AN hp ERROR ANALYSIS OF SOME GALERKIN LEAST SQUARES METHODS FOR THE ELASTICITY EQUATIONS

ERIC BOILLAT [1] and ROLF STENBERG [2]

Abstract

Until now, the technique of stabilizing mixed finite element methods by augmenting the Galerkin formulation with least square terms calculated separately on each element has always been considered in a context where the appropriate accuracy is obtained by letting the characteristic size h of the mesh tend to zero. We now want to give a generalisation of these methods in such a way that the accuracy can also be obtained by increasing the degree p of the shape functions.

1 Introduction

Our article is concerned with the approximation of the two dimensional elasticity equations with some finite element methods. In that context, It is well known that the convergence property of a standard Galerkin method is in general slowed down in the incompressible limit, when the Poisson ratio ν is close to $\frac{1}{2}$. This phenomenon is called *"Poisson locking"* and, if we are only interested in displacement (and not in stresses), it can be avoided by selecting the discretisation spaces according to a criterion established in [?]. However this criterion imply severe restriction on meshes. In [?], the examples of method free from locking are mostly based on quasiuniform triangular meshes and a generalization to three dimensional problems or a generalization to curved elements seem to be far from trivial.

During the last decade, the *mixed-method* (see [?], [?] and the references therein) has been succesfully studied. In that context, we introduce the pressure as a new variable and we discretize it. The obtained numerical scheme falls into the class of saddlepoint problem (see [?], [?]). They give, independently of the Poisson ratio, an optimal order of convergence for both displacement and pressure under the so-called *"inf-sup"* or *"Babuška-Brezzi"* condition, which is a divergence stability property between the discrete spaces used for the two unknowns. At present, the performances of many combinations are known (see [?], [?], [?]) and other stable methods can be designed by using the *"bubble-functions"* techniques (see [?], [?], [?]). But, for a particular choice of displacement and pressure approximations, it can often be a difficult task to verify this *"Babuška-Brezzi"* condition.

Recently the so-called *"Galerkin Least-Squares"* or in short *"GLS"* methods have received considerable attention (see [?] and the references therein). The idea is to combine the usual bilinear form of the mixed methods with a least-squares form of the differential equation. The least-squares terms are evaluated separately on each element. On the one hand, the disadvantage of this method is that it is not consistent if the exact solution has only the basic regularity demanded by the variational formulation of the elasticity problem. It needs, at least, the regularity required by the strong formulation. On the other hand, in a pure h-version, where the accuracy is obtained by only refining the meshes, it has been proved by Franca and Stenberg (see [?]) that this method has uniform stability properties. It namely provides, independently of the Poisson ratio, an optimal order of convergence for both displacement and pressure, without any conditions on the discretization spaces.

The aim of our work is to study an *"h-p"* version of the *GLS* method, where accuracy is achieved not only by decreasing the mesh size h but also by increasing (usually at the same rate) the degrees

[1] Département de mathématiques, EPFL, CH-1015 Lausanne, Switzerland. The work of this author was supported by the Swiss National Research Funds.

[2] Faculty of Mechanical Engineering, Helsinki University of Technology, 02150 Espoo, Finland.

p and q of the piecewise polynomial approximations of the displacement and pressure. Our main result is an error estimate, independent of the Poisson ratio, for the *GLS* method. It proves that the order of convergence for the displacement is optimal in h and not worse than the optimal order minus one in p. As far as the approximation of the pressure is concerned, the order of convergence in h is optimal and the order of convergence in q is at least equal to the optimal order minus 2.

According to this analysis, there do exist schemes in *h-p* version with better performances. But, unlike the *GLS* methods, they are usually constructed with severe restriction on meshes and their extension to three dimensional situations or to problems requiring curved elements is not always possible.

In this article, we also obtain an other relevant result. We show that the *GLS* strategy of adding least squares terms to the classical bilinear form will never slow down the convergence rate. Assume, for instance, that some combination of discrete spaces for displacement and pressure provides an *h-p* mixed method whose convergence is optimal in displacement. In that context, we prove that the *GLS* method, built with these spaces, presents also an optimal convergence rate for the displacement. Moreover, it turns out that the convergence in pressure is good, as well, since we do not lose more then one power of p with regard to the optimal order. We thus can propose the *GLS* method as a worthwile alternative to the usual techniques of recovering pressure by post-processing the displacement.

We now present the plan of our paper.

In Section **2** we introduce our notations we used to denote different functional spaces. In Section **3**, we describe brievly the continuous problem of elasticity to be approximated. In Section **4** we introduce the *h-p* version of the *GLS* method and we establish a general error estimate independent of the Poisson ratio. In Section **5**, we take a method which has optimal order of convergence for the displacement and we analyze its property when it is stabilized with *GLS* strategy.

2 Notation

In this section we present the standard notation to be used for different function spaces (cf. [**?**], [**?**]).

Let S be a subset of $R^d, d = 1, 2$. We denote by $L^2(S)$ the usual set of all square-integrable real-valued functions on S. For $k \in N$, we introduce $H^k(S)$, the set of all elements of $L^2(S)$ whose derivatives up to order k are also elements of $L^2(S)$. For $k > 0, k \notin N$, we define $H^k(S)$ by interpolation between $H^{[k]}(S)$ and $H^{[k]+1}(S)$ (cf. [**?**]), $[k]$ being the biggest integer less than k.

For any $p \in N$, we denote by $P_p(S)$ the set of all real-valued polynomials defined on S, with total degree $\leq p$. In the same way, $Q_p(S)$ represents the set of all the real-valued polynomials defined on S, with separate degree $\leq p$ in each variable.

If m, n are some integers and V a class of functions, $(V = L^2(S), H^r(S), P_p(S)....)$, we call V^m (respectively $V^{m\times n}$) the set of all R^m-valued (respectively $R^{m\times n}-$valued) functions of S the components of which are all elements of the class V.

We will represent by $\|\cdot\|_{r,S}$ the norm of $H^r(S)$ as well as the one of $H^r(S)^m$ or of $H^r(S)^{m\times n}$. In the same way, $(\cdot,\cdot)_S$ will denote the scalar product of $L^2(S), L^2(S)^m$ or of $L^2(S)^{m\times n}$. We will drop the subscript S from $\|\cdot\|_{r,S}$ and $(\cdot,\cdot)_S$ each time it represents Ω.

If $\mathbf{u} \in H^1(S)^2$, we use the special notation $\varepsilon(\mathbf{u})$ for the symmetric part of the Jacobian matrix of $\mathbf{u}$. $\varepsilon(\mathbf{u})$ is the *strain tensor.*

3 The continuous problem

In this section we introduce the continuous problem to be approximated. For a more detailed discussion on this subject, we refer to [**?**].

Let Ω be a two dimensional domain with a polygonal boundary Γ. We suppose that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_1$, where Γ_1 and Γ_2 are disjoint and satisfy $\Gamma_1 \neq \emptyset, \Gamma_2 \neq \emptyset$. We then introduce the set of functions

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^2 \quad | \quad \mathbf{v}_{|\Gamma_1} = 0 \quad \}. \tag{3.1}$$

As in [?], we consider that the domain Ω is occupied by an isotropic body clamped along Γ_1, subjected to a load $\mathbf{f} \in L^2(\Omega)^2$ and a boundarytraction $\mathbf{g} \in L^2(\Gamma_2)^2$ on Γ_2. Assuming, for simplicity, that the Lame coefficient μ is equal to $\frac{1}{2}$ and setting

$$\epsilon = \frac{1-2\nu}{\nu} \tag{3.2}$$

where ν is the Poissons ratio, we get that the displacement $\mathbf{u}$ of the body is solution to the following problem formulated in the displacement-pressure form. Find $\mathbf{u} \in \mathbf{V}$ and $P \in L^2(\Omega)$ such that

$$B_1(\mathbf{u}, P; \mathbf{v}, Q) = F_1(\mathbf{v}),\ \forall\ (\mathbf{v}, Q) \in \mathbf{V} \times L^2(\Omega) \tag{3.3}$$

where B_1 is the standard bilinear form defined on $\left[\mathbf{V} \times L^2(\Omega)\right]^2$,

$$B_1(\mathbf{u}, P; \mathbf{v}, Q) = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) - (P, \nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{u}, Q) - \epsilon(P, Q) \tag{3.4}$$

and where the second member is defined by

$$F_1(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\mathbf{g}, \mathbf{v})_{\Gamma_2},\ \forall \mathbf{v} \in \mathbf{V} \tag{3.5}$$

Because of the assumed regularity of the data, the problem (3.3) is easily proven to have a unique solution and is well posed with respect to the parameter $\epsilon > 0$ (cf. [?]).

4 The h-p version of the GLS method

In this section, we first describe the h-p version of the GLS method. We then establish our main result which is a general error estimate independent of the parameter $\epsilon > 0$.

Following [?] and using the standard notations of [?] and [?] , we consider a regular family of partitioning $\{\mathcal{C}_h\}$ of $\overline{\Omega}$, h denoting the mesh size of $\mathcal{C}_h$. For each h, $\mathcal{C}_h$ is assumed to consist of triangles or parallelograms. A mixing of triangles and parallelograms is not excluded. Moreover, for h and $m = 0, 1, 2..$ given, we define the space $H_h^m(\Omega)$:

$$H_h^m(\Omega) = \{Q \in H^{m-1}(\Omega) | \quad Q_{|K} \in H^m(K),\ \forall\ K \in \mathcal{C}_h\}, \tag{4.1}$$

as well as the bilinear form b_h acting on $\left[H_h^2(\Omega) \times H_h^1(\Omega)\right]^2$ and the linear form f_h acting on $H_h^2(\Omega)$.

$$b_h(\mathbf{u}, P; \mathbf{v}; Q) = -\sum_{K \in \mathcal{C}_h} h_K^2(-\nabla \cdot \varepsilon(\mathbf{u}) + \nabla P, -\nabla \cdot \varepsilon(\mathbf{v}) + \nabla Q)_K, \tag{4.2}$$

$$f_h(\mathbf{v}) = \sum_{K \in \mathcal{C}_h} h_K^2(\mathbf{f}, -\nabla \cdot \varepsilon(\mathbf{v}) + \nabla Q)_K. \tag{4.3}$$

In (4.2) and (4.3), h_K denotes the diameter of $K \in \mathcal{C}_h$. In (4.3), $\mathbf{f} \in L^2(\Omega)^2$ is the load which appears in the continuous problem (3.3).

For convenience we adopt the notation

$$R_m(K) = \begin{cases} P_m(K) \text{ if } K \text{ is a triangle,} \\ Q_m(K) \text{ if } K \text{ is a parallelogram} \end{cases}$$

and we introduce the standard discrete subspaces $\mathbf{V}_{h,p} \subset \mathbf{V}$ and $W_{h,q} \subset L^2(\Omega)$ for $p, q = 1, 2, 3....$ as follows.

$$\mathbf{V}_{h,p} = \{\mathbf{v} \in \mathbf{V} | \quad \mathbf{v}_{|K} \in R_p(K)^2, K \in \mathcal{C}_h\}, \tag{4.4}$$

$$W_{h,q} = \{P \in L^2(\Omega) | \quad P_{|K} \in R_q(K), K \in \mathcal{C}_h\}. \tag{4.5}$$

Remark 4.1 *Our analysis is restricted to meshes consisting only of triangles and parallelograms since we intend to use the approximation properties of the space $\mathbf{V}_{h,p}$ described by Theorem 4.6 in [?]. If the conclusion of this Theorem is valid for more general meshes (e.g. meshes consisting of triangles and convex quadrilaterals), the content of the present work can immediately be extended.*

Note that Theorem 4.6 in [?] is also established under the assumption of quasiuniform meshes. But this hypothesis is actually never used in the proof and the Theorem holds without it.

We will now describe the h-p version of the GLS method by Franca and Stenberg. It amounts to find $\mathbf{u}_{h,p,q} \in \mathbf{V}_{h,p}$ and $P_{h,p,q} \in W_{h,q}$ such that

$$B(\mathbf{u}_{h,p,q}, P_{h,p,q}; \mathbf{v}, Q) = F(\mathbf{v}),\ \forall\ (\mathbf{v}, Q) \in \mathbf{V}_{h,p} \times W_{h,q}, \tag{4.6}$$

where the bilinear form B is defined on $\left[H_h^2(\Omega)^2 \times H_h^1(\Omega)\right]^2$ and the linear form F on $H_h^2(\Omega)^2$ by perturbing the corresponding forms B_1 and F_1 of the continuous problem (3.3). We set

$$B(\mathbf{u}, P; \mathbf{v}, Q) = B_1(\mathbf{u}, P; \mathbf{v}, Q) + \frac{\alpha}{(p+q)^4} b_h(\mathbf{u}, P; \mathbf{v}, Q), \tag{4.7}$$

$$F(\mathbf{v}) = F_1(\mathbf{v}) + \frac{\alpha}{(p+q)^4} f_h(\mathbf{v}), \tag{4.8}$$

b_h and f_h being the mesh dependent forms introduced in (4.2) and (4.3). In (4.7) and (4.8), α is a positive parameter which we will choose properly. Of course, B and F depend on h, p and q, but we do not indicate it, to keep the notations as simple as possible.

Remark 4.2 *The expression $B(\mathbf{u}, P; \mathbf{v}, Q)$ with $(\mathbf{v}, Q) \in \mathbf{V}_{h,p} \times W_{h,q}$ cannot be defined for any element $(\mathbf{u}, P)$ of $\mathbf{V} \times L^2(\Omega)$. As a consequence, the stabilized scheme (4.6) is not always consistent with the continuous problem (3.3). However, if the solution $(\mathbf{u}, P)$ to (3.3) happens to satisfy*

$$(\mathbf{u}, P) \in H^{k+1}(\Omega)^2 \cap \mathbf{V} \times H^k(\Omega), \tag{4.9}$$

with $k \geq 1$, then our scheme becomes consistent. The reason is classical. Under (4.9), we know that the strong equality $-\nabla \cdot \varepsilon(\mathbf{u}) + \nabla P = \mathbf{f}$ is valid in $L^2(\Omega)^2$. Due to (4.2) and (4.3), we thus have

$$b_h(\mathbf{u}, P; \mathbf{v}, Q) = f_h(\mathbf{v}), \tag{4.10}$$

for any $h > 0$ and any $(\mathbf{v}, Q) \in H_h^2(\Omega)^2 \times H_h^1(\Omega)$. The consistency relation,

$$B(\mathbf{u}, P; \mathbf{v}, Q) = F(\mathbf{v}),\ \forall (\mathbf{v}, Q) \in \mathbf{V}_{h,p} \times W_{h,q},\ h > 0,\ p, q \in N$$

follows from (4.10) and from the definitions (4.7) of B and (4.8) of F.

We now turn our attention to the approximation properties of (4.6) when (4.9) holds with $k \geq 1$.

Franca and Stenberg (cf. Theorem 3.1 in [?]) have already shown that the scheme (4.6) converges optimally, for any $\epsilon \geq 0$, when h tends to zero, provided $\Gamma_2 \neq \emptyset$, p and q are fixed, $p \geq 2$ and

$$0 < \alpha < C_I^{-1} \tag{4.11}$$

where C_I is the constant in the inverse inequality,

$$\sum_{K \in \mathcal{C}_h} \frac{h_K^2}{p^4} \|\nabla \cdot \varepsilon(\mathbf{v})\|_{0,K}^2 \leq C_I \|\varepsilon(\mathbf{v})\|_0^2, \quad \forall\ \mathbf{v} \in \mathbf{V}_{h,p}. \tag{4.12}$$

(Note that (4.12) is due to the regularity of the family $\mathcal{C}_h$, (cf. [?])).

The general analysis, when h tends to zero and p, q to infinity, can now be sumarized as follows.

Theorem 4.1 *We assume that condition (4.11) is fulfilled by the parameter α. We take $k \geq 1$ and we suppose that the solution $(\mathbf{u}, P)$ to (3.3) satisfies the regularity property (4.9). Then, for any $h > 0$, $p \geq 2, q \in N$, the solution $(\mathbf{u}_{h,p,q}, P_{h,p,q})$ to (4.6), exists, is unique and satisfies*

$$\|\mathbf{u}-\mathbf{u}_{h,p,q}\|_1 + \frac{p}{(p+q)^2}\|P-P_{h,p,q}\|_0 \leq C(p+q)\,[\left(\frac{h^{\min\{k,p-1\}}}{p^k}\right)\|\mathbf{u}\|_{k+1} + \left(\frac{h^{\min\{k,q\}}}{q^k}\right)\|P\|_k], \tag{4.13}$$

with a constant C independent of $h, p, q, \mathbf{u}, P$ and of the parameter $\epsilon \geq 0$.

Remark 4.3 *Theorem 4.1 contains the result by Franca and Stenberg that the GLS method has optimal order of convergence with respect to h for both displacement and pressure. Unfortunately, it does not indicate that this method is also optimal with respect to p and q. The estimate (4.13) can be interpreted as follows. Suppose that we let the degrees p and q grow simultaneously, according to the law $q = p - r$ where r is fixed (e.g. $r = 1$). In that case, (4.13) shows that the convergence rate for the displacement of the GLS-method is not more than p times the optimal rate. As far as the rate of convergence for the pressure is concerned, it is not more that p^2 times the optimal rate. In section 5, it will be discussed how this result can be improved in particular situations.*

We now give a serie of Lemmas, used in the proof of Theorem 4.1. We will need some more notation. In particular, we introduce the abreviation $|\cdot|_h$ for the semi-norm defined on $H^1_h(\Omega)$ (cf. (4.1)),

$$|Q|_h = (\alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla Q\|^2_{0,K})^{1/2}.$$

Note that we maintain the parameter α in $|\cdot|_h$ for convenience. Because of the regularity of $\mathcal{C}_h$, standard inverse inequalities imply that

$$\frac{1}{q^4}|Q|_h^2 \leq \alpha C_I \|Q\|_0^2 \tag{4.14}$$

is valid for any $Q \in W_{h,q}$, $h > 0, q \geq 1$, with C_I as in (4.12).

Our first Lemma is a boundedness result for the bilinear form b defined in (4.2).

Lemma 4.1 *For any $k \geq 1$, there is a constant C with the following property. Let $h > 0$ and $p, q \geq 1$ be given and set $\kappa_1 = \min\{k, p\}$ and $\kappa_2 = \min\{k, q\}$. Let $\mathbf{u} \in H^{k+1}(\Omega)$, $P \in H^k(\Omega)$ and $\tilde{\mathbf{u}} \in \mathbf{V}_{h,p}$, $\tilde{P} \in W_{h,q}$. Then, the inequality*

$$\frac{\alpha}{(p+q)^4}|b_h(\mathbf{u}-\tilde{\mathbf{u}}, P-\tilde{P}; \mathbf{v}, Q)| \leq C\Big(\{\|\mathbf{u}-\tilde{\mathbf{u}}\|_1^2 + \|P-\tilde{P}\|_0^2\}^{\frac{1}{2}} + \left(\frac{h^{\kappa_1}}{p^k}\right)\|u\|_{k+1} + \left(\frac{h^{\kappa_2}}{q^k}\right)\|P\|_k\Big) \tag{4.15}$$

is valid as soon as the test functions $\mathbf{v} \in \mathbf{V}_{h,p}$ and $Q \in W_{h,q}$ satisfy $\|v\|_1^2 + \frac{1}{(p+q)^4}|Q|_h^2 \leq 1$.

Proof. We need some preliminary results. Using the triangle and the Schwarz inequalities, it follows from the definition (4.2) of b that the estimate

$$\frac{\alpha}{(p+q)^4}|b_h(\mathbf{u}, P; \mathbf{v}, Q)| \leq 2\Big\{\alpha \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{p^4}\|\nabla\cdot\varepsilon(\mathbf{u})\|^2_{0,K} + \frac{1}{q^4}|P|_h^2\Big\}^{\frac{1}{2}}$$
$$\Big\{\alpha \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{p^4}\|\nabla\cdot\varepsilon(\mathbf{v})\|^2_{0,K} + \frac{1}{(p+q)^4}|Q|_h^2\Big\}^{\frac{1}{2}} \tag{4.16}$$

is valid for any value of h, p, q and for any couples $(\mathbf{u}, P), (\mathbf{v}, Q)$ belonging to $H^2_h(\Omega)^2 \times H^1_h(\Omega)$. If it happens that $\mathbf{v} \in \mathbf{V}_{h,p}$, one can use the inverse inequality (4.12) in (4.16) and get that

$$\frac{\alpha}{(p+q)^4}|b_h(\mathbf{u}, P; \mathbf{v}, Q)| \leq C\Big\{\alpha \sum_{K\in\mathcal{C}_h} \frac{h_K^2}{p^4}\|\nabla\cdot\varepsilon(\mathbf{u})\|^2_{0,K} + \frac{1}{q^4}|P|_h^2\Big\}^{\frac{1}{2}}\Big\{\|\mathbf{v}\|_1^2 + \frac{1}{(p+q)^4}|Q|_h^2\Big\}^{\frac{1}{2}} \tag{4.17}$$

where C is independent of $h, p, q, \mathbf{u}, \mathbf{v}, P$ and Q. Finally, if $\mathbf{u}, \mathbf{v}$ are both in $\mathbf{V}_{h,p}$ and if $P \in W_{h,q}$, with $q \leq p$, one can combine the inverse inequality (4.12) and (4.14) to deduce from (4.17) that

$$\frac{\alpha}{(p+q)^4}|b_h(\mathbf{u}, P; \mathbf{v}, Q)| \leq C\Big\{\|\mathbf{u}\|_1^2 + \|P\|_0^2\Big\}^{\frac{1}{2}}\Big\{\|\mathbf{v}\|_1^2 + \frac{1}{(p+q)^4}|Q|_h^2\Big\}^{\frac{1}{2}} \tag{4.18}$$

where C does not depend on $h, p, \mathbf{u}, \mathbf{v}, P$ and Q.

Let us now prove the Lemma. We take $k \geq 1$, $\mathbf{u} \in H^{k+1}(\Omega)^2$, $P \in H^k(\Omega)$, $h > 0$, $p, q \geq 1$. According to Theorem 4.6 in [?], there is $\bar{\mathbf{u}} \in \mathbf{V}_{h,p}$ such that

$$\|\mathbf{u} - \bar{\mathbf{u}}\|_1 \leq C\left(\frac{h^{\min\{k,p-1\}}}{p^k}\right)\|\mathbf{u}\|_{k+1} \tag{4.19}$$

where C does not depend on h, p or $\mathbf{u}$. One can also prove that there is another constant C independent of h, p and $\mathbf{u}$ such that

$$\Big\{\sum_{K\in\mathcal{C}_h}\|\mathbf{u} - \bar{\mathbf{u}}\|_{2,K}^2\Big\}^{\frac{1}{2}} \leq C\left(\frac{h^{\min\{k,p-1\}-1}}{p^{k-1}}\right)\|\mathbf{u}\|_{k+1}. \tag{4.20}$$

We then define $\bar{P} \in W_{h,q}$ by requiring that $\bar{P}_{|K}$ is the H^1- projection of $P_{|K}$ onto $P_q(K)$, for any $K \in C_h$. From Lemma 4.5 in [?], there is a constant C, independent of h, q and $\mathbf{u}$ such that

$$\Big\{\sum_{K\in\mathcal{C}_h}|P - \bar{P}|_{1,K}^2\Big\}^{\frac{1}{2}} \leq C\left(\frac{h^{\min\{k,q\}-1}}{q^{k-1}}\right)\|P\|_k. \tag{4.21}$$

With the classical duality principle, one can also prove that

$$\|P - \bar{P}\|_0 \leq C\left(\frac{h^{\min\{k,q\}}}{q^k}\right)\|P\|_k. \tag{4.22}$$

Let now $\tilde{\mathbf{u}}$ be an element of $\mathbf{V}_{h,p}$ and $\tilde{P}$ an element of $W_{h,q}$. Introducing $\bar{\mathbf{u}}$ between $\mathbf{u}$ and $\tilde{\mathbf{u}}$, $\bar{P}$ between P and $\tilde{P}$ and using the triangle inequality as well as (4.17) and (4.18) we get that

$$\begin{aligned}\frac{\alpha}{(p+q)^4}|b_h(\mathbf{u}-\tilde{\mathbf{u}}, P-\tilde{P}; \mathbf{v}, Q)| &\leq \frac{\alpha}{(p+q)^4}(|b_h(\mathbf{u}-\bar{\mathbf{u}}, P-\bar{P}; \mathbf{v}, Q)| + |b_h(\bar{\mathbf{u}}-\tilde{\mathbf{u}}, \bar{P}-\tilde{P}; \mathbf{v}, Q)|) \\ &\leq C\Big\{\alpha\sum_{K\in\mathcal{C}_h}\frac{h_K^2}{p^4}\|\nabla\cdot\varepsilon(\mathbf{u}-\bar{\mathbf{u}})\|_{0,K}^2 + \frac{1}{q^4}|P-\bar{P}|_h^2\Big\}^{\frac{1}{2}} + C\Big\{\|\bar{\mathbf{u}}-\tilde{\mathbf{u}}\|_1^2 + \|\bar{P}-\tilde{P}\|_0^2\Big\}^{\frac{1}{2}}\end{aligned} \tag{4.23}$$

is true for any $\mathbf{v} \in \mathbf{V}_{h,p}$ and $Q \in W_{h,q}$ such that $\|v\|_1^2 + \frac{1}{(p+q)^4}|Q|_h^2 \leq 1$. Since the triangle inequality implies that

$$\Big\{\|\bar{\mathbf{u}}-\tilde{\mathbf{u}}\|_1^2 + \|\bar{P}-\tilde{P}\|_0^2\Big\}^{\frac{1}{2}} \leq \Big\{\|\mathbf{u}-\bar{\mathbf{u}}\|_1^2 + \|P-\bar{P}\|_0^2\Big\}^{\frac{1}{2}} + \Big\{\|\mathbf{u}-\tilde{\mathbf{u}}\|_1^2 + \|P-\tilde{P}\|_0^2\Big\}^{\frac{1}{2}}.$$

We conclude with (4.19)-(4.23). ■

We then give a stability result for the divergence operator in $\mathbf{V}_{h,p}$. This statement is the h-p version of the lemma 3.3 in [?].

Lemma 4.2 *There exists constants C_1 and C_2 such that, for any $h > 0$, $p \geq 2$ and any $P \in H_h^1(\Omega)$, one can find $\mathbf{v} \in \mathbf{V}_{h,p}$ satisfying*

$$\|\mathbf{v}\|_1 \leq C_1\|P\|_0, \tag{4.24}$$

$$(\nabla\cdot\mathbf{v}, P) \geq \|P\|_0(\|P\|_0 - \frac{C_2}{p}|P|_h). \tag{4.25}$$

Proof. We take $P \in H_h^1(\Omega)$ and we have to construct C_1, C_2 and $\mathbf{v}$. First, we define $\Pi_h P$ as the $L^2(\Omega)-$ projection of P onto the piecewise constant,

$$\Pi_h P_{|K} = \frac{1}{\text{meas}(K)} \int_K P dK, \quad K \in \mathcal{C}_h.$$

Since p is assumed to be more than 2, it is a classical result (cf. [?], [?]) that there is $\mathbf{z} \in \mathbf{V}_{h,p}$ satisfying

$$\nabla \cdot \mathbf{z} = \Pi_h P, \tag{4.26}$$

as well as $\|\mathbf{z}\|_1 \leq C_3 \|P\|_0$ for a constant C_3 independent of h. Using the estimate $\|P\|_0 \geq \|\Pi_h P\|_0$ we conclude that

$$\|\mathbf{z}\|_1 \leq C_3 \|P\|_0. \tag{4.27}$$

We then observe that $P - \Pi_h P$ is of zero mean value on each $K \in \mathcal{C}_h$. It is again a classical result that there is a displacement $\overline{\mathbf{w}}$ such that

$$\overline{\mathbf{w}}_{|K} \in (H_0^1(K))^2, \quad K \in \mathcal{C}_h, \tag{4.28}$$

$$\nabla \cdot \overline{\mathbf{w}} = P - \Pi_h P, \tag{4.29}$$

$$\|\overline{\mathbf{w}}\|_{1,K} \leq C_4 \|P - \Pi_h P\|_{0,K}, \quad K \in \mathcal{C}_h. \tag{4.30}$$

¿From a scaling argument and from the minimum angle condition on $\mathcal{C}_h$, we notice that the constant C_4, in (4.31), can be choosen independently on the mesh size h. Moreover, if we use the standard estimate $\|P - \Pi_h P\|_{0,K} \leq \|P\|_{0,K}$ we get

$$\|\overline{\mathbf{w}}\|_{1,K} \leq C_4 \|P\|_{0,K}, \quad K \in \mathcal{C}_h. \tag{4.31}$$

We now define an interpolant $\mathbf{w} \in \mathbf{V}_{h,p}$ of $\overline{\mathbf{w}}$ by applying the following rule: for each $K \in \mathcal{C}_h$, $\mathbf{w}_{|K}$ is the $H_0^1(K)$- projection of $\overline{\mathbf{w}}_{|K}$ onto $(H_0^1(K))^2 \cap (R_p(K))^2$. If we use the estimate (4.31) for $\|\overline{\mathbf{w}}\|_{1,K}$, we have

$$\mathbf{w}_{|K} \in H_0^1(K), \quad K \in \mathcal{C}_h, \tag{4.32}$$

$$\|\mathbf{w}\|_1 \leq C_5 \|P\|_0, \tag{4.33}$$

with a constant C_5 independent of h and p. With the usual duality principle, we also get a convenient $L^2(K)-$estimate of $\overline{\mathbf{w}} - \mathbf{w}$. We give all the details for completness. We introduce $\overline{\boldsymbol{\phi}}_K \in H_0^1(K)^2$ through the Poisson equation

$$(\overline{\boldsymbol{\phi}}_K, \theta)_{1,K} = (\overline{\mathbf{w}} - \mathbf{w}, \boldsymbol{\theta})_{0,K}, \ \forall \, \boldsymbol{\theta} \in H_0^1(K)^2. \tag{4.34}$$

Since K is a triangle, we know by the shift inequality that $\overline{\boldsymbol{\phi}}_K$ belongs to $H^2(K)^2$ and satisfies

$$\|\overline{\boldsymbol{\phi}}_K\|_{2,K} \leq C_6 \|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K}, \quad K \in \mathcal{C}_h. \tag{4.35}$$

Combining a scaling argument with the minimum angle condition on the triangulation $\mathcal{C}_h$, we prove that the constant C_6 in (4.35) can be choosen independently of the particular triangle $K \in \mathcal{C}_h$ and of the parameter h. Let us denote by $\boldsymbol{\phi}_K$ the $H_0^1(K)-$projection of $\overline{\boldsymbol{\phi}}_K$ onto $(H_0^1(K))^2 \cap (R_p(K))^2$. From (4.35) and from Lemma 4.5 in [?], it follows that there is C_7, independent of h and p, such that

$$\|\overline{\boldsymbol{\phi}}_K - \boldsymbol{\phi}_K\|_{1,K} \leq C_7 \left(\frac{h_K}{p} \right) \|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K}, \quad K \in \mathcal{C}_h. \tag{4.36}$$

We now use (4.34) to write $\|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K}^2$ as $(\overline{\boldsymbol{\phi}}_K, \overline{\mathbf{w}} - \mathbf{w})_{1,K}$. Because of the orthogonality properties and of the Schwarz inequality we get

$$\|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K}^2 = (\overline{\boldsymbol{\phi}}_K - \boldsymbol{\phi}_K, \overline{\mathbf{w}} - \mathbf{w})_{1,K} \leq \|\overline{\boldsymbol{\phi}}_K - \boldsymbol{\phi}_K\|_{1,K} \|\overline{\mathbf{w}} - \mathbf{w}\|_{1,K} \leq \|\overline{\boldsymbol{\phi}}_K - \boldsymbol{\phi}_K\|_{1,K} \|\overline{\mathbf{w}}\|_{1,K}.$$

With this relation, (4.36) and (4.31) yield a constant C_8, independent of h and p, such that

$$\|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K} \leq C_8 \left(\frac{h_K}{p}\right) \|P\|_0, \quad K \in \mathcal{C}_h. \tag{4.37}$$

We are in a position to conclude. We set $\mathbf{v} = \mathbf{w} + \mathbf{z}$. By construction $\mathbf{v}$ belongs to $\mathbf{V}_{h,p}$ and satisfies (4.24) (see (4.27) and (4.33)). It remains to check (4.25). Due to (4.26) and (4.29), we have

$$\begin{aligned}(\nabla \cdot \mathbf{v}, P) = (\nabla \cdot \mathbf{w} + \nabla \cdot \mathbf{z}, P) = (\nabla \cdot \overline{\mathbf{w}} + \nabla \cdot \mathbf{z}, P) + (\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P) = \\ (p - \Pi_h P + \Pi_h P, P) + (\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P) = \|P\|_0^2 + (\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P).\end{aligned} \tag{4.38}$$

We can write the last term of (4.38) in the following form,

$$(\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P) = \sum_{K \in C_h} (\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P)_{0,K}.$$

Since $P_{|K} \in H^1(K), K \in \mathcal{C}_h$ and since the functions $\mathbf{w}$ and $\overline{\mathbf{w}}$ both vanish on each $\partial K, K \in \mathcal{C}_h$, (see (4.28) and (4.32)) we can integrate $(\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P)_{0,K}$ by part. We obtain

$$(\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P) = -\sum_{K \in C_h} (\mathbf{w} - \overline{\mathbf{w}}, \nabla P)_{0,K}.$$

Finally, using the Schwarz inequality as well as (4.37), we get that

$$|(\nabla \cdot \mathbf{w} - \nabla \cdot \overline{\mathbf{w}}, P)| \leq \sum_{K \in C_h} \|\overline{\mathbf{w}} - \mathbf{w}\|_{0,K} \|\nabla P\|_{0,K} \leq C_8 \sum_{K \in C_h} \frac{h_K}{p} \|P\|_{0,K} \|\nabla P\|_{0,K}.$$

Carrying this relation in (4.38) we get that

$$(\nabla \cdot \mathbf{v}, P) \geq \|P\|_0^2 - C_8 \sum_{K \in C_h} \frac{h_K}{p} \|P\|_{0,K} \|\nabla P\|_{0,K}.$$

Applying once more the Schwarz inequality, we deduce that (4.25) is true. This completes the proof of the Lemma. ■

¿From Lemma 4.2, one can deduce a stability result for the form B defined in (4.7).

Lemma 4.3 *Assume that α satisfies (4.11). Then there is a positive constant C which is independent of the parameter ϵ and has the following property. Given $h > 0$, $p \geq 2$, $q \geq 1$ and $(\mathbf{u}, P) \in (\mathbf{V}_{h,p}, W_{h,q})$ one can find $(\mathbf{w}, Q) \in (\mathbf{V}_{h,p}, W_{h,q})$ satisfying*

$$\|\mathbf{w}\|_1^2 + (\epsilon + \frac{p^2}{(p+q)^4})\|Q\|_0^2 + \frac{1}{(p+q)^4}|Q|_h^2 = 1, \tag{4.39}$$

$$B(\mathbf{u}, P; \mathbf{w}, Q) \geq C\Big\{\|\mathbf{u}\|_1^2 + (\epsilon + \frac{p^2}{(p+q)^4})\|P\|_0^2 + \frac{1}{(p+q)^4}|P|_h^2)\Big\}^{\frac{1}{2}}. \tag{4.40}$$

Remark 4.4 *In an h-version context (where p and q are fixed), we can drop the term $\frac{1}{(p+q)^4}|Q|_h^2$ from the normalisation equation (4.39) as well as the term $\frac{1}{(p+q)^4}|P|_h^2$ from the stability inequality (4.40) and still get an equivalent formulation of our Lemma. This is due to the inverse inequality (4.14). Note that this simplification has been used by Franca-Stenberg in their h-version of Lemma 3.2 (cf. [?], lemma 3.2).*

We only sketch the proof of Lemma 4.3, since the proof of Lemma 3.2 in [?] is based on the very same arguments, we refer to it for more details.

Proof of Lemma 4.3. We take $(u,P) \in (\mathbf{V}_{h,p}, W_{h,q})$. From the definition (4.7) of B, we observe that

$$B(\mathbf{u},P;\mathbf{u},-P) \geq (1-\alpha C_I)\|\varepsilon(\mathbf{u})\|_0^2 + \epsilon\|P\|_0^2 + \frac{1}{(p+q)^4}|P|_h^2, \tag{4.41}$$

by using (4.12) as well as Schwarz inequality. We then construct the element $\mathbf{v} \in \mathbf{V}_{h,p}$ corresponding to P in Lemma 4.2. Because of (4.25), (4.12) and Schwarz inequality, we arrive at the conclusion that

$$B(\mathbf{u},P;-\mathbf{v},0) \geq \|P\|_0^2 - \frac{C_2}{p}\|P\|_0|P|_h - (1+\alpha C_I)\|\varepsilon(\mathbf{u})\|_0\|\varepsilon(\mathbf{v})\|_0 - \frac{\alpha C_I^{1/2}}{p^2}\|\varepsilon(\mathbf{v})\|_0|P|_h.$$

Taking into account that $\|\varepsilon(\mathbf{v})\|_0 \leq \|\mathbf{v}\|_1 \leq C_1\|P\|_0$ (see (4.24)) and applying Youngs inequality we deduce from the above relation that

$$B(\mathbf{u},P;-\mathbf{v},0) \geq \frac{1}{2}\|P\|_0^2 - \frac{C_9}{p}\|P\|_0|P|_h - C_{10}\|\varepsilon(\mathbf{u})\|_0^2. \tag{4.42}$$

whith some constants C_9 and C_{10} independent of p and h. We now sum (4.41) and $\frac{\eta p^2}{(p+q)^4}$ times (4.42), $\eta \geq 0$. We also use Korn inequality,

$$\|\varepsilon(\mathbf{u})\|_0 \geq C_{11}\|\mathbf{u}\|_1, \ \forall\ \mathbf{u} \in \mathbf{V},$$

and we get that

$$\begin{aligned} B(\mathbf{u},P;\mathbf{u}-\frac{\eta p^2}{(p+q)^4}\mathbf{v},-P) &\geq \Big[1-\alpha C_I - \eta C_{10}\Big]\|\mathbf{u}\|_1^2 \\ &+\Big(\epsilon + [\eta(\frac{1}{2} - C_9\frac{|P|_h}{p\|P\|_0}) + \Big(\frac{|P|_h}{2p\|P\|_0}\Big)^2]\frac{p^2}{(p+q)^4}\Big)\|P\|_0^2 + \frac{1}{2(p+q)^4}|P|_h^2. \end{aligned} \tag{4.43}$$

Combining (4.24) and the triangle inequality, we note that

$$\begin{aligned} \|\mathbf{u} - \frac{\eta p^2}{(p+q)^4}\mathbf{v}\|_1^2 + (\epsilon + \frac{p^2}{(p+q)^4})\|P\|_0^2 + \frac{1}{(p+q)^4}|P|_h^2 \leq \\ 2\|\mathbf{u}\|_1^2 + \Big(\epsilon + \big[2C_1^2\eta^2 + 1\big]\frac{p^2}{(p+q)^4}\Big)\|P\|_0^2 + \frac{1}{(p+q)^4}|P|_h^2. \end{aligned} \tag{4.44}$$

Because of (4.43) and (4.44), it is possible to find two numbers $\eta_0 > 0$ and ρ_0 such that the following choice for η

$$\eta = \begin{cases} \eta_0 & \text{if } \frac{|P|_h}{p\|P\|_0} \leq \rho_0 \\ 0 & \text{in any other situation} \end{cases}$$

make sure that

$$(\mathbf{w},Q) = \frac{(\mathbf{u} - \frac{\eta p^2}{(p+q)^4}\mathbf{v}, -P)}{(\|\mathbf{u} - \frac{\eta p^2}{(p+q)^4}\mathbf{v}\|_1^2 + \big(\epsilon + \frac{p^2}{(p+q)^4}\big)\|P\|_0^2 + \frac{1}{(p+q)^4}|P|_h^2)^{\frac{1}{2}}}$$

satisfies (4.39) and (4.40) with a constant C, independent of $h,p,q,\mathbf{u},P$ and ϵ. This completes the proof of Lemma 4.3. ■

Theorem 4.1 will now be deduced in the usual way, by combining the boundedness properties of b, the stability result established above and the consistency of the scheme (4.6).

Proof of Theorem 4.1. We define $\tilde{\mathbf{u}} \in \mathbf{V}_{h,p}$, as the H^1−projection of $\mathbf{u}$ and $\tilde{P} \in W_{h,q}$, as the L^2-projection of P. We proceed in four steps.

Step 1. We estimate the distance between $(\mathbf{u},P)$ and $(\tilde{\mathbf{u}},\tilde{P})$. We use the results of Theorem 4.6 and Lemma 4.5 of [?]. We get that there is a constant C, independent of $h,p,q,\mathbf{u}$ and P, such that

$$\{\|\mathbf{u}-\tilde{\mathbf{u}}\|_1^2 + \|P-\tilde{P}\|_0^2)\}^{\frac{1}{2}} \leq C\,\Big[\Big(\frac{h^{\min\{k,p-1\}}}{p^k}\Big)\|\mathbf{u}\|_{k+1} + \Big(\frac{h^{\min\{k,q\}}}{p^k}\Big)\|P\|_k\Big]. \tag{4.45}$$

Step 2. The stability of the form B (see Lemma 4.3), implies the existence of $(\mathbf{w}, Q) \in (\mathbf{V}_{h,p}, W_{h,q})$ such that (4.39) as well as

$$(\|\mathbf{u}_{h,p,q} - \tilde{\mathbf{u}}\|_1^2 + \frac{p^2}{(p+q)^4}\|P_{h,p,q} - \tilde{P}\|_0^2)^{1/2} \leq \beta B(\mathbf{u}_{h,p,q} - \tilde{\mathbf{u}}, P_{h,p,q} - \tilde{P}; \mathbf{w}, Q) \tag{4.46}$$

hold with a constant β independent of h, p and $\epsilon \geq 0$. Since we assume the regularity property (4.9) of the solution $(\mathbf{u}, P)$ to (3.3) with $k \geq 1$, we know that the scheme (4.6) is consistent (cf. Remark 4.2). Hence, $B(\mathbf{u}_{h,p,q} - \tilde{\mathbf{u}}, P_{h,p,q} - \tilde{P}; \mathbf{w}, Q) = B(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q)$. Carrying this relation in (4.46), we get that

$$(\|\mathbf{u}_{h,p,q} - \tilde{\mathbf{u}}\|_1^2 + \frac{p^2}{(p+q)^4}\|P_{h,p,q} - \tilde{P}\|_0^2)^{1/2} \leq \beta B(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q). \tag{4.47}$$

Step 3. By (4.7) and the triangle inequality, we have

$$B(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q) \leq |B_1(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q)| + |b_h(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q)|. \tag{4.48}$$

Because of the obvious boundedness properties of the form B_1, and since $(P - \tilde{P}, Q) = 0$, by definition of $\tilde{P}$, the first term on the left hand-side of (4.48) is less than

$$\{\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \|P - \tilde{P}\|_0^2)\}^{\frac{1}{2}} \{\|\mathbf{w}\|_1^2 + \|Q\|_0^2)\}^{\frac{1}{2}} \leq \frac{(p+q)^2}{p}\{\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \|P - \tilde{P}\|_0^2)\}^{\frac{1}{2}},$$

the second inequality following from (4.39). Combining Lemma 4.1 with (4.39), we get that the second term on the left hand-side of (4.48) is less than

$$C\Big(\{\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \|P - \tilde{P}\|_0^2)\}^{\frac{1}{2}} + \Big[\left(\frac{h^{\min\{k,p-1\}}}{p^k}\right)\|\mathbf{u}\|_{k+1} + \left(\frac{h^{\min\{k,q\}}}{q^k}\right)\|P\|_k\Big]\Big),$$

where C is independent of $h, p, q, \mathbf{u}, P$. As a conclusion and using the approximation property (4.45), we get in (4.48) that

$$B(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; \mathbf{w}, Q) \leq C\frac{(p+q)^2}{p}\Big[\left(\frac{h^{\min\{k,p-1\}}}{p^k}\right)\|\mathbf{u}\|_{k+1} + \left(\frac{h^{\min\{k,q\}}}{q^k}\right)\|P\|_k\Big]. \tag{4.49}$$

Step 4. Theorem 4.1 now follows from (4.49), (4.47) and (4.45). ■

Remark 4.5 *The results of Theorem 4.1 can be extended to a three dimensional context, without changing the definition of the GLS-method (4.6). The only new argument that would be needed in the proof is a three dimensional version of the approximation results (Lemma 4.5 and Theorem 4.6 in [?]) established in two dimensions by Babuška and Suri.*

In the same way, Theorem 4.1 will also be valid for a stabilized method built with curved elements, if the approximation results of [?] hold for such meshes.

5 The *GLS* strategy with special combinations of discrete spaces

Let us consider a particular sequence

$$\{\mathbf{V}_{h_n,p_n}, \mathbf{W}_{h_n,q_n}\} \tag{5.1}$$

of discrete spaces. We turn our attention to the usual mixed method, which is to find $\mathbf{u}_n \in \mathbf{V}_{h_n,p_n}$ and $P_n \in \mathbf{W}_{h_n,p_n}$ such that

$$B_1(\mathbf{u}_n, P_n; \mathbf{v}, Q) = F_1(\mathbf{v}),\ \forall\ (\mathbf{v}, Q) \in \mathbf{V}_{h_n,p_n} \times \mathbf{W}_{h_n,p_n},\ n \in N \tag{5.2}$$

(cf. (3.4) and (3.5) for the meaning of B_1 and F_1) and we assume that this scheme have an *"optimal order of convergence for the displacement"*. The aim of this section is to analyse how the general error estimate (4.13) can be improved for the corresponding *GLS* method (see (4.6)), which amounts to finding $\mathbf{u}_{h_n,p_n} \in \mathbf{V}_{h_n,p_n}$ and $P_{h_n,p_n} \in \mathbf{W}_{h_n,p_n}$ such that

$$B(\mathbf{u}_{h_n,p_n}, P_{h_n,p_n}; \mathbf{v}, Q) = F(\mathbf{v}),\ \forall\, (\mathbf{v}, Q) \in \mathbf{V}_{h_n,p_n} \times \mathbf{W}_{h_n,p_n},\ n \in N. \tag{5.3}$$

First, we give a precise definition of what we mean by a mixed method with *optimal order of convergence for the displacement.*

Definition 5.1 *We say that the mixed method (5.2) has an optimal order of convergence for the displacement if there is a constant C, independent of $\epsilon \geq 0$, with the following property. If the solution $(\mathbf{u}, P)$ to (3.3) fulfills (4.9) for some $k \geq 0$, the solution $(\mathbf{u}_n, P_n)$ to (5.2) satisfies*

$$\|\mathbf{u} - \mathbf{u}_n\|_1 \leq C \left[\left(\frac{h_n^{\min\{k, p_n-1\}}}{p_n^k} \right) + \left(\frac{h_n^{\min\{k, q_n\}}}{q_n^k} \right) \right] \left[\|\mathbf{u}\|_{k+1} + \|P\|_k \right],\ \forall n. \tag{5.4}$$

We are now in position to state the main result of this section.

Theorem 5.1 *We assume that condition (4.11) is fulfilled by the parameter α. We also assume that the mixed method (5.2) has an optimal order of convergence for the displacement. We take $k \geq 1$ and we suppose that the solution $(\mathbf{u}, P)$ to (3.3) satisfies the regularity property (4.9). Then there is a constant C, independent of $(\mathbf{u}, P)$ and of the parameter $\epsilon \geq 0$, such that the solution $(\mathbf{u}_{h_n,p_n}, P_{h_n,p_n})$ to (5.3) satisfies*

$$\|\mathbf{u} - \mathbf{u}_{h_n,p_n}\|_1 + \frac{p_n}{(p_n + q_n)^2} \|P - P_{h_n,p_n}\|_0 \leq$$
$$C \left[\left(\frac{h_n^{\min\{k, p_n-1\}}}{p_n^k} \right) + \left(\frac{h_n^{\min\{k, q_n\}}}{q_n^k} \right) \right] \left[\|\mathbf{u}\|_{k+1} + \|P\|_k \right], \tag{5.5}$$

for any n such that $p_n \geq 2$.

The meaning of Theorem 5.1 is clear: if the convergence in displacement of the mixed method (5.2) is optimal, the *GLS*-method (5.3) has the same property. But (5.5) tells also that the convergence in pressure is good. It is optimal in h and, roughly speaking, at most one order less than optimal in p. Note that this is true without any assumption on the convergence in pressure of the corresponding mixed scheme (5.2).

To help us in the proof of Theorem 5.1, we first establish an intermediate lemma. It shows that the spaces $\{\mathbf{V}_{h_n,p_n}, \mathbf{W}_{h_n,p_n}\}$ have a particular approximation property when the convergence in $\mathbf{u}$ of the mixed method (5.2) is optimal.

Lemma 5.1 *Assume that the mixed method (5.2) has an optimal order of convergence for the displacement (cf. definition 5.1). Then, for any $k \geq 1$, there is a constant C such that the relation*

$$\inf_{\substack{\mathbf{v} \in \mathbf{V}_{h_n,p_n} \\ (\nabla \cdot \mathbf{v}, Q) = 0, \forall Q \in \mathbf{W}_{h_n,p_n}}} \|\mathbf{u} - \mathbf{v}\|_1 \leq C \left[\left(\frac{h_n^{\min\{k, p_n-1\}}}{p_n^k} \right) + \left(\frac{h_n^{\min\{k, q_n\}}}{q_n^k} \right) \right] \|\mathbf{u}\|_{k+1},\ \forall n \tag{5.6}$$

is fulfilled by any $\mathbf{u} \in H^{k+1}(\Omega)^2 \cap \mathbf{V}$ with vanishing divergence (the space $\mathbf{V}$ is defined in (3.1)).

Proof. We will adapt some arguments by Babuska and Suri (cf. [**?**]).

Let $k \geq 1$ and $\mathbf{u}$ be such that

$$\mathbf{u} \in H^{k+1}(\Omega)^2 \cap V \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0. \tag{5.7}$$

Due to (5.7), it is possible to find $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{g} \in L^2(\Gamma_2)^2$ such that $(\mathbf{u}, 0)$ is solution to the elasticity problem (3.3) with any value of the Poisson ratio ν. In particular, we can choose $\nu = \frac{1}{2}$, which means (see (3.2)) that

$$\epsilon = 0. \tag{5.8}$$

Let us now consider the approximations $\{(\mathbf{u}_n, P_n)\}$ of $(\mathbf{u}, 0)$ obtained with the mixed method (5.2). Using that (5.2) is assumed to have optimal order of convergence for the displacement, we have

$$\|\mathbf{u} - \mathbf{u}_n\|_1 \leq C \left[\left(\frac{h_n^{\min\{k, p_n - 1\}}}{p_n^k} \right) + \left(\frac{h_n^{\min\{k, q_n\}}}{q_n^k} \right) \right] \|\mathbf{u}\|_{k+1}, \ \forall n \tag{5.9}$$

where C is as in (5.4). We now take (5.8) into account. Setting $\mathbf{v} = 0$, we get from (5.2) that

$$(\nabla \cdot \mathbf{u}_n, Q) = 0, \ \forall Q \in \mathbf{W}_{h_n, p_n}.$$

Therefore it holds

$$\inf_{\substack{\mathbf{v} \in \mathbf{V}_{h_n, p_n} \\ (\nabla \cdot \mathbf{v}, Q) = 0, \forall Q \in \mathbf{W}_{h_n, p_n}}} \|\mathbf{u} - \mathbf{v}\|_1 \leq \|\mathbf{u} - \mathbf{u}_n\|_1.$$

We conclude with (5.9). ∎

Remark 5.1 *It should be noticed that the converse of Lemma 5.1 is also true. If the spaces* $\{\mathbf{V}_{h_n, p_n}, \mathbf{W}_{h_n, p_n}\}$ *have the approximation property (5.6), then the convergence of the mixed method (5.2), with respect to the displacement, is optimal, in the sense of definition 5.1. This statement can be proved with arguments developed in [?].*

The proof of Theorem 5.1 is now based on the above Lemma, on the boundedness result for b (cf. Lemma 4.1) and on the stability of B (cf. Lemma 4.3).

Proof of Theorem 5.1. Let $(\mathbf{u}, P)$ be a solution to (3.3) fulfilling (4.9) with $k > 1$. We fix n such that $p_n \geq 2$ and we prove (5.5) in two steps.

Step 1. The first step is concerned with the construction of a particular interpolant $\tilde{\mathbf{u}} \in \mathbf{V}_{h_n, p_n}$ to $\mathbf{u}$. We observe that (3.3) implies

$$\nabla \cdot \mathbf{u} = -\epsilon P. \tag{5.10}$$

Combining this relation with (4.9), we conclude that

$$-\epsilon P \in H^k(\Omega) \cap \nabla \cdot (H^{k+1}(\Omega)^2 \cap V).$$

Since $k > 1$, Theorem 3.1 in [?] proves that there is $\mathbf{u}^D \in H^{k+1}(\Omega)^2 \cap V$ as well as a constant C, independent of P, such that

$$\nabla \cdot \mathbf{u}^D = -\epsilon P, \tag{5.11}$$

$$\|\mathbf{u}^D\|_{k+1} \leq C\epsilon \|P\|_k. \tag{5.12}$$

¿From (5.10) and (5.12), we also get that

$$\|\mathbf{u}^D\|_{k+1} \leq C \|\mathbf{u}\|_{k+1}. \tag{5.13}$$

The three relations (5.10), (5.11) and (5.13) allow us to decompose $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{u}^D + \mathbf{u}^{ND} \tag{5.14}$$

where $\mathbf{u}^{ND} = \mathbf{u} - \mathbf{u}^D$ has a vanishing divergence

$$\nabla \cdot \mathbf{u}^{ND} = 0 \tag{5.15}$$

and satisfies

$$\|\mathbf{u}^D\|_{k+1} \leq C\|\mathbf{u}\|_{k+1} \tag{5.16}$$

with a constant C independent of $\mathbf{u}$. We now use that the method (5.2) has an optimal order of convergence for the displacement. Because of the conclusion of Lemma 5.1, (5.15) and (5.16) yield $\tilde{\mathbf{u}}^{ND} \in \mathbf{V}_{h_n,p_n}$ and C, independent of $\mathbf{u}$ and n, such that

$$(\nabla \cdot \tilde{\mathbf{u}}^{ND}, Q) = 0,\ \forall Q \in \mathbf{W}_{h_n,p_n}, \tag{5.17}$$

$$\|\mathbf{u}^{ND} - \tilde{\mathbf{u}}^{ND}\|_1 \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\|\mathbf{u}\|_{k+1}. \tag{5.18}$$

Let us now define $\tilde{\mathbf{u}}^D$ as the H^1-projection of $\mathbf{u}^D$ onto $\mathbf{V}_{h_n,p_n}$. ¿From Theorem 4.6 in [?] and from (5.12), (5.13) it follows that

$$\|\mathbf{u}^D - \tilde{\mathbf{u}}^D\|_1 \leq C\epsilon\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right)\|P\|_k, \tag{5.19}$$

$$\|\mathbf{u}^D - \tilde{\mathbf{u}}^D\|_1 \leq C\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right)\|\mathbf{u}\|_{k+1}, \tag{5.20}$$

where C does not depend on n and $\mathbf{u}$. We finally propose

$$\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^D + \tilde{\mathbf{u}}^{ND} \tag{5.21}$$

as an interpolant to $\mathbf{u}$ in $\mathbf{V}_{h_n,p_n}$. Due to (5.14), to (5.18) and to (5.20), there is C, independent of n and $\mathbf{u}$, such that

$$\|\mathbf{u} - \tilde{\mathbf{u}}\|_1 \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\|\mathbf{u}\|_{k+1}. \tag{5.22}$$

In addition to the above estimate, the interpolant $\tilde{\mathbf{u}}$ also has a particular property which we now establish. We combine (5.15) and (5.17) to get that

$$|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)| = |(\nabla \cdot (\mathbf{u}^D - \tilde{\mathbf{u}}^D), Q)|,\ \forall Q \in \mathbf{W}_{h_n,p_n}.$$

Hence, because of Schwarz inequality and (5.19), it holds

$$|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)| = \leq C\epsilon\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right)\|P\|_k\|Q\|_0,\ \forall Q \in \mathbf{W}_{h_n,p_n}. \tag{5.23}$$

Step 2. We prove (5.5) with the usual techniques. We take $\tilde{\mathbf{u}}$ in (5.21) and we define $\tilde{P}$ as the L^2-projection of P onto $\mathbf{W}_{h_n,p_n}$. Because of Lemma 4.5 in [?], we have

$$\|P - \tilde{P}\|_0 \leq C\left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\|P\|_k \tag{5.24}$$

with C, independent of n and P.

We then use the stability Lemma 4.3. Since $p_n \geq 2$, it proves that there is a constant β, independent of $n, \mathbf{u}, P$ and $\epsilon \geq 0$, and an element $(\mathbf{w}, Q) \in \mathbf{V}_{h_n,p_n} \times \mathbf{W}_{h_n,p_n}$ such that

$$\|\mathbf{w}\|_1^2 + \epsilon\|Q\|_0^2 + \frac{1}{(p_n+q_n)^4}|Q|_h^2 = 1 \tag{5.25}$$

$$\|\mathbf{u}_{h_n,p_n} - \tilde{\mathbf{u}}\|_1 + \frac{p_n}{(p_n+q_n)^2}\|P_{h_n,p_n} - \tilde{P}\|_0 \leq \beta B(\mathbf{u}_{h_n,p_n} - \tilde{\mathbf{u}}, P_{h_n,p_n} - \tilde{P}; w, Q). \tag{5.26}$$

The regularity property (4.9) of $(\mathbf{u}, P)$ being true with $k > 1$, we know (cf. Remark 4.2) that (5.3) is a consistant method. Therefore, it holds

$$B(\mathbf{u}_{h_n,p_n} - \tilde{\mathbf{u}}, P_{h_n,p_n} - \tilde{P}; w, Q) = B(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; w, Q) = (\varepsilon(\mathbf{u} - \tilde{\mathbf{u}}), \varepsilon(\mathbf{w}))$$
$$-(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q) - (\nabla \cdot \mathbf{w}, P - \tilde{P}) - \epsilon(P - \tilde{P}, Q) + \frac{\alpha}{(p_n + q_n)^4} b_h(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; w, Q),$$

where we have used the definition (4.7) of B. $\tilde{P}$ being the L^2-projection of P, the term $(P - \tilde{P}, Q)$ vanishes. Combining (5.25) with Lemma 4.1 to estimate $(\varepsilon(\mathbf{u} - \tilde{\mathbf{u}}), \varepsilon(\mathbf{w}))$, $(\nabla \cdot \mathbf{w}, P - \tilde{P})$ and $(\alpha/(p_n + q_n)^4) b_h(\mathbf{u} - \tilde{\mathbf{u}}, P - \tilde{P}; w, Q)$, we come to the conclusion that

$$B(\mathbf{u}_{h_n,p_n} - \tilde{\mathbf{u}}, P_{h_n,p_n} - \tilde{P}; w, Q) \leq$$
$$C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\Big[\|\mathbf{u}\|_{k+1} + \|P\|_k\Big] + |(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)|, \tag{5.27}$$

where C does not depend on $n, \mathbf{u}$ or P. The crucial point is to get a bound for $|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)|$. We have to distinguish the cases wether $\epsilon \leq 1$ or $\epsilon > 1$. If $\epsilon \leq 1$, we notice that (5.25) implies

$$\epsilon\|Q\|_0 = \left\{\epsilon^2\|Q\|^2\right\}^{\frac{1}{2}} \leq \left\{\epsilon\|Q\|^2\right\}^{\frac{1}{2}} \leq 1.$$

We then use (5.23) and get

$$|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)| \leq C\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right)\|P\|_k. \tag{5.28}$$

If $\epsilon > 1$, (5.25) and (5.22) directly give

$$|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)| \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\|\mathbf{u}\|_{k+1}. \tag{5.29}$$

We can sumarize (5.28) and (5.29). It turns out that

$$|(\nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), Q)| \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\Big[\|\mathbf{u}\|_{k+1} + \|P\|_k\Big], \tag{5.30}$$

for any value of $\epsilon \geq 0$. Substituting (5.30) and (5.27) in (5.26) we obtain that

$$\|\mathbf{u}_{h_n,p_n} - \tilde{\mathbf{u}}\|_1 + \frac{p_n}{(p_n + q_n)^2}\|P_{h_n,p_n} - \tilde{P}\|_0 \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^k}\right) + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\Big[\|\mathbf{u}\|_{k+1} + \|P\|_k\Big].$$

The Theorem now follows from this inequality and from (5.22) and (5.24). ∎

Remark 5.2 *Note that the optimal convergence in* $\mathbf{u}$ *of the mixed method (5.2) has not been used dircetly in the proof of Theorem 5.1. We have only employed the approximation property (5.6) of the spaces* $\{\mathbf{V}_{h_n,p_n}, \mathbf{W}_{h_n,p_n}\}$*. But this observation does not allow us to weaken the hypothesis of Theorem 5.1. The approximation property (5.6) and the optimal convergence in* $\mathbf{u}$ *of the method (5.2) are indeed equivalent statements (see Lemma 5.1 and Remark 5.1).*

Remark 5.3 *A result, similar to the conclusion of Theorem 5.1, also exists for more general combinations* $\{\mathbf{V}_{h_n,p_n}, \mathbf{W}_{h_n,p_n}\}$ *of spaces. Let us assume that there is constants* C *and* $\eta \in [0, 1)$*, independent of the parameter* $\epsilon \geq 0$*, with the following property. If* $(\mathbf{u}, P)$ *is solution to (3.3) and satisfies (4.9) with* $k \geq 1$*, then the solution* $\mathbf{u}_n$ *to (5.2) is such that*

$$\|\mathbf{u} - \mathbf{u}_n\|_1 \leq C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^{k-\eta}}\right)\|\mathbf{u}\|_{k+1} + \left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\|P\|_k\right],\ \forall n. \tag{5.31}$$

Suppose also that the condition (4.11) is fulfilled by the parameter α. In that context, it can be shown that the GLS-method (5.3) has also better convergence properties than those announced in the general theorem 4.1: The solution $(\mathbf{u}_{h_n,p_n}, P_{h_n,p_n})$ to (5.3) satisfies

$$\|\mathbf{u}-\mathbf{u}_{h_n,p_n}\|_1 + \frac{p_n}{(p_n+q_n)^2}\|P-P_{h_n,p_n}\|_0 \le C\left[\left(\frac{h_n^{\min\{k,p_n-1\}}}{p_n^{k-\eta}}\right)+\left(\frac{h_n^{\min\{k,q_n\}}}{q_n^k}\right)\right]\left[\|\mathbf{u}\|_{k+1}+\|P\|_k\right],$$

where η is as in (5.31) and where C does not depend on $\epsilon \ge 0, n$ and $(\mathbf{u}, P)$.

Theorem 5.1 appears as the particular case $\eta = 0$ of this general statement.

We now conclude this section by two important observations.

At first we point out that it can be interesting to use the stabilized method (5.3), even if the corresponding mixed method (5.2) has an optimal convergence in displacement. The reasons for that is as follows. Suppose that the mixed method (5.2) presents a strong locking in pressure. In that case, it is usual to recover an approximation of P by post-processing the displacement $\mathbf{u}_n$. But this technique is difficult to apply and the *GLS*-strategy is certainly a worthwhile alternative, since the error estimate (5.5) shows that the solution P_{h_n,p_n} of the scheme (4.6) converges to P with a reasonable order. Roughly speaking, this order is optimal in h and, at least, equal to the optimal order minus one with respect to the degree p.

We finally consider the case where the mixed method (5.2) is optimal in both unknown, because the spaces $\{\mathbf{V}_{h_n,p_n}, \mathbf{W}_{h_n,p_n}\}$ satisfy the *"Babuška-Brezzi"* condition. Under this assumption, we can improve the result of Theorem 5.1. At first we notice that the stability result of Lemma 4.3 is no longer optimal. We actually have a full stability for the form B: there is $\beta > 0$ such that

$$\sup_{(\mathbf{w},Q)\in\mathbf{V}_{h_n,p_n}\times\mathbf{W}_{h_n,p_n}} \frac{B(\mathbf{v},P;\mathbf{w},Q)}{\{\|\mathbf{w}\|_1^2+\|Q\|_0^2\}^{\frac{1}{2}}} \ge \beta\{\|\mathbf{v}\|_1^2+\|P\|_0^2\}^{\frac{1}{2}},\ \forall(\mathbf{v},P)\in\mathbf{V}_{h_n,p_n}\times\mathbf{W}_{h_n,p_n},\ \forall n.$$

It thus follows from usual arguments that the *GLS*-method (5.3) is also optimal in both unknown.