Defeating Noise in Communication

A Brief Explanation and Demo on Coding

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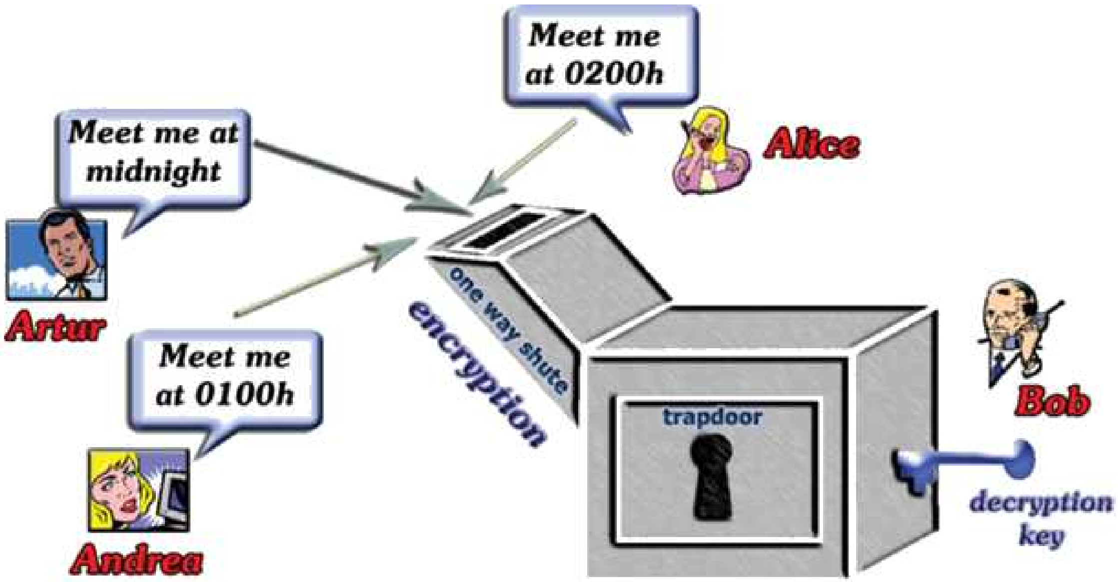
What we are **not** *talking about*



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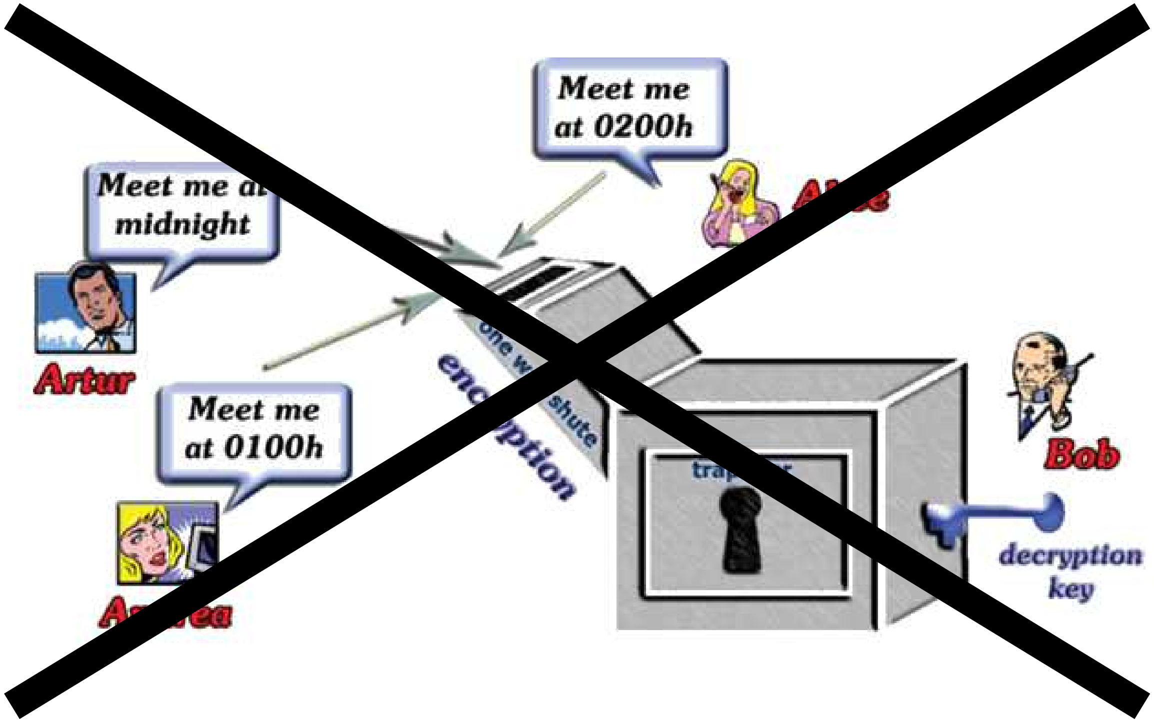

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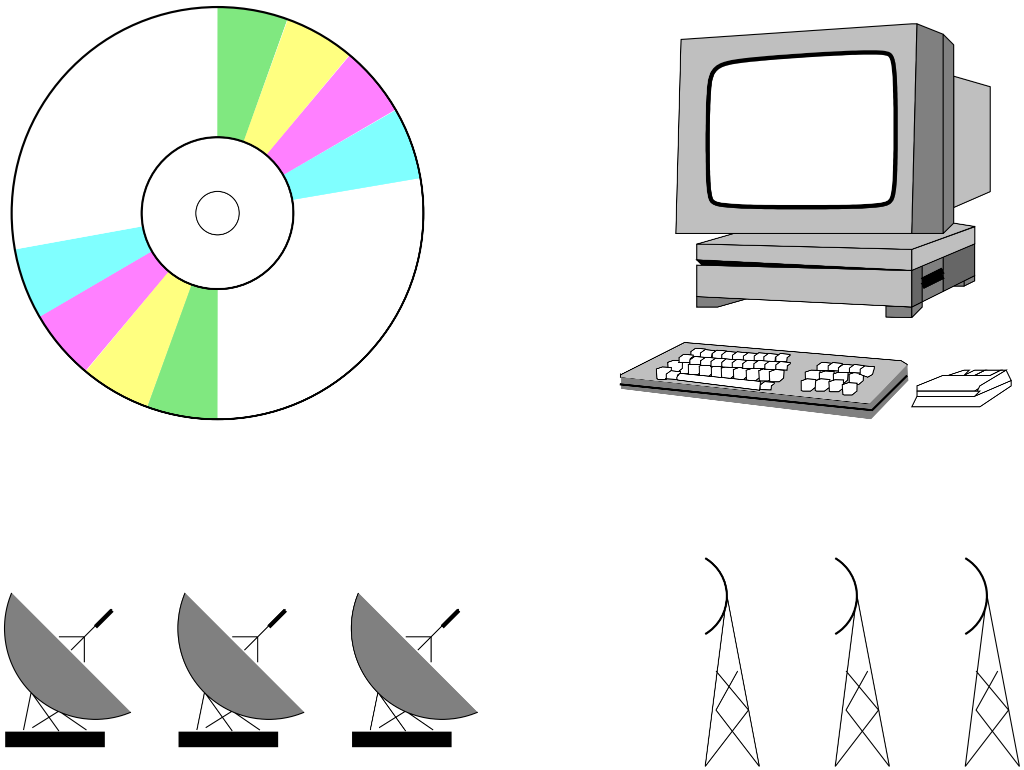


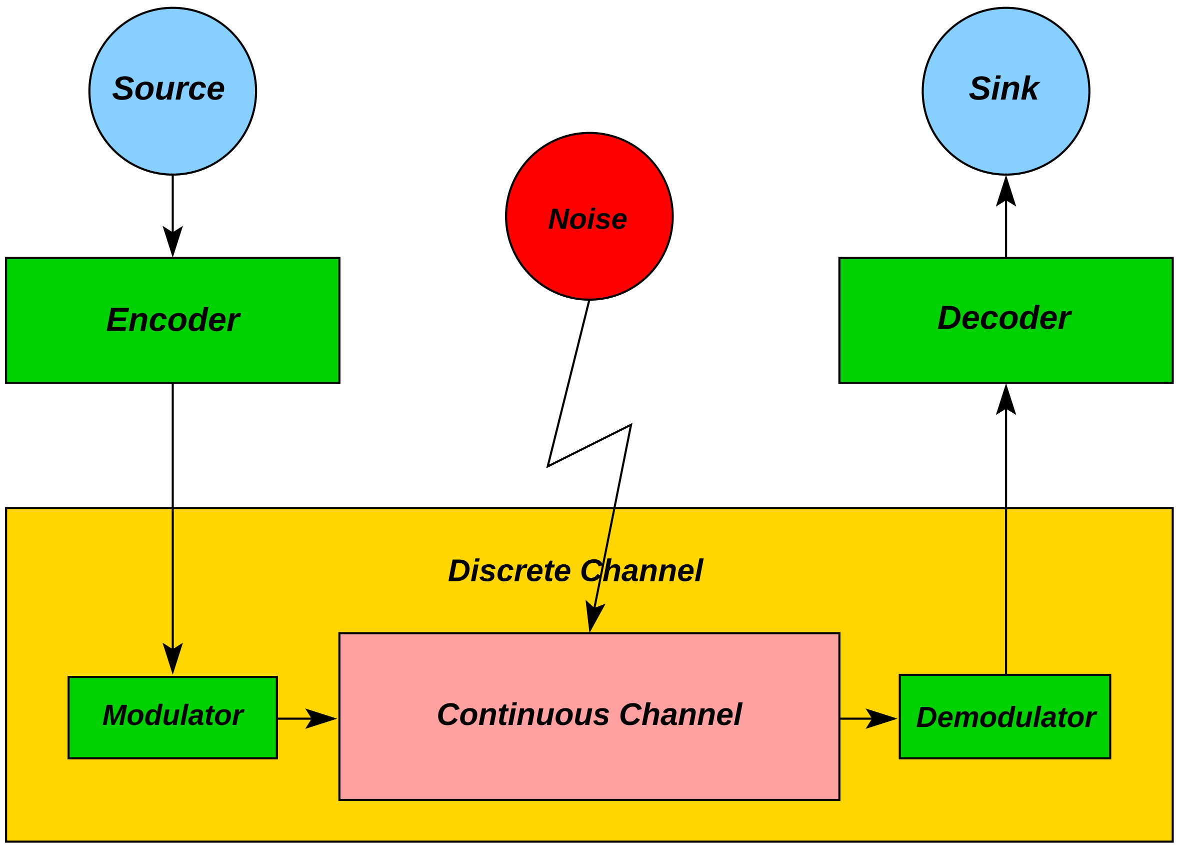
Codes protect against noise. There is no such thing as *cracking a code*.

A Quotation

The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point.

Claude Shannon 1948

Basic Problem: Noisy Transmission



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- A componentwise sum (agreeing to $1+1=0$) is defined and yields the new word 11100111.

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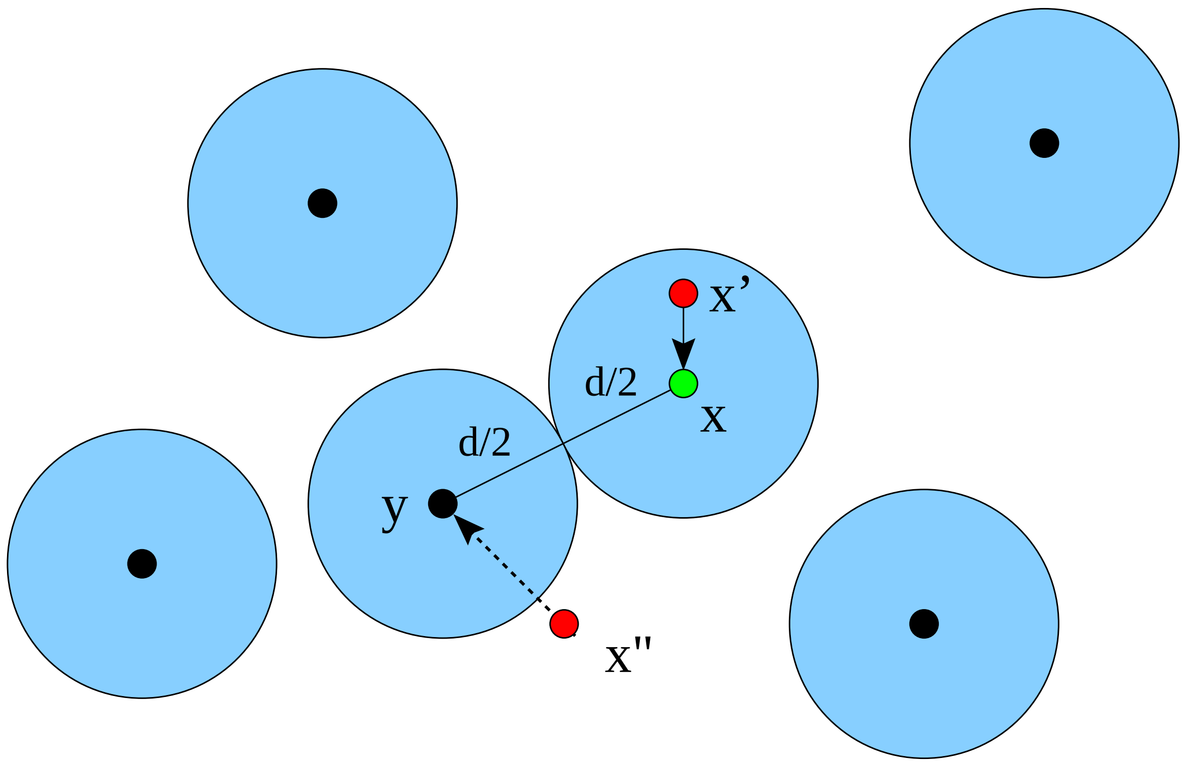
- As this code contains 4 words, mathematicians say that C is an $(8, 4, 5)$-code.

Illustration

- **Remark:** A code of minimum distance d can be used to correct errors of weight smaller than $\frac{d}{2}$, and sometimes even more!



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- Then, assigning these four messages the four codewords is what we mean by *encoding*:

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- As we encode every single bit essentially by 4 bits, we say the *rate* of C is $1/4$.

A Question

- Why did we not use the assignment

$$\begin{array}{rclcrcl} 00 & \mapsto & 00000000 & \quad & 01 & \mapsto & 00001111 \\ 10 & \mapsto & 11110000 & \quad & 11 & \mapsto & 11111111, \end{array}$$

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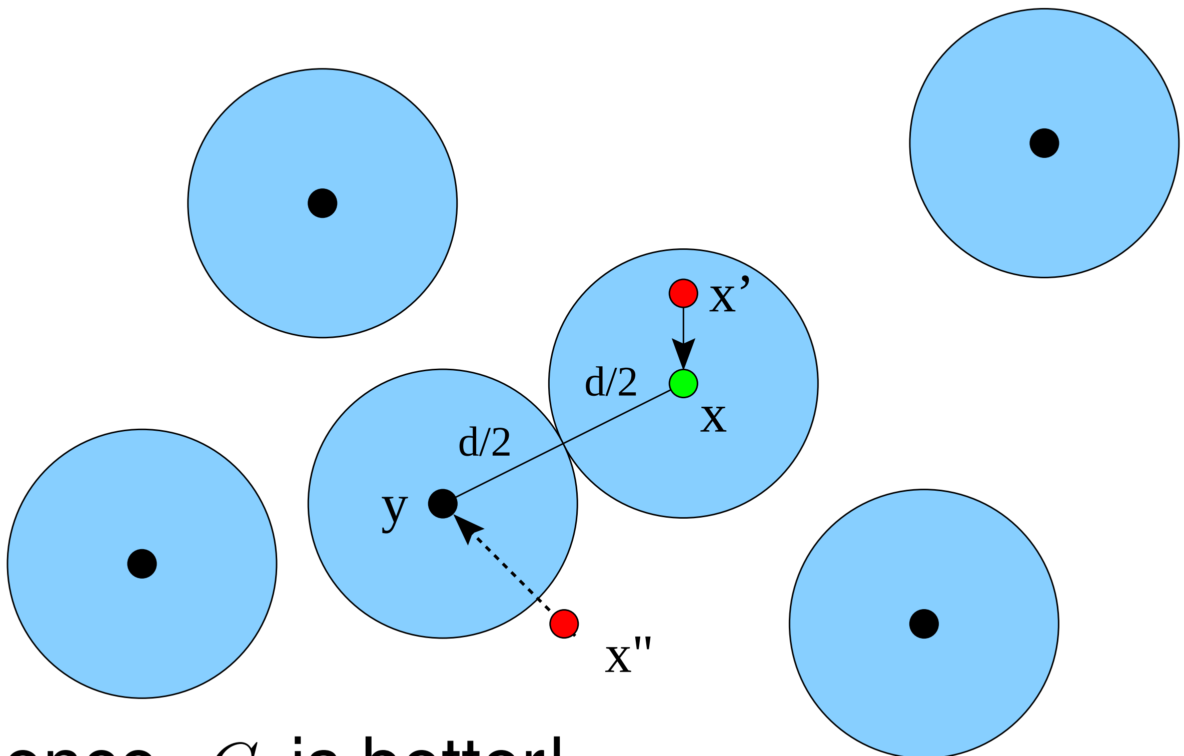
- Hint: Determine the parameters of D and think of the packing illustration.

- Observation: D is an $(8, 4, 4)$ code, hence . . .

Which Code is Better?

- **Recall:** A code of minimum distance d can be used to correct errors of weight smaller than $\frac{d}{2}$, and sometimes even more!



$\ldots$ hence, C is better!

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- Assuming that a lower number of errors is more likely than a larger number, we decide that the word 11100111 was the one originally sent.
- A *decoder* is a device that performs the task of finding the closest codeword to a given received word.

The Other Example

- We have seen that a decoder can correct up to 2 bit changes when C is used. What if we had used

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- The decoder will fail, because it finds two equally likely choices, 11110000 and 11111111.

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- **Recipe A:** Divide the stream into pieces of length 4, encode these to length 8 and send them off.
- **Recipe B:** Divide the stream into pieces of length 32, encode these to length 64 and send them off.
- What is better, given the same noise level?

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- **Theorem:** Extending the the length (and keeping the rate) of the used codes we can achieve arbitrary reliability of the communication process.
- In other words, recipe B is preferable to recipe A. By going up to higher length, communication errors will become less and less likely.

A Puzzle: Pascal's Triangle

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
?	?	?	?	?	?	?	?

What are the next entries in this table?

The Plotkin Sum of Codes

- Let C and D be the codes studied earlier. We define

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- Proceeding in the same way with all choices of words in C and D we see that $C \oplus D$ contains 16 words.

Reed-Muller Codes

- **Definition:** For $m \in \mathbb{N}$ and $0 \leq r \leq m$ we define a family $\mathrm{RM}(r,m)$ of linear codes by:
  - $\star$ $\mathrm{RM}(0,m) = \{000\ldots0, 111\ldots1\}$ of length 2^m.
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  - $\star$ For all $m \geq 1$ and $1 \leq r \leq m-1$

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Another Pascal Triangle

$$\begin{array}{cccccc}
\mathrm{RM}(0,0) & & & & & \\
\mathrm{RM}(0,1) & \mathrm{RM}(1,1) & & & & \\
\mathrm{RM}(0,2) & \mathrm{RM}(1,2) & \mathrm{RM}(2,2) & & & \\
\mathrm{RM}(0,3) & \mathrm{RM}(1,3) & \mathrm{RM}(2,3) & \mathrm{RM}(3,3) & & \\
\mathrm{RM}(0,4) & \mathrm{RM}(1,4) & \mathrm{RM}(2,4) & \mathrm{RM}(3,4) & \mathrm{RM}(4,4) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The recursion of Pascal’s triangle is underlying!

- According to the previous slide

$$\begin{aligned}
\mathrm{RM}(1,5) &= \mathrm{RM}(0,4) \oplus \mathrm{RM}(1,4), \\
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Here, $\mathrm{RM}(0,1) = \{00, 11\}$, $\mathrm{RM}(0,2) = \{0000, 1111\}$, and so forth, and $\mathrm{RM}(1,1) = \{00, 01, 10, 11\}$.

Example $\mathrm{RM}(1,5)$

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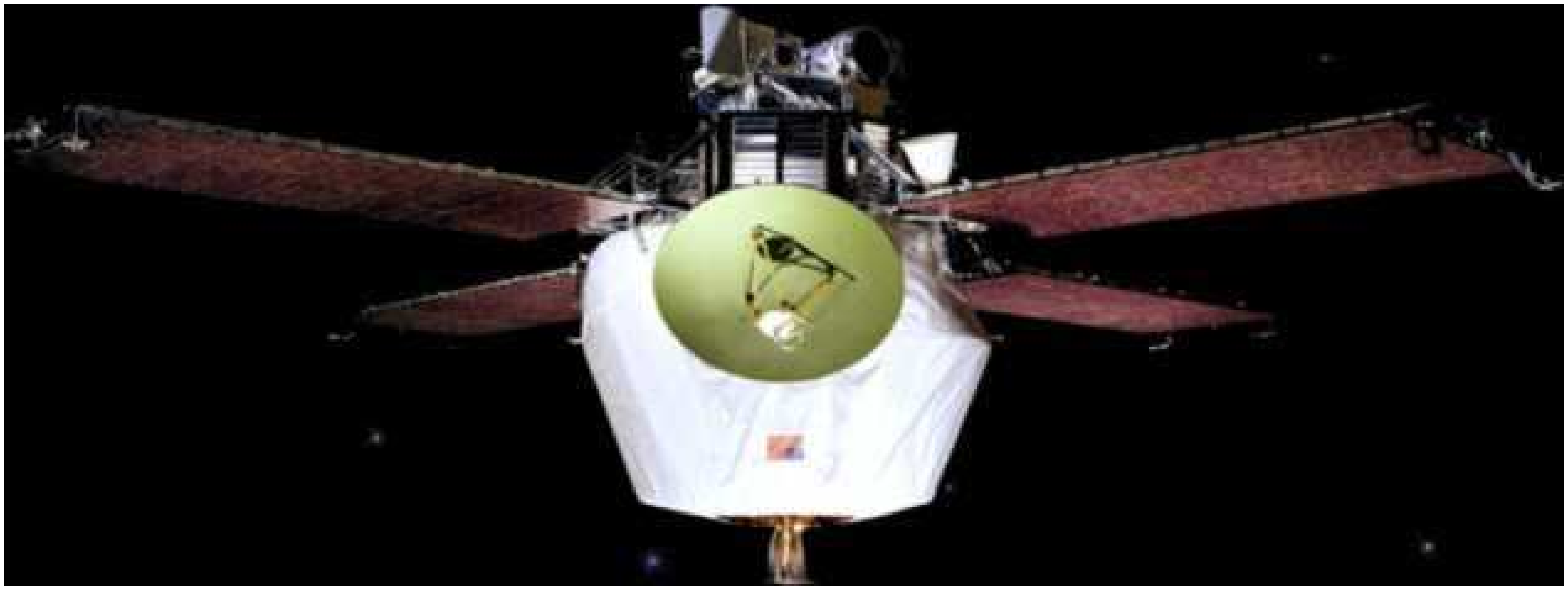
- The entire code consists of 64 words of length 32, has minimum distance 16, hence hence can correct up to 7 errors.

The Mariner-9 Mission

- The code $\mathrm{RM}(1,5)$ was used in the Mariner-9 program of NASA.



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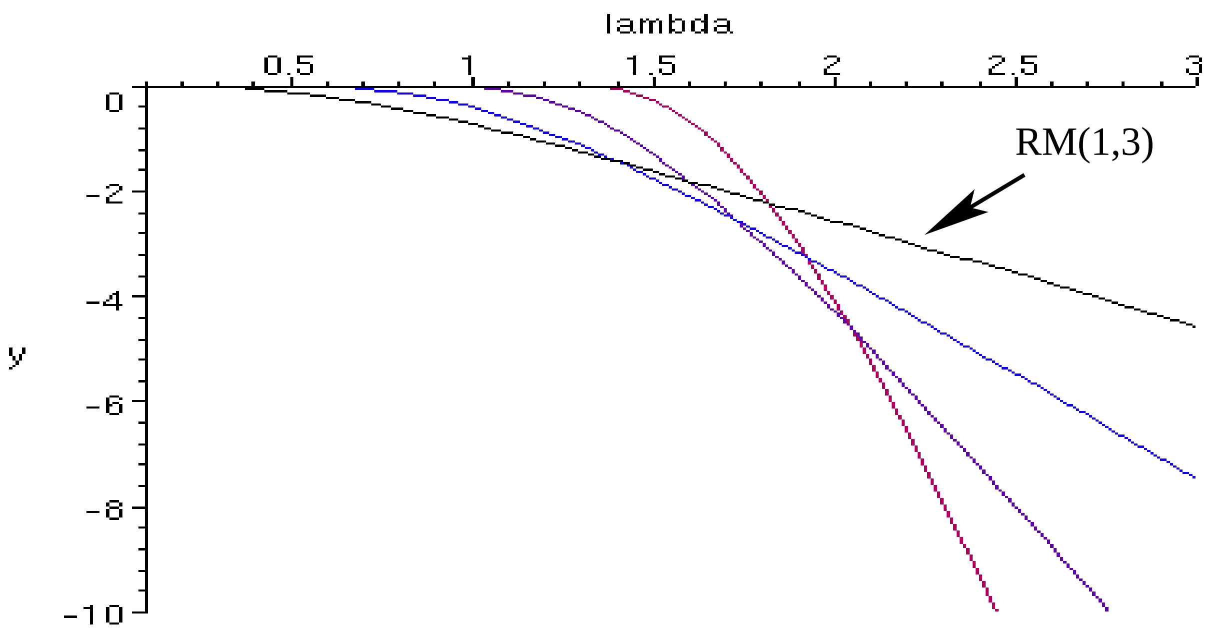
- Even if we use an error-correcting code C there will still a probability $P_{BE}(C, \lambda)$ that a transmitted word is decoded wrongly.
- Performance comparison of different codes is usually done in a logarithmic plot of $P_{BE}(C, \lambda)$ against λ.
- We will however not go deeper into the mathematics behind this.

Performance Questions (cntd)

We do this with the codes $\mathrm{RM}(r,m)$ where $(r,m) = (1,3),(2,5),(3,7)$ and $(4,9)$.



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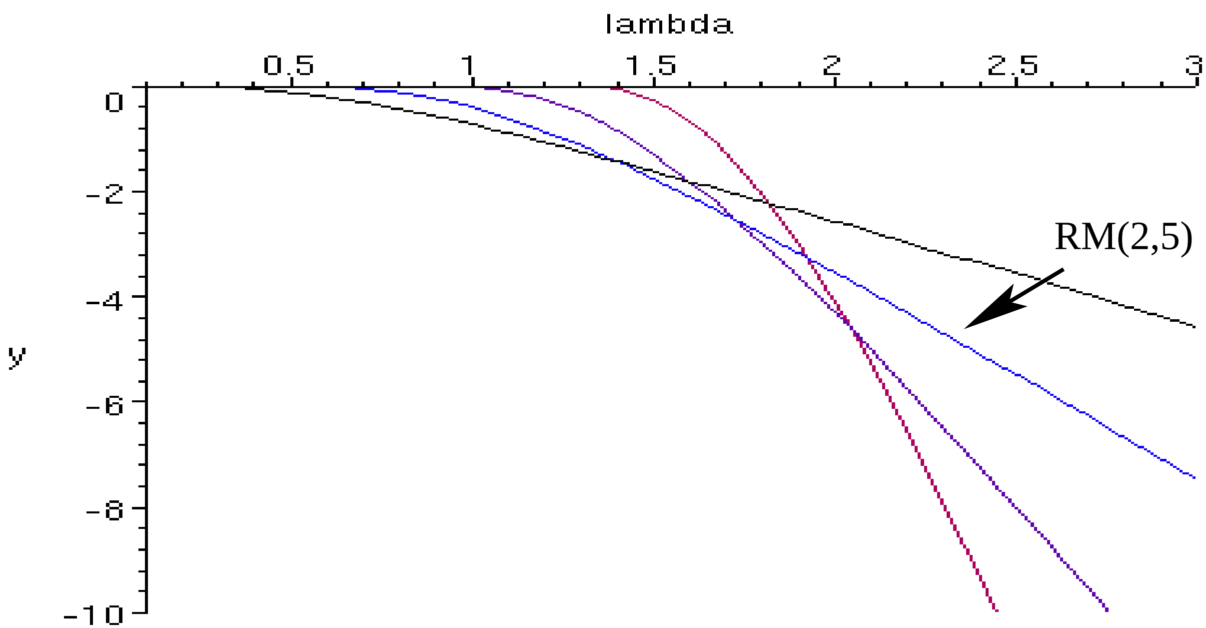

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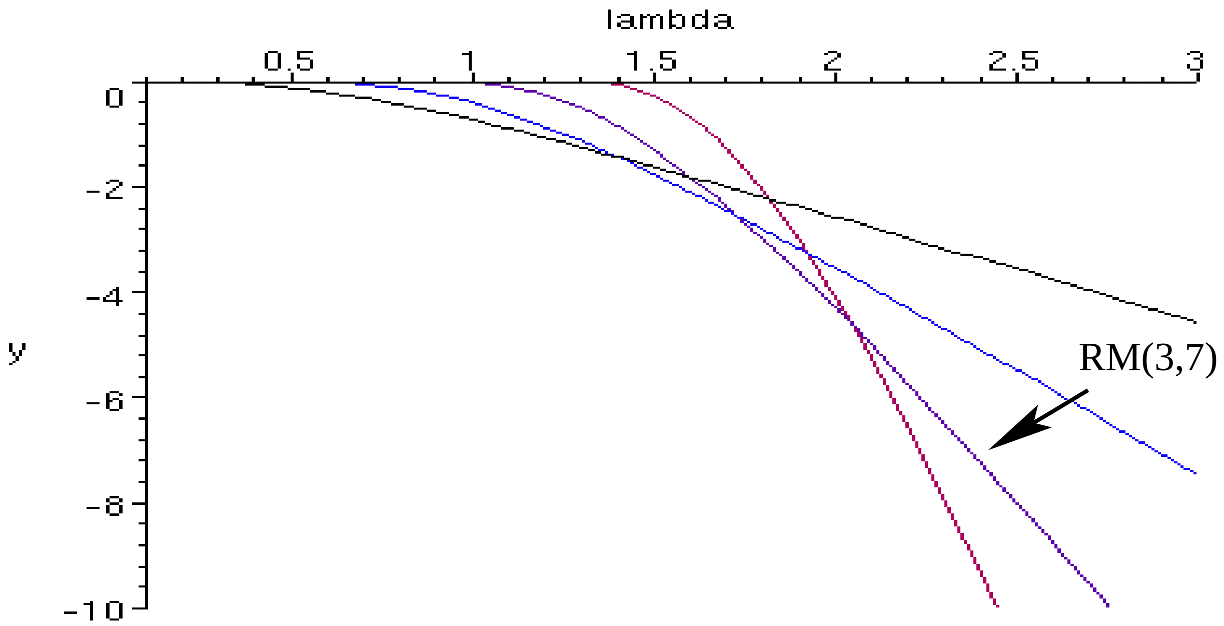

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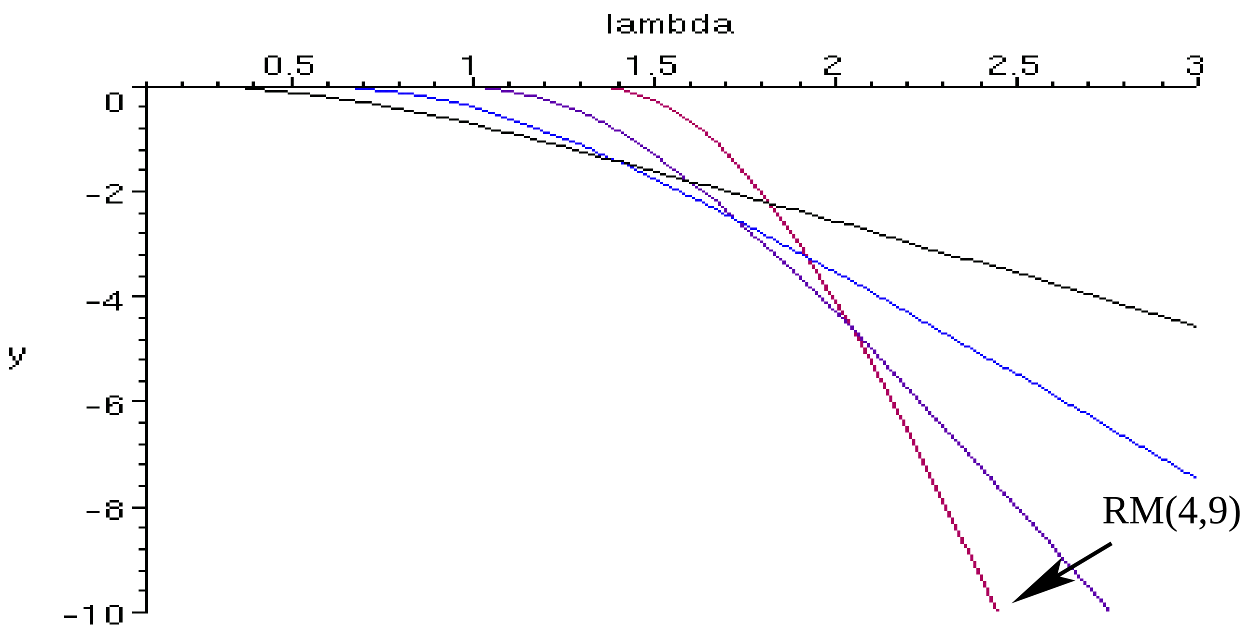

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- Thanks for your attention!