ERROR BOUNDS FOR THE APPROXIMATION OF THE STOKES
PROBLEM USING BILINEAR/CONSTANT ELEMENTS
ON IRREGULAR QUADRILATERAL MESHES

J. Pitkäranta and R. Stenberg

Institute of Mathematics, Helsinki
University of Technology

1. INTRODUCTION

One of the most popular methods to numerically solve the
Stokes equations in fluid mechanics is to use a mixed finite
element method where the velocities are approximated with
continuous isoparametric bilinear elements on quadrilateral
meshes whereas a piecewise constant approximation is used for
the pressure. After eliminating the pressure by simple pertur-
bation techniques one obtains a positive definite system for the
velocities alone. Another way to obtain the same method is to
apply penalty techniques with reduced/selective integration
(cf. [6], [8]). In numerical computations this method has been
shown to give excellent results for the computed velocities
and also for the pressure provided that the latter has been
"smoothed" in an appropriate way (cf. e.g. [6]). From a
theoretical point of view this success has been considered
somewhat surprising since it is well known that the method is
not uniformly stable in the sense of Babuška [1] and Brezzi [3].
For rectangular meshes, however, it has been possible to analyze
the method, cf. [7]. The analysis of [7] relies on a weak
Babuška-Brezzi-type stability condition together with a careful
consistency estimate and shows that the method converges with
the optimal rate provided the exact solution is smooth enough.
In this note we will extend and improve the analysis of [7].
We will show that the method in fact converges (after a pres-
sure smoothing) for a very general class of meshes and that this
happens without any extra smoothness assumptions on the exact
solution. The fact that the extra smoothness assumption of [7]
is not required was also observed by Boland and Nicolaides [2]
in the case of a rectangular grid.

2. NOTATION AND PRELIMINARIES

Let $\Omega$ be a polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma$. The
problem under consideration consists of the stationary Stokes equations for an incompressible viscous fluid:

\[-\nabla \Delta u + \nabla p = f \quad \text{in} \; \Omega,\]
\[\text{div} \; u = 0 \quad \text{in} \; \Omega,\]
\[u = 0 \quad \text{on} \; \Gamma,\]  \hspace{1cm} (2.1)

where \( u \) is the fluid velocity, \( p \) is the pressure, \( f \) is the body force and \( \nu > 0 \) is the kinematic viscosity.

We denote by \( |\cdot|_{s,T} \) and \( \|\cdot\|_{s,T} \), respectively, the semi-norm and norm of the Sobolev space \([H^s(T)]^\alpha\) where \( s \) and \( \alpha \) are integers. As usual \( H^1_0(T) \) denotes the subspace of \( H^1(T) \) consisting of functions with vanishing trace on \( \partial T \). We will also introduce the space

\[L^2_0(T) = \{ p \in L^2(T) \mid \int_T p \; dx = 0 \}.\]

The inner product in \([L^2(T)]^\alpha\), for integral \( \alpha \), is denoted by \((\cdot,\cdot)_T\). The subscript \( T \) will be dropped if \( T = \Omega \). As usual we will denote by \( C \) and \( C_j \) positive constants, possibly different at different occurrences, which are independent of the mesh parameter \( h \).

In variational form (2.1) reads: Find \( u \in [H^1_0(\Omega)]^2 \) and \( p \in L^2_0(\Omega) \) such that

\[\nu(\nabla u, \nabla v) - (\text{div} \; v, p) = (f, v) \quad \forall v \in [H^1_0(\Omega)]^2,\]
\[(\text{div} \; u, u) = 0 \quad \forall u \in L^2_0(\Omega).\]  \hspace{1cm} (2.2)

In the finite approximation of (2.2) the spaces \([H^1_0(\Omega)]^2\) and \( L^2_0(\Omega) \) are replaced by the finite dimensional subspaces \( V_h \) and \( P_h \), respectively. Below we define the subspaces as

\[V_h = \{ v \in [H^1_0(\Omega)]^2 \mid v|_K \in [Q_1(K)]^2 \quad \forall K \in \mathcal{C}_h \}\]

and

\[P_h = \{ p \in L^2_0(\Omega) \mid p|_K \text{ is constant } \forall K \in \mathcal{C}_h \},\]

where \( \mathcal{C}_h \) stands for a partitioning of \( \Omega \) into convex quadrilaterals and \( Q_1(K) \) is the space of (isoparametrically) transformed bilinear functions \([4] \). As usual, the mesh para-
mometer \( h \) is defined as \( h = \max_{K \in C_h} h_K \), where \( h_K \) denotes the diameter of \( K \).

We now specify our assumptions on the partitioning \( C_h \).

First, we assume that \( C_h \) is a refinement of a coarser partitioning \( C_{2h} \), obtained by subdividing each \( \hat{K} \in C_{2h} \) into four quadrilaterals by joining the midpoints of the opposite sides of \( \hat{K} \) by straight lines. Second, we assume that \( C_{2h} \) is also a similar refinement of a still coarser partitioning \( C_{4h} \).

Third, regarding \( C_{4h} \), we merely assume that \( C_{4h} \) is regular. By this we mean that there are the constants \( \sigma > 1 \) and \( 0 < \gamma < 1 \) independent of \( h \) such that

\[
h_K \leq \sigma \rho_K, \quad |\cos \theta_{iK}| \leq \gamma; \quad i = 1, 2, 3, 4, \quad \forall K \in C_{4h},
\]

where \( h_K, \rho_K \) and \( \theta_{iK} \) are respectively the diameter of \( K \), the diameter of the largest circle contained in \( K \), and the angles of \( K \).

Below we refer to the quadrilaterals of \( C_{2h} \) or \( C_{4h} \) as "macroelements" and denote them by \( M \). We also introduce the subspace

\[
V_{2h} = \{ v \in [H^1_0(\Omega)]^2 \mid v |_M \in [Q_1(M)]^2, \forall M \in C_{2h} \}
\]

where \( Q_1(M) \) is as above. The space \( P_h \) will be written as the sum of three subspaces. The unit square \( \hat{K} \) is partitioned into subdomains \( \hat{K}_{ij} = \{(x_1, x_2) \in \hat{K} \mid \frac{(i-1)}{2} \leq x_1 \leq \frac{i}{2}, \frac{(j-1)}{2} \leq x_2 \leq \frac{j}{2}\}, i, j = 1, 2 \) and on \( \hat{K} \) we define the function \( \eta \) through

\[
\eta |_{\hat{K}_{ij}} = (-1)^{i+j}, \quad i, j = 1, 2.
\]

We then define the subspaces

\[
P_{h1} = \{ p \in P_h \mid p |_M \text{ is constant } \forall M \in C_{2h} \},
\]

\[
P_{h3} = \{ p \in P_h \mid p |_M = c_M \eta o F_M^{-1}, c_M \in \mathbb{R}, \forall M \in C_{2h} \}
\]

where \( F_M \) is the bilinear mapping of \( \hat{K} \) onto \( M \). The orthogonal complement of \( P_h \) with respect to \( P_{h1} \perp P_{h3} \) is denoted by \( P_{h2} \). Finally we introduce a "pressure smoothing
operator $\pi: P_h \to P_{h1} \oplus P_{h2}$. Every $p \in P_h$ can be written uniquely as $p = \sum_{i=1}^3 p_i$, $p_i \in P_{hi}$. The filtered pressure $\pi p$ is then defined as $\pi p = p_1 + p_2$.

3. ERROR ANALYSIS

Let us start with a consistency estimate which is crucial for the analysis in this paper.

**Lemma 3.1.** For each $v \in V_{2h}$ and $p \in P_{h3}$ we have

$$(\text{div } v, p) = 0.$$ 

**Proof.** Consider a macroelement $M \in C_{2h}$ with nodes $x^i$, $i = 1, 2, 3, 4$, as in the figure below and suppose $p_M = \eta \circ F_M^{-1}$ takes the values $\pm 1$ as in the figure.

![Diagram](image)

Denote by $v^i = v(x^i)$, $i = 1, 2, 3, 4$, the degrees of freedom of $v \in V_{2h}|M$ and write

$$x^i x^j = x^i - x^j, \quad i, j = 1, 2, 3, 4.$$ 

It will further be convenient to use the (scalar valued) vector product in $\mathbb{R}^2$, i.e. if $a = (a_1, a_2)$ and $b = (b_1, b_2)$ we define

$$a \wedge b = a_1 b_2 - a_2 b_1.$$ 

Using Green's formula and integrating over the sides in $M$ one obtains
\[
\begin{align*}
\left( \text{div} \, \mathbf{v}, p_M \right)_M &= \frac{1}{4} \left[ \left( \mathbf{v}^4 - \mathbf{v}^1 \right) \wedge x^x x^1 \\
+ 2 \left( -\left( \mathbf{v}^1 + \mathbf{v}^2 \right)^2 \right) \wedge \left( \mathbf{v}^3 + \mathbf{v}^4 \right)^2 \wedge \left( \frac{\mathbf{x}^4}{2} \mathbf{x}^1 + \mathbf{x}^3 \mathbf{x}^2 \right) \\
+ \left( \mathbf{v}^2 - \mathbf{v}^3 \right) \wedge x^x x^2 + \left( -\mathbf{v}^1 + \mathbf{v}^2 \right) \wedge x^x x^1 \\
+ \left( \left( \mathbf{v}^1 + \mathbf{v}^2 \right)^2 \right) \wedge \left( \mathbf{v}^3 + \mathbf{v}^4 \right)^2 \wedge \left( \frac{\mathbf{x}^2}{2} \mathbf{x}^1 + \mathbf{x}^3 \mathbf{x}^4 \right) \\
+ \left( -\mathbf{v}^1 + \mathbf{v}^2 + \mathbf{v}^4 - \mathbf{v}^3 \right) \wedge \left( \frac{\mathbf{x}^2}{2} \mathbf{x}^1 - \mathbf{x}^3 \mathbf{x}^4 \right)
\right]
= \frac{1}{8} \left[ \left( \mathbf{v}^1 - \mathbf{v}^4 - \mathbf{v}^2 + \mathbf{v}^3 \right) \wedge \left( \frac{\mathbf{x}^4}{2} \mathbf{x}^1 - \mathbf{x}^3 \mathbf{x}^2 \right) \\
+ \left( -\mathbf{v}^1 + \mathbf{v}^2 + \mathbf{v}^4 - \mathbf{v}^3 \right) \wedge \left( \frac{\mathbf{x}^2}{2} \mathbf{x}^1 - \mathbf{x}^3 \mathbf{x}^4 \right) \right] = 0,
\end{align*}
\]

since by the definition of \(x^i x^j\), \(i, j = 1, 2, 3, 4\) we have \(x^1 x^2 x^3 x^4 = 2 1 - 3 4\).

Since \(p|_M = c_M p_M\), \(c_M \in \mathbb{R}\), for every \(p \in \mathbb{P}_{h3}\), the assertion is proved.

Next we will turn to the stability estimate, the proof of which will only be sketched since the arguments are very similar to those given in [9].

**Lemma 3.2.** There is a constant \(C > 0\) such that

\[
\sup_{\substack{u \in \mathbb{V}_h \\ u \neq 0}} \frac{(\text{div} \, u, p)}{|u|_1} \geq C \|p\|_0 \quad \forall p \in \mathbb{P}_h.
\]

**Proof.** Consider a macroelement \(M \in \mathbb{C}_{4h}\) and define

\[
\mathbb{V}_{0,M} = \{ v \in [H^1_0(M)]^2 \mid v|_K \in [Q_1(K)]^2 \quad \forall K \subset M, K \in \mathbb{C}_h \}
\]

and

\[
\mathbb{N}_M = \{ p \in \mathbb{P}_h|_M \mid (\text{div} \, v, p)_M = 0 \quad \forall v \in \mathbb{V}_{0,M} \}.
\]

A straightforward calculation shows that \(\mathbb{N}_M = \{ c_1 \psi_1^M + c_2 \psi_3^M \mid c_1, c_2 \in \mathbb{R} \}\), where \(\psi_1^M\) is constant on \(M\) and \(\psi_3^M\) takes the values \(\pm 1\) in a chessboard-like manner on the subrectangles of \(M\). Let \(\widetilde{\mathbb{P}}_{hi} = \{ p \in \mathbb{P}_h \mid p|_M = c_M \psi_1^M, c_M \in \mathbb{R} \}\), \(i = 1, 3\), and let \(\widetilde{\mathbb{P}}_{h2}\) be the orthogonal complement of \(\mathbb{P}_h\) to \(\widetilde{\mathbb{P}}_{h1} \cap \widetilde{\mathbb{P}}_{h3}\).

By the same arguments as those leading to the macroelement principle introduced in [9] (cf. Lemma 3.1 and Lemma 3.2 of
one now concludes that for every \( p \in P_h \), 
\[
p = \sum_{i=1}^{3} \tilde{p}_i,
\]
\( \tilde{p}_i \in \tilde{P}_h \), there is a \( v \in V_h \) such that \( v \big|_{M} \in V_{0,M} \forall M \in \mathcal{C}_{4h} \) and
\[
(\text{div} v, p) \geq C_1 \| \tilde{p}_2 \|_0^2
\]
and
\[
|v|_1 \leq \| \tilde{p}_2 \|_0.
\]
By the same reasoning as in Lemma 3.3 of [9] one can also show that for every \( \tilde{p}_1 \in \tilde{P}_h \) there is a \( g \in V_{2h} \) such that
\[
(\text{div} g, \tilde{p}_1) = \| \tilde{p}_1 \|_0^2
\]
and
\[
|g|_1 \leq C_2 \| \tilde{p}_1 \|.
\]
Since \( g \in V_{2h} \) we have by Lemma 3.1
\[
(\text{div} g, \tilde{p}_3) = 0.
\]
Let now \( p \in P_h \) be arbitrary and write \( p = \sum_{i=1}^{3} \tilde{p}_i \). Define
\[
z = v + \frac{1}{2 C_1} g \text{, where } v, g, C_1 \text{ and } C_2 \text{ are as above. A straightforward calculation, using the relations (3.1) to (3.5), then gives (cf. the proof of Theorem 3.1 in [9])}
\[
\frac{(\text{div} z, p)}{|z|_1} \geq C(\| \tilde{p}_1 \|_0 + \| \tilde{p}_2 \|_0) \geq C\|p\|_0.
\]

As a final preparation for our error estimate we will introduce a seminorm on \( P_h \) defined through
\[
|p|_h = \sup_{v \in V_h} \frac{(\text{div} v, p)}{|v|_1} \quad \forall p \in P_h.
\]
The following estimate is an immediate consequence of Lemma 3.2 and the definition of the seminorm \( |\cdot|_h \)
\[
\|p\|_0 \geq |p|_h \geq C \|p\|_0 \quad \forall p \in P_h.
\]
We are now ready to prove
Theorem 3.1. Let \((u,p)\) be the solution to (2.1) and let 
\((u_h,p_h) \in V_h \times P_h\) be its finite element approximation defined
as above. Then we have the error estimate
\[
|u-u_h|_1 + \|p-p_h\|_0 \leq Ch(|u|_2 + |p|_1),
\]
provided \(u \in [H^2(\Omega)]^2\) and \(p \in H^1(\Omega)\).
Moreover, if \(\Omega\) is a convex region, we have the additional
estimate
\[
\|u-u_h\|_0 \leq Ch^2(|u|_2 + |p|_1).
\]

Proof. Let \(\tilde{u} \in V_h\) be the interpolant to \(u\) and let \(\tilde{p}\) be
the \(L^2\)-projection of \(p\) onto \(P_h\). By the general theory of
Babuška [1] and Brezzi [3] (cf. also [7]) one concludes that
there exists \(v \in V_h\) and \(\mu \in P_h\) such that
\[
|v|_1 + |\mu|_h \leq C,
\]
and
\[
|u_h-\tilde{u}|_1 + |P_h-\tilde{p}|_h \leq C\{(\nabla(u-\tilde{u}),\nabla v) \, |(\text{div} \, v,p-\tilde{p})| + \text{ (3.7)}
\]
\[
+ \, (\text{div}(u-\tilde{u}),\mu)\}.
\]
By standard interpolation theory [4] the first two terms on the
right hand side of (3.7) can be estimated as
\[
|\nabla(u-\tilde{u}),\nabla v| \leq |u-\tilde{u}|_1 |v|_1 \leq Ch |u|_2, \quad (3.8)
\]
\[
|\text{div} \, v,p-\tilde{p}| \leq |v|_1 \|p-\tilde{p}\|_0 \leq Ch |p|_1. \quad (3.9)
\]
To estimate the third term we write \(\mu = \pi \mu + (I-\pi)\mu\) so as to
obtain
\[
|(\text{div}(u-\tilde{u}),\mu)| \leq |(\text{div}(u-\tilde{u}),\pi \mu)| + \quad (3.10)
\]
\[
+ \, |(\text{div}(u-\tilde{u}),(I-\pi)\mu)|
\]
For the first term on the right hand side of (3.10) we obtain,
using the estimate (3.6),
\[
|(\text{div}(u-\tilde{u}),\pi \mu)| \leq |u-\tilde{u}|_1 \|\pi \mu\|_0 \leq Ch |u|_2. \quad (3.11)
\]
To estimate the second term on the right hand side of (3.10)
we introduce the interpolant \(\tilde{u} \in V_{2h}\) to \(u\). Since \(\text{div} \, u = 0\),
\[(I-\pi)u \in PH_3 \quad \text{and} \quad \tilde{u} \in V_{2h} \quad \text{we have by Lemma 3.1} \]
\[
|\langle \text{div}(u-\tilde{u}), (I-\pi)u \rangle | = |\langle \text{div}(u-\tilde{u}), (I-\pi)\mu \rangle | \leq \]
\[
\leq |u-\tilde{u}|_1 (|\mu|_h + \|\pi \mu\|_0) \leq C|u-\tilde{u}|_1 |\mu|_h \leq C|u-\tilde{u}|_1 \leq Ch|u|_2. \quad (3.12)
\]

Here the last inequality is a consequence of standard interpolation error estimates. Upon collecting the estimates (3.7) through (3.12) we obtain
\[
|u_h-\tilde{u}|_1 + |p_h-\tilde{p}|_h \leq Ch(|u|_2 + |p|_1). \quad (3.13)
\]

The asserted estimate for $|u-u_h|_1$ now follows using the triangle inequality. To obtain the estimate for $\|p-\pi p_h\|_0$ we use (3.6) and get
\[
\|\pi p_h - \tilde{p}\|_0 \leq Ch(|u|_2 + |p|_1). \quad (3.14)
\]

The asserted estimate is now obtained upon applying the triangle inequality together with the estimate
\[
\|p-\tilde{p}\|_0 \leq Ch|p|_1. \quad (3.15)
\]

In order to obtain the $L^2$-estimate for the velocity we first note that if we replace $\tilde{p}$ by $\pi \tilde{p}$ in (3.7) through (3.12) we obtain
\[
|p_h-\pi \tilde{p}|_h \leq Ch(|u|_2 + |p|_1). \quad (3.16)
\]

From (3.16), (3.6) and (3.14) we then obtain
\[
|(I-\pi)p_h|_h \leq |p_h-\pi \tilde{p}|_h + |\pi \tilde{p}-\pi p_h|_h \quad \leq |p_h-\pi \tilde{p}|_h + \|\pi \tilde{p}-\pi p_h\|_0 \leq Ch(|u|_2 + |p|_1). \quad (3.17)
\]

We now proceed using the Aubin-Nitsche trick. Let $(z, \lambda) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega)$ be the solution to the problem

\[
\nabla (\nabla z, \nabla v) - \langle \text{div} \lambda, v \rangle = \langle u-u_h, v \rangle \quad \forall v \in [H^1_0(\Omega)]^2,
\]
\[
\langle \text{div} z, \mu \rangle = 0 \quad \forall \mu \in L^2_0(\Omega).
\]

In the usual way (cf. e.g. [5]) we then obtain
\[
\|u-u_h\|_0^2 \leq Ch\|u-u_h\|_1 (|z|_2 + |\lambda|_1) + |\langle \text{div}(z-\tilde{z}), p-p_h \rangle |, \quad (3.18)
\]
where $\tilde{z} \in V_h$ is the interpolant to $z$. To estimate the second term on the right hand side of (3.18) we repeat the arguments used in proving the estimates (3.10) through (3.12).

Let $\tilde{\tilde{z}}$ be $V_{2h}$-interpolant to $z$. Since $\text{div} \ z = 0$ and $\tilde{\tilde{z}} \in V_{2h}$ we obtain

$$
|\text{div}(\tilde{z} - \tilde{\tilde{z}}), p-p_h| \leq |\text{div}(z-\tilde{\tilde{z}}), p-\pi p_h| + 
$$

$$
+ |\text{div}(\tilde{z} - z), (I-\pi)p_h| \leq C \|z\|_2 (|p-\pi p_h|_0 + 
$$

$$
+ |(I-\pi)p_h|_h).
$$

(3.19)

The asserted estimate now follows upon combining (3.18), (3.19) and (3.17) and using the regularity estimate

$$
|z|_2 + |\lambda|_1 \leq C \|u-u_h\|_0.
$$

\[ \Box \]

Remark. We could have simplified the above analysis by choosing the interpolants $\tilde{u}$ and $\tilde{z}$ in $V_{2h}$, i.e. setting $\tilde{u} = \tilde{u}$, $\tilde{z} = \tilde{z}$. The reason for not doing this was to show that the error constants in the final estimates are not substantially larger than what they would be if the method were uniformly stable in the classical sense - a fact that has also been confirmed by numerous numerical calculations.

REFERENCES


