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Geometric implications of the Poincaré inequality

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Metrisen avaruuden analyysissä mitan tuplaavuus ja Poincarén epäyhtälö ovat vakiintuneita oletuksia. Karkeasti ottaen tuplaavuusehto antaa ylärajan avaruuden dimensiolle ja Poincarén epäyhtälö takaa sen, että mitä tahansa avaruuden pisteparia yhdistää riittävä määrä suoristuvia polkuja. Näistä oletuksista tiedetään kuitenkin hyvin vähän. Tämän työn tarkoituksena on tutkia näiden ehtojen geometrisia seurauksia.

Tiedetään, että jos täydellinen metrinen avaruus kantaa Poincarén epäyhtälön, niin avaruus on kvasikonveksi, eli mikä tahansa pistepari voidaan yhdistää polulla, jonka pituus on verrannollinen pisteiden etäisyyteen.

Tämän työn päätuloksena parannamme kvasikonveksisuustulosta: Jos avaruus kantaa riittävän vahvan Poincarén epäyhtälön, niin renkaat ovat lähes kvasikonvekseja. Tämä tulos on kvantitatiivinen ja saamme arvion pisteiden pieniä ympäristöjä yhdistävien polkuperheiden modulille. On huomattavaa, että sen sijaan avaruuden kvasikonveksisuuden klassinen todistus antaa vain yhden polun, joka yhdistää kutakin pisteparia.

Päätuloksessa todistettava ehto on vahvempi kuin kirjallisuudessa paljon käytetty LLC–ehto. Eräissä tutkimuksissa on jouduttu olettamaan LLC–ehto Poincarén epäyhtälön lisäksi. Tuloksiemme nojalla tämä oletus voidaan poistaa.

Lopuksi täydennämme tuloksia ja osoitamme, että Poincarén epäyhtälön avulla saadaan johdettua alarajoja Hausdorffin kontentille ja pallonkuorien läpimitoille. Päätuloksen sovellutuksena laajennamme version Sobolevin upotuslauseesta melkein kaikille pallonkuorille. Tämä on yleistys vastaavasta tuloksesta Ahlfors–säännöllisissä avaruuksissa yleisempiin metrisiin avaruuksiin.

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Standard assumptions in analysis on metric measure spaces include that the measure is doubling and that the space supports a Poincaré inequality. Roughly speaking the doubling condition gives an upper bound for the dimension of the metric space and the Poincaré inequality implies that there has to be a number of rectifiable curves connecting any two points in the space. However, very little is known about these assumptions. The objective of this work is to obtain geometric consequences of the Poincaré inequality under some conditions for the measure.

It is known that if a complete doubling metric measure space supports a $(1, p)$–Poincaré inequality, then the space is quasiconvex i.e. there exists a constant such that every pair of points can be connected with a curve whose length is at most the constant times the distance between the points.

In this work, we improve this result: If the space supports a weak $(1, p)$–Poincaré inequality with p sufficiently small, then annuli are almost quasiconvex. This result is of a quantitative nature, and we obtain an estimate for the modulus of curve families joining small neighbourhoods of a pair of points. Observe that the proof of quasiconvexity gives only one curve joining any pair of points.

The condition in the main result is related to a weaker condition, which is called local linear connectivity. In some papers, the standing assumption is that the space is locally linearly connected and that it supports a $(1, p)$–Poincaré inequality. It follows from the main theorem that the assumption on local linear connectivity can be removed.

Finally, we complement the main result by proving that if the space supports a $(1, p)$–Poincaré inequality for a sufficiently small p, then we obtain lower bounds for the Hausdorff s–content and the diameter of the spheres. As an application, we prove an Sobolev embedding theorem for almost all spheres centered at a given point. This is a generalization of a similar result in Ahlfors-regular spaces to more general metric spaces.

Number of pages: 46 Keywords: Poincaré inequality, quasiconvexity, capacity, modulus, metric spaces, doubling measure

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Espoo, October 5, 2006

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Contents

1 Introduction

Standard assumptions in analysis on metric measure spaces include that the measure is doubling and that the space supports a Poincaré inequality, see for example Björn–Björn–Shanmugalingam [3], Cheeger [5] and Semmes [21]. Roughly speaking, the doubling condition gives an upper bound for the dimension of the metric space, and the Poincaré inequality implies that there has to be a number of rectifiable curves connecting any two points in the space. However, very little is known about these assumptions. The objective of this work is to obtain geometric consequences of the Poincaré inequality under some conditions for the measure.

It is known that if a complete doubling metric measure space supports a $(1,p)$–Poincaré inequality, that is

$$\fint_B |u - u_B| \,\mathrm{d}\mu \leq C \operatorname{diam}(B) \left(\fint_B |g|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}$$

for every ball $B \subset X$ and measurable function u with upper gradient g, then the space is quasiconvex i.e. there exists a constant such that every pair of points can be connected with a curve whose length is at most the constant times the distance between the points, see Semmes [21], Cheeger [5] and Keith [14]. We have included the proof here, because it is somewhat difficult to find it in the literature.

In this thesis, we improve this result: If the space supports a weak $(1,p)$–Poincaré inequality with p sufficiently small, then annuli are almost quasiconvex. This result is of a quantitative nature, and we obtain an estimate for the modulus of curve families joining small neighbourhoods of a pair of points. Observe that the proof of quasiconvexity gives only one curve joining any pair of points. This result is also a partial converse of the result by Semmes about families of curves implying Poincaré inequality, see Semmes [20]. We discuss these results in Section 2.1.

The condition in the main result is related to a weaker condition called local linear connectivity, see for example Heinonen–Koskela [11]. In Björn–MacManus–Shanmugalingam [4] and Holopainen–Shanmugalingam [12], the standing assumption is that the space is locally linearly connected and that it supports a $(1,p)$–Poincaré inequality. It follows from the main theorem that the assumption on local linear connectivity can be removed.

This thesis is based on a paper Korte [16] and it is organized as follows: Section 2 contains preliminaries. In Section 2.1, we discuss the definition of the Poincaré inequality, its invariance under bi–Lipschitz mappings and sufficient conditions to ensure that the space supports a Poincaré inequality. Section 2.2 is devoted to the relation of modulus and capacity and to a fundamental equality between them. In Section 2.3, we introduce a fractional sharp maximal function, which is an important tool in analysis on metric spaces. We present a useful pointwise estimate and construct a Lipschitz extension with useful properties. In Section 2.5, it is shown that Poincaré inequality implies Sobolev–Poincaré inequality.

Section 3 contains the proof of the quasiconvexity of annuli. We also present a proof for the fact that quasiconvexity of the space follows from the Poincaré inequality. This result is needed also in the proof of the quasiconvexity of the annuli. We also present some examples that show sharpness of our results, that is, any weaker version of the Poincaré is not enough to guarantee quasiconvexity of annuli.

In Section 4, we complement the main result by proving that if the space supports a $(1, p)$–Poincaré inequality for a sufficiently small p, then we obtain lower bounds for the Hausdorff s–content and the diameter of the spheres. If the measure is Ahlfors Q–regular, for some $Q > 1$, then the results of Ambrosio–Miranda–Pallara [1] and Miranda [17] yield upper bounds for the Hausdorff $(Q - 1)$–dimensional content of spheres. Our result therefore completes this picture.

Finally, in Section 5, we prove an Sobolev embedding theorem for almost all spheres centered at a given point. This result is a generalization of results in Hajłasz–Koskela [9]. As an application of our main results in this work, we extend the embedding theorem from Ahlfors–regular spaces to more general metric spaces.

2 Preliminaries

In this work (X, d, μ) denotes a metric measure space and μ is a Borel regular outer measure such that the measure of bounded open sets is positive and finite. The ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B = B(x, r) = \{y \in X \,:\, d(x, y) < r\}.$$

We write

$$u_A = \fint_A u \,\mathrm{d}\mu = \frac{1}{\mu(A)} \int_A u \,\mathrm{d}\mu$$

for every measurable set $A \subset X$ with $0 < \mu(A) < \infty$ and measurable function $u \,:\, X \to [-\infty, \infty]$.

The measure is said to be *doubling* if there is a constant $C_\mu \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$

for every $x \in X$ and $r > 0$. A complete doubling metric measure space is always *proper*, that is its closed and bounded subsets are compact.

Let $s > 0$. The *restricted Hausdorff s–content* of a set $E \subset X$ is

$$\mathcal{H}_s^R(E) = \inf \sum_{i=1}^{\infty} r_i^s$$

where the infimum is taken over all countable covers of E by balls B_i of radius $r_i \leq R$. The *Hausdorff s-content* of E is $\mathcal{H}_s^\infty(E)$ and the Hausdorff measure

$$\mathcal{H}_s(E) = \lim_{R \to 0} \mathcal{H}_s^R(E).$$

Thus the s–content of E is less than or equal to the Hausdorff s–measure of the set, and it is finite for bounded sets.

A *curve* in X is a continuous map γ of an intervel $I \subset \mathbb{R}$ into X. If $I = [a, b]$ is a closed interval, then the length of a curve $\gamma : I \to X$ is

$$l(\gamma) = \sup \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i+1})),$$

where the supremum is over all sequences $a = t_1 \leq t_2 \leq \ldots \leq t_n \leq t_{n+1} = b$. If I is not closed, then we define

$$l(\gamma) = \sup_J l(\gamma_{|J}),$$

where the supremum is taken over all closed intervals $J \subset I$. A curve is rectifiable, if its length is finite. We say that the space is *quasiconvex* if there exists a uniform constant $C_q \geq 1$ such that every pair of points $x, y \in X$ can be connected with a rectifiable curve γ_{xy}, whose length satisfies $l(\gamma_{xy}) \leq C_q d(x,y)$. Moreover, the space is *locally quasiconvex* if each point has a quasiconvex neighbourhood.

A metric space is said to be *linearly locally connected* if there is a constant $C \geq 1$ so that for each $x \in X$ and $r > 0$ any pair of points in $B(x,r)$ can be joined in $B(x, Cr)$ by a rectifiable curve, and any pair of points in $X \setminus \overline{B}(x,r)$ can be joined in $X \setminus \overline{B}(x, r/C)$ with a rectifiable curve.

Let U be an open set in X. We say that a Borel function $g : U \to [0,\infty]$ is an upper gradient of u in U if

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} g \,\mathrm{d}s \tag{2.1}$$

whenever γ_{xy} is a rectifiable curve joining two points x and y in U. If (2.1) holds for p-almost every path, then g is a *p-weak upper gradient* of u. By saying that something holds for p-almost every path, we mean that it fails only for a path family with zero p-modulus. In $\mathbb{R}^n$ the modulus of the gradient is an upper gradient of every Sobolev function, but unlike the gradient, an upper gradient is not unique.

We will need the following basic covering theorem:

Theorem 2.1. *Every family $\mathcal{F}$ of balls of uniformly bounded diameter in a metric space X contains a disjointed subfamily $\mathcal{G}$ such that*

$$\bigcup_{B(x,r)\in\mathcal{F}} B(x,r) \subset \bigcup_{B(x,r)\in\mathcal{G}} B(x,5r)$$

Proof. Let Ω denote the set consisting of all disjointed subfamilies ω of $\mathcal{F}$ with the following property: If a ball B from $\mathcal{F}$ meets some ball from ω, then it meets one whose radius is at least half the radius of B. We define to Ω the natural partial ordering (by inclusion).

If $\mathcal{C} \subset \Omega$ is a chain, then

$$w_0 = \bigcup_{w \in \mathcal{C}} w$$

belongs to Ω. Therefore, by Zorn's lemma there is a maximal element $\mathcal{G}$ in Ω. By construction, $\mathcal{G}$ is disjointed.

If there is a ball in $\mathcal{F}$ that does not intersect any ball from $\mathcal{G}$, then pick a ball B_0 from $\mathcal{F}$ such that the radius of B_0 is larger than half of the radius of any other ball that does not meet the balls from $\mathcal{G}$. Then, if a ball B from $\mathcal{F}$ meets a ball from the collection $\tilde{\mathcal{G}} = \mathcal{G} \cup \{B_0\}$, by construction it meets one whose radius is at least half that of B, showing that $\tilde{\mathcal{G}}$ belongs to Ω. But this contradicts tha maximality of $\mathcal{G}$.

Thus, every ball $B = B(x,r)$ from $\mathcal{F}$ meets a ball $\tilde{B} = B(\tilde{x}, \tilde{r})$ from $\mathcal{G}$ so that $r \leq 2\tilde{r}$, and by the triangle inequality $B \subset B(\tilde{x}, 5\tilde{r})$. $\square$

2.1 Poincaré inequality

We will use the following definition for the Poincaré inequality:

Definition 2.2. We say that a metric measure space X supports a weak $(1,p)$–Poincaré inequality, $1 \leq p < \infty$, if there exist constants $C > 0$ and $\lambda \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, \mathrm{d}\mu \leq Cr \left(\fint_{B(x,\lambda r)} g^p \, \mathrm{d}\mu \right)^{1/p} \tag{2.2}$$

for every $x \in X$ and $0 < r < \operatorname{diam}(X)$, for every function $u : X \to \mathbb{R}$, and for every upper gradient g of u. The word *weak* refers to the possibility that λ may be strictly greater than 1.

Remark 2.3. Poincaré inequality is qualitatively invariant under bi–Lipschitz mappings, if the measure is doubling: Let (X, d, μ) be a metric measure space and μ doubling with a constant C_μ and let $(\tilde{X}, \tilde{d})$ be a metric space. Let $T : X \longrightarrow \tilde{X}$ be a L–bi–Lipschitz mapping. Let $\tilde{\mu}$ be a measure on $\tilde{X}$ coinduced by T. Assume that (X, d, μ) supports a weak $(1,p)$–Poincaré inequality with constants C and λ. Fix a point $\tilde{x} \in \tilde{X}$, a function $\tilde{u} : \tilde{X} \longrightarrow \mathbb{R}$ and $r > 0$. Denote

$$\begin{aligned} &x = T^{-1}\tilde{x}, \\ &u = \tilde{u} \circ T, \qquad \text{and} \\ &\tilde{B}(\tilde{x}, r) = \{y \in \tilde{X} \,:\, \tilde{d}(y, \tilde{x}) < r\}. \end{aligned}$$

First we use the definition of $\tilde{\mu}$ and L–bi–Lipschitz continuity of T to obtain

$$
\begin{aligned}
\fint_{\tilde{B}(\tilde{x},r)} |\tilde{u} - \tilde{u}_{\tilde{B}(\tilde{x},r)}| \,\mathrm{d}\tilde{\mu} &\leq 2 \fint_{\tilde{B}(\tilde{x},r)} |\tilde{u} - u_{B(x,Lr)}| \,\mathrm{d}\tilde{\mu} \\
&= 2 \fint_{T^{-1}\tilde{B}(\tilde{x},r)} |u - u_{B(x,Lr)}| \,\mathrm{d}\mu \\
&\leq 2 \frac{\mu(B(x,Lr))}{\mu(T^{-1}\tilde{B}(\tilde{x},r))} \fint_{B(x,Lr)} |u - u_{B(x,Lr)}| \,\mathrm{d}\mu \\
&\leq 2 \frac{\mu(B(x,Lr))}{\mu(B(x,\frac{r}{L}))} \fint_{B(x,Lr)} |u - u_{B(x,Lr)}| \,\mathrm{d}\mu.
\end{aligned}
$$

Then we use the fact that X supports a weak $(1,p)$–Poincaré inequality to conclude that

$$
\begin{aligned}
&2 \frac{\mu(B(x,Lr))}{\mu(B(x,\frac{r}{L}))} \fint_{B(x,Lr)} |u - u_{B(x,Lr)}| \,\mathrm{d}\mu \\
\leq & 2C_{\mu,L}^2 CLr \left[\fint_{B(x,\lambda Lr)} (g_u)^p \,\mathrm{d}\mu \right]^{\frac{1}{p}} \\
\leq & 2C_{\mu,L}^2 CLr \left[\fint_{TB(x,\lambda Lr)} (Lg_{\tilde{u}})^p \,\mathrm{d}\tilde{\mu} \right]^{\frac{1}{p}} \\
\leq & 2C_{\mu,L}^2 CL^2 r \left[\frac{\tilde{\mu}(\tilde{B}(\tilde{x},\lambda L^2 r))}{\tilde{\mu}(TB(x,\lambda Lr))} \fint_{\tilde{B}(\tilde{x},\lambda L^2 r)} (g_{\tilde{u}})^p \,\mathrm{d}\tilde{\mu} \right]^{\frac{1}{p}} \\
\leq & 2C_{\mu,L}^2 CL^2 r \left[\frac{\mu(B(x,\lambda L^3 r))}{\mu(B(x,\lambda r))} \fint_{\tilde{B}(\tilde{x},\lambda L^2 r)} (g_{\tilde{u}})^p \,\mathrm{d}\tilde{\mu} \right]^{\frac{1}{p}} \\
\leq & 2C_{\mu,L}^{2+\frac{3}{p}} CL^2 r \left[\fint_{\tilde{B}(\tilde{x},\lambda L^2 r)} (g_{\tilde{u}})^p \,\mathrm{d}\tilde{\mu} \right]^{\frac{1}{p}}.
\end{aligned}
$$

Thus the space $\tilde{X}$ satisfies the weak $(1,p)$–Poincaré inequality with constants $\tilde{C} = 2C_{\mu,L}^{2+\frac{3}{p}} CL^2$ and $\tilde{\lambda} = L^2\lambda$. Here $C_{\mu,L}$ is a constant that satisfies

$$\mu(LB) \leq C_{\mu,L}\mu(B)$$

for every ball $B \subset X$. It is finite because μ is doubling.

There are several possible definitions for the Poincaré inequality. Most of them are equivalent if the measure is doubling and the space is complete. For example, instead of all measurable functions, it is enough to require inequality (2.2) for compactly supported Lipschitz functions with Lipschitz upper gradients. Or we may replace the upper gradient by the local Lipschitz constant, see Keith [14]. For more information about the Poincaré inequality, see also for example Bakry–Coulhon–Ledoux–Saloff-Coste [2], Franchi–Pérez–Wheeden [6] and Heinonen [10].

Poincaré inequality enables us to derive other results that are analogous to the Euclidean spaces, like Sobolev embedding theorems and isoperimetric inequalities. Geometrically Poincaré inequality means that any pair of points can be connected by many curves that are not too long and that the curves can be distributed nicely.

Notice that if X supports a weak $(1, p)$–Poincaré inequality, then it supports a weak $(1, q)$–Poincaré inequality for all $q > p$. This follows easily from Hölder's inequality. Recently Keith and Zhong proved in Keith–Zhong [15] that there exists also $\varepsilon > 0$ such that X supports a $(1, q)$–Poincaré inequality for all $q > p - \varepsilon$. We will discuss this question in Section 2.3.

Semmes has studied conditions on a metric measure space under which Sobolev and Poincaré inequalities make sense and are valid. He studied this problem first in Ahlfors-regular spaces in [19] and then in more general metric spaces in [20]. The conditions restrict the behavior of the Hausdorff measure of the space and the behavior of its topology. Separately these assumptions permit fractal phenomena which are incompatible with analytical results like Sobolev embedding theorems.

In the Ahlfors–regular case, the conditions include additionally rectifiability and restrictions on topology. We say that an Q–regular metric space (X, d, μ) is *rectifiable* if Q is an integer and if there is a sequence of compact sets $\{E_j\}_j$ in $\mathbb{R}^Q$ and a sequcence of Lipschitz mappings $f_j : E_j \to X$ such that

$$\mu\left(X \setminus \bigcup_j f_j(E_j)\right) = 0.$$

There is not necessarily a uniform bound on the Lipschitz constants of the f_j's, which means that rectifiability is not a quantitative condition. For example self-similar fractals are not rectifiable. If $(M, d(x, y))$ is a metric space, then its snowflaked version $(M, d(x, y)^s)$, $0 < s < 1$, is never rectifiable. Actually snowflaked metric spaces have no rectifiable curves and therefore they cannot support a Poincaré inequality.

Restrictions on topology are done by assuming that the space is *locally linearly contractable*. By local linear contractability we mean that the space X is complete and there exists a constant C so that if $x \in X$ and $0 < r < \operatorname{diam} X/C$, then $B(x, r)$ can be contracted to a point inside $B(x, Cr)$. (That is, there should exist a continuous one-parameter family of mappings from $B(x, r)$ into $B(x, Cr)$ which begins with the identity and ends with a constant mapping.)

Euclidean spaces and compact smooth manifolds are locally linearly contractable. The standard self-similar fractal trees and snowflake curves are locally linearly contractable as well. Local linear contractability is also invariant under snowflaking, but it prevents some bad behaviour, like cusps, bubbles and long thin tubes.

Now we are able to formulate Semmes's conditions that ensure the Poincaré inequality:

Theorem 2.4. *Suppose that (X, d, μ) is an Ahlfors n-regular metric space which is also rectifiable, locally linearly contractable, and a connected orientable topological manifold of dimension n. Then X supports a weak $(1, 1)$–Poincaré inequality.*

2.2 Modulus and capacity

Let Γ be a family of curves in X and let $1 \le p < \infty$. The *p–modulus* of Γ is defined as

$$\operatorname{mod}_p \Gamma = \inf \int_X \rho^p \, \mathrm{d}\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0, \infty]$ satisfying

$$\int_\gamma \rho \, \mathrm{d}s \ge 1 \tag{2.3}$$

for all rectifiable curves $\gamma \in \Gamma$. Functions ρ satisfying (2.3) are called *admissible (metrics)* for Γ.

The p–modulus enjoys the following properties (see for example Heinonen [10]):

- $\operatorname{mod}_p \emptyset = 0$.
- $\operatorname{mod}_p \Gamma_1 \le \operatorname{mod}_p \Gamma_2$ if $\Gamma_1 \subset \Gamma_2$.

- $\operatorname{mod}_p\left(\bigcup_{i=1}^{\infty}\Gamma_i\right) \le \sum_{i=1}^{\infty}\operatorname{mod}_p\Gamma_i$.
- $\operatorname{mod}_p\Gamma \le \operatorname{mod}_p\widetilde{\Gamma}$ if Γ and $\widetilde{\Gamma}$ are two curve families such that each curve $\gamma\in\Gamma$ has a subcurve $\widetilde{\gamma}\in\widetilde{\Gamma}$.

Suppose that E and F are closed subsets of an open set $U \subset X$. The triple (E, F, U) is called a *condenser* and its *p–capacity* for $1 \le p < \infty$ is defined as

$$\operatorname{cap}_p(E, F, U) = \inf \int_U g^p \, \mathrm{d}\mu,$$

where the infimum is taken over all upper gradients g of all functions u in U such that $u_{|E} \ge 1$ and $u_{|F} \le 0$. Such a function u is called *admissible* for the condenser (E, F, U). If $U = X$, we write $(E, F, U) = (E, F)$. If X supports a $(1, p)$–Poincaré inequality then any function u that has an upper gradient in $L^p(X)$ must be measurable and be in class $L^1_{loc}(X)$ (see Järvenpää–Järvenpää–Rogovin–Rogovin–Shanmugalingam [13]) and therefore is in class $L^p_{loc}(X)$ (see Hajłasz–Koskela [9]).

There is a fundamental equality between the modulus and the capacity.

Proposition 2.5. *Let (X, d, μ) be a metric measure space. Then*

$$\operatorname{cap}_p(E, F, U) = \operatorname{mod}_p(E, F, U),$$

where the modulus on the right-hand side is the modulus of all curves joining the sets E and F in U.

Proof. If u is a function on X with $u_{|E} \le 0$ and $u_{|F} \ge 1$, and if g is any upper gradient of u, then

$$1 \le |u(x) - u(y)| \le \int_\gamma g \, \mathrm{d}s$$

for any rectifiable curve γ joining a point $x \in E$ and a point $y \in F$. Therefore

$$\operatorname{mod}_p(E, F) \le \operatorname{cap}_p(E, F).$$

On the other hand, if ρ is an admissible function for the condenser (E, F), then define

$$u(x) = \inf \int_{\gamma_x} \rho \, \mathrm{d}s, \tag{2.4}$$

where the infimum is taken over all curves γ_x joining E to the point $x \in X$. Then, $u_{|E} = 0$ and $u_{|F} \geq 1$.

Let us show that ρ is an upper gradient of u. Without loss of generality, we may assume that $u(x) \geq u(y)$. Let γ_{xy} be a rectifiable curve joining x and y. Fix $\varepsilon > 0$, and let γ_ε be a curve joining E to y such that

$$u(y) > \int_{\gamma_\varepsilon} \rho \, \mathrm{d}s - \varepsilon.$$

Now $\gamma_\varepsilon \cup \gamma_{xy}$ is a curve joining E to x and hence

$$\begin{aligned} & |u(x) - u(y)| \\ = & \, u(x) - u(y) \\ \leq & \int_{\gamma_\varepsilon \cup \gamma_{xy}} \rho \, \mathrm{d}s - \int_{\gamma_\varepsilon} \rho \, \mathrm{d}s + \varepsilon \\ = & \int_{\gamma_{xy}} \rho \, \mathrm{d}s + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary

$$|u(x) - u(y)| \leq \int_{\gamma_{xy}} \rho \, \mathrm{d}s$$

for any curve joining x and y, and hence ρ is an upper gradient of u. This implies that

$$\mathrm{cap}_p(E, F) \leq \mathrm{mod}_p(E, F),$$

and the proposition follows. $\square$

It is possible to estimate modulus through capacity. Because we want estimates from below, the following result is useful.

Proposition 2.6. *Let (X, d, μ) be a proper and locally quasiconvex metric measure space. Suppose that X supports a weak $(1, p)$–Poincaré inequality, $1 \leq p < \infty$, and that E and F are two compact disjoint subsets of X. Then, in the definition of $\mathrm{cap}_p(E, F)$, we can restrict to locally Lipschitz functions u.*

Proof. By Proposition 2.5 it suffices to show that the capacity, if defined over continuous functions, does not exceed the modulus. To this end, fix an admissible metric ρ for (E, F). We would like to build an appropriate continuous admissible function u using ρ, under the additional contiditon that X is locally quasiconvex. The definition (2.4) may not work as ρ need not be bounded, and to avoid this possiblity, an approximation argument is needed.

We may assume that ρ is lower semicontinuous, because it follows from the Vitali–Carathéodory theorem (see for example Theorem 2.25 in Rudin [18]) that every function f in $L^p(X)$ can be approximated in $L^p(X)$ by a lower semicontinuous function g with $g \geq f$. This requires X to be locally compact.

We may also clearly assume that $\rho_{|F} = 0$. By considering the lower semicontinuous functions $\max\{\rho, 1/m\}$, $m = 1, 2, \ldots$ we may further assume that $\rho_{|X \setminus F}$ is bounded away from zero: $\rho \geq \xi$ in $X \setminus F$, where $\xi > 0$ is a positive constant (use Lebesgue's monotone convergence theorem). Fix a positive integer k and consider the function $\rho_k = \min(\rho, k)$. Then ρ_k is a bounded lower semicontinuous function in X, $\rho_k \geq \xi$ in $X \setminus F$, and vanishes in F.

Define

$$u_k(x) = \inf \int_{\gamma_x} \rho_k \, \mathrm{d}s$$

for $x \in X$, where the infimum is taken over all rectifiable curves γ_x joining x to F. As above, we find that $u_{k|F} = 0$ and that ρ_k is an upper gradient of u_k. It is not difficult to see that u_k is continuous in X. Indeed, pick a point in X and let x and y be points in some open quasiconvex neighbourhood U of the point. We can choose a curve γ_{xy} such that $l(\gamma_{xy}) \leq C_q d(x, y)$. By the definition of u_k, we have

$$|u_k(x) - u_k(y)| \leq \int_{\gamma_{xy}} \rho_k \, \mathrm{d}s \leq k l(\gamma_{xy}) \leq k C_q d(x, y),$$

and we conclude that u_k is Lipschitz continuous in U.

Denote

$$m_k = \min_{x \in E} u_k(x).$$

Then the function $v_k = u_k / m_k$ satisfies $v_{k|E} \geq 1$ and $v_{k|F} = 0$. Because ρ_k / m_k is an upper gradient of the continuous function v_k, and because

$$\int_X \rho_k^p \, \mathrm{d}\mu \leq \int_X \rho^p \, \mathrm{d}\mu,$$

it suffices to show that

$$\sup_k m_k \geq 1.$$

Note also that the Lipschitz continuity of u_k implies that also v_k is Lipschitz.

Suppose on the contrary that for each k there are points $x_k \in E$ and $y_k \in F$ and curves γ_k joining x_k to y_k such that

$$\int_{\gamma_k} \rho_k \, \mathrm{d}s \leq 1 - \delta$$

for some $\delta > 0$.

We may assume that each y_k belongs to the compact set ∂F, and that each x_k belongs to the compact set ∂E, and thus by passing to a subsequence, we may assume that $y_k \to y \in \partial F$ and $x_k \to x \in \partial E$, as $k \to \infty$. Because ρ_k is bounded away from zero in $X \setminus F$ by ξ, the lengths of the curves γ_k remain bounded from above by $M = (1-\delta)/\xi$. We assume that each curve $\gamma_k : [0, l(\gamma_k)] \to X$ is parametrized by its arc length, and then extend $\gamma_k(t) = \gamma_k(l(\gamma_k))$ for $l(\gamma_k) \leq t \leq M$. We obtain a family of 1–Lipschitz maps $\gamma_k : [0, M] \to X$ with images lying in a fixed compact set. The Arzela–Ascoli theorem implies, by passing to a subsequence if necessary, that γ_k converges uniformly on $[0, M]$ to a 1–Lipschitz map $\gamma : [0, M] \to X$. In particular, γ is a rectifiable curve in X joining the points $x \in \partial E$ and $y \in \partial F$. We may also assume at this point that $l(\gamma_k) \to M$. Hence, for a fixed positive integer k_0, we have that

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\gamma_k} \rho_{k_0} \, \mathrm{d}s &= \liminf_{k\to\infty} \int_0^{l(\gamma_k)} \rho_{k_0} \circ \gamma_k(t) \, \mathrm{d}t \\
&\geq \liminf_{k\to\infty} \int_0^{M-\varepsilon} \rho_{k_0} \circ \gamma_k(t) \, \mathrm{d}t \\
&\geq \int_0^{M-\varepsilon} \liminf_{k\to\infty} \rho_{k_0} \circ \gamma_k(t) \, \mathrm{d}t \\
&\geq \int_0^{M-\varepsilon} \rho_{k_0} \circ \gamma(t) \, \mathrm{d}t,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. Note that the lower semicontinuity of ρ_{k_0} was needed here. Thus

$$\liminf_{k\to\infty} \int_{\gamma_k} \rho_{k_0} \, \mathrm{d}s \geq \int_0^M \rho_{k_0} \circ \gamma(t) \, \mathrm{d}t \geq \int_\gamma \rho_{k_0} \, \mathrm{d}s.$$

In conclusion

$$1 - \delta \geq \liminf_{k\to\infty} \int_{\gamma_k} \rho_k \, \mathrm{d}s \geq \int_\gamma \rho_{k_0} \, \mathrm{d}s.$$

for all $k_0 = 1, 2 \ldots$. But this is a contradiction as

$$\lim_{k_0 \to \infty} \int_\gamma \rho_{k_0} \, \mathrm{d}s = \int_\gamma \rho \, \mathrm{d}s \geq 1. \qquad \square$$

2.3 Fractional sharp maximal function and pointwise estimates

Given a Lipschitz function $u : X \to \mathbb{R}$ and $x \in X$, we set

$$M^{\#}u(x) = \sup_B \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| \, \mathrm{d}\mu,$$

where the supremum is taken over all balls $B \subset X$ that contain x. With this sharp fractional maximal operator we obtain a pointwise estimate for the oscillation of functions. See for example Hajłasz–Kinnunen [8].

Proposition 2.7. *Let (X, d, μ) be a metric measure space with μ doubling, and let $u : X \to \mathbb{R}$ be Lipschitz. Then there exists $C \geq 0$ that depends only on the doubling constant of μ such that*

$$|u(x) - u(y)| \leq C d(x, y) \left(M^{\#}u(x) + M^{\#}u(y) \right),$$

whenever $x, y \in X$.

Proof. Let us fix $x \in X$ and $r > 0$. Let $B_i = B(x, 2^{-i}r)$. Because u is Lipschitz continuous, we have

$$\lim_{i \to \infty} u_{B_i} = u(x). \tag{2.5}$$

By inequality (2.5) and telescope sums, we obtain

$$\begin{aligned} &\left| u(x) - u_{B(x,r)} \right| \\ \leq\ & \sum_{i=0}^{\infty} \left| u_{B_{i+1}} - u_{B_i} \right| \\ \leq\ & \sum_{i=0}^{\infty} \fint_{B_{i+1}} |u - u_{B_i}| \, \mathrm{d}\mu, \end{aligned} \tag{2.6}$$

and consequently

$$\begin{aligned}\left|u(x) - u_{B(x,r)}\right| &\leq \sum_{i=0}^{\infty} \frac{\mu(B_i)}{\mu(B_{i+1})} \fint_{B_i} |u - u_{B_i}| \,\mathrm{d}\mu \\ &\leq C_\mu \sum_{i=0}^{\infty} 2^{-i} r \frac{1}{2^{-i}r} \fint_{B_i} |u - u_{B_i}| \,\mathrm{d}\mu \\ &\leq 2C_\mu r M^{\#} u(x).\end{aligned} \tag{2.7}$$

Let us fix $y \in B(x,r)$. It follows that $B(y,r) \subset B(x,2r)$. Hence

$$\begin{aligned}&|u(x) - u_{B(y,r)}| \\ &\leq \left|u(x) - u_{B(x,2r)}\right| + \left|u_{B(x,2r)} - u_{B(y,r)}\right| \\ &\leq 4C_\mu r M^{\#} u(x) + \fint_{B(y,r)} \left|u - u_{B(x,2r)}\right| \,\mathrm{d}\mu \\ &\leq 4C_\mu r M^{\#} u(x) + \frac{\mu(B(x,2r))}{\mu(B(y,r))} \fint_{B(x,2r)} \left|u - u_{B(x,2r)}\right| \,\mathrm{d}\mu \\ &\leq \left(4C_\mu + 2C_\mu^2\right) r M^{\#} u(x).\end{aligned} \tag{2.8}$$

Here we used equation (2.7) and the fact that the doubling property of the measure implies that the measures of balls $B(y,r)$ and $B(x,2r)$ are comparable.

Let $r = 2d(x,y)$. By equation (2.8), we find that

$$\begin{aligned}&|u(x) - u(y)| \\ &\leq \left|u(x) - u_{B(x,r)}\right| + \left|u_{B(x,r)} - u(y)\right| \\ &\leq C_\mu r M^{\#} u(x) + C r M^{\#} u(y) \\ &\leq C\, d(x,y) \left(M^{\#} u(x) + M^{\#} u(y)\right).\end{aligned}$$

□

The sharp fractional maximal operator $M^{\#}$ is an important tool in the analysis of metric spaces. For example, it was used by Keith and Zhong [15] in the proof of self-improvement of the $(1,p)$–Poincaré inequality.

Theorem 2.8. *Let $p > 1$ and let (X,d,μ) be a complete metric measure space with μ Borel and doubling, that admits a $(1,p)$–Poincaré inequality. Then there exists $\varepsilon > 0$ such that (X,d,μ) admits a $(1,q)$–Poincaré inequality for every $q > p - \varepsilon$, quantitatively.*

Part of the proof is a construction of a Lipschitz–extension that has some useful properties. We present the construction here.

Theorem 2.9. *Let $u : X \longrightarrow \mathbb{R}$ be a Lipschitz function and $\lambda > 0$. Let us denote*

$$U_\lambda := \{x \in X \ : \ M^\# u(x) > \lambda\}. \tag{2.9}$$

Then there exists a $C\lambda$–Lipschitz extension of $u_{|X\setminus U_\lambda}$ to the whole space X such that

$$M^\# f(x) \leq CM^\# u(x) \tag{2.10}$$

for every $x \in X \setminus U_\lambda$. The constant C depends only on the doubling constant C_μ.

Proof. By Proposition 2.7, $u_{|X\setminus U_\lambda}$ is $C\lambda$–Lipschitz. We use a Whitney-type partition of the set U_λ to extend the function to the whole space.

Because $M^\#$ is not centered, the set U_λ is clearly open. By Theorem 2.1, we can choose from the family

$$\left\{ B(x, r_x) \ \middle| \ x \in U_\lambda,\, r_x = \frac{1}{100}\operatorname{dist}(x, X \setminus U_\lambda) \right\}$$

a countable disjoint subset $\{B(x_i, r_{x_i})\}_{i=1}^\infty,\ i = 1, 2, \ldots$ satisfying

$$U_\lambda \subset \bigcup_{i=1}^\infty B(x_i, 5r_{x_i}).$$

The cover may be finite, but this doesn't affect the proof.

If U_λ is not bounded, the diameters of the balls may not be uniformly bounded. However, this is not a problem, because the radii of any two balls intersecting each other are always approximately equal.

We denote $B_i := B(x_i, r_i) := B(x_i, 5r_{x_i})$ and $\mathcal{F} = \{B_i\}_{i=1}^\infty$. Because $B_i \subset U_\lambda$ for every $i \in \mathbb{N}$, we conclude that $U_\lambda = \bigcup_{i=1}^\infty B_i$.

Because X is doubling and the balls near each other have approximately same radii, we have

$$\sum_{i=1}^\infty \chi_{2B_i} < C. \tag{2.11}$$

We fix a ball B_i. Let

$$J := \{j \in \mathbb{N} \ : \ 2B_i \cap 2B_j \neq \emptyset\}.$$

Then $x_j \in B(x_i, 24r_{x_i})$ for every $j \in J$. Because balls $B(x_j, r_{x_j})$ are disjoint and their radii are at least $\frac{3}{4}r_{x_i}$, we have $|J| \leq N^5 \leq C_\mu^{20}$, where N is the doubling constant of X (that is, every ball of radius r in X can be covered by at most N balls of radius at most $r/2$) . Thus equality (2.11) is satisfied $C = C_\mu^{20}$.

We construct a partition of unity subordinate to $\mathcal{F}$. We define $\hat{\psi}_i : X \longrightarrow \mathbb{R}$ as follows

$$\hat{\psi}_i(x) := \begin{cases} 1, & d(x, x_i) < r_i, \\ 2 - \frac{d(x,x_i)}{r_i}, & r_i \leq d(x, x_i) \leq 2r_i, \\ 0, & d(x, x_i) > 2r_i. \end{cases} \tag{2.12}$$

Cleary $\hat{\psi}_i$ is $\frac{1}{r_i}$–Lipschitz, $\hat{\psi}_i = 1$ in B_i and $\hat{\psi}_i = 0$ in $X \setminus 2B_i$.

We define

$$\psi_i(x) := \frac{\hat{\psi}_i(x)}{\sum_{j=1}^{\infty} \hat{\psi}_j(x)} \tag{2.13}$$

for every $x \in U_\lambda$. It follows from (2.11) that fuctions ψ_i are well defined.

Lemma 2.10. *Functions ψ_i are $\frac{C}{r_i}$–Lipschitz for every $i \in \mathbb{N}$.*

The proof of Lemma 2.10. We may assume $x, y \in 2B_i$, because $\psi_i = 0$ outside $2B_i$. If $\{x, y\} \cap 2B_j \neq \emptyset$, then $\frac{1}{2}r_j \leq r_i \leq 2r_j$. Because $\sum_{j\in\mathbb{N}} \hat{\psi}_j \geq 1$, we have

$$\begin{aligned} |\psi_i(x) - \psi_i(y)| &= \left| \frac{\hat{\psi}_i(x) \sum_{j\in\mathbb{N}} \hat{\psi}_j(y) - \hat{\psi}_i(y) \sum_{j\in\mathbb{N}} \hat{\psi}_j(x)}{\sum_{j\in\mathbb{N}} \hat{\psi}_j(y) \sum_{j\in\mathbb{N}} \hat{\psi}_j(x)} \right| \\ &\leq \left| \hat{\psi}_i(x) \sum_{j\in\mathbb{N}} \hat{\psi}_j(y) - \hat{\psi}_i(y) \sum_{j\in\mathbb{N}} \hat{\psi}_j(x) \right| \\ &= \left| \sum_{j\in\mathbb{N}} \left(\hat{\psi}_i(x) \left(\hat{\psi}_j(y) - \hat{\psi}_j(x) \right) + \hat{\psi}_j(x) \left(\hat{\psi}_i(x) - \hat{\psi}_i(y) \right) \right) \right|. \end{aligned}$$

And by (2.11) and triangle inequality, we conclude that

$$\begin{aligned}
&\left|\sum_{j\in\mathbb{N}}\left(\hat{\psi}_i(x)\left(\hat{\psi}_j(y)-\hat{\psi}_j(x)\right)+\hat{\psi}_j(x)\left(\hat{\psi}_i(x)-\hat{\psi}_i(y)\right)\right)\right| \\
\leq&\sum_{j\in\mathbb{N}}\left(\hat{\psi}_i(x)\left|\hat{\psi}_j(y)-\hat{\psi}_j(x)\right|+\hat{\psi}_j(x)\left|\hat{\psi}_i(x)-\hat{\psi}_i(y)\right|\right) \\
\leq&\sum_{\substack{j\in\mathbb{N},\\ 2B_i\cap 2B_j\neq\emptyset}}\left(\hat{\psi}_i(x)\frac{1}{r_j}+\hat{\psi}_j(x)\frac{1}{r_i}\right)d(x,y) \\
\leq&\frac{C}{r_i}d(x,y).
\end{aligned}$$

□

Now we define $f : X \longrightarrow \mathbb{R}$

$$f(x) := \begin{cases} \sum_{i=1}^{\infty} u_{B_i}\psi_i(x), & x \in U_\lambda, \\ u(x), & x \in X \setminus U_\lambda. \end{cases} \tag{2.14}$$

We prove that f satisfies the required conditions. We start by proving that f is $C\lambda$–Lipschitz, that is

$$|f(x) - f(y)| \leq C\lambda\, d(x, y) \tag{2.15}$$

for every $x, y \in X$.

By Proposition 2.7 equation (2.15) is satisfied when $x, y \in X \setminus U_\lambda$.

Next we consider the case where $x \in U_\lambda$ and $y \in X \setminus U_\lambda$. First we assume that $d(x, y) \leq 2\,\mathrm{dist}(x, X \setminus U_\lambda)$.

As $x \in U_\lambda$ there exists $B \in \mathcal{F}$ such that $x \in 2B = B(w, 2r)$ for some $w \in U_\lambda$ and $r > 0$. Because radius of the balls in $\mathcal{F}$ is $1/20$ of the distance of the center to the complement of U_λ, we have

$$\begin{aligned}
d(x, y) \leq& 2\,\mathrm{dist}(x, X \setminus U_\lambda) \leq 2 \cdot 22r, \\
d(x, y) \geq& \,\mathrm{dist}(x, X \setminus U_\lambda) \geq 18r,
\end{aligned}$$

that is

$$18r \leq d(x, y) \leq 44r.$$

Because $x \in B(w, 2r)$ and $d(x, y) \leq 44r$, we have $y \in B(w, 50r)$. By Proposition 2.7 we conclude

$$\begin{aligned}
&\left|u(y) - u_{B(w,r)}\right| \\
\leq &\left|u(y) - u_{B(w,50r)}\right| + \left|u_{B(w,50r)} - u_{B(w,r)}\right| \\
\leq &CrM^{\#}u(y) + C_\mu^6 \fint_{B(w,50r)} \left|u - u_{B(w,50r)}\right| \, \mathrm{d}\mu \\
\leq &CrM^{\#}(y) \\
\leq &C\lambda d(x, y).
\end{aligned}$$

And now we get the Lipschitz condition (2.15):

$$\begin{aligned}
&|f(x) - f(y)| \\
= &\left|\sum_{i=0}^{\infty} u_{B_i}\psi_i(x) - u(y)\right| \\
\leq &\sum_{i=0}^{\infty} \psi_i(x) \left|u_{B_i} - u(y)\right| \\
\leq &C\lambda d(x, y).
\end{aligned}$$

We have proved the case $d(x, y) \leq 2 \operatorname{dist}(x, X \setminus U_\lambda)$. Let us consider the case $d(x, y) > 2 \operatorname{dist}(x, X \setminus U_\lambda)$. There exists $z \in X \setminus U_\lambda$ such that $d(x, z) \leq 2 \operatorname{dist}(x, X \setminus U_\lambda)$. Hence $d(x, z) < d(x, y)$ and $d(z, y) < d(x, y) + d(x, z) < 2d(x, y)$, and it follows from the previous cases that

$$\begin{aligned}
|f(x) - f(y)| \leq &|f(x) - f(z)| + |f(z) - f(y)| \\
\leq &C\lambda(d(x, z) + d(z, y)) \\
\leq &C\lambda d(x, y).
\end{aligned}$$

It remains to prove the case $x, y \in U_\lambda$. We may assume that

$$\operatorname{dist}(y, X \setminus U_\lambda) \leq \operatorname{dist}(x, X \setminus U_\lambda) =: r.$$

Let us show that we may assume without loss of generality that

$$\operatorname{dist}(y, X \setminus U_\lambda) \geq \frac{r}{5} \quad \text{and} \tag{2.16}$$

$$d(x, y) \leq r. \tag{2.17}$$

If $\operatorname{dist}(y, X\setminus U_\lambda) < \frac{r}{5}$, then $d(x,y) \geq \operatorname{dist}(x, X\setminus U_\lambda) - \operatorname{dist}(y, X\setminus U_\lambda) \geq \frac{4}{5}r$ and there exists $z \in X\setminus U_\lambda$ such that $d(y,z) \leq \frac{2r}{5}$. Thus we find that

$$d(x,z) + d(z,y) \leq d(x,y) + 2d(y,z) \leq d(x,y) + \frac{4}{5}r \leq 2d(x,y).$$

If $d(x,y) > r$, we choose a point $z \in X\setminus U_\lambda$ so that $d(x,z) < 2r$. Then

$$d(x,z) + d(z,y) \leq 2d(x,z) + d(x,y) \leq 4r + d(x,y) \leq 5d(x,y).$$

Hence, in both cases, there exists $z \in X\setminus U_\lambda$ such that

$$d(x,z) + d(z,y) \leq 5d(x,y)$$

and the problem is reduced to the case where one of the points is in U_λ and the other in $X\setminus U_\lambda$. Thus we may assume that (2.16) and (2.17) are satisfied.

Let $B = B(x,2r)$ and $B_i = B(z,s) \in \mathcal{F}$ such that

$$B(z,2s) \cap \{x,y\} \neq \emptyset. \tag{2.18}$$

Then

$$18s \leq r \leq 110s, \tag{2.19}$$

$$\psi_i \text{ is } \frac{C}{r}\text{–Lipschitz, and} \tag{2.20}$$

$$B(z,s) \subset B. \tag{2.21}$$

Let $p \in \{x,y\} \cap B(z,2s)$. Now (2.19) follows from the fact that $\frac{r}{5} \leq \operatorname{dist}(p, X\setminus U_\lambda) \leq r$ and $18s \leq \operatorname{dist}(p, X\setminus U_\lambda) \leq 22s$. Equation (2.20) is a consequence of Lemma 2.10. Equation (2.21) is satisfied, because $3s < r$ and balls $B(z,2s)$ and $B(x,r)$ intersect.

Now let us fix any point $w \in B\setminus U_\lambda$. We have

$$\begin{aligned}
&|u_B - u_{B_i}| \\
&\leq \fint_{B_i} |u - u_B| \,\mathrm{d}\mu \\
&\leq C \fint_B |u - u_B| \,\mathrm{d}\mu \\
&\leq CrM^{\#}u(w) \\
&\leq C\lambda r.
\end{aligned} \tag{2.22}$$

The second inequality follows from (2.19) and (2.21). The third inequality is a consequence of the definition of the maximal operator and the last inequality is a consequence of the fact that $w \in X \setminus U_\lambda = \{x \in X \mid M^{\#}u(x) \leq \lambda\}$.

Now we can estimate $|f(x) - f(y)|$:

$$\begin{aligned}
|f(x) - f(y)| &= \left| \sum_{i=1}^{\infty} \psi_i(x) u_{B_i} - \sum_{i=1}^{\infty} \psi_i(y) u_{B_i} \right| \\
&= \left| \sum_{i=1}^{\infty} \psi_i(x) u_{B_i} - \sum_{i=1}^{\infty} \psi_i(x) u_B + \sum_{i=1}^{\infty} \psi_i(y) u_B - \sum_{i=1}^{\infty} \psi_i(y) u_{B_i} \right| \\
&= \left| \sum_{i=1}^{\infty} (\psi_i(x) - \psi_i(y))(u_{B_i} - u_B) \right| \\
&\leq C\,\frac{C}{r} d(x,y)\, C\lambda r \\
&\leq C\lambda d(x,y).
\end{aligned}$$

On the second line we have added and subtracted u_B, because $\sum_{i=1}^{\infty} \psi_i \equiv 1$ in U_λ. On the fourth line we used equations (2.11) and (2.22) and Lemma 2.10.

Now we have proved that f is $C\lambda$–Lipschitz. We still need to prove that the equation (2.10) is satisfied.

We fix a point $x \in X \setminus U_\lambda$ and assume that $M^{\#}f(x) > \delta > 0$. Then there exists a ball $B = B(y,r) \subset X$ such that $x \in B$ and

$$\fint_B |f - f_B| \,\mathrm{d}\mu > \delta r. \tag{2.23}$$

We would like to show that

$$\fint_{aB} |u - u_{aB}| \,\mathrm{d}\mu > C\delta r \tag{2.24}$$

for some $a \geq 1$ and $C > 0$. Notice that (2.23) and (2.24) are invariant under adding a constant to the functions. Therefore, without loss of generality, we may assume that $u_{5B} = 0$. Because $u = f$ in the set $X \setminus U_\lambda$, we may also assume that $B \cap U_\lambda \neq \emptyset$. Otherwise (2.24) is an easy consequence of (2.23).

We have

$$
\begin{aligned}
\int_B |f - f_B| \, \mathrm{d}\mu &\leq 2 \int_B |f| \, \mathrm{d}\mu \\
&\leq 2 \int_{B \setminus U_\lambda} |f| \, \mathrm{d}\mu + 2 \sum_{i=1}^{\infty} \int_B |\psi_i u_{B_i}| \, \mathrm{d}\mu \\
&\leq 2 \int_{B \setminus U_\lambda} |f| \, \mathrm{d}\mu + 2 \sum_{i=1}^{\infty} \int_{B \cap 2B_i} |u_{B_i}| \, \mathrm{d}\mu
\end{aligned} \tag{2.25}
$$

If $B(z, s) \in \mathcal{F}$ for some $z \in U_\lambda$, $s > 0$ and $B(z, 2s) \cap B \neq \emptyset$, we know that $s < r$. Thus $B(z, s) \subset 5B$ and

$$
\begin{aligned}
&2 \int_{B \setminus U_\lambda} |f| \, \mathrm{d}\mu + 2 \sum_{i=1}^{\infty} \int_{B \cap 2B_i} |u_{B_i}| \, \mathrm{d}\mu \\
=& 2 \int_{B \setminus U_\lambda} |f| \, \mathrm{d}\mu + 2 \sum_{i=1}^{\infty} \frac{\mu(B \cap 2B_i)}{\mu(B_i)} \int_{B_i} |u| \, \mathrm{d}\mu \\
\leq& 2 \int_{B \setminus U_\lambda} |f| \, \mathrm{d}\mu + C \sum_{\substack{i \in \mathbb{N} \\ 2B_i \cap B \neq \emptyset}} \int_{B_i} |u| \, \mathrm{d}\mu \\
\leq& C \int_{5B} |u| \, \mathrm{d}\mu.
\end{aligned} \tag{2.26}
$$

Here we used (2.11).

Because we assumed that $u_{5B} = 0$ and because μ is doubling, we have

$$
\fint_B |f - f_B| \, \mathrm{d}\mu \leq C \fint_{5B} |u - u_{5B}| \, \mathrm{d}\mu,
$$

and thus (2.10) is proved. $\square$

2.4 Sobolev spaces

The classical theory of Sobolev spaces is based on the notion of distributional derivatives, which are defined via integration by parts. In general metric spaces, an alternative way of defining Sobolev spaces is needed. There are various approaches to define Sobolev type spaces in metric spaces. First, there is an approach due to Hajłasz [7]. It is based on the fact that for $p > 1$, a p-integrable function in $\mathbb{R}^n$ is in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if and only if there exists a nonnegative p-integrable function g such that for almost all points $x, y \in \mathbb{R}^n$,

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)). \tag{2.27}$$

This definition can be easily adapted to the metric space setting and it can be used to define first order Sobolev type spaces.

Definition 2.11. Let (X, d, μ) be a metric measure space. For $1 \leq p \leq \infty$, the *Hajłasz space* $M^{1,p}(X)$ is the collection of L^p-equivalence classes of functions u that together with p-integrable function g satisfy equation (2.27).

Second, there is a geometric approach via the notion of an upper gradient, see Shanmugalingam [22].

Definition 2.12. Let $1 \leq p < \infty$. For a p-integrable function u, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, \mathrm{d}\mu + \inf_g \int_X g^p \, \mathrm{d}\mu \right)^{\frac{1}{p}},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u \, : \, \|u\|_{N^{1,p}(X)} < \infty\} / \sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

These definitions yield the same spaces when $1 < p < \infty$. In the definition of $M^{1,p}(X)$, the function g corresponds to the Hardy–Littlewood-maximal function of an upper gradient. When $p > 1$, the maximal function of an upper gradient is in $L^p(X)$ if and only if the upper gradient is in $L^p(X)$. However, that is not true if $p = 1$, and in this case, the definitions give different spaces.

2.5 Sobolev–Poincaré inequality

Most of this work concentrates on geometric questions concerning the Poincaré inequality. In this section, we will also consider some analytic consequences of the Poincaré inequality. We will prove that a weak $(1, p)$–Poincaré inequality implies a weak (t, p)–Poincaré inequality for some $t > q$. The proof follows the presentation in Heinonen [10]. For the proof, we will need the following definition:

Definition 2.13 (Chain condition). Let $\lambda \geq 1$ and $M \geq 1$. Then a bounded subset A of a metric space X is said to satisfy a (λ, M)–*chain condition*, if there exists a ball B_0 such that for each $x \in A$ there exists a sequence of balls $\{B_i : i = 1, 2, \ldots\}$, satisfying the following conditions:

(i) $\lambda B_i \subset A$ for every $i \geq 0$,

(ii) the center of B_i is x for all sufficiently large i,

(iii) the radius r_i of B_i satisfies

$$M^{-1}2^{-i}\operatorname{diam}(A) \leq r_i \leq M2^{-i}\operatorname{diam}(A)$$

for every $i \geq 0$, and

(iv) the intersection $B_i \cap B_{i+1}$ contains a ball R_i such that $B_i \cup B_{i+1} \subset MR_i$ for every $i \geq 0$.

Theorem 2.14. *Let (X, d, μ) be a metric space with a doubling measure μ and $A \subset X$ bounded subset satisfying a (λ, M)–chain condition. Suppose that $u \in L^1_{loc}(A)$ and $\rho > 0$, $\rho \in L^p(A)$, $p > 0$, satisfy the Poincaré inequality* (2.2) *for every ball B, with $\lambda B \subset A$. Then there exists $k > 1$, that depends only on the doubling constant C_μ and p, such that the following Sobolev inequality is satisfied*

$$\left(\fint_A |u - u_A|^{kp}\, d\mu\right)^{\frac{1}{kp}} \leq C\operatorname{diam}(A)\left(\fint_A \rho^p\, d\mu\right)^{\frac{1}{p}}. \tag{2.28}$$

If $kp < 1$, u_A is replaced by u_{B_0}. The constant C depends on all parameters related to the chain condition, doubling and the Poincaré inequality.

Remark 2.15. In (2.28) one can choose k so that kp is the Sobolev conjugate of p that is $kp = ps/(p - s)$, where $s = \log_2 C_\mu$, see Heinonen [10].

By Theorem 3.1, every doubling metric measure space supporting a Poincaré inequality is bi–Lipschitz equivalent to a geodesic space. In geodesic spaces all balls satisfy the chain condition 2.13. Now by Remark 2.3 and Theorem 2.14, we can conclude that $(1, p)$–Poincaré inequality implies a weak (kp, p)–Poincaré inequality.

We need the following technical lemma to prove Theorem 2.14.

Lemma 2.16. *Let* (X,μ) *be a measure space and* $u : X \longrightarrow \mathbb{R}$ *a measurable function. If* $s > 1$ *and*

$$\mu(\{x \in X \,:\, |u(x)| > t\}) \le C_0 t^{-s}$$

for every $t > 0$*, then we have*

$$\left(\fint_X |u|^q \, d\mu\right)^{\frac{1}{q}} \le 2^{\frac{1}{q}} \left(\frac{C_0 q}{s-q}\right)^{\frac{1}{s}} \mu(X)^{-\frac{1}{s}}.$$

for all $q < s$.

Proof. Fix $t_0 > 0$. Then

$$\begin{aligned} \mu(\{|u| > t\}) &\le \mu(X), \quad \text{if } t \le t_0, \\ \mu(\{|u| > t\}) &\le C_0 t^{-q}, \quad \text{if } t > t_0, \end{aligned}$$

and thus we have

$$\begin{aligned} \int_X |u|^q \,\mathrm{d}\mu &= q \int_0^\infty t^{q-1} \mu(\{|u| > t\}) \,\mathrm{d}t \\ &\le q \int_0^{t_0} t^{q-1} \mu(X) \,\mathrm{d}t + q C_0 \int_{t_0}^\infty t^{q-s-1} \,\mathrm{d}t \\ &= t_0^q \mu(X) + \frac{q C_0}{s-q} t_0^{q-s}. \end{aligned}$$

The claim follows now by choosing $t_0 = \left(\frac{C_0 q}{s-q}\right)^{\frac{1}{s}} \mu(X)^{-\frac{1}{s}}$. □

The proof of Theorem 2.14 in the case $kp \ge 1$. We may assume that $u_{B_0} = 0$. We need to estimate the integral

$$\left(\fint_A |u - u_A|^{kp} \,\mathrm{d}\mu\right)^{\frac{1}{kp}}.$$

Let $x \in A_t := \{x \in A \,:\, |u(x)| > t\}$ be a Lebesgue point of u and let $B_0, B_1, B_2, \ldots$ be a sequence of balls related to x as in Definition 2.13. As x is a Lebesgue point, we have

$$|u_{B_j}| \longrightarrow |u(x)| > t.$$

By the doubling property and the triangle inequality, we have

$$\begin{aligned} t &\le |u(x) - u_{B_0}| \\ &\le \sum_{i=0}^{\infty} \left(|u_{B_i} - u_{R_i}| + |u_{B_{i+1}} - u_{R_i}| \right) \\ &\le \sum_{i=0}^{\infty} \left(\fint_{R_i} |u - u_{B_i}| \, \mathrm{d}\mu + \fint_{R_i} |u - u_{B_{i+1}}| \, \mathrm{d}\mu \right). \end{aligned}$$

Now it follows from the Poincaré inequality that

$$\begin{aligned} t &\le \sum_{i=0}^{\infty} \left(\fint_{R_i} |u - u_{B_i}| \, \mathrm{d}\mu + \fint_{R_i} |u - u_{B_{i+1}}| \, \mathrm{d}\mu \right) \\ &\le C \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}| \, \mathrm{d}\mu \\ &\le C \sum_{i=0}^{\infty} r_i \left(\fint_{\lambda B_i} \rho^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}. \end{aligned}$$

Given $0 < \varepsilon < 1$, we have

$$\sum_{i=0}^{\infty} r_i \left(\fint_{\lambda B_i} \rho^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \ge Ct = Ct \sum_{i=0}^{\infty} 2^{-i\varepsilon} \ge Ct(\mathrm{diam}(A))^{-\varepsilon} \sum_{i=0}^{\infty} r_i^{\varepsilon}.$$

Clearly, in the sum on the left-hand side, there must be at least one term that is equal to, or greater than, the corresponding term in the sum on the right-hand side, that is there exists $j \in \mathbb{N}$ such that

$$r_j \left(\fint_{\lambda B_j} \rho^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \ge Ct(\mathrm{diam}(A))^{-\varepsilon} r_j^{\varepsilon}.$$

If we combine this with the fact that there exists a constant C that is independent of j such that $\lambda B_j \subset B(x, Cr_j)$, and use the doubling property, we may conclude that there exists $r_x > 0$ such that

$$(\mathrm{diam}(A))^{p\varepsilon} \int_{B(x,r_x)\cap A} \rho^p \, \mathrm{d}\mu \ge Ct^p r_x^{p(\varepsilon-1)} \mu(B(x, r_x)). \tag{2.29}$$

It follows from the doubling condition that

$$\mu(B(x,r)) \ge 2^{-\delta} \mu(A) \left(\frac{r}{\mathrm{diam}(A)} \right)^{\delta} \tag{2.30}$$

for every $\delta \geq \delta_0 := \log_2 C_d$, $x \in A$ and $r \leq \operatorname{diam}(A)$. As $\varepsilon < 1$, we have $p(\varepsilon - 1) < 0$, and there fore it follows from the equation (2.29) that

$$r_x^{p(\varepsilon-1)} \geq C\mu(B(x,r_x))^{\frac{p(\varepsilon-1)}{\delta}}\mu(A)^{\frac{-p(\varepsilon-1)}{\delta}} \operatorname{diam}(A)^{p(\varepsilon-1)}.$$

If we combine this with equation (2.30), we get

$$Ct^p\mu(B(x,r_x))^{1+\frac{p(\varepsilon-1)}{\delta}} \leq \mu(A)^{\frac{p(\varepsilon-1)}{\delta}} \operatorname{diam}(A)^p \int_{B(x,r_x)\cap A} \rho^p \,\mathrm{d}\mu. \tag{2.31}$$

By Theorem 2.1 there exists a collection of disjointed balls $B_{x_i} = B(x_i, r_{x_i})$, $i = 1, 2, \dots$ satisfying

$$A_t \subset \bigcup_{i=1}^{\infty} 5B_{x_i}.$$

We may assume that $\delta > p$. Then $0 < \frac{\delta+p(\varepsilon-1)}{\delta} < 1$. Now it follows from the doubling and equation (2.31) that

$$\begin{aligned} \mu(A_t)^{1+\frac{p(\varepsilon-1)}{\delta}} &\leq C\sum_{i=1}^{\infty} \mu(B_{x_i})^{1+\frac{p(\varepsilon-1)}{\delta}} \\ &\leq Ct^{-p}L\sum_{i=1}^{\infty}\int_{B_{x_i}\cap A} \rho^p \,\mathrm{d}\mu \\ &\leq Ct^{-p}L\int_A \rho^p \,\mathrm{d}\mu, \end{aligned}$$

where

$$L = (\operatorname{diam}(A))^p\mu(A)^{\frac{p(1-\varepsilon)}{\delta}},$$

and consequently

$$\mu(A_t) \leq CL^{\frac{\delta}{\delta+p(\varepsilon-1)}}t^{\frac{-\delta p}{\delta+p(\varepsilon-1)}}\left(\int_A \rho^p \,\mathrm{d}\mu\right)^{\frac{\delta}{\delta+p(\varepsilon-1)}}.$$

For every $1 \leq q < \frac{\delta p}{\delta - p}$ there exists $\varepsilon > 0$ such that $q < \frac{\delta p}{\delta+p(\varepsilon-1)}$. Thus, by Lemma 2.16, we conclude

$$\left(\fint_A |u|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} \leq C\operatorname{diam}(A)\left(\fint_A |\rho|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Note that C depends on q.

$\square$

3 Quasiconvexity of annuli

In this section, we prove Theorem 3.4, which is the main result of this thesis. It is shown that when the space supports a $(1,p)$–Poincaré inequality with p sufficiently small, then all annuli are almost quasiconvex. We will also present examples demonstrating that any weaker Poincaré inequality is not enough to guarantee the quasiconvexity of annuli.

Theorem 3.1 concerning the quasiconvexity of the space is needed to prove the main theorem. We start with a proof of Theorem 3.1. See also Semmes [21], Cheeger [5] and Keith [14].

Theorem 3.1. *Suppose that (X,d,μ) is a complete metric measure space with μ a doubling measure. If X supports a weak $(1,p)$–Poincaré inequality for some $1 \leq p < \infty$, then X is quasiconvex with a constant depending only on the constants of the Poincaré inequality and the doubling constant.*

Proof. Let $\varepsilon > 0$. We say that $x, z \in X$ lie in the same ε–component of X if there exists a finite chain $z_0, z_1, \ldots, z_N$ such that

$$\begin{aligned} &z_0 = x, \\ &z_N = z \text{ and} \\ &d(z_i, z_{i+1}) \leq \varepsilon \text{ for all } i = 0, \ldots, N-1. \end{aligned} \tag{3.1}$$

Clearly, lying in the same ε–component defines an equivalence relation, and the distance between two different ε–components is at least ε. If x and y lie in different ε–components, then it is obvious that there does not exist a rectifiable curve joining x and y. Thus, the function $g \equiv 0$ is an upper gradient for the characteristic function of any component. By applying the $(1,p)$–Poincaré inequality to the characteristic function of any component, it follows that all the points of X lie in the same ε–component. Hence for every $\varepsilon > 0$ the space X consists of only one ε–component.

Let us fix $x, y \in X$ and prove that there exists a curve γ joining x and y with length at most $Cd(x,y)$, where C depends only on the doubling constant and the constants in the Poincaré inequality. We define the ε–distance of x and z to be

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite chains $\{z_i\}$ satisfying (3.1). Note that

$$\rho_{x,\varepsilon}(z) < \infty$$

for all $x, z \in X$. If $d(z, w) \le \varepsilon$, then

$$|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \le d(z, w).$$

Clearly, for all $\varepsilon > 0$, the function $g \equiv 1$ is an upper gradient of $\rho_{x,\varepsilon}$. Thus by Proposition 2.7, we have

$$\begin{aligned}
\rho_{x,\varepsilon}(y) &= |\rho_{x,\varepsilon}(x) - \rho_{x,\varepsilon}(y)| \\
&\le Cd(x, y)\left(M^{\#}\rho_{x,\varepsilon}(x) + M^{\#}\rho_{x,\varepsilon}(y)\right) \\
&\le Cd(x, y) \sup_{z\in X} M^{\#}\rho_{x,\varepsilon}(z),
\end{aligned}$$

and now it follows from the Poincaré inequality that

$$\begin{aligned}
\rho_{x,\varepsilon}(y) &\le Cd(x, y) \sup_{B} \left(\fint_B g^p \,\mathrm{d}\mu\right)^{1/p} \\
&\le Cd(x, y).
\end{aligned}$$

Note that C does not depend on ε.

Now we take a sequence $\varepsilon_j \to 0$. For every ε_j, there exists a chain $z_{j,0} = x, \dots, z_{j,N_j} = y$ such that $d(z_{j,i}, z_{j,i+1}) \le \varepsilon_j$ for all $i = 0, \dots, j_{N_j} - 1$.

Let

$$s_{j,i} = \sum_{k=0}^{i} d(z_{j,k}, z_{j,k+1}).$$

We define mappings

$$\gamma_j : [0, Cd(x, y)] \longrightarrow \{z_{j,0}, \dots, z_{j,N_j}\}$$

so that

$$\begin{aligned}
\gamma_j\left([s_{j,i-1}, s_{j,i})\right) &= z_{j,i}, \quad \text{if} \quad i = 0, \dots, N_j - 1 \text{ and} \\
\gamma(t) &= y, \quad \text{if} \quad t \ge s_{j,N_j-1}.
\end{aligned} \tag{3.2}$$

Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of $[0, Cd(x, y)]$. We define $\gamma_{0,i} = \gamma_i$ and choose $\{\gamma_{j,i}\}_i \subset \{\gamma_{j-1,i}\}_i$ for every j so that

$$\lim_{i\to\infty} \gamma_{j,i}(a_j) = x_j$$

for some $x_j \in X$. Because bounded and closed sets are compact in X, such a subsequence exists for every j.

Now we can define $\widetilde{\gamma} : \{a_1, a_2, \dots\} \to X$ so that

$$\widetilde{\gamma}(a_j) = x_j$$

for every $j \in \mathbb{N}$.

Let us prove that $\tilde{\gamma}$ is 1–Lipschitz:

Let us denote the values of $\gamma_{i,i}$ by $w_{i_0}, w_{i_1}, \dots, w_{i_{M_i}}$ and let

$$b_{i_j} := \min\{r \in [0, Cd(x, y)] : \gamma_{i,i}(r) = w_{i_j}\}.$$

Let us fix $i, j \in \mathbb{N}$ so that $a_i < a_j$.

Let $\varepsilon > 0$. Because

$$\begin{aligned} x_i &= \lim_{n\to\infty} \gamma_{n,n}(a_i) \quad \text{and} \\ x_j &= \lim_{n\to\infty} \gamma_{n,n}(a_j), \end{aligned}$$

there exists N_ε such that

$$\begin{aligned} &\gamma_{n,n}(a_i) \in B(x_i, \varepsilon) \quad \text{and} \\ &\gamma_{n,n}(a_j) \in B(x_j, \varepsilon) \end{aligned}$$

for every $n > N_\varepsilon$.

We may choose N_ε to be sufficiently big so that $\varepsilon_{N_\varepsilon} < \varepsilon$.

For every $n > N_\varepsilon$, there exist i' and j' such that

$$\begin{aligned} \gamma_{n,n}(a_i) =& \gamma_{n,n}(b_{n_{i'}}) \quad \text{and} \\ \gamma_{n,n}(a_j) =& \gamma_{n,n}(b_{n_{j'}}). \end{aligned}$$

It follows from the parametrization of $\gamma_{i,i}$ and the choice of N_ε, that

$$\begin{aligned} &|b_{n_{i'}} - a_i| < \varepsilon \quad \text{and} \\ &|b_{n_{j'}} - a_j| < \varepsilon. \end{aligned}$$

Let us fix $n > N_\varepsilon$. Then

$$\begin{aligned} d(\tilde{\gamma}(a_i), \tilde{\gamma}(a_j)) &= d(x_i, x_j) \\ &\leq d(\gamma_{n,n}(a_i), \gamma_{n,n}(a_j)) + d(\gamma_{n,n}(a_i), x_i) + d(\gamma_{n,n}(a_j), x_j) \\ &= d\left(\gamma_{n,n}\left(b_{n_{i'}}\right), \gamma_{n,n}\left(b_{n_{j'}}\right)\right) + 2\varepsilon, \end{aligned}$$

and

$$\begin{aligned} d\left(\gamma_{n,n}\left(b_{n_{i'}}\right), \gamma_{n,n}\left(b_{n_{j'}}\right)\right) &\leq \left|b_{n_{i'}} - b_{n_{j'}}\right| \\ &\leq \left|b_{n_{i'}} - a_i\right| + |a_i - a_j| + \left|a_j - b_{n_{j'}}\right| \\ &\leq |a_i - a_j| + 2\varepsilon. \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, it follows that

$$d(\tilde{\gamma}(a_i), \tilde{\gamma}(a_j)) = |x_i - x_j| \leq |a_i - a_j|.$$

Thus $\tilde{\gamma}$ is 1–Lipschitz continuous.

Because $\{a_i\}_i$ is dense and X is complete, there exists a unique 1–Lipschitz extension γ for $\tilde{\gamma}$ to the set $[0, Cd(x,y)]$. Hence γ is a curve connecting x and y, and its length is at most $Cd(x,y)$. As x and y were arbitrary points in X, this proves that X is quasiconvex.

□

Remark 3.2. Quasiconvexity is almost as good as *geodecity*, which means that every pair of points can be joint with a curve whose length is the distance of the points: if the space (X,d) is quasiconvex, complete and doubling, we can define a new metric

$$d'(x,y) = \inf\{l(\gamma) \,:\, \gamma(0) = x, \gamma(1) = y\}.$$

The new metric d' is geodesic and bi–Lipschitz equivalent to the original metrid d. For example the Poincaré inequality and doubling are invariant under bi–Lipschitz mappings, see Remark 2.3.

The following result is a modification of Theorem 5.9 in Heinonen–Koskela [11].

Lemma 3.3. *Let (X,d,μ) be a metric measure space. Suppose that the measure μ is doubling and that X supports a weak $(1,p)$–Poincaré inequality for some*

$1 \leq p < \infty$. *Let E and F be two compact subsets of a ball $B(z, R) \subset X$ and assume that for some $0 < \kappa \leq 1$, we have*

$$\min\{\mu(E), \mu(F)\} \geq \kappa\mu(B(z, R)). \tag{3.3}$$

Then there is a constant $C \geq 1$ so that

$$\int_{B(z,10\lambda R)} g^p \, d\mu \geq C^{-1}\kappa\mu(B(z, R))R^{-p},$$

whenever u is a continuous function in the ball $B(z, 10\lambda R)$ with $u_{|E} \leq 0$ and $u_{|F} \geq 1$, and g is an upper gradient of u in B. Here, λ is the same constant that appears in the weak Poincaré inequlity (2.2).

Proof. Let u be a continuous function in the ball $B(z, 10\lambda R)$. Assume that $u_{|E} \leq 0$ and $u_{|F} \geq 1$, and let g be an upper gradient of u in $B(z, 10\lambda R)$.

The proof splits into two cases depending on whether or not there are points x in E and y in F so that neither

$$|u(x) - u_{B(x,R)}|$$

nor

$$|u(y) - u_{B(y,R)}|$$

exceeds $1/5$. If such points can be found, then

$$1 \leq |u(x) - u(y)| \leq 1/5 + |u_{B(x,R)} - u_{B(y,R)}| + 1/5,$$

and hence

$$1 \leq C \fint_{B(y,5R)} |u - u_{B(y,5R)}| \, \mathrm{d}\mu \leq CR \left(\fint_{B(z,10\lambda R)} g^p \, \mathrm{d}\mu \right)^{1/p}$$

from which the claim follows. Note that $B(x, R) \subset B(y, 5R) \subset B(z, 10R)$.

By symmetry, the second alternative is that for all points x in E we have that

$$1/5 \leq |u(x) - u_{B(x,R)}|.$$

Because u is continuous, and hence x is a Lebesgue point of u,

$$\begin{aligned}1/5 &\leq \sum_{j=0}^{\infty} |u_{B(x,2^{-j}R)} - u_{B(x,2^{-j-1}R)}| \\ &\leq C \sum_{j=0}^{\infty} \fint_{B(x,2^{-j}R)} |u - u_{B(x,2^{-j}R)}| \,\mathrm{d}\mu \\ &\leq C \sum_{j=0}^{\infty} (2^{-j}R) \left(\fint_{B(x,\lambda 2^{-j}R)} g^p \,\mathrm{d}\mu \right)^{1/p}.\end{aligned}$$

Hence there exists j_x such that

$$C\, 2^{-j_x} R \left(\fint_{B(x,\lambda 2^{-j_x}R)} g^p \,\mathrm{d}\mu \right)^{1/p} \geq 2^{-j_x}.$$

By Theorem 2.1 and the fact that X is doubling, we find a pairwise disjoint collection of balls of the form $B(x_k, \lambda r_k)$ with $r_k = 2^{-j_{x_k}} R$ such that

$$E \subset \bigcup_k B(x_k, 5\lambda r_k)$$

and

$$C \int_{B(x_k,\lambda r_k)} g^p \,\mathrm{d}\mu \geq \mu(B(x_k, \lambda r_k)) R^{-p}. \tag{3.4}$$

Hence using equations (3.4) and (3.3) we get

$$\begin{aligned}\int_{B(z,10\lambda R)} g^p \,\mathrm{d}\mu &\geq \sum_{k=1}^{\infty} \int_{B(x_k,\lambda r_k)} g^p \,\mathrm{d}\mu \\ &\geq (1/C) \sum_{k=1}^{\infty} \mu(B(x_k, \lambda r_k)) R^{-p} \\ &\geq (1/C) \sum_{k=1}^{\infty} \mu(B(x_k, 5\lambda r_k)) R^{-p} \\ &\geq (1/C) \mu(E) R^{-p} \geq (\kappa/C) \mu(B(z, R)) R^{-p}\end{aligned}$$

as desired. This completes the proof. □

The following theorem is our main result.

Theorem 3.4. *Let (X, d, μ) be a complete metric measure space with a doubling measure μ that satisfies*

$$\frac{\mu(B(x,r))}{\mu(B(x,R))} \le C\left(\frac{r}{R}\right)^Q$$

for every $x \in X$ and $0 < r < R$ for some $Q > 1$. If X supports a weak $(1,p)$–Poincaré inequality for some $p \le Q$, then there exists a constant $C > 1$ such that for all $z \in X$, $r > 0$, every pair of points in $B(z,r) \setminus B(z,r/2)$ can be joined in $B(z,Cr) \setminus B(z,r/C)$ with a curve whose length is at most C times the distance between the points.

Proof. Fix a ball $B = B(z,r)$ and points $x, y \in B(z,r) \setminus B(z,r/2)$. By Theorem 3.1 X is quasiconvex. Hence there exists a curve connecting x and y whose length is at most $C_q d(x,y)$. If $d(x,y) < r/2C_q$, the shortest curve connecting x and y in X cannot intersect $B(z,r/4)$ or leave $B(z,2r)$. So we may assume that

$$d(x,y) \ge \frac{r}{2C_q}.$$

Consider the sets

$$E = \overline{B}(x,ar) \quad \text{and} \quad F = \overline{B}(y,ar).$$

If we choose $a = 1/(8C_q)$, then $\operatorname{dist}(E,F)$ is comparable to r, and from each point of E and F there exists a curve in the annulus connecting it to x and y respectively with length no more than r. So it is enough to prove that the sets E and F can be connected with a curve in $B(z,Cr) \setminus B(z,r/C)$ with length at most Cr for some uniform constant $C > 1$.

Let Γ be the family of rectifiable curves joining E and F and not leaving the ball $B(z,10\lambda r)$. Given $A > 1$, Γ_1^A is the subset of Γ consisting of all the curves intersecting $B(z,r/A)$ and Γ_2^A is the subset of Γ consisting of all the curves not intersecting $B(z,r/A)$.

Because the measure μ is doubling, it follows that

$$\min\{\mu(E),\mu(F)\} \ge \frac{1}{C}\mu(B(z,r)).$$

By Lemma 3.3 with $p = Q$,

$$\begin{aligned} \operatorname{mod}_Q(\Gamma) &= \operatorname{cap}_Q(E, F, B(z, 10\lambda r)) \\ &= \inf_g \int_{B(z,10\lambda r)} g^Q \, \mathrm{d}\mu \\ &\geq \frac{1}{C_0} \mu(B(z, r)) r^{-Q}, \end{aligned}$$

where the infimum is taken over all functions g that satisfy the conditions of Lemma 3.3. The constant C_0 is independent of r. On the other hand,

$$\operatorname{mod}_Q(\Gamma_1^A) \leq \operatorname{mod}_Q(B(z, r/A), X \setminus B(z, r/2))$$

and we can control the right–hand side of the above inequality using the following admissible metric

$$\rho(x) = (|z - x| \log_2(A/2))^{-1} \chi_{B(z,r/2)\setminus B(z,r/A)}(x).$$

Without loss of generality, we may assume that $\log_2 A$ is an integer. By using

$$B(z, r/2) \setminus B(z, r/A) = \bigcup_{j=1}^{\log_2(A/2)} B(z, 2^{-j}r) \setminus B(z, 2^{-j-1}r),$$

we have

$$\begin{aligned} \operatorname{mod}_Q(\Gamma_1^A) \leq & \int_X \rho^Q \, \mathrm{d}\mu \\ \leq & (\log_2(A/2))^{-Q} \sum_{j=1}^{\log_2(A/2)} \mu(B(z, 2^{-j}r))(2^{-j}r)^{-Q} \\ \leq & C(\log_2(A/2))^{-Q} \mu(B(z, r)) r^{-Q} \sum_{j=1}^{\log_2(A/2)} (2^{-j})^Q 2^{jQ} \\ = & C(\log_2(A/2))^{1-Q} \mu(B(z, r)) r^{-Q}. \end{aligned}$$

Because $1 - Q < 0$, by choosing

$$A \geq \exp\left((4C_0)^{1/(Q-1)}\right),$$

then $\operatorname{mod}_Q(\Gamma_1^A)$ is small compared to $\operatorname{mod}_Q(\Gamma)$. Hence

$$\operatorname{mod}_Q(\Gamma_2^A) \geq \operatorname{mod}_Q(\Gamma) - \operatorname{mod}_Q(\Gamma_1^A) \geq \frac{1}{2C_0} \mu(B(z, r)) r^{-Q}.$$

Let

$$\Gamma_2^{A,L} = \{\gamma \in \Gamma_2^A \,:\, l(\gamma) > Lr\}.$$

Then $\rho = 1/(Lr)\chi_{B(z,10\lambda r)}$ is an admissible metric for $\Gamma_2^{A,L}$, and hence we have that

$$\begin{aligned}\operatorname{mod}_Q(\Gamma_2^{A,L}) &\leq \int_X \rho^Q \,\mathrm{d}\mu = \mu(B(z,10\lambda r))r^{-Q} \cdot L^{-Q} \\ &\leq C\mu(B(z,r))r^{-Q} \cdot L^{-Q}\end{aligned}$$

If $L \geq (4C_0C)^{1/Q}$, then

$$\operatorname{mod}_Q(\Gamma_2^A \setminus \Gamma_2^{A,L}) \geq \frac{1}{4C_0}\mu(B(z,r))r^{-Q},$$

and hence there exists a curve connecting E and F in $B(z,10\lambda r) \setminus B(z,r/A)$, with length less than Lr. This completes the proof. $\square$

The following theorem shows that Theorem 3.4 does not hold if $p > Q$, because in that case, the modulus of a curve family going through one point may be positive. If we have two metric measure spaces (X_1, d_1, μ_1) and (X_2, d_2, μ_2) that satisfy the $(1,p)$–Poincaré inequality, and we glue X_1 and X_2 together by identifying points $x_1 \in X_1$ and $x_2 \in X_2$ with positive p–capacity, we get a space that supports the $(1,p)$–Poincaré inequality but where the annuli around the point x_1 are disconnected.

Theorem 3.5. *Let (X,d,μ) be a metric measure space with a doubling measure μ. If the local growth bound*

$$Q(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

at a point x_0 is strictly less than p and the space supports $(1,p)$–Poincaré inequality, then there exists r_{x_0} such that

$$\operatorname{cap}_p(\{x_0\}, X \setminus B(x_0, r_{x_0})) > 0.$$

On the other hand, if $p < Q(x_0)$, then

$$\operatorname{cap}_p(\{x_0\}, X \setminus B(x_0, r)) = 0$$

for all $r > 0$.

Proof. If $p > q$ where $q = Q(x_0)$, then there exists $0 < t_{x_0} < \min\{1, \operatorname{diam}(X)/3\}$ such that for all $r \le t_{x_0}$ we have

$$\mu(B(x_0, r)) \ge r^s, \tag{3.5}$$

where $s = (p+q)/2$. Let u be a continuous function such that $u(x_0) = 1$ and $u = 0$ in the complement of $B(x_0, r_{x_0})$, where $0 < r_{x_0} < t_{x_0}$ is chosen so that $\mu(B(x_0, r_{x_0})) \le \mu(B(x_0, t_{x_0}))/2$, and let g be an upper gradient of u. Let $r_i = 2^{-i} t_{x_0}$ and $B_i = B(x_0, r_i)$. Because X supports a $(1,p)$–Poincaré inequality and x_0 is a Lebesgue point, we have

$$\begin{aligned}
1/2 \le |u(x_0) - u_{B_0}| \le & \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\
\le & C \sum_{i=0}^{\infty} r_i \left(\frac{1}{\mu(B_i)} \int_{B(x_0, \lambda r_i)} g^p \, \mathrm{d}\mu \right)^{1/p} \\
\le & C \sum_{i=0}^{\infty} r_i \left(r_i^{-s} \int_{B(x_0, \lambda r_i)} g^p \, \mathrm{d}\mu \right)^{1/p} \\
\le & C \sum_{i=0}^{\infty} r_i^{1-s/p} \left(\int_{B(x_0, \lambda t_{x_0})} g^p \, \mathrm{d}\mu \right)^{1/p} \\
\le & C t_{x_0}^{1-s/p} \left(\int_{B(x_0, \lambda t_{x_0})} g^p \, \mathrm{d}\mu \right)^{1/p} .
\end{aligned}$$

Here we used also equation (3.5) and the fact that μ is a doubling measure.

It follows that

$$\int_{B(x_0, \lambda t_{x_0})} g^p \, \mathrm{d}\mu \ge C t_{x_0}^{s-p} \ge C$$

and we have a lower bound for the capacity.

If $1 < p < q$, then there exist a real number $p < q_0 < q$ and a sequence of positive numbers $s_i \to 0$ such that

$$\mu(B(x_0, s_i)) \le s_i^{q_0}.$$

Then $\mu(B(x_0, r)) \le r^p$ for all $s_i^{q_0/p} < r < s_i$ and by a standard way of estimating the capacity of an annulus we get

$$\begin{aligned}
\operatorname{cap}_p(\{x_0\}, X \setminus B(x_0, 1)) &\le \operatorname{cap}_p(B(x_0, s_i^{q_0/p}), X \setminus B(x_0, s_i)) \\
&\le C(-\log s_i)^{1-p},
\end{aligned}$$

which converges to 0 as $s_i \to 0$. $\square$

The following example shows that the condition $Q > 1$ is necessary in Theorem 3.4.

Example 3.6. Let $X = \mathbb{R}$ with euclidean metric and Lebesgue measure. Then X supports a $(1, 1)$–Poincaré inequality and satisfies the mass bound with $Q = 1$, but all the annuli are disconnected and hence they are not quasiconvex.

4 Size of spheres

In this section, we complement the main result by proving that if the space satisfies the assumptions of Theorem 3.4, we obtain lower bounds for the Hausdorff s–content and the diameter of the spheres. The proofs are based on methods similar to that in Theorem 3.4.

Theorem 4.1. *Let (X, d, μ) satisfy the same assumptions as in Theorem 3.4. Then there exists $c > 0$ such that if $3r \leq \operatorname{diam} X$, then*

$$\operatorname{diam}(\{x \in X \,:\, d(x, x_0) = r\}) \geq cr$$

for every $x_0 \in X$.

Proof. Fix $x_0 \in X$ and $r \leq \operatorname{diam}(X)/3$. Let

$$G = \{x \in X \,:\, d(x, x_0) = r\}.$$

Fix $z \in G$ and $0 < a$ such that $G \subset B(z, ar)$. Suppose that $a < 1/4$. Let

$$E = \overline{B}(x_0, r/2) \quad \text{and} \quad F = X \setminus B(x_0, r).$$

By the Poincaré inequality, as $r \leq \frac{1}{3}\operatorname{diam}(X)$, we have $B(x_0, 2r) \setminus B(x_0, 3r/2)$ nonempty. Because the measure μ is doubling, there exists $\lambda = \lambda(C_\mu) > 0$ such that

$$\min\{\mu(E), \mu(F \cap \overline{B}(x_0, 2r))\} \geq \lambda\mu(B(x_0, 3r)).$$

By Lemma 3.3 we have

$$\operatorname{cap}_Q(E, F) = \operatorname{cap}_Q(E, F \cap \overline{B}(x_0, 2r), B(x_0, 3r)) \geq \frac{1}{C_1}\mu(B(x_0, r))r^{-Q}.$$

On the other hand, we can estimate $\operatorname{cap}_Q(X \setminus B(z, r/2), B(z, ar))$ in the same way as in Theorem 3.4 and obtain

$$\begin{aligned}\operatorname{cap}_Q(E, F) =& \operatorname{cap}_Q(E, G)\\ \leq& \operatorname{cap}_Q(X \setminus B(z, r/4), B(z, ar))\\ \leq& C_2(\log 1/a)^{1-Q}\mu(B(x_0, r))r^{-Q}.\end{aligned}$$

Here we used the fact that $E \subset X \setminus B(z, r/4)$ and $G \subset B(z, ar)$.

Hence

$$C_2(\log 1/a)^{1-Q}\mu(B(x,r))r^{-Q} \geq \operatorname{cap}_Q(E,F) \geq \frac{1}{C_1}\mu(B(x_0,r))r^{-Q},$$

and this gives the lower bound

$$a \geq \exp\left(-(C_1C_2)^{\frac{1}{Q-1}}\right).$$

□

Theorem 4.2. *Let (X,d,μ) satisfy the same assumptions as in Theorem 3.4. Then if $x_0 \in X$, $3R \leq \operatorname{diam}(X)$ and $s \leq Q-p$, we have*

$$\mathcal{H}_s^\infty(\{x \in X \,:\, d(x,x_0)=R\}) \geq cR^s.$$

Proof. Let

$$\begin{aligned} E =& \overline{B}(x_0,R/2),\\ F =& \overline{B}(x_0,2R)\setminus B(x_0,R), \qquad \text{and}\\ \widetilde{F} =& \{x \in X \,:\, d(x,x_0)=R\}. \end{aligned}$$

Let $\{B(x_i,r_i)\}_i$ be a covering of $\widetilde{F}$. If r_i is comparable to R for some i, then

$$\mathcal{H}_s^\infty(\bigcup_i B(x_i,r_i)) \geq cR^s,$$

so we may assume that all r_i are small compared to R.

By using the admissible metric

$$g_i(x) = Cr_i^{(Q-p)/(p-1)}d(x,x_i)^{(1-Q)/(p-1)}\chi_{B(x_i,R/2)\setminus B(x_i,r_i)}(x)$$

we get the estimate

$$\operatorname{cap}_p(B(x_i,r_i),E) \leq C\mu(B(x_0,R))R^{-Q}r_i^{Q-p},$$

and hence

$$\begin{aligned} \operatorname{cap}_p(\widetilde{F},E) \leq& \sum_{i=1}^\infty \operatorname{cap}_p(B(x_i,r_i),E)\\ \leq& C\mu(B(x_0,R))R^{-Q}\sum_{i=1}^\infty r_i^{Q-p}\\ \leq& C\mu(B(x_0,R))R^{-p}\sum_{i=1}^\infty (r_i/R)^s. \end{aligned}$$

On the other hand using Lemma 3.3 in the same way as in the proof of Theorem 4.1 we get the following lower bound for the capacity

$$\operatorname{cap}_p(\widetilde{F}, E) = \operatorname{cap}_p(F, E) \geq c\mu(B(x_0, R))R^{-p}. \tag{4.1}$$

Therefore, by the previous series of inequalities,

$$C \sum_i (r_i/R)^s \mu(B(x_0, R))R^{-p} \geq c\mu(B(x_0, R))R^{-p},$$

that is,

$$\sum_i r_i^s \geq cR^s.$$

Finally, by taking infimum over all coverings of $\widetilde{F}$ we get $\mathcal{H}_s^\infty(\widetilde{F}) \geq cR^s$. $\square$

5 Sobolev embedding theorem on spheres

In this section we prove an embedding theorem for almost all spheres centered at a given point. This result is a generalization of Theorem 7.2 and Corollary 7.3 in Hajłasz–Koskela [9], where a similar result was proved in Ahlfors–regular spaces. We use Theorem 3.4 to generalize their result to more general metric spaces.

In order to state our version of the Sobolev embedding theorem on spheres, we first have to deal with the problem that u is only defined almost everywhere. To take care of this matter we define $u(x)$ everywhere by the formula

$$u(x) = \limsup_{r\to 0} \fint_{B(x,r)} u \,\mathrm{d}\mu. \tag{5.1}$$

By Lebesgue's differentation theorem (see for example Heinonen [10]), we know that the limit above exists and is finite almost everywhere, if u is a locally integrable function.

Now we can state the Sovolev embedding theorem.

Theorem 5.1. *Let (X, d, μ) be a complete metric measure space with a measure μ that satisfies*

$$\frac{1}{C}\left(\frac{r}{R}\right)^{Q_1} \leq \frac{\mu(B(x,r))}{\mu(B(x,R))} \leq C\left(\frac{r}{R}\right)^{Q_2}$$

for every $x \in X$ and $0 < r < R < \operatorname{diam}(X)$ for some $1 < Q_2 \leq Q_1$. Suppose that X supports a weak $(1, Q_2)$–Poincaré inequality and that $p > Q_1 - 1$ and $p \geq Q_2$. Then there is a constant C_1 and a radius $r_0/2 < r < r_0$ such that

$$|u(x) - u(y)| \leq C_1 d(x,y)^{1-(Q_1-1)/p} r_0^{(Q_1-1)/p} \left(\fint_{B(x_0,C_1r_0)\setminus B(x_0,r_0/C_1)} g^p \,d\mu\right)^{1/p}$$

whenever $d(x, x_0) = d(y, x_0) = r$.

Proof. Fix $x_0 \in X$, $r > 0$, $x, y \in B(x_0, r_0) \setminus B(x_0, r_0/2)$ and $0 \leq \alpha < 1$. By Theorem 3.4 there exists a rectifiable curve joining x to y such that $l(\gamma) \leq Cd(x, y)$ and $\gamma \subset B(x_0, Cr_0) \setminus B(x_0, r_0/C)$. The constant C is independent of the choice of the points x, y and x_0 and the radius r_0.

Let $r_C = d(x, y)/(100C\lambda)$. There exists a collection of balls $\{V_i = B(y_i, r_C)\}_{i=1}^k$ such that $x \in V_1$, $y \in V_k$, $V_i \cap V_{i+1} \neq \emptyset$ for every $i = 1, \ldots, k-1$ and every ball intersects γ. The number of balls is uniformly bounded by a constant that depends only on C_d, λ and C.

Set

$$\begin{cases} B_i = V_i, & i = 1, 2, \ldots, k, \\ B_i = B(y, 2^{-i+l} r_C), & i > k, \\ B_i = B(x, 2^i r_C), & i < 0. \end{cases} \tag{5.2}$$

Then, by the Poincaré inequality and the triangle inequality, we have

$$\begin{aligned} |u(x) - u(y)| &\le \sum_{i=-\infty}^{\infty} |u_{B_i} - u_{B_{i+1}}| \\ &\le C \sum_{i=-\infty}^{\infty} \fint_{2B_i} |u - u_{2B_i}| \, \mathrm{d}\mu \\ &\le C \sum_{i=-\infty}^{\infty} r_i \left(\fint_{2\lambda B_i} g^p \, \mathrm{d}\mu \right)^{1/p} \\ &= C \sum_{i=-\infty}^{\infty} r_i^{1-\alpha} r_i^{\alpha} \left(\fint_{2\lambda B_i} g^p \, \mathrm{d}\mu \right)^{1/p} \\ &\le C d(x,y)^{1-\alpha} \left(M_{2\lambda d(x,y),p,\alpha} g(x) + M_{2\lambda d(x,y),p,\alpha} g(y) \right), \end{aligned}$$

where

$$M_{R,p,\alpha} g(x) = \sup_{r<R} r^{\alpha} \left(\fint_{B(x,r)} g^p \, \mathrm{d}\mu \right)^{1/p}.$$

Observe that according to (5.1) the above inequality holds everywhere.

Write

$$G_t = \{ x \in B(x_0, r_0) \, : \, M_{4\lambda r_0, p, \alpha} g(x) < t \}.$$

Then

$$|u(x) - u(y)| \le C t d(x,y)^{1-\alpha} \tag{5.3}$$

for all $x, y \in G_t$.

By Theorem 2.1, there exists a countable collection of disjoint balls $\{B(z_i, s_i)\}_i$, such that

$$B(x_0, r_0) \setminus G_t \subset \bigcup_i B(z_i, 5s_i)$$

and
$$s_i^{\alpha}\left(\fint_{B(z_i,s_i)} g^p \,\mathrm{d}\mu\right)^{1/p} \geq t \tag{5.4}$$
for every i. By the lower bound on $\mu(B(z_i,s_i))$ and (5.4), we get

$$\begin{aligned}
\mathcal{H}_{Q_1-\alpha p}(B(x_0,r_0)\setminus G_t) &\leq \sum_i s_i^{Q_1-\alpha p}\\
&\leq C\sum_i \frac{\mu(B(z_i,s_i))}{\mu(B(x_0,r_0))} r_0^{Q_1} s_i^{-\alpha p}\\
&\leq C\sum_i \frac{\mu(B(z_i,s_i))}{\mu(B(x_0,r_0))} r_0^{Q_1}\left(\fint_{B(z_i,s_i)} g^p \,\mathrm{d}\mu\right) t^{-p}\\
&\leq C\sum_i \frac{r_0^{Q_1} t^{-p}}{\mu(B(x_0,r_0))}\left(\int_{B(z_i,s_i)} g^p \,\mathrm{d}\mu\right)\\
&\leq C\frac{r_0^{Q_1} t^{-p}}{\mu(B(x_0,r_0))}\left(\int_{B(x_0,5\lambda r_0)} g^p \,\mathrm{d}\mu\right).
\end{aligned}$$

Define $v : B(x_0,r_0) \to [0,r_0)$ by the formula $v(x) = d(x,x_0)$. Then v is Lipschitz continuous with constant 1, and hence, for each set $E \subset B(x_0,r_0)$, we have
$$\mathcal{H}_{Q_1-\alpha p}(v(E)) \leq \mathcal{H}_{Q_1-\alpha p}(E).$$
Let $\alpha = (Q-1)/p$. Then

$$\mathcal{H}_1(v(B(x_0,r_0)\setminus G_t)) \leq C t^{-p} r_0^{Q_1} \fint_{B(x_0,5\lambda r_0)} g^p \,\mathrm{d}\mu.$$

This implies that the length of the set $v(B(x_0,r_0)\setminus G_t) \subset [0,r_0)$ goes to 0 as t goes to ∞. Now the theorem follows from the observation that for $r \in [0,r_0)\setminus v(B(x_0,r_0)\setminus G_t)$ the sphere $\{x \in X : d(x,x_0) = r\}$ is contained in G_t and hence inequality (5.3) applies.

□

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