A CACCIOPPOLI ESTIMATE AND FINE CONTINUITY FOR SUPERMINIMIZERS ON METRIC SPACES

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Abstract. We prove a Caccioppoli estimate for p–superminimizers on metric spaces. As an application, we provide a new proof for the fine continuity of p–superminimizers.

1. Introduction

We study superminimizers of the p–Dirichlet integral

$$\int_\Omega |Du|^p \, \mathrm{d}\mu$$

on metric measure spaces. In the Euclidean case, minimizing this p–energy functional is equivalent to solving the p–harmonic equation. In general metric spaces, it is not clear how to define the p–harmonic equation, but the variational approach is available.

Our main result is a Caccioppoli type estimate for p–superminimizers, Theorem 3.4. It answers to a question that was motivated in [5] by Kinnunen and Latvala. They were able to prove a weaker estimate that is sufficient to show that the infinity set of any p–superharmonic function is of zero capacity. It is well known that the sharp estimate holds in the Euclidean case, see for example [9], and it is also one of the main ingredients in proving that the Wiener condition is sufficient for regularity at the boundary, see for example [4].

The difficulties in the proof of Theorem 3.4 arise from the fact that the equation is not available and we can use only the minimizing property. We have developed a method to overcome this difficulty, and it enables us to extend the classical proof also to this situation.

Our method can be used in the metric space setting to obtain simpler proofs also for other estimates that are classically proved exploiting the equation. These include for example some Caccioppoli type estimates, see Lemma 3.1 in [7] and Lemma 4.1 in [8], as well as an integrability estimate, see Theorem 7.45 in [4].

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As an application of Theorem 3.4, we present a new proof for the fact that p–superharmonic functions are p–finely continuous. The proof follows ideas in [5], where q–fine continuity of p–superharmonic functions was proved for all $q < p$ with weaker estimates. Recently, Björn proved the p–fine continuity using a different approach by obstacle problem technique, see [3].

2. Preliminaries

Let X be a metric space with a Borel measure μ. The measure is said to be *doubling* if the measure of every open ball is positive and finite, and there exists a constant $c_\mu > 0$ such that

$$\mu(B(x,2r)) \le c_\mu \mu(B(x,r))$$

for every $x \in X$ and $r > 0$.

Let $1 \le p < \infty$. The space X is said to support *a weak* $(1,p)$*–Poincaré inequality* if there exist positive constants c_P and τ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, \mathrm{d}\mu \le c_P r \left(\fint_{B(z,\tau r)} g_u^p \, \mathrm{d}\mu \right)^{1/p}$$

for all balls $B(z,r) \subset X$ and for all measurable functions u with upper gradients g_u. Function $g_u : X \to [0,\infty]$ is an upper gradient of u if

$$|u(x) - u(y)| \le \int_\gamma g_u \, \mathrm{d}s,$$

for every $x, y \in X$ and every rectifiable path γ joining x and y. If u is a function that is integrable to power p in X, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, \mathrm{d}\mu + \inf_{g_u} \int_X {g_u}^p \, \mathrm{d}\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. Following [10], we define the *Newtonian space* on X to be the quotient space

$$N^{1,p}(x) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Let $E \subset X$. We define $N_0^{1,p}(E)$ to be the set of functions that can be extended to a function in $N^{1,p}(X)$ that is zero p–quasieverywhere in $X \setminus E$.

The relative p*–capacity* of a set $E \subset B(z,r)$ is defined by

$$\operatorname{cap}_p(E, B(z,2r)) = \inf_u \int_{B(z,2r)} g_u^p \, \mathrm{d}\mu,$$

where the infimum is taken over all upper gradients g_u of functions $u \in N_0^{1,p}(B(z,2r))$, whose restriction to E is bounded below by 1. A property is said to hold p*–quasieverywhere* if it holds outside a set of

p–capacity zero. Moreover, a function u is said to be p–quasicontinuous if for every $\varepsilon > 0$, there exists an open set U with p–capacity less than ε such that $u_{X\setminus U}$ is continuous.

Let $1 < p < \infty$. A set $E \subset X$ is called *p–thin* at $z \in X$ if

$$\int_0^\infty \left(\frac{\operatorname{cap}_p(E \cap B(z,r), B(z,2r))}{\operatorname{cap}_p(B(z,r), B(z,2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty.$$

A set $U \subset X$ is said to be *p–finely open* if $X \setminus U$ is p–thin at each point $x \in U$. The p–finely open sets define a topology, which we call the *p–fine topology.* We say that a function is *p–finely continuous* if it is continuous with respect to the p–fine topology.

Let $1 < p < \infty$. Suppose that $\Omega \subset X$ is an open set and let $\vartheta \in N^{1,p}(\Omega)$. A function $u \in N^{1,p}(\Omega)$ such that $u - \vartheta \in N_0^{1,p}(\Omega)$ is a p–minimizer with boundary values ϑ in Ω, if

$$\int_\Omega g_u^p \, \mathrm{d}\mu \leq \int_\Omega g_v^p \, \mathrm{d}\mu \tag{2.1}$$

for every $v \in N^{1,p}(\Omega)$ such that $v - \vartheta \in N_0^{1,p}(\Omega)$. A function $u \in N_{loc}^{1,p}(X)$ is called a *p–minimizer* in Ω, if (2.1) holds in every open set $\Omega' \Subset \Omega$ for all v such that $v - u \in N_0^{1,p}(\Omega')$. A function $u \in N_{loc}^{1,p}(\Omega)$ is a *p–superminimizer* in Ω, if (2.1) holds in every open set $\Omega' \Subset \Omega$ for all v such that $v - u \in N_0^{1,p}(\Omega')$ and $v \geq u$ μ–almost everywhere in Ω'. Observe that u is a p–minimizer if and only if u and $-u$ are p–superminimizers. If a p–minimizer is continuous, we call it *p–harmonic.*

3. A Caccioppoli estimate for p–superminimizers

We will need the following Caccioppoli and Harnack type estimates. For the proofs, see for example Lemma 3.1 and Theorem 4.3 in [7].

Lemma 3.1. *Suppose that $u > 0$ is a p–superminimizer in Ω and let $\beta < 0$. Let η be a compactly supported Lipschitz continuous function in Ω such that $0 \leq \eta \leq 1$. Then*

$$\int_\Omega g_u^p u^{\beta-1} \eta^p \, d\mu \leq c \int_\Omega u^{p+\beta-1} g_\eta^p \, d\mu,$$

where $c = (p/|\beta|)^p$.

Lemma 3.2. *Let $u \geq 0$ be a p–superminimizer in Ω. If $0 < s < \kappa(p-1)$, then for every ball $B(z,R)$ with $B(z, 10\tau R) \subset \Omega$, we have*

$$\left(\fint_{B(z,R)} u^s \, d\mu \right)^{1/s} \leq c \inf_{B(z,R)} u,$$

where $c < \infty$ depends only on p, c_μ and the constants in the Poincaré inequality, and κ depends on p and the data associated to the space.

Lemma 3.3 is a straightforward generalization of Lemma 2.117 in [9].

Lemma 3.3. *Let $u \geq 0$ be a p–superminimizer in Ω and let η be a compactly supported Lipschitz continuous function such that $0 \leq \eta \leq 1$, $\mathrm{supp}(\eta) \subset B(z,R)$ with $B(z,10\tau R) \subset \Omega$ and $g_\eta \leq c/R$. Then*

$$\int_{B(z,R)} g_u^{p-1}\eta^{p-1} g_\eta \, d\mu \leq c\mu(B(z,R))R^{-p}(\inf_{B(z,R)} u)^{p-1}.$$

Proof. Fix β so that $\max\{1-p, 1-\kappa\} < \beta < 0$. By Lemma 3.1,

$$\begin{aligned}\int_{B(z,R)} g_u^p u^{\beta-1}\eta^p \,\mathrm{d}\mu &\leq c\int_{B(z,R)} u^{p+\beta-1} g_\eta^p \,\mathrm{d}\mu \\ &\leq cR^{-p}\int_{B(z,R)} u^{p+\beta-1}\,\mathrm{d}\mu.\end{aligned} \tag{3.1}$$

Then by Hölder's inequality, (3.1) and Lemma 3.2, we have

$$\begin{aligned}&\int_{B(z,R)} g_u^{p-1}\eta^{p-1} g_\eta \,\mathrm{d}\mu \\ &\leq \left(\int_{B(z,R)} g_u^p u^{\beta-1}\eta^p \,\mathrm{d}\mu\right)^{(p-1)/p} \left(\int_{B(z,R)} u^{(1-\beta)(p-1)} g_\eta^p \,\mathrm{d}\mu\right)^{1/p} \\ &\leq \left(R^{-p}\int_{B(z,R)} u^{p+\beta-1}\,\mathrm{d}\mu\right)^{(p-1)/p} \left(\int_{B(z,R)} u^{(1-\beta)(p-1)} g_\eta^p \,\mathrm{d}\mu\right)^{1/p} \\ &\leq \left(R^{-p}\int_{B(z,R)} (\inf_{B(z,R)} u)^{p+\beta-1}\,\mathrm{d}\mu\right)^{(p-1)/p} \times \\ &\qquad \times \left(\int_{B(z,R)} (\inf_{B(z,R)} u)^{(1-\beta)(p-1)} R^{-p}\,\mathrm{d}\mu\right)^{1/p} \\ &= c\mu(B(z,R))R^{-p}(\inf_{B(z,R)} u)^{p-1}.\end{aligned}$$

□

Now we are ready to prove our main estimate.

Theorem 3.4. *Suppose that $0 < u \leq k$ is p–superminimizer in an open set $\Omega \subset X$. Let η be a Lipschitz continuous function with the properties $0 \leq \eta \leq 1$, $\eta = 0$ in $\Omega \setminus B(z,R)$, and $g_\eta \leq c/R$, $B(z,10\tau R) \subset \Omega$. Then there exists a constant c such that*

$$\int_{B(z,R)} g_u^p \eta^p \, d\mu \leq ck\mu(B(z,R))R^{-p}(\inf_{B(z,R)} u)^{p-1}.$$

Proof. Let

$$v_\varepsilon = u + \varepsilon(k-u)\eta^p.$$

Then for every $0 < \varepsilon < 1$, we have $v_\varepsilon \geq u$ and $v_\varepsilon - u \in N_0^{1,p}(B(z,R))$. Moreover, since v_ε is absolutely continuous outside a path family of

p–modulus zero, we have

$$\begin{aligned} g_{v_\varepsilon} &\le g_u(1-\varepsilon\eta^p)+\varepsilon p(k-u)\eta^{p-1}g_\eta \\ &= g_u+\varepsilon(-g_u\eta^p+p(k-u)\eta^{p-1}g_\eta). \end{aligned}$$

Fix $x\in B(z,R)$. We apply mean value theorem to function

$$f(\varepsilon)=\big(g_u(x)+\varepsilon\left(-g_u(x)\eta(x)^p+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\big)^p$$

to conclude that

$$g^p_{v_\varepsilon}(x)\le f(\varepsilon)=f(0)+\varepsilon f'(\xi)$$

for some $\xi\in(0,\varepsilon)$ that may depend on x. It follows that

$$\begin{aligned} g^p_{v_\varepsilon}(x) \le & g_u(x)^p+\varepsilon p\left(-g_u(x)\eta^p(x)+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\cdot \\ & \big(g_u(x)+\xi\left(-g_u(x)\eta(x)^p+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\big)^{p-1} \\ \le & g_u(x)^p+\varepsilon p\left(-g_u(x)\eta^p(x)+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\cdot \\ & \big(g_u(x)+\varepsilon\left(-g_u(x)\eta(x)^p+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\big)^{p-1}. \end{aligned}$$

Because u is p-superminimizer, we have

$$\int_{B(z,R)} g_u(x)^p\,\mathrm{d}\mu(x)\le\int_{B(z,R)} g_{v_\varepsilon}(x)^p\,\mathrm{d}\mu(x),$$

and consequently

$$\begin{aligned} 0\le\int_{B(z,R)} & \left(-g_u(x)\eta^p(x)+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\cdot \\ & \big(g_u(x)+\varepsilon\left(-g_u(x)\eta(x)^p+p(k-u(x))\eta(x)^{p-1}g_\eta(x)\right)\big)^{p-1}\,\mathrm{d}\mu(x). \end{aligned}$$

Now by using Lebesgue's Dominated Convergence Theorem and by letting $\varepsilon\to0$, it follows that

$$0\le\int_{B(z,R)}\left(-g_u\eta^p+p(k-u)\eta^{p-1}g_\eta\right)g_u^{p-1}\,\mathrm{d}\mu.$$

Hence by Lemma 3.3,

$$\begin{aligned} \int_{B(z,R)} g_u^p\eta^p\,\mathrm{d}\mu \le & \int_{B(z,R)} p(k-u)\eta^{p-1}g_\eta g_u^{p-1}\,\mathrm{d}\mu \\ \le & pk\int_{B(z,R)}\eta^{p-1}g_\eta g_u^{p-1}\,\mathrm{d}\mu \\ \le & ck\mu(B(z,R))R^{-p}(\inf_{B(z,R)} u)^{p-1}. \end{aligned}$$

□

Remark 3.5. The proof of Theorem 3.6 in [5] combined with Theorem 3.4 shows the capacity of level sets of p–supersolutions decreases at the following rate

$$\begin{aligned} &\mathrm{cap}_p\left(\{x\in B(z,R)\,:\,u(x)\ge\lambda\},B(z,2R)\right) \\ &\qquad\le c\,\lambda^{-(p-1)}\mu(B(z,R))R^{-p}(\inf_{B(z,R)} u)^{p-1}. \end{aligned}$$

This estimate is optimal, as can be seen by considering the fundamental p–superharmonic function $u(x) = |x|^{(p-n)/(p-1)}$, $1 < p < n$, in $\mathbb{R}^n$.

4. Fine continuity

Definition 4.1. We say that a function $u : \Omega \to (-\infty, \infty]$ is *p–superharmonic* if

(1) u is lower semicontinuous in Ω,
(2) u is not identically ∞ in any component of Ω, and
(3) for every open $\Omega' \Subset \Omega$ the comparison principle holds: if $h \in C(\overline{\Omega'})$ is p–harmonic in Ω' and $h \leq u$ on $\partial\Omega'$, then $h \leq u$ in Ω'.

Every bounded p–superharmonic function is a p–superminimizer by [6]. In this section, we use Theorem 3.4 to prove that p–superharmonic functions are p–finely continuous. The proof follows closely ideas in [5], where it is shown that p–superharmonic functions are q–finely continuous for every $q < p$. With the sharp estimate, we are able to obtain the optimal result. See also Theorem 2.121 in [9] for the Euclidean case.

First, we recall Lemma 3.3 in [2].

Lemma 4.2. *There exists $c > 0$ such that if $E \subset B(z, r)$ with $0 < r < \operatorname{diam}(X)/6$, then*

$$\frac{1}{c}\frac{\mu(E)}{r^p} \leq \operatorname{cap}_p(E, B(z, 2r)) \leq c\,\frac{\mu(B(z, r))}{r^p}.$$

Theorem 4.3. *Let $u > 0$ be p–superharmonic in Ω. Then u is p–finely continuous in Ω.*

Proof. By lower semicontinuity, u is continuous at $z \in \Omega$ if $u(z) = \infty$. Suppose that $u(z) < \infty$ for $z \in \Omega$. Fix R with $B(z, 20R) \subset \Omega$. Denote $E_k = \{u \geq k\}$ and $u_k = \min\{u, k\}$ for $k \in \mathbb{R}$. It is enough to show that E_k is p–thin at z whenever $u(z) < k$. By the lower semicontinuity of u in Ω and Theorem 5.1 in [6], we have

$$u(z) = \lim_{r \to 0} m(r),$$

where

$$m(r) = \inf_{B(z,r)} u_k.$$

Let $0 < r < R$ and denote

$$v = u_k - m(20r).$$

Let η be a Lipschitz cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B(z, r)$, $\eta = 0$ in $\Omega \setminus B(z, 2r)$ and $g_\eta \leq c/r$. Since the function $(k - u(z))^{-1} v\eta$ is a test function for the capacity $\operatorname{cap}_p(E_k \cap B(z, r), B(z, 2r))$, we have

$$\operatorname{cap}_p(E_k \cap B(z, r), B(z, 2r)) \leq (k - u(z))^{-p} \int_{B(z,2r)} g_{v\eta}^p \,\mathrm{d}\mu.$$

Theorem 3.4 implies that

$$\begin{aligned}\int_{B(z,2r)} g_v^p \eta^p \, \mathrm{d}\mu &\le c\mu(B(z,2r))r^{-p}(\inf_{B(z,2r)} v)^{p-1} \sup_{B(z,2r)} v \\ &\le ck\mu(B(z,2r))r^{-p}\left(m(2r)-m(20r)\right)^{p-1}\end{aligned}$$

By Lemma 3.2,

$$\begin{aligned}\int_{B(z,2r)} v^p g_\eta^p \, \mathrm{d}\mu &\le cr^p \le (k-m(20R)) \int_{B(z,2r)} v^{p-1} \, \mathrm{d}\mu \\ &\le c\mu(B(z,2r))r^{-p}(k-m(20R))(m(2r)-m(20r))^{p-1}.\end{aligned}$$

Combining the estimates, we obtain

$$\int_{B(z,2r)} g_{v\eta}^p \, \mathrm{d}\mu \le ck\mu(B(z,2r))r^{-p}\left(m(2r)-m(20r)\right)^{p-1}.$$

By Lemma 4.2, it follows that

$$\begin{aligned}\varphi(r) &= \frac{\operatorname{cap}_p(E_k \cap B(z,r), B(z,2r))}{\operatorname{cap}_p(B(z,r), B(z,2r))} \\ &\le c(k-u(z))^{-p} \frac{\int_{B(z,2r)} g_{v\eta}^p \, \mathrm{d}\mu}{\mu(B(z,2r))r^{-p}} \le c(m(2r)-m(20r))^{p-1}.\end{aligned}$$

Since $m(20R) \le m(r) \le u(z)$ for $r \in (0, 20R)$, we have

$$\begin{aligned}\int_\rho^R \varphi(r)^{\frac{1}{p-1}} \frac{\mathrm{d}r}{r} &\le c \int_\rho^R \left(m(2r)-m(20r)\right) \frac{\mathrm{d}r}{r} \\ &= \int_{2\rho}^{2R} m(r)\frac{\mathrm{d}r}{r} - \int_{20\rho}^{20R} m(r)\frac{\mathrm{d}r}{r} \\ &= \int_{2\rho}^{20\rho} m(r)\frac{\mathrm{d}r}{r} - \int_{2R}^{20R} m(r)\frac{\mathrm{d}r}{r} \\ &\le (u(z)-m(20R))\ln(10).\end{aligned}$$

Letting $\rho \to 0$ proves that E_k is p–thin at z. □

We obtain the following corollary. Note that by [1], all Newtonian functions are p–quasicontinuous.

Corollary 4.4. *Let $u : \Omega \to [-\infty, \infty]$ be p–quasicontinuous. Then u is p–finely continuous outside a set of p–capacity zero.*

Proof. It is enough to prove the claim for any given ball $B(z,R) \Subset \Omega$ with small radius. Let $(E_i)_i$ be a sequence of subsets of $B(z,R)$ such that

$$\lim_{i\to\infty} \operatorname{cap}_p(E_i, B(z,2R)) = 0$$

and the restriction of u to $B(z,R) \setminus E_i$ is continuous. Let $\overline{E}_i^p$ be the p–fine closure of E_i. It is enough to show that

$$\operatorname{cap}_p(\cap_i \overline{E}_i^p, B(z,2R)) = 0.$$

By Theorem 3.2 in [6], there is a function $u_i \in N_0^{1,p}(B(z,2R))$ such that

$$\operatorname{cap}_p(E_i, B(z,2R)) = \int_{B(z,2R)} g_{u_i}^p \, d\mu$$

and $u_i \geq 1$ p–quasieverywhere in E_i. It is easy to see that u_i is a p–superminimizer as a solution of a obstacle problem. Hence Theorem 4.3 implies that u_i is p–finely continuous in $B(z,2R)$. By the p–fine continuity, $u_i \geq 1$ quasieverywhere in $\overline{E}_i^p$. Thus

$$\operatorname{cap}_p(\overline{E}_i^p, B(z,2R)) \leq \operatorname{cap}_p(E_i, B(z,2R))$$

and the claim follows. □

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