Inverse problems for wave equation

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Motivation

Let $\Omega \subset \mathbb{R}^m$,

$$u(x,t) \text{ satisfy a wave equation in } \Omega \times \mathbb{R}$$

Inverse problem:

Can we determine the coefficients of the wave equation, i.e., physical model in Ω by observing

$$u(x,t) \text{ near } \partial\Omega \times \mathbb{R}$$

for all possible solutions $u(x,t)$?

The inverse problem has no unique solution as

- We can change definition of x-coordinate: Let

$$v(x,t) = u(\phi(x),t)$$

  where

$$\phi : \Omega \to \Omega, \quad \phi|_{\partial\Omega} = id$$

- We can change scale of u-coordinate: Let

$$w(x,t) = \kappa(x)u(x,t)$$

  where $\kappa(x) > 0$.

All functions u, v and w model the same physical process.

Let us consider Ω as Riemannian manifold

$$d_g(x, y) = \text{travel time between } x \text{ and } y.$$

Let us identify all isometric Riemannian manifolds, that is, we ask following question

Do the boundary measurements determine uniquely the isometry type of the Riemannian manifold?

Let u satisfy the wave equation

$$u_{tt} + a(x, D)u = 0.$$

Then the gauge transformation of u,

$$w(x, t) = \kappa(x)u(x, t)$$

satisfy

$$w_{tt} + a_\kappa(x, D)w = 0,$$

where

$$a_\kappa(x, D)w = \kappa a(x, D)(\kappa^{-1} w)$$

We say that the gauge equivalence class of $a(x, D)$ is

$$[a(x, D)] = \{a_\kappa(x, D) \ : \quad \kappa > 0\}$$

Can the equivalence class be uniquely determined?

1 Setting of the problem I

Let us consider the wave equation

$$\begin{aligned} u_{tt}(x,t) + Au(x,t) &= 0, \quad \text{in} \quad M \times \mathbb{R}_+, \\ u|_{t=0} &= 0, \quad u_t|_{t=0} = 0, \\ u|_{\partial M \times \mathbb{R}^+} &= f \end{aligned}$$

where M is a m-dimensional manifold and

$$Au = -\sum_{j,k=1}^{m} a^{jk} \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{m} b^j \frac{\partial u}{\partial x^j} + cu,$$

where a^{jk}, b^j, c are real, smooth, $[a^{jk}(x)] > 0$.
In addition ...

Assume that there is dV such that A is selfadjoint in $L^2(M, dV)$ with

$$\mathcal{D}(A) = H^2(M) \cap H_0^1(M).$$

Now

$$g^{jk} = a^{jk} \text{ defines a metric tensor on } M.$$

This makes (M, g) a Riemannian manifold.

1.1 Invariant inverse problem

The Robin-to-Dirichlet map is

$$\Lambda : (\partial_\nu u + \sigma u)|_{\partial M \times \mathbb{R}_+} \mapsto u|_{\partial M \times \mathbb{R}_+}.$$

Dynamical inverse problem:

Let ∂M and the map Λ be given. Can we determine

(M, g) and $[A(x, D)]$?

Energy flux through boundary The energy of the wave at time t is

$$E(u,t) = \int_M \big(|\partial_t u(t)|^2 + |\,\mathrm{Grad}\, u(t)|_g^2 + q|u(t)|^2\big)dV + \\ + \int_{\partial M} \sigma |u(t)|^2 dS.$$

For $h = u|_{\partial M \times \mathbb{R}_+} \in C_0^\infty(\partial M \times \mathbb{R}_+)$ let

$$\Pi(h) = \lim_{t\to\infty} E(u,t).$$

Inverse problem for energy flux:

Let ∂M and map Π be given. Can we determine

$$(M,g) \text{ and } [A(x,D)]?$$

Inverse boundary spectral problem:
Operator A has in $L^2(M, dV)$ orthonormal eigenfunctions φ_j,

$$\begin{aligned} (-\Delta_g + P + q - \lambda_j)\varphi_j &= 0, \\ \partial_\nu \varphi_j|_{\partial M} = 0. \end{aligned}$$

Let boundary spectral data

$$\{\partial M, \quad \lambda_j, \quad \varphi_j|_{\partial M}, \quad j = 1, 2, \dots\}$$

be given. Can we determine

(M, g) and $[A(x, D)]$?

- The above inverse problems are equivalent.
- Consider gauge equivalence class $[A(x,D)]$ of operator $A(x,D)$. Then there is a unique Schrödinger operator

$$-\Delta_g + q \in [A(x,D)].$$

Because of this we next restrict ourselves to the case $A = -\Delta_g + q$.

2 Setting of the problem II

Denote by

$$u^f = u^f(x,t)$$

the solutions of

$$\begin{aligned}
&u_{tt} - \Delta_g u + qu = 0 \quad \text{on } M \times \mathbb{R}_+,\\
&-\partial_\nu u|_{\partial M\times\mathbb{R}_+} = f,\\
&u|_{t=0} = 0, \quad u_t|_{t=0} = 0,
\end{aligned}$$

where ν is unit interior normal of ∂M. Define

$$\Lambda_T f = u^f|_{\partial M\times(0,T)}.$$

We denote $\Lambda = \Lambda_\infty$. Assume that we are given the boundary data $(\partial M, \Lambda)$.

Results on the problem:

- Nachman-Sylvester-Uhlmann '88.
- $c(x)^2\Delta$ in $\mathbb{R}^m$ by boundary control method, Belishev '87 , Belishev-Kurylev '87.
- Δ_g on manifold, Belishev-Kurylev '92.
- Local controllability, Tataru '95.
- Equivalence of above inverse problems Katchalov-Kurylev-L.-Mandache 2004
- Maxwell's equations Kurylev-L.-Somersalo 2006.
- Dirac system Kurylev-L.-Somersalo 2006.
- Reconstruction based on iterated time reversal Bingham-Kurylev-L.-Siltanen 2007.

In the following we consider the geometric version of the Belishev-Kurylev-Tataru method, or Boundary Control method, see references [1-7].

2.1 Blagovestchenskii identity

Lemma 2.2 *Let* $f, h \in L^2(\partial M \times [0, 2T])$*. Then*

$$\int_M u^f(x,T)u^h(x,T)\,dV_\mu(x) =$$

$$\int_{[0,2T]^2}\int_{\partial M} J(t,s)\big[f(t)(\Lambda_{2T}h)(s) - (\Lambda_{2T}f)(t)h(s)\big]\,dS_g(x)dtds,$$

where $J(t,s) = \frac{1}{2}\chi_L(s,t)$ *and* χ_L *being the characteristic function of the triangle*

$$L = \{(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ t + s \le 2T,\ \ s < t\}.$$

Proof. Let $w(t,s) = \int_M u^f(t) u^h(s)\, dV_\mu$. Integrating by parts, we see that

$$
\begin{aligned}
(\partial_t^2 - \partial_s^2) w(t,s) &= -\int_M \left[A u^f(t) u^h(s) - u^f(t) A u^h(s) \right] dV_\mu(x) \\
&= -\int_{\partial M} \left[\partial_\nu u^f(t) u^h(s) - u^f(t) \partial_\nu u^h(s) \right] dS_g \\
&= \int_{\partial M} \left[f(t) \Lambda h(s) - \Lambda f(t) h(s) \right] dS_g.
\end{aligned}
$$

Moreover,

$$
w|_{t=0} = w|_{s=0} = 0, \quad \partial_t w|_{t=0} = \partial_s w|_{s=0} = 0.
$$

Thus we can find $w(s,t)$ by solving a wave equation with known initial data and right side. $\square$

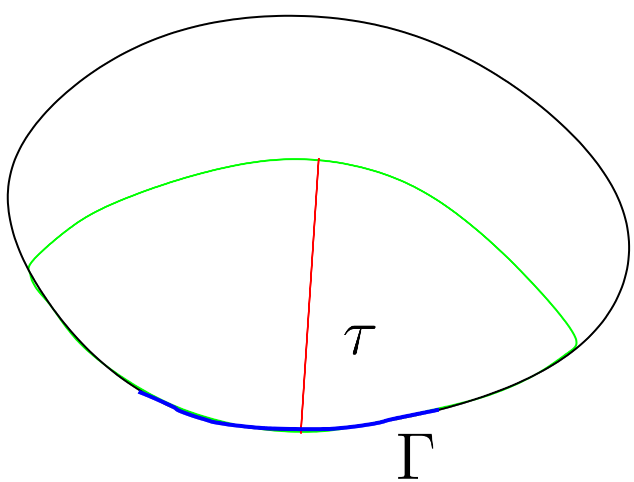
2.3 Domains of influence

Let $\Gamma \subset \partial M$ be a non-empty open set. We denote by $L^2(\Gamma \times [0,T])$ the subspace of $L^2(\partial M \times [0,T])$ that consists of the functions f with supp $(f) \subset \overline{\Gamma} \times [0,T]$.

Definition 2.4 *The subset $M(\Gamma,\tau) \subset M$, $\tau > 0$,*

$$M(\Gamma,\tau) = \{x \in M : \ d(x,\Gamma) \leq \tau\}$$

is called the domain of influence of Γ at time τ.

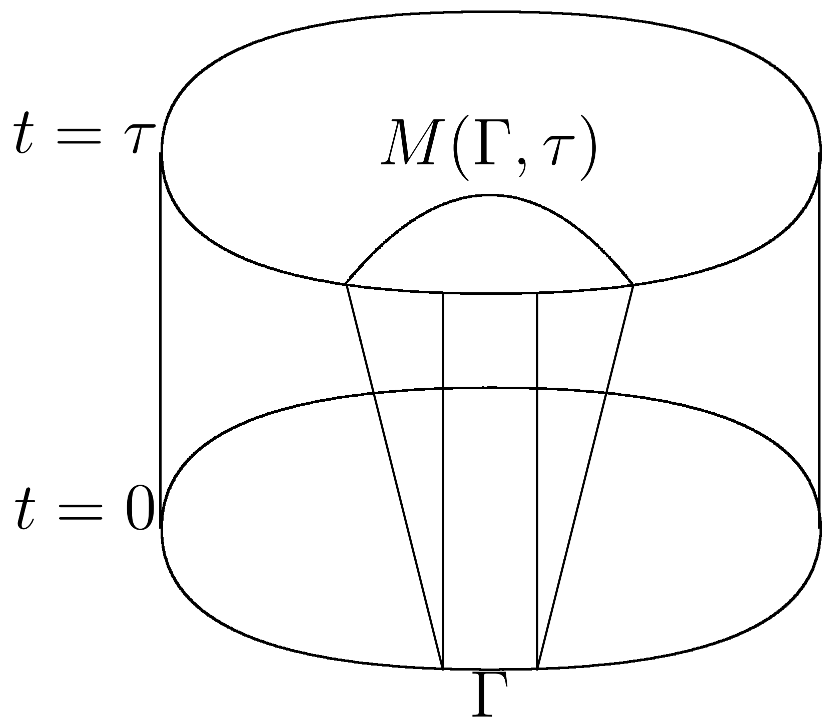


$$M(\Gamma,\tau) = \{x \in M : \ d(x,\Gamma) \leq \tau\}.$$

Lemma 2.5 *Let* $f \in L^2(\Gamma \times [0, T])$*. Then*

$$\textit{supp}\,(u^f(\tau)) \subset M(\Gamma, \tau).$$

Proof. The result follows finite velocity of wave propagation. $\square$



We denote by $L^2(\Omega),\ \Omega \subset M$, the subspace of $L^2(M)$, which consists of all functions $f \in L^2(M)$ that are equal to zero in $M \setminus \Omega$. We prove following Tataru-type controllability type theorem.

Theorem 1 *Let $\tau > 0$. The linear subspace,*

$$\{u^f(\tau) \in L^2(M(\Gamma, \tau)) :\ f \in C_0^\infty(\Gamma \times [0, \tau])\},$$

is dense in $L^2(M(\Gamma, \tau))$.

Proof. Let $\psi \in L^2(M(\Gamma, \tau))$ be such that

$$\langle u^f(\cdot\,, \tau), \psi\rangle = 0$$

for all $f \in C_0^\infty(\Gamma \times [0, \tau])$.
To prove the claim, it is sufficient to show that $\psi = 0$.

We consider the wave equation,

$$(\partial_t^2 - \Delta_g + q)e = 0, \quad \text{in} \quad M \times (0,\tau),$$
$$\partial_\nu e|_{\partial M \times (0,\tau)} = 0, \quad e|_{t=\tau} = 0, \quad \partial_t e|_{t=\tau} = \psi.$$

Integrating by parts we obtain

$$\begin{aligned} 0 &= \int_{M\times(0,\tau)} [u^f(\partial_t^2 - \Delta_g + q)e - ((\partial_t^2 - \Delta_g + q)u^f)e]\, dV_g\, dt \\ &= \int_M u^f(\tau)\,\psi\, dV_g + \int_{\partial M\times(0,\tau)} f\, e\, dS_g\, dt \\ &= \int_{\partial M\times(0,\tau)} f\, e\, dS_g\, dt, \end{aligned}$$

for all $f \in C_0^\infty(\Gamma \times [0,\tau])$.
This yields that the Cauchy data of e vanish on $\Gamma \times (0,\tau)$.

Recall that $e(x,\tau)=0$. We continue e onto $t\in[\tau,2\tau]$ as

$$E(x,t)=\begin{cases} e(x,t), & \text{for } t\le\tau,\\ -e(x,2\tau-t), & \text{for } t>\tau.\end{cases}$$

Then $E\in C([0,2\tau];H^1(M))\cap C^1([0,2\tau];L^2(M))$ and

$$(\partial_t^2-\Delta_g+q)E=0 \qquad \text{in } M\times(0,\tau).$$

The Cauchy data of E vanish on $\Gamma\times([0,2\tau]\setminus\{\tau\})$. Since $\partial_\nu E\in L^2(\partial M\times(0,2\tau))$, we see that

$$E|_{\Gamma\times(0,2\tau)}=0,\;\; \partial_\nu E|_{\Gamma\times(0,2\tau)}=0.$$

Then $\psi=0$ by the following Tataru-Holmgren-John theorem.

Theorem 2 *Let u be a solution in $M \times (0, 2\tau)$ of the wave equation*

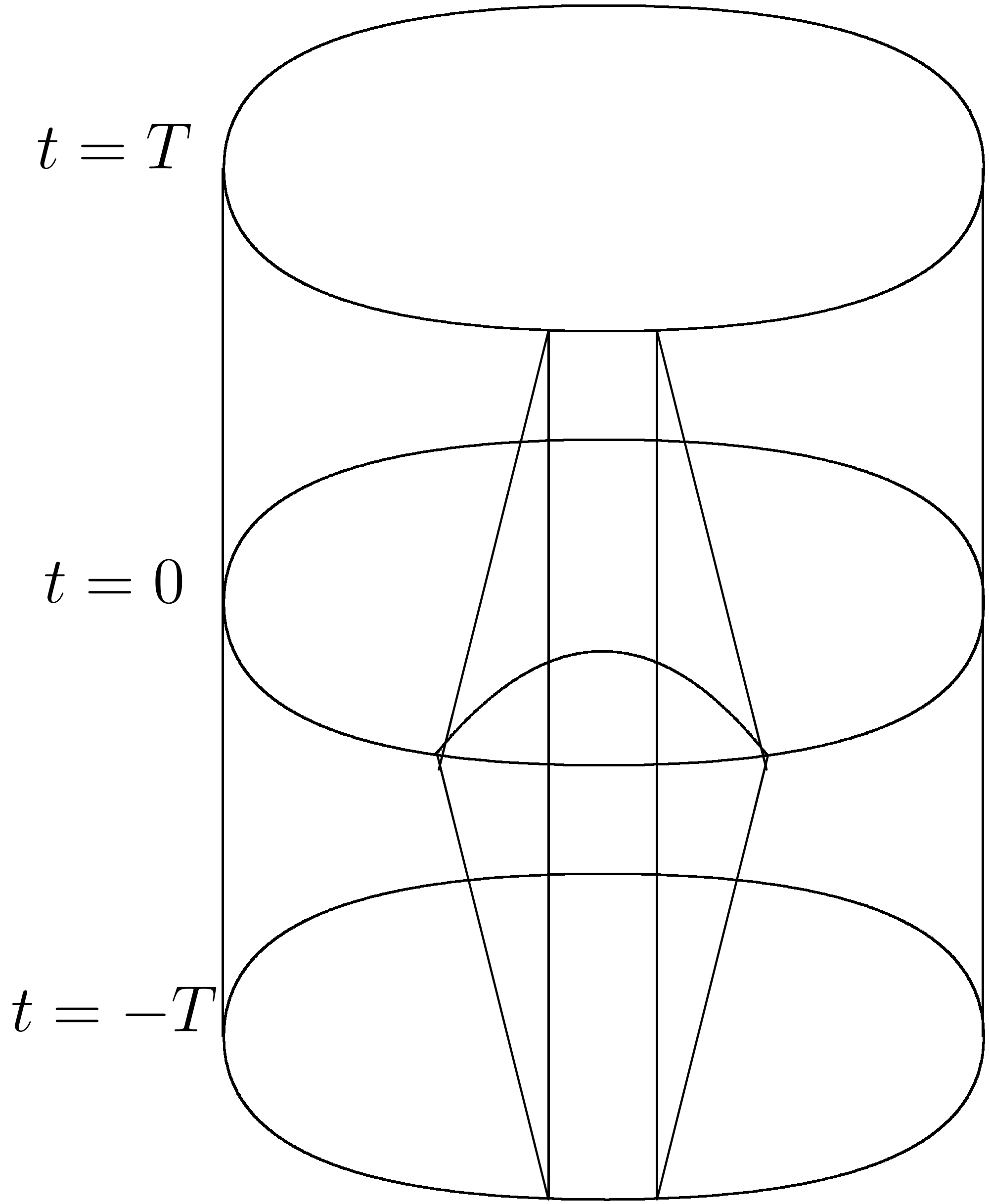
$$(\partial_t^2 - \Delta_g + q)u = 0 \qquad \textit{in } M \times (0, 2\tau).$$

such that for an open set $\Gamma \subset \partial M$,

$$u|_{\Gamma \times [0,2\tau]} = 0,\ \partial_\nu u|_{\Gamma \times (0,2\tau)} = 0.$$

Then, at $t = \tau$, the function u and its derivative $\partial_t u$ vanish in the domain of influence of Γ,

$$u(x, \tau) = 0,\ \partial_t u(x, \tau) = 0 \qquad \textit{for } x \in M(\Gamma, \tau).$$


$t = T$
$t = 0$
$t = -T$

2.6 Wave basis

The set

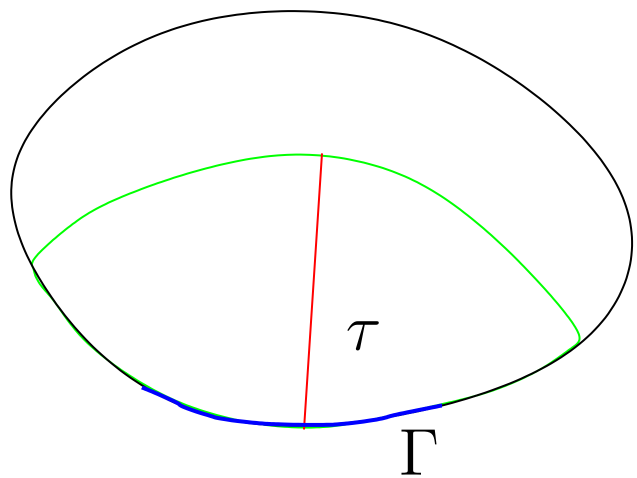
$$\{u^f(\tau) \in L^2(M(\Gamma,\tau)) : \ f \in L^2(\Gamma \times [0,\tau])\}$$

is dense in $L^2(M(\Gamma,\tau))$. Thus, there are functions f_j, $j = 1, 2, \ldots$, such that $\{u^{f_j}(\tau)\}_{j=1}^{\infty}$ form an orthonormal basis in the space $L^2(M(\Gamma,\tau))$.
We will construct such functions f_j from the boundary data. The corresponding basis $\{u^{f_j}(\tau)\}_{j=1}^{\infty}$ is called the wave basis.

Lemma 2.7 *Let $\tau > 0$. Given the boundary data it is possible to construct boundary sources $f_j \in L^2(\Gamma \times [0, \tau])$ such that*

$$v_j = u^{f_j}(\tau),\ j = 1, 2, \ldots,$$

form an orthonormal basis of $L^2(M(\Gamma, \tau))$.



Proof. Let $\{h_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\Gamma \times (0,\tau))$ be a complete set in $L^2(\Gamma \times [0,\tau])$.
We can compute that inner products

$$c_{jk} = \langle u^{h_j}(\tau), u^{h_k}(\tau)\rangle.$$

Next we use the Gram-Schmidt orthogonalization procedure to construct f_j. More precisely, we define $f_j \in L^2(\Gamma \times [0,\tau])$ recursively by

$$g_j = h_j - \sum_{k=1}^{j-1} \langle u^{h_j}(\tau), u^{f_k}(\tau)\rangle f_k,$$

$$f_j = \frac{g_j}{\langle u^{g_j}(\tau), u^{g_j}(\tau)\rangle^{1/2}}.$$

When $g_j = 0$, we remove the corresponding h_j from the original sequence and continue the procedure. $\square$

Since $\{h_j\} \subset C_0^\infty(\Gamma \times [0,\tau])$, we have $f_j \in C_0^\infty(\Gamma \times [0,\tau])$. Thus $u^{f_j}(\tau) \in C^\infty(M)$.

Let $T >$ diam$\,(M)$. Then $M(\partial M, T) = M$, and the corresponding wave basis

$$\{u^{\eta_j}(T)\}_{j=1}^{\infty}$$

is the orthonormal basis in $L^2(M)$. Next we reserve the notation $\eta_j \in C^\infty(\partial M \times (0,T))$ for such boundary values.

2.8 Projectors

Denote by $P_{\Gamma,\tau}$ the orthogonal projector in $L^2(M)$ onto the space $L^2(M(\Gamma,\tau))$,

$$P_{\Gamma,\tau} : L^2(M) \to L^2(M(\Gamma,\tau)),$$

$$(P_{\Gamma,\tau}a)(x) = \chi_{M(\Gamma,\tau)}(x)a(x),$$

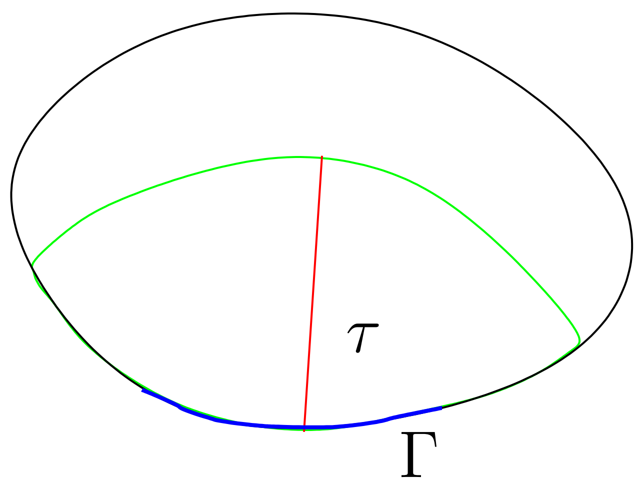
where $\chi_{M(\Gamma,\tau)}$ is the characteristic function of the domain of influence $M(\Gamma,\tau)$,

$$\chi_{M(\Gamma,\tau)}(x) = \begin{cases} 1, & \text{for } x \in M(\Gamma,\tau), \\ 0, & \text{for } x \notin M(\Gamma,\tau). \end{cases}$$

Lemma 2.9 *Let* $f, h \in L^2(\partial M \times [0, T])$ *and* $\Gamma \subset \partial M$ *be an open set. Then, given the the map* Λ*, it is possible to find the inner product*

$$\langle P_{\Gamma,\tau} u^f(t), u^h(s) \rangle = \int_{M(\Gamma,\tau)} u^f(x,t)\, u^h(x,s)\, dV_g$$

for any $0 \leq t, s, \tau \leq T$*.*



Proof. We can find $f_j \in C_0^\infty(\Gamma \times [0,\tau])$ such that $v_j = u^{f_j}(\tau)$ is an orthonormal basis in $L^2(M(\Gamma,\tau))$,
Then, for any $a \in L^2(M(\Gamma,\tau))$,

$$a = \sum_{j=1}^{\infty} \langle a, v_j \rangle \, v_j.$$

As $\langle P_{\Gamma,\tau} u^f(t), v_j \rangle = \langle u^f(t), v_j \rangle$, we have

$$\langle P_{\Gamma,\tau} u^f(t), u^h(s) \rangle = \sum_{j=1}^{\infty} \langle u^f(t), v_j \rangle \langle u^h(s), v_j \rangle.$$

Here $\langle u^f(t), v_j \rangle$ and $\langle u^h(s), v_j \rangle$ can be computed using boundary data. $\square$

Denote by $M(y,\tau)$ the domain of influence of a point $y \in \partial M$,

$$M(y,\tau) = \{x \in M : \ d(x,y) \le \tau\},$$

and by $P_{y,\tau}$ the orthoprojector

$$P_{y,\tau} : L^2(M) \to L^2(M(y,\tau)).$$

Corollary 2.10 *Let* $f, h \in L^2(\partial M \times [0,T])$ *and* $y \in \partial M$ *be given. Then the boundary data determine the inner product*

$$\langle P_{y,\tau} u^f(t), u^h(s) \rangle = \int_{M(y,\tau)} u^f(x,t)\, u^h(x,s)\, dV_g$$

for any $0 \le t, s, \tau \le T$*.*

Proof. Let $\Gamma_l,\ l = 1, 2, \ldots$ be open sets such that

$$\Gamma_{l+1} \subset \Gamma_l, \quad \bigcap_{l=1}^{\infty} \Gamma_l = \{y\}.$$

Then,

$$\lim_{l\to\infty} \chi_{M(\Gamma_l,\tau)}(x) = \chi_{M(y,\tau)}(x)$$

pointwise. By the Lebesgue dominated convergence theorem,

$$\lim_{l\to\infty} \langle P_{\Gamma_l,\tau} u^f(t), u^h(s)\rangle = \langle P_{y,\tau} u^f(t), u^h(s)\rangle.$$

$\square$

Corollary 2.11 *Let $f \in L^2(\partial M \times [0,T])$ and $y \in \partial M$. Then the boundary data determine uniquely the inner product*

$$\langle P_{y,\tau} u^{\eta_k}(T), u^{\eta_l}(T)\rangle = \sum_{j=1}^{\infty} \langle u^{\eta_k}(T), u^{f_j}(\tau)\rangle \langle u^{\eta_l}(T), u^{f_j}(t)\rangle,$$

where $\{u^{f_j}(\tau)\}_{j=1}^{\infty}$ form an orthonormal basis in $L^2(M(y,\tau))$.

Corollary 2.12 *Let* $f \in L^2(\partial M \times [0,T])$ *and* $y_j \in \partial M$, $\tau_j > 0$. *Then the boundary data determine the inner product*

$$\langle Q_N u^f(s), u^{\eta_l}(T)\rangle$$

where

$$Q_N = \prod_{j=1}^{N} P_{y_j,\tau_j}$$

and $\{u^{f_j}(\tau)\}_{j=1}^{\infty}$ *form an orthonormal basis in* $L^2(M(y,\tau))$.

Proof. For $N = 1$ the claim follows from Corollary 2.11. Assume now that it is valid for $N - 1$.
We can write

$$Q_{N-1}u^f(s) = \sum_{k=1}^{\infty} \langle Q_{N-1}u^f(s), u^{\eta_k}(T)\rangle u^{\eta_k}(T)$$

and

$$\langle Q_N u^f(T), u^{\eta_l}(T)\rangle = \langle P_{y_N,\tau_N} Q_{N-1}u^f(T), u^{\eta_l}(T)\rangle$$

$$= \sum_{k=1}^{\infty} \langle P_{y_N,\tau_N} u^{\eta_k}(T), u^{\eta_l}(T)\rangle \langle Q_{N-1}u^f(s), u^{\eta_k}(T)\rangle.$$

From this the claim follows by induction.
□

Observations:

- We can compute the Gram matrix $[q_{jk}]_{j,k=1}^{\infty}$,

$$q_{jk} = \langle Q u^{\eta_j}(T), u^{\eta_k}(T) \rangle$$

where $\{u^{\eta_j}(T)\}_{j=1}^{\infty}$ is an orthonormal basis in $L^2(M)$ and

$$Q = \left(\prod_{j=1}^{N} P_{y_j,\tau_j^+} \right) \left(\prod_{j=1}^{N} (1 - P_{y_j,\tau_j^-}) \right)$$

- The projector $Q : L^2(M) \to L^2(M)$ is

$$Qv(x) = \chi_I(x)\, v(x), \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

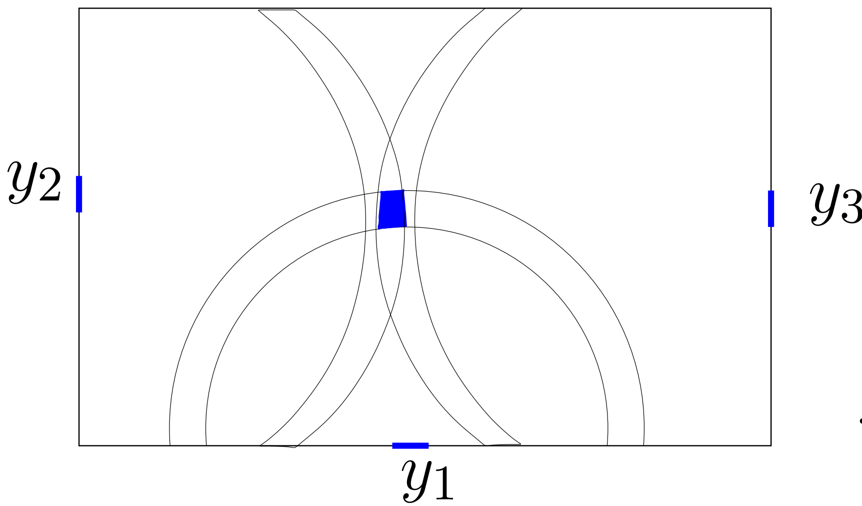
- The projector $Q : L^2(M) \to L^2(M)$ is

$$Qv(x) = \chi_I(x)\, v(x), \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

- The projector $Q : L^2(M) \to L^2(M)$ vanishes, that is, its Gram matrix is zero if and only if

$$m(I) = 0, \quad I = \bigcap_{j=1}^{N} (M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-)).$$

Thus we can check using boundary data if $m(I) = 0$.



$$\bigcap_{j=1}^{N} \left(M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-) \right)$$

Boundary distance functions. For $x \in M$ define

$$r_x(y) = d(x,y),\ y \in \partial M.$$

Let

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.

Theorem 3 *Using boundary data we can determine*

$$R(M) = \{r_x \in C(\partial M) : \quad x \in M\}.$$

Thus the constructed set $R(M)$ can be identified with M.

By previous observations, it is enough to prove the following result:

Lemma 2.13 *Let* $\{z_n\}_{n=1}^{\infty}$ *be a dense set on* ∂M*. Then* $r(\cdot) \in C(\partial M)$ *lies in* $R(M)$ *if and only if, for any* $N > 0$*,*

$$I_N = \bigcap_{n=1}^{N} M(z_n, r(z_n) + \frac{1}{N}) \cap \bigcap_{n=1}^{N} (M(z_n, r(z_n) - \frac{1}{N}))^c.$$

satisfies

$$m(I_N) \neq 0 \qquad (1)$$

Moreover, condition (1) can be verified using the boundary data.

Proof *"If"–part.* Assume that $r(\cdot) = r_x(\cdot)$ with some $x \in M$. Consider a ball $B_{1/N}(x)$. Then,

$$B_{1/N}(x) \subset M(z, r(z) + \frac{1}{N}) \setminus M(z, r(z) - \frac{1}{N}).$$

Thus if $B_{1/N}(x) \subset I_N$ and $m(I_N) \neq 0$.

"Only if"–part. Assume that $m(I_N) \neq 0$. Then there exists

$$x_N \in \bigcap_{n=1}^{N} \left(M(z_n, r(z_n) + \frac{1}{N}) \setminus M(z_n, r(z_n) - \frac{1}{N}) \right).$$

Since M is compact, we can choose a subsequence of x_N (denoted also by x_N), so that there exists a limit

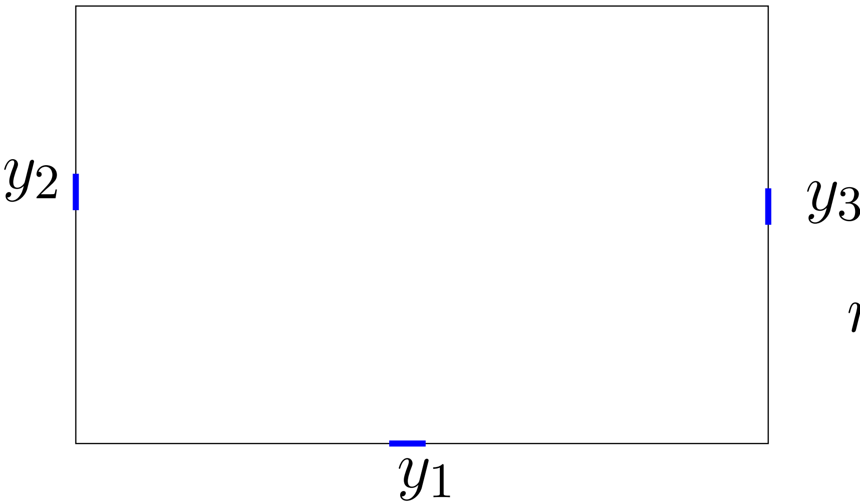
$$x = \lim_{n \to \infty} x_N.$$

By continuity of the distance function, it follows from (2) that

$$d(x, z_n) = r(z_n), \quad n = 1, 2, \ldots.$$

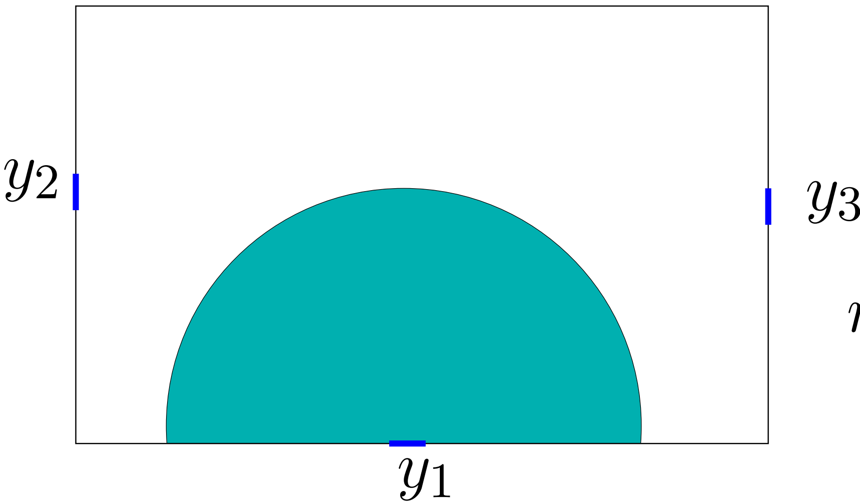
Since $\{z_n\}$ are dense in ∂M, we see that $r(z) = d(x, z)$ for all $z \in \partial M$. Thus $r = r_x$. $\square$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



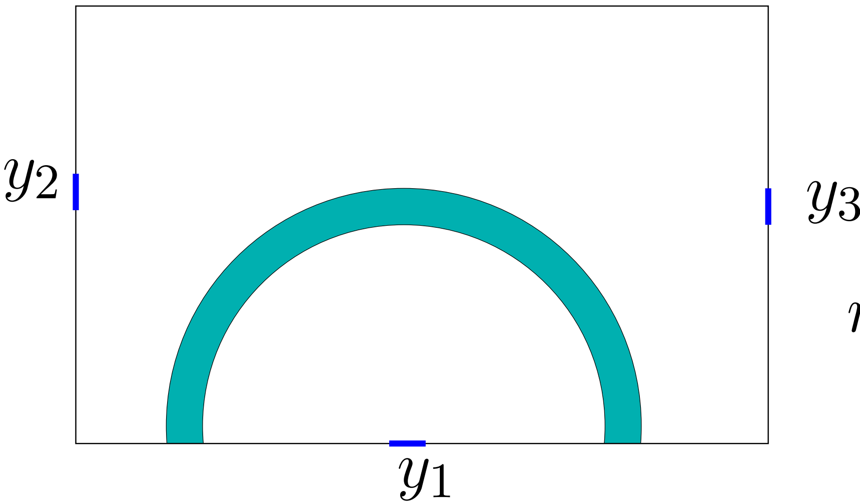
$$m\Big(\bigcap_{j=1}^{N}\big(M(y_j,\tau_j^{+}) \setminus M(y_j,\tau_j^{-})\big)\Big) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



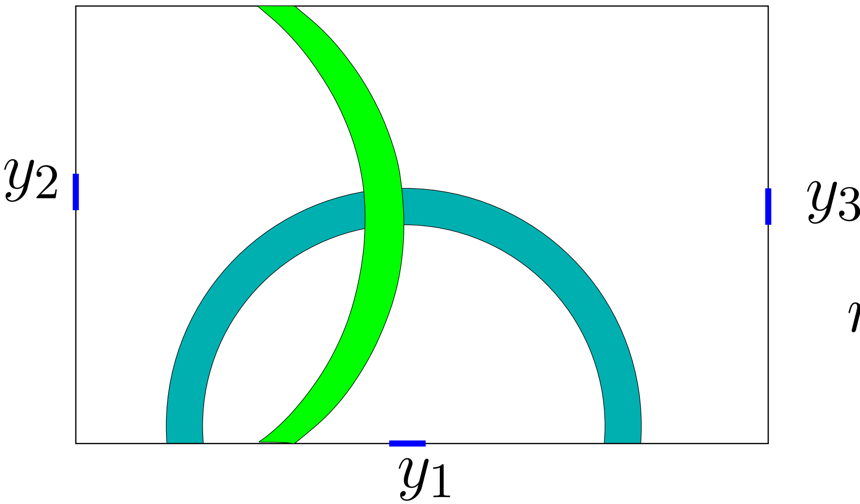
$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)\right)\right) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



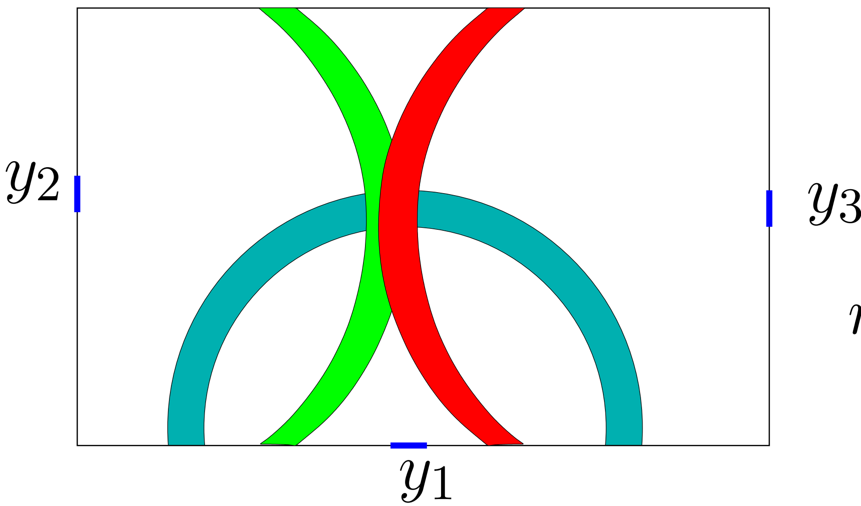
$$m(\bigcap_{j=1}^{N}(M(y_j,\tau_j^{+}) \setminus M(y_j,\tau_j^{-}))) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



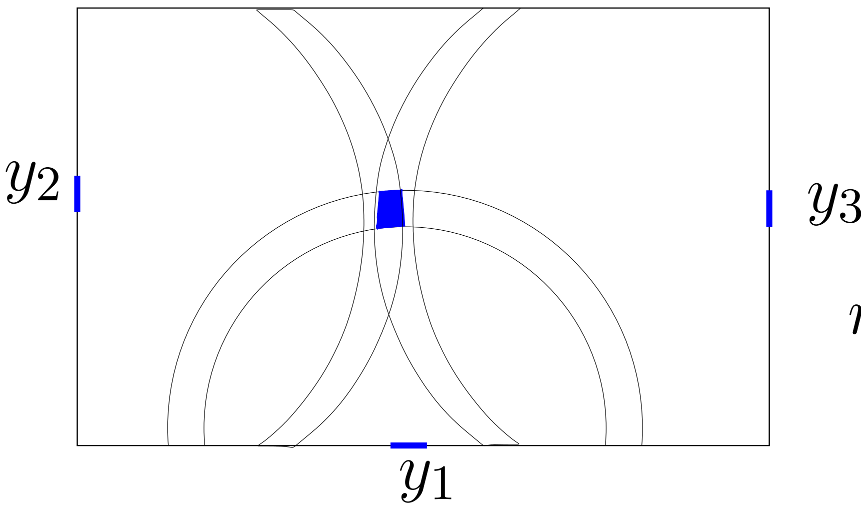
$$m(\bigcap_{j=1}^{N}(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)))=0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



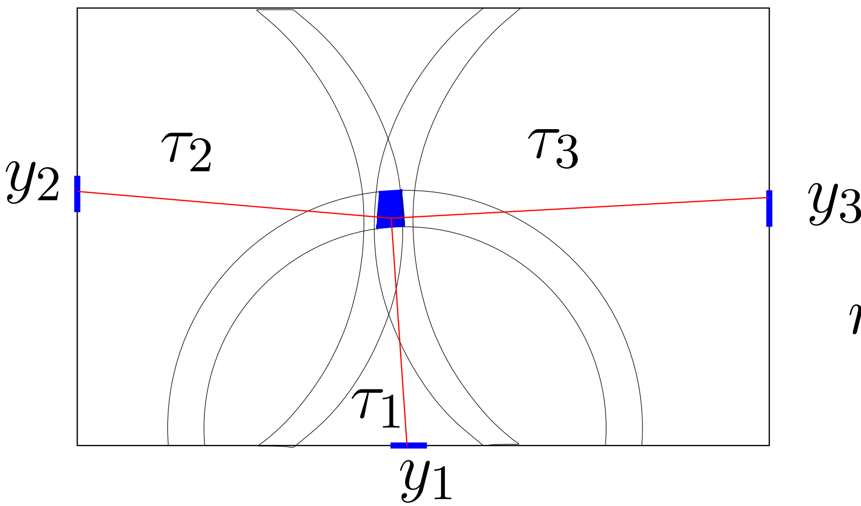
$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)\right)\right)=0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



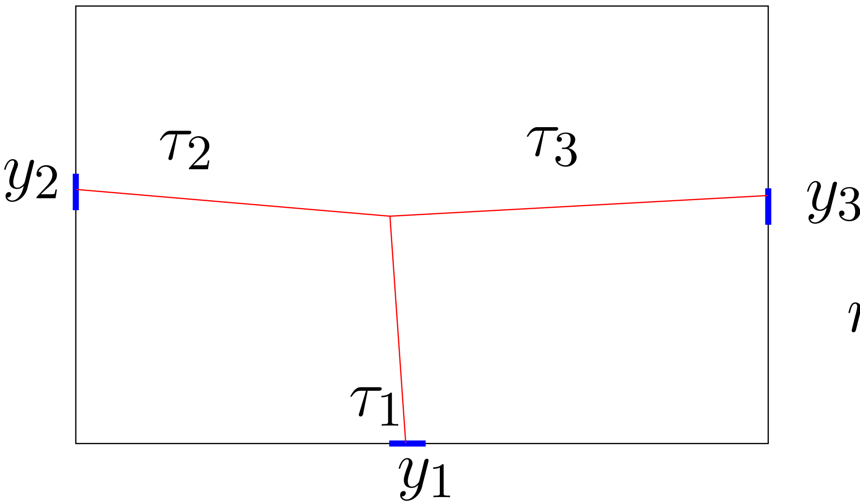
$$m(\bigcap_{j=1}^{N}(M(y_j, \tau_j^+) \setminus M(y_j, \tau_j^-))) = 0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)\right)\right)=0?$$

Visualization how to check if $r(\cdot)$ is in $R(M)$.



$$m\left(\bigcap_{j=1}^{N}\left(M(y_j,\tau_j^+)\setminus M(y_j,\tau_j^-)\right)\right) = 0?$$

2.14 Reconstruction of (M, g) from $R(M)$.

Theorem 4 *The set* $R(M)$ *has a Riemannian manifold structure which is isometric to* (M, g).

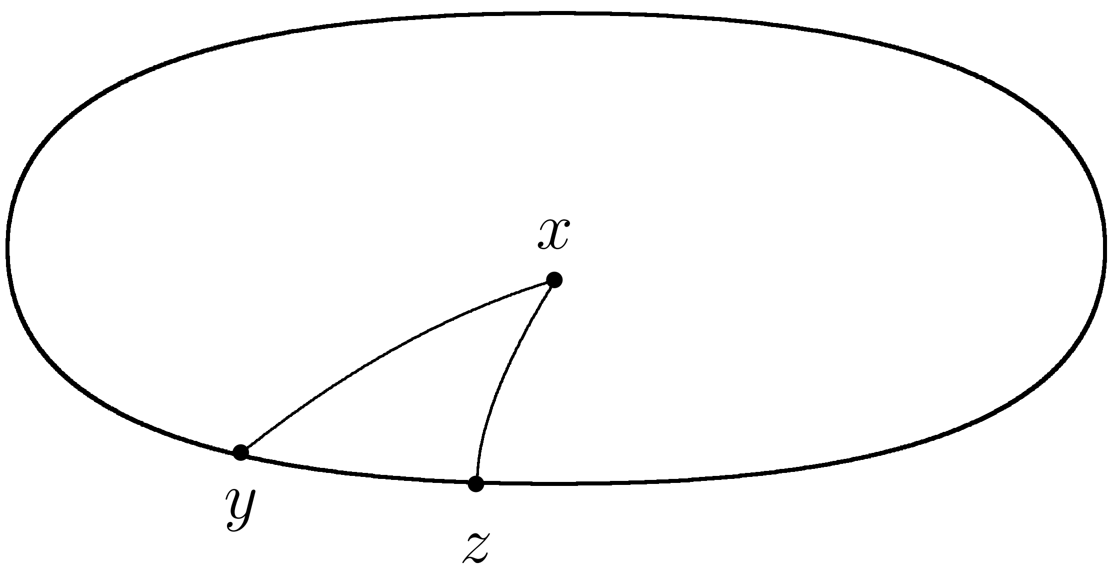
Recall that for $x \in M$

$$r_x(z) = d(x, z),\ \ z \in \partial M$$

and that

$$R : M \to C(\partial M), \quad R(x) = r_x.$$

Next we consider $R(M)$ as a submanifold on $C(\partial M)$.



By triangular inequality we have

$$\|r_x - r_y\|_{C(\partial M)} \leq d(x,y), \quad x,y \in M.$$

Example: Consider that case when all geodesics of a compact manifold (M,g) are the shortest curves between their endpoints and all geodesics can be continued to geodesics that hit the boundary. Then for any $x,y \in M$ the geodesic from x to y hits later to $z \in \partial M$. Then

$$\|r_x - r_y\|_{C(\partial M)} \geq |r_x(z) - r_y(z)| = d(x,y)$$

Then (M,d) is isometric to $(R(M), \|\cdot\|_\infty)$.

Lemma 2.15 *The set $R(M)$ is homeomorphic to (M, g).*

Proof.

Recall the following simple result from topology:

Assume that X and Y are Hausdorff spaces, X is compact and $F : X \to Y$ is a continuous, bijective map from X to Y. Then F is a homeomorphism.

Clearly, $R : M \to R(M)$ is surjective and continuous. Next we prove that it is one-to-one. Assume that $r_x(\cdot) = r_y(\cdot)$. Denote by z_0 any point where

$$d(x, \partial M) = \min_{z \in \partial M} r_x(z) = r_x(z_0) \quad \text{or}$$
$$d(y, \partial M) = \min_{z \in \partial M} r_y(z) = r_y(z_0).$$

Then z_0 is a nearest boundary point to x implying that the shortest geodesic from z_0 to x is normal to ∂M. The same is true for y with the same point z_0.
Thus $x = \gamma_{z_0}(s) = y$ for $s = d(x, z_0)$. $\square$

Boundary normal coordinates.

Consider a normal geodesic $\gamma_z(s)$ starting from z. For small s,

$$d(\gamma_z(s), \partial M) = s, \tag{2}$$

and z is the unique nearest point to $\gamma_z(s)$ on ∂M. Let $\tau(z)$ be the largest value for which (2) is valid. Then for $s > \tau(z)$,

$$d(\gamma_z(s), \partial M) < s,$$

and z is no more the nearest boundary point.

$\tau(z) \in C(\partial M)$ is the cut locus distance function.
The cut locus is

$$\omega = \{x_z : \ z \in \partial M, \ x_z = \gamma_z(\tau(z))\}.$$

In domain $M \setminus \omega$ we can use the coordinates

$$x \mapsto (z(x), t(x)),$$

where $z(x) \in \partial M$ is the unique nearest point to x and $t(x) = d(x, \partial M)$.

We will now use boundary normal coordinates to introduce a differential structure and metric tensor, g_R, on $R(M)$ to have an isometry

$$R : (M, g) \to (R(M), g_R).$$

We will concentrate mainly on doing so for $R(M) \setminus R(\omega)$. (For the general case, see [KKL])

First, observe that we can identify those $r = r_x \in R(M)$ with $x \in M \setminus \omega$.

Indeed, $r = r_x$ with $x = \gamma_z(s),\ s < \tau(z)$ if and only if

i. $r(\cdot)$ has a unique global minimum at some point $z \in \partial M$.

ii. there is $\widetilde{r} \in R(M)$ having a unique global minimum at the same z and $r(z) < \widetilde{r}(z)$.

A differential structure on $R(M \setminus \omega)$ can be defined by introducing coordinates near each $r^0 \in R(M \setminus \omega)$.
In a sufficiently small neighbourhood $V \subset R(M)$ of r^0 the coordinates

$$r \mapsto (Y(r), T(r)) = (y(\operatorname*{argmin}_{z \in \partial M} r), \min_{z \in \partial M} r)$$

are well defined. The

$$x \mapsto (Y(r_x), T(r_x))$$

coincides with the boundary normal coordinates $x \mapsto (y(x), t(x))$ on (M, g).
These coordinate determine the differential structure on $R(M \setminus \omega)$.

Construction of the metric $g_{_R}$ on $R(M)$.

Let $r^0 \in R(M \setminus \omega)$, $V \subset R(M)$ be its neighbourhood, and $Y : V \to U \subset \mathbb{R}^m$ be local coordinates, $Y(r^0) = 0$

For $z \in \partial M$ we define an evaluation function

$$K_z : V \to \mathbb{R}, \quad K_z(r) = r(z).$$

The function $E_z = K_z \circ Y^{-1} : U \to \mathbb{R}$ satisfies

$$E_z(y) := d(z, Y^{-1}(y)), \quad y \in U.$$

Consider the function $E_z(y)$ as a function of y with a fixed z. The differential dE_z at point 0 is a covector in T_0^*U. Since the gradient of a distance function has length one, we see that

$$\|dE_z\|^2_{g_R} := (g_R)^{jk}\frac{\partial E_z}{\partial y^j}\frac{\partial E_z}{\partial y^k} = 1, \quad j,k = 1,\dots,m.$$

Varying $z \in \partial M$ we obtain a set of covectors $dE_z(0)$ in the unit ball of (T_0^*U, g_R) which contains an open set.

This determines uniquely the tensor g_R.

Hence we have proven

Theorem 5 *The boundary data* $(\partial M, \Lambda)$ *determine the manifold* (M, g) *upto isometry.*

Also the potential $q(x)$ of the operator $-\Delta_g + q$ can be uniquely determined.

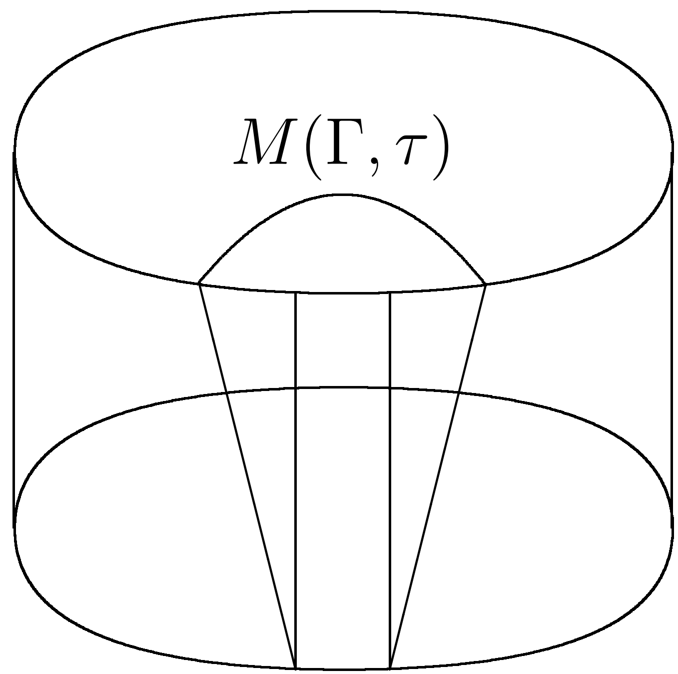
2.16 New results: Time reversal

On formal level, the the previous algorithm is based on the following task: Let f be given. Can we find h such that

$$u^h(x,T) = \chi_{M(\Gamma,\tau)}(x)u^f(x,T).$$

This is equivalent of the minimization of

$$\|u^f(T) - u^h(T)\|_{L^2(M)} : \quad h \in C_0^\infty(\Gamma \times [0,\tau]).$$



Generally, the minimization problem has no solution and is ill-posed. We consider the regularized minimization problem

$$\min_{h \in L^2(\partial M \times [0,2T])} F(h, \alpha)$$

where $\alpha \in (0,1)$ and

$$F(h, \alpha) = \langle K(Ph - f), Ph - f \rangle_{L^2(\partial M \times [0,2T], dS_g)} + \alpha \|h\|_{L^2}^2.$$

Let us recall the Blagovestchenskii identity

$$\begin{aligned}
&\int_M u^f(x,T)u^h(x,T)\,dV_\mu(x) \\
= &\int_{[0,2T]^2}\int_{\partial M} J(t,s)[f(t)(\Lambda_{2T}h)(s)-(\Lambda_{2T}f)(t)h(s)]dS_g dt ds \\
= &\int_{\partial M\times[0,2T]} (Kf)(x,t)h(x,t)\,dS_g(x)dt,
\end{aligned}$$

where $J(t,s)=\frac{1}{2}\chi_L(s,t)$ and

$$L=\{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+:\ t+s\le 2T,\ \ s>t\}.$$

Here

$$K = R_{2T}\Lambda_{2T}R_{2T}J - J\Lambda_{2T},$$

where

$$Rf(x,t) = f(x, 2T - t),$$

is the time reversal operator and

$$Jf(x,t) = \frac{1}{2}\int_0^{\min(2T-t,t)} f(x,s)ds,$$

is the time filter. Note that

$$\Lambda_{2T}^* = R_{2T}\Lambda_{2T}R_{2T} \quad \text{as} \quad G(x,x',t'-t) = G(x',x,-(t)-(-t')).$$

We also use the restriction operator

$$P_B f(x,t) = \chi_B(x,t)u(x,t),$$

The *processed time reversal iteration* is

$$
\begin{aligned}
F &:= \frac{1}{\omega} P(R\Lambda_{2T}RJ - J\Lambda_{2T})f,\\
a_n &:= \Lambda_{2T}(h_n),\\
b_n &:= \Lambda_{2T}(RJh_n),\\
h_{n+1} &:= (1-\frac{\alpha}{\omega})h_n - \frac{1}{\omega}(PRb_n - PJa_n) + F,
\end{aligned}
$$

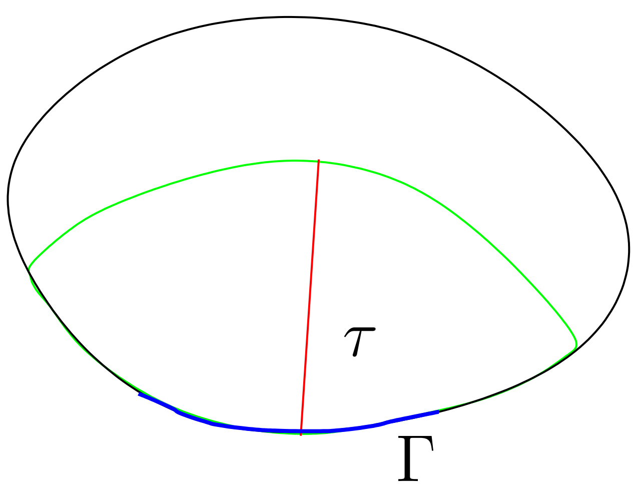
where $f \in L^2(\partial M \times [0,2T])$ and $\alpha, \omega > 0$ are parameters.
Iteration starts at $h_0 = 0$.

Theorem 6 (Bingham-Kurylev-L.-Siltanen 2007) *Let $\Gamma_1 \subset \partial M$, $0 \leq T_1 \leq T$, and $B = \Gamma_1 \times [T - T_1, T]$. Let $f \in L^2(\partial M \times \mathbb{R}_+)$ and $h_n = h_n(\alpha)$ be defined by the processed time reversal iteration. Then*

$$h(\alpha) = \lim_{n \to \infty} h_n(\alpha)$$

and the limits satisfy in $L^2(M)$

$$\lim_{\alpha \to 0} u^{h(\alpha)}(x, T) = \chi_{M(\Gamma_1, T_1)}(x) u^f(x, T).$$



$$M(\Gamma, \tau) = \{x \in M : \ d(x, \Gamma) \leq \tau\}.$$

Proof. The minimization problem

$$\min_{h \in L^2(\partial M \times [0,2T])} F(h,\alpha)$$

with $\alpha \in (0,1)$ and

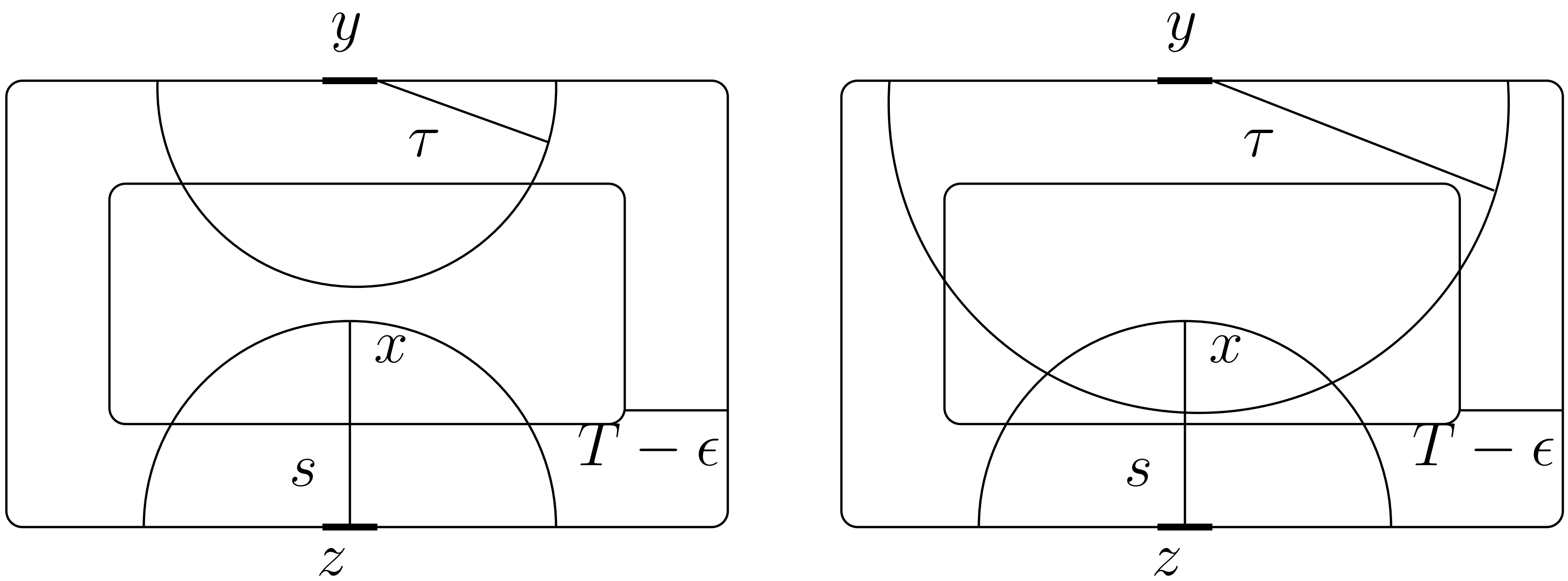
$$\begin{aligned} F(h,\alpha) &= \langle K(Ph-f), Ph-f \rangle_{L^2(\partial M \times [0,2T], dS_g)} \\ &\quad + \alpha \|h\|^2_{L^2} \end{aligned}$$

leads to a linear equation

$$(PKP+\alpha)h = PKf.$$

This can be solved using iteration. $\square$

Corollary 2.17 *Assume we are given the boundary* ∂M *and the response operator* Λ*. Then using the the processed time reversal iteration we can find constructively the manifold* (M, g) *upto an isometry and on it the operator* A *uniquely.*



Let $x = \gamma_{z,\nu}(s)$.
The distance $\text{dist}\,(x, z)$ is the infimum of all τ that satisfy the condition

(A) The set

$$(M(z, s) \cap M(y, \tau)) \setminus M(\partial M, s - \epsilon)$$

is non-empty for all $\epsilon > 0$.

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