tfrac{1}{2} C_2^2 (\|\nabla v\|_0 + \|\mu\|_{-1/2,h})^2 - \tfrac{1}{2} \|\Pi_h v\|_{1/2,h}^2 \\
&\geqslant \tfrac{1}{2} \|\Pi_h v\|_{1/2,h}^2 - C_3 (\|\nabla v\|_0^2 + \|\mu\|_{-1/2,h}^2).
\end{aligned} \tag{3.28}$$

Let now $(z,\eta) = (v, -\mu + \delta\bar{\mu})$ with $\delta > 0$. Using (3.26) and (3.28) we obtain

$$\begin{aligned}
\mathscr{B}_h(v,\mu;z,\eta) &= \mathscr{B}_h(v,\mu;v,-\mu) + \delta \mathscr{B}_h(v,\mu;0,\bar{\mu}) \\
&\geqslant (C_1 - \delta C_3) \|\nabla v\|_0^2 + \tfrac{1}{2}\delta \|\Pi_h v\|_{1/2,h}^2 + (C_1 - \delta C_3) \|\mu\|_{-1/2,h}^2 \\
&\geqslant C(\|\nabla v\|_0^2 + \|\Pi_h v\|_{1/2,h}^2 + \|\mu\|_{-1/2,h}^2),
\end{aligned} \tag{3.29}$$

when choosing $\delta < C_1/C_3$. Now, by scaling one can prove that

$$\|\nabla v\|_0^2 + \|\Pi_h v\|_{1/2,h}^2 \geqslant C \|v\|_{1,h}^2. \tag{3.30}$$

Since (3.27) gives

$$\|z\|_{1,h} + \|\eta\|_{-1/2,h} \leqslant C(\|v\|_{1,h} + \|\mu\|_{-1/2,h}), \tag{3.31}$$

we have proved the asserted stability estimate (which is optimal in view of Lemma 5). □

Theorem 2. *Let* $(u_h, \lambda_h) \in V_h \times \Lambda_h$ *be the solution to the problem* (3.21) *and suppose that* $0 < \alpha < C_I$. *With* $u \in H^{k+1}(\Omega)$ *and* $\lambda \in H^{l+1}(\Gamma)$ *it then holds*

$$\|u - u_h\|_{1,h} + \|\lambda - \lambda_h\|_{-1/2,h} \leqslant C(h^k \|u\|_{k+1} + h^{l+3/2} \|\lambda\|_{l+1,\Gamma}). \tag{3.32}$$

The big advantage with this formulation compared to the original method of Babuška is that the finite element subspaces can be chosen completely arbitrarily.

Let us next have a closer look at the method. We note that since the functions of Λ_h are discontinuous, the variable λ_h can be eliminated locally on each boundary element. By testing with $\mu \in \Lambda_h$ in (3.21), we get the following expression for λ_h:

$$\lambda_{h|E} = -\left(\Pi_h \frac{\partial u_h}{\partial n}\right)\bigg|_E + (\alpha h_E)^{-1}(\Pi_h u - \Pi_h g)|_E \quad \forall E \in \mathscr{E}_h, \tag{3.33}$$

where (as before) Π_h is the L^2-projection onto Λ_h. Now, substituting this into the equation we get from (3.21) when testing by $v \in V_h$, and using the basic property of an orthogonal projection (i.e. $\Pi_h = \Pi_h^2 = \Pi_h^*$) we get the following symmetric (and positive definite, cf. below) system for solving the unknown u_h:

$$\begin{aligned}(\nabla u_h, \nabla v) &- \left\langle \Pi_h \frac{\partial u_h}{\partial n}, \Pi_h v\right\rangle - \left\langle \Pi_h \frac{\partial v}{\partial n}, \Pi_h u_h\right\rangle \\ &+ \sum_{E \in \mathscr{E}_h} (\alpha h_E)^{-1} \langle \Pi_h u_h, \Pi_h v\rangle_E + \alpha \sum_{E \in \mathscr{E}_h} h_E \left\langle (\Pi_h - I)\frac{\partial u_h}{\partial n}, (\Pi_h - I)\frac{\partial v}{\partial n}\right\rangle_E \\ &= (f, v) - \left\langle \Pi_h \frac{\partial v}{\partial n}, \Pi_h g\right\rangle + \sum_{E \in \mathscr{E}_h} (\alpha h_E)^{-1} \langle \Pi_h g, \Pi_h v\rangle_E. \end{aligned} \tag{3.34}$$

Now, since the space Λ_h can be chosen arbitrarily, we may think that we choose $\Lambda_h = L^2(\Gamma)$. Then the projection Π_h becomes the identity and we observed that we have rediscovered a classical method.

Method 3: *Nitsche's method* [18]

Find $u_h \in V_h$ such that

$$\mathscr{B}_h(u_h; v) = \mathscr{F}_h(v) \quad \forall v \in V_h, \tag{3.35}$$

with

$$\mathscr{B}_h(u; v) = (\nabla u, \nabla v) - \left\langle \frac{\partial u}{\partial n}, v\right\rangle - \left\langle \frac{\partial v}{\partial n}, u\right\rangle + \gamma \sum_{E \in \mathscr{E}_h} h_E^{-1} \langle u, v\rangle_E, \tag{3.36}$$

$$\mathscr{F}_h(v) = (f, v) + \langle g, v\rangle - \left\langle \frac{\partial v}{\partial n}, g\right\rangle + \gamma \sum_{E \in \mathscr{E}_h} h_E^{-1} \langle g, v\rangle_E. \tag{3.37}$$

By the way we have arrived at this formulation, it is clear that we have an optimal error estimate for it. That is, however, more easily obtained directly.

Theorem 3. *Let $u_h \in V_h$ be the solution to the problem* (3.35) *and suppose that $\gamma > C_I^{-1}$. With $u \in H^{k+1}(\Omega)$ it then holds*

$$\| u - u_h \|_{1,h} \leqslant C h^k \| u \|_{k+1}. \tag{3.38}$$

Proof. The consistency of the method is immediately seen from the formulation. The stability is proved by Schwartz, Young and Lemma 3:

$$\begin{aligned}\mathscr{B}_h(v;v) &= \|\nabla v\|_0^2 - 2\left\langle v, \frac{\partial v}{\partial n}\right\rangle + \gamma\|v\|_{1/2,h}^2 \\ &\geqslant \|\nabla v\|_0^2 - 2\|v\|_{1/2,h}\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,h} + \gamma\|v\|_{1/2,h}^2 \\ &\geqslant \|\nabla v\|_0^2 - \frac{1}{\varepsilon}\left\|\frac{\partial v}{\partial n}\right\|_{-1/2,h}^2 + (\gamma-\varepsilon)\|v\|_{1/2,h}^2 \\ &\geqslant \left(1 - \frac{1}{\varepsilon C_I}\right)\|\nabla v\|_0^2 + (\gamma-\varepsilon)\|v\|_{1/2,h}^2 \\ &\geqslant C\|v\|_{1,h}^2, \end{aligned} \tag{3.39}$$

when we choose $C_I^{-1} < \varepsilon < \gamma$.

We have thus established the stability and the consistency. The assertion then follows from Lemma 1. □

In view of our analysis it seems that the Nitsche method is the most straightforward method to use. Unfortunately, this method seems to be quite unknown. We think, however, that it would be worthwhile to explore it in applications such as contact problems, for fictitious domain methods and for domain decomposition.

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