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Mori's Technique

Exploring Hidden Gems in Birational Geometry

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Abstract

Shigefumi Mori made ground-breaking discoveries in the 1980s in the field of birational geometry, which launched the development of the Minimal Model Program for higher-dimensional varieties. The program is a tool that is used in the classification of varieties into birational equivalence classes. One of Mori's key results that is needed for the Minimal Model Program is his cone theorem, and this thesis is centred around proving this remarkable theorem. The theorem describes the structure of the cone of curves of a smooth projective variety, which encodes intersection-theoretic information about curves on the variety. The proof relies on another result by Mori, called the bend-and-break theorem, which we lay out carefully. This theorem is used to produce certain types of curves by "bending and breaking" a fixed initial curve on a smooth variety. These results were extended in 2009 using Deligne-Mumford stacks, and we provide a general outline of this development in the story. I hope to present these beautiful mathematical gems—which are hidden to the students not specialising in birational geometry—to a wider audience.

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The work contained in this thesis is my own work unless otherwise stated.

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Introduction

Classification problems constitute an important part of any branch of mathematics. In algebraic geometry the study of these problems forms a subfield of its own: *birational geometry*. The aim of birational geometry is to develop tools to classify spaces by their birational equivalence class and to study the properties shared by the members of the classes. The first step in the classification is to find a suitable representative for each equivalence class. This is achieved by the Minimal Model Program (MMP), which attempts to construct a *minimal model* for a given variety. We will concern ourselves with the theory underlying the MMP for higher-dimensional varieties, which is inspired by the classification of surfaces.

Let us briefly sketch the relevant parts in the classification of surfaces [Bea96]. Suppose S is a smooth projective surface and p is a point on S. Recall that one can construct the *blow-up* of S at the point p:

$$\pi\colon \operatorname{Bl}_p S \to S.$$

The fibre $E := \pi^{-1}(p)$—which is called the *exceptional divisor*—is a curve on $\operatorname{Bl}_p S$ isomorphic to $\mathbb{P}^1$, and the map π is an isomorphism away from E. The exceptional divisor always satisfies a specific intersection-theoretic property: its self-intersection number $E \cdot E$ is -1 (Definition 1.6, Proposition 1.8). Curves that are isomorphic to $\mathbb{P}^1$ and have self-intersection number equal to -1 are called (-1)-curves. Thus, we say that the blow-up creates a (-1)-curve on the surface. In fact, the converse is true:

Castelnuovo's Contractibility Criterion. *If S is a smooth surface admitting a (-1)-curve E, then there is a smooth surface T such that*

1. *S is the blow-up of T at a point, and*
2. *E is the exceptional divisor of the blow-up.*

This result allows us to "contract" any (-1)-curves we find on our surface to produce a simpler surface representing the same birational equivalence class. Then, given a smooth surface S, we may construct a sequence of surfaces by contracting (-1)-curves. If we reach a surface $\bar{S}$ with no (-1)-curves, we call it a minimal model of S, because any birational morphism from $\bar{S}$ is an isomorphism. Indeed, any birational morphism that is not an isomorphism factors through a sequence of blow-ups and blow-downs, but since there are no (-1)-curves, this leads to a contradiction.

The first breakthroughs in the development of the MMP for higher-dimensional varieties were achieved by Shigefumi Mori in the 1980s [Mor82; Mor86], and these brilliant insights awarded him the Fields Medal in 1990. For a smooth projective variety X, he considered a convex cone $\overline{\mathrm{NE}}(X)$ sitting in some finite-dimensional $\mathbb{R}$-vector space. He proved in the case of 3-folds over an algebraically closed field of characteristic 0 that certain parts of the cone—satisfying some intersection-theoretic property—correspond to curves on X that can be contracted by a morphism of varieties. This theorem was later extended to higher dimensions [Sho86; Rei83], and it is called the **contraction theorem** (Theorem 1.23). In general terms, these parts of the cone $\overline{\mathrm{NE}}(X)$ replace the role of (-1)-curves in the MMP, and the contraction theorem generalises Castelnuovo's criterion.

We will give a definition of $\overline{\mathrm{NE}}(X)$ and study its properties in Chapter 1. In Chapter 4, we meet the **cone theorem** (Theorem 4.1), which describes the structure of the cone $\overline{\mathrm{NE}}(X)$, which is essential for the functioning of the MMP. Indeed, we can see the cone and contraction theorems appear at the centre of the flow chart describing the MMP in Figure 1.

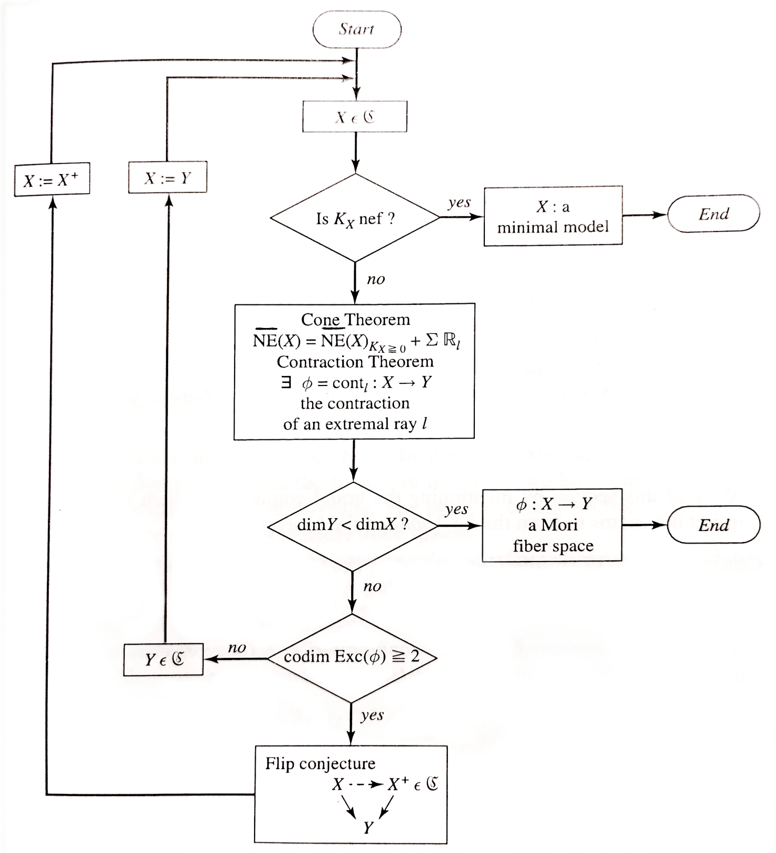

Figure 1 MMP flowchart, scanned from [Mat02].

Mori proved the cone theorem using his **bend-and-break theorem** [Mor82; Mor86], which is the topic of Chapter 3. The proof of the theorem is a beautiful combination of a deformation-theoretic argument with reduction to positive characteristic. Unfortunately, the proof does not allow for any singularities on the variety. This is a serious drawback, because even if we start the MMP with a smooth variety, the program may produce a mildly singular variety, which we need to be able to feed back into it. Another approach had to be found, so the cone theorem was proved [Kaw84; Kol84] using cohomological methods, which are more powerful—but not very geometric.

Even though there is no known way of fully rescuing the deformation-theoretic approach, I will provide a short expository account of [CT09], which extends the bend-and-break and cone theorems to LCIQ varieties using *Deligne-Mumford stacks*. What motivates me the most is the beauty of the argument, but there is also a pragmatic justification for generalising the original bend-and-break: the cohomological methods work only in characteristic 0, while the bend-and-break works in every characteristic.

I have not produced any original results in this thesis, but instead the goal is to provide a clear explanation of Mori's results. In the chapters excluding the last one, I have used [Deb01] as the main resource.

Cone of Curves

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Higher-dimensional varieties increase in complexity as one moves up in dimension. Amid this complexity we find something to hold on to, if we concentrate on the simple pieces we do understand, namely 1-dimensional subvarieties. Mori's idea was to introduce a useful abstraction—the cone of curves—that neatly packages the data of curves on a variety in an object that can be studied by the means of convex geometry. After laying out the relevant definitions, we discuss two results, where this formalism proves to be rather brilliant. The first theorem that should be highlighted is Kleiman's criterion (Theorem 1.18), which states a condition for certain line bundles to be *ample* (see Definition 1.17) in terms of the cone of curves. Ampleness itself is somewhat of a tricky notion to deal with, but this criterion provides a clean way of working with ampleness. The second result is the contraction theorem mentioned in the introduction, and it motivates the rest of this the thesis. Unless stated otherwise, the results and definitions are from [Deb01, Chapter 1].

1.1 Basic Intersection Theory

Restricting attention to curves on a variety is a good idea, but an even better idea is to form an Abelian group by taking formal sums of curves, which is exactly what we do in the following definition.

Definition 1.1. A 1-cycle on a variety X is a formal sum

$$\sum_{i=1}^{k} n_i C_i,$$

where C_i are irreducible 1-dimensional subvarieties of X and n_i are integers. Moreover, we say a 1-cycle is effective, when the coefficients are positive integers.

Now, the set of 1-cycles has the structure of a free Abelian group, but the obvious problem is that it has infinite rank. We will soon see that this problem is overcome by introducing the *intersection product*, which provides additional structure that we can work with. Namely, it leads to the definition of *numerical equivalence*, which we can quotient by to obtain a free group of finite rank. But to define the intersection product, we need to first define the object we are intersecting with 1-cycles:

Definition 1.2. A (Weil) divisor on a variety X is a formal sum

$$\sum_{i=1}^{k} n_i D_i,$$

where D_i are irreducible codimension 1 subvarieties of X and n_i are integers. Moreover, we say a divisor is effective, when the coefficients are positive integers.

It turns out that divisors are linked with sheaves, so let us describe this correspondence before we discuss the intersection product. We should first define divisors associated to rational functions, and then define the sheaf of a divisor. Thus, we make the following two definitions [Vak17, Section 14.2].

Definition 1.3. Let X be a normal variety and f a rational function on X. The divisor of zeroes and poles of f is defined as follows.

$$\operatorname{div}(f) := \sum_{Y} \operatorname{val}_Y(f) Y,$$

where the sum is taken over all irreducible codimension subvarieties Y of X, and val_Y denotes the valuation of the DVR $\mathcal{O}_{X,\eta}$, where η is the generic point of Y.

Definition 1.4. Suppose X is an normal, irreducible variety. The sheaf $\mathcal{O}_X(D)$ of the Weil divisor D on X is defined on an open set $U \subseteq X$ by

$$\Gamma\big(U, \mathcal{O}_X(D)\big) := \big\{ \varphi \in K(X)^\times \mid \operatorname{div}_U(\varphi) + D|_U \geq 0 \big\} \cup \{0\},$$

where div_U is defined by taking the sum over subvarieties of U. The definition extends directly to normal, reducible varieties.

The way to think about this definition is that the sections of $\mathcal{O}_X(D)$ are rational functions the zeroes and poles of which are controlled by D. A section of $\mathcal{O}_X(D)$ must vanish along the components of D with negative coefficients, and the section can have poles along components with positive coefficients. Furthermore, the short exact sequence below relates $\mathcal{O}_X(-D)$ to concrete objects we already understand in the case when D is an irreducible hypersurface.

When working with singular varieties, most often we take "divisors" to mean *Cartier divisors* instead of Weil divisors. Thus, we will implicity assume all divisors are Cartier unless otherwise stated. On locally factorial—and in particular smooth—varieties these two notions agree [Har97]. Moreover, Cartier divisors on normal varieties are always Weil divisors. Thus, there is no harm in sticking with Weil divisors for now. Since working with singularities will be relevant only in Chapter 5, we omit the definition of Cartier divisors in the interest of avoiding the potential confusion resulting from the two competing definitions. The following three facts are easy to prove from the definition of Cartier divisors [Har97, Section II.6].

1. For any divisor D, the sheaf $\mathcal{O}_X(D)$ is a line bundle.
2. The following two identities hold: $\mathcal{O}_X(D_1 + D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ and $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^\vee$.
3. When D is an irreducible hypersurface, $\mathcal{O}_X(-D)$ fits into the following SES.

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

Every smooth variety comes with a specific divisor, which will play an important role through out this thesis. The reason why this specific divisor is so important is that the corresponding line bundle is the "dualising sheaf" for Serre duality [Vak17, Section 18.5], and Serre duality is used in the Riemann-Roch theorem, which will be needed later.

Definition 1.5 ([Vak17, Definition 21.5.3])**.** If X is a smooth variety of dimension n, then there is a divisor K_X on X called (the) canonical divisor such that $\mathcal{O}_X(K_X)$ is the line bundle Ω_X^n of algebraic volume forms.

Note that the canonical divisor is not unique, but the line bundle $\mathcal{O}_X(K_X)$ is. When D_1 and D_2 are two divisors such that $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, we say D_1 and D_2 are linearly equivalent, and denote it by $D_1 \sim D_2$. Thus, the canonical divisor is defined up to linear equivalence.

We are now ready to define the intersection product between 1-cycles and divisors, which will be the single most important operation that we will see through out this thesis.

Definition 1.6. Suppose X is a normal, proper variety. Let C be an irreducible curve and D a divisor on X. Then, define their intersection product as

$$(D \cdot C) := \deg\big(\mathcal{O}_C(D)\big).$$

Here $\mathcal{O}_C(D)$ denotes the pullback of the line bundle $\mathcal{O}_X(D)$ along the closed embedding $C \hookrightarrow X$. Extend the definition to 1-cycles linearly.

The geometric meaning behind this definition is somewhat obscured. One can show that if the intersection of C and D is proper, then the product $(D \cdot C)$ is the number of intersection points counted with multiplicity [Deb01]. We will use this fact freely.

When calculating intersection products, it will be useful to be able to transport curves and divisors along morphisms. Thus, we define the pullback of divisors and the pushforward of curves. First, suppose $f\colon X \to Y$ is a proper morphism of proper varieties and C is an irreducible curve on X. Then, the pushforward of C is defined as follows.

$$f_*C := \begin{cases} 0, & f(C) \text{ is a point} \\ \deg\Big(X \xrightarrow{f} Y\Big) f(C), & f(C) \text{ is a curve} \end{cases}$$

Next, suppose further that X and Y are normal, and D is a divisor on Y. Recall that the pullback $f^*\mathscr{O}_Y(D)$ is a line bundle [Vak17, Theorem 16.3.7]. We define f^*D as any divisor such that $\mathscr{O}_X(f^*D) \cong f^*\mathscr{O}_Y(D)$. The existence of such a divisor is guaranteed by [Vak17, Section 14.2]. One must again be a bit careful here, because the divisor f^*D is only unique up to lineal equivalence. But knowing divisors up to linear equivalence is sufficient for us, since the intersection product is invariant under linear equivalence, as one can easily see from the definition. Having defined these functorial constructions, we can state one of the most useful formulas for computing intersection products.

Proposition 1.7 (Projection Formula). *Let $\pi\colon X \to Y$ be a proper morphism of normal, proper varieties. For an irreducible curve C on X and a divisor D on Y, we have*

$$(\pi^*D \cdot C) = (D \cdot \pi_*C).$$

Proof. Suppose first that π contracts C to a point. By adding $\operatorname{div}(f)$ to D, where f is some suitable rational fuction, we may assume no component of D contains the point $\pi(C)$. Note that adding $\operatorname{div}(f)$ does not change the linear equivalence class of D [Vak17, Section 14.2]. Then, no component of π^*D intersects C, so $(\pi^*D \cdot C) = 0 = (D \cdot \pi_*C)$.

Next, suppose $\pi(C)$ is a curve, so $(D \cdot \pi_*C) = \deg(\pi)(D \cdot \pi(C))$. Using [Har97, Proposition 6.9], we see that

$$(\pi^*D \cdot C) = \deg \mathscr{O}_C(\pi^*D) = (\deg(\pi)) \deg \mathscr{O}_{\pi(C)}(D) = (\deg(\pi))(D \cdot \pi(C)) = (D \cdot \pi_*C). \qquad \square$$

As an example, we can use this formula to verify that the exceptional divisor of the blow-up of a point on a surface has self-intersection number equal to -1.

Proposition 1.8 ([Bea96, Proposition II.3]). *Suppose S is a smooth, proper surface and fix a point $p \in S$. Let $\pi\colon \widehat{S} \to S$ be the blow-up of S at p and denote by E the exceptional divisor of π. Then $(E \cdot E) = -1$.*

Proof. Let C be a curve on S intersecting p. If the multiplicity of C at p is 1, then $\pi^*C = \widehat{C} + E$, where $\widehat{C}$ is the strict transform of C. The strict transform intersects E once at the point corresponding to the tangent direction of C with multiplicity 1. Thus, by the projection formula,

$$1 = \widehat{C} \cdot E = \Big(\pi^*C - E\Big) \cdot E = (\pi^*C \cdot E) - E^2 = (C \cdot \pi_*E) - E^2 = 0 - E^2. \qquad \square$$

1.2 Numerical Equivalence and the Cone

The intersection product provides now new structure attached to 1-cycles, which allows us to solve the problem of 1-cycles forming a group of infinite rank. The solution is to take the quotient of the group by an equivalence relation induced by the intersection product.

Definition 1.9. Two 1-cycles Z_1 and Z_2 on some normal, proper variety X are said to be numerically equivalent, if $(D \cdot Z_1) = (D \cdot Z_2)$ for every divisor D on X. When Z_1 and Z_2 are numerically equivalent, we write $Z_1 \equiv Z_2$.

It is easy to see that numerically trivial 1-cycles (those numerically equivalent to 0) form a subgroup of the group of all 1-cycles. Therefore, the following quotient is a (free) Abelian group.

$$N_1(X)_{\mathbb{Z}} := \{ \text{ 1-cycles on } X \} / \equiv$$

We can make the analogous definitions for divisors:

Definition 1.10. Two divisors D_1 and D_2 on some normal, proper variety X are said to be numerically equivalent, if $(D_1 \cdot C) = (D_2 \cdot C)$ for every irreducible curve C on X. When D_1 and D_2 are numerically equivalent, we write $D_1 \equiv D_2$.

As for 1-cycles, we obtain an Abelian group of divisors.

$$N^1(X)_{\mathbb{Z}} := \{ \text{ divisors on } X \} / \equiv$$

The intersection product descends to a perfect pairing between these two $\mathbb{Z}$-modules.

Proposition 1.11. *For a normal, proper variety X, the map*

$$\begin{array}{ccc} N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} & \longrightarrow & \mathbb{Z} \\ (D, Z) & \longmapsto & (D \cdot Z) \end{array}$$

is a well-defined and non-degenerate $\mathbb{Z}$-bilinear morphism.

Proof. One immediately sees that it is well-defined by the definition of numerical equivalence. Now, suppose Z is a 1-cycle such that $(D \cdot Z) = 0$ for every divisor D. Then, $Z \equiv 0$. Therefore, $N^1(X)_{\mathbb{Z}} \times N_1(X)_{\mathbb{Z}} \to \mathbb{Z}$ is non-degenerate on the right. By the same argument, it is also non-degenerate on the left. □

In particular, this implies that $N_1(X)_{\mathbb{Z}}$ and $N^1(X)_{\mathbb{Z}}$ have the same rank. In fact, taking the quotient by numerical equivalence always results in a free Abelian group of finite rank! The proof is difficult, so we omit it.

Theorem 1.12 ([Kle66])**.** *If X is a proper variety, then the group $N^1(X)_{\mathbb{Z}}$ has finite rank.*

Remark. Here is another motivation for studying 1-cycles modulo numerical equivalence in addition to the above theorem: Recall that contractions of curves play an important role in the MMP. Thus, suppose $\pi : X \to Y$ is a proper morphism of normal, projective varieties that contracts an irreducible curve C on X. Fix an irreducible curve C' that is numerically equivalent to C. Then, for every divisor D on Y, we have

$$0 = D \cdot \pi_* C = \pi^* D \cdot C = \pi^* D \cdot C' = D \cdot \pi_* C'.$$

Now, suppose Y embeds into $\mathbb{P}^n$ and fix a point p on $\pi_* C'$. If C' is not contracted by π, then $\pi_* C'$ is a positive multiple of an irreducible curve. Then, one can find a hyperplane H in $\mathbb{P}^n$ through p that does not contain $\pi_* C'$, implying $(H \cdot \pi_* C') > 0$, which is a contradiction. Therefore, if we want to understand the curves that are contracted by a morphism, we may as well talk about numerical equivalence classes contracted by it.

Since we know that the two $\mathbb{Z}$-modules have finite rank, we can extend scalars by taking tensor products with fields to obtain finite dimensional vector spaces. Thus, we define

$$N_1(X)_{\mathbb{Q}} := N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \quad \text{and} \quad N_1(X)_{\mathbb{R}} := N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}.$$

The elements of $N_1(X)_{\mathbb{Q}}$ are called rational 1-cycles and the elements of $N_1(X)_{\mathbb{R}}$ are called real 1-cycles. When we want to emphasise that a 1-cycle Z belongs to $N_1(X)_{\mathbb{Z}}$, we call it an integral 1-cycle. The spaces $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ are defined similarly, and the elements of $N^1(X)_{\mathbb{Q}}$ (resp. $N^1(X)_{\mathbb{R}}$) are called $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-divisors). The intersection product defines a duality between $N_1(X)_{\mathbb{R}}$ and

$N^1(X)_{\mathbb{R}}$. Thus, an $\mathbb{R}$-divisor D can be thought of as a functional on $N_1(X)_{\mathbb{R}}$, and the subset of real 1-cycles Z with $(D \cdot Z) = 0$, is a linear subspace denoted by $\{D = 0\}$. Similarly, we denote $\{D > 0\}$ and $\{D < 0\}$ for the subsets where the intersection product with the real 1-cycles is positive and negative respectively. We say a an element of $\{D > 0\}$ is D-positive. Similarly for $\{D < 0\}$.

We now come to the main definition.

Definition 1.13. When X is a normal, proper variety, let $\mathrm{NE}(X)$ be the convex cone in $N_1(X)_{\mathbb{R}}$ generated by numerical equivalence classes of effective 1-cycles. In other words,

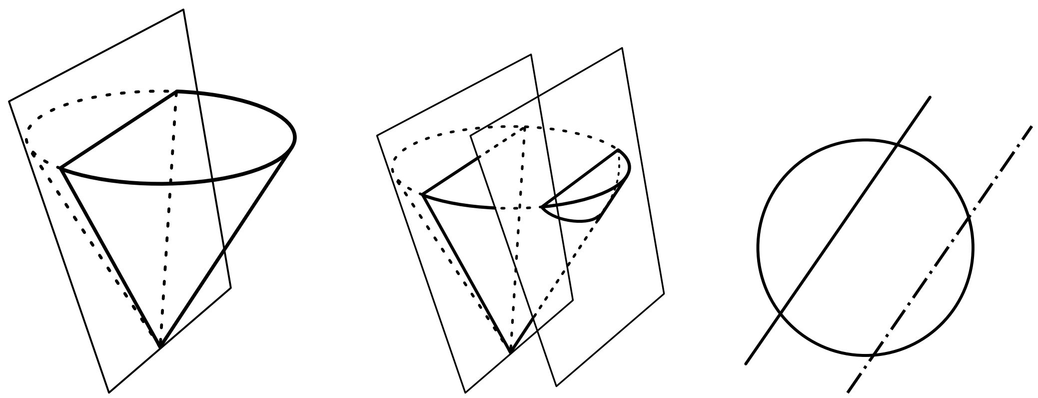
$$\mathrm{NE}(X) := \left\{ \sum_{i=1}^{k} r_i[C_i] \;\middle|\; r_i \in \mathbb{R}^+, C_i \subset X \text{ irreducible curve} \right\}.$$

The cone of curves $\overline{\mathrm{NE}}(X)$ of X is the closure of $\mathrm{NE}(X)$ in $N_1(X)_{\mathbb{R}}$ with respect to the standard topology.

We will be splitting the cone into pieces based on where a fixed divisor is positive or negative. Thus, we use the following notations.

$$\overline{\mathrm{NE}}(X)_{D>0} := \overline{\mathrm{NE}}(X) \cap \{D > 0\} \quad \text{and} \quad \overline{\mathrm{NE}}(X)_{D<0} := \overline{\mathrm{NE}}(X) \cap \{D < 0\}.$$

Let us now fix some conventions regarding the visualisation of cones and functionals. In Figure 1.1b is an illustration of a cone along with two planes passing through it. One of the planes contains the origin and the other plane is a copy of the first one but shifted away from the origin. Instead of drawing this type of 3-dimensional illustrations, it is more convenient to draw a cross-section as in Figure 1.1c.



(a) Cone and one plane. **(b)** Cone and two planes. **(c)** Cone and two planes.

Figure 1.1 Visualising cones.

Now, if we are intersecting the cone only with planes passing through the origin, then the cross-section will look more or less the same regardless of where the section is taken from. But if a plane intersecting the cone does not contain the origin, then the cross-sections can look drastically different. Thus, the plane that does not pass through the origin is drawn as a dashed line in the cross-section to remind us that if the cross-section is taken from a "lower part of the cone", then the plane will not even be visible in the cross-section. When we are working with $\overline{\mathrm{NE}}(X)$, a plane containing the origin corresponds to the subspace $\{D = 0\}$ for some divisor D, where as a plane that does not intersect the origin might correspond—for example—to the set $\{D = 1\}$.

Note that these pictures are accurate only when the cross-sections are compact.

Definition 1.14. We say a closed convex cone $C \subset \mathbb{R}^n$ has compact cross-sections, when there is a functional $h \in (\mathbb{R}^n)^\vee$ such that h is positive on $C \setminus 0$.

This definition makes sense due to the following result.

Proposition 1.15. *Suppose $C \subset \mathbb{R}^n$ is a closed convex cone. For any functional $h \in (\mathbb{R}^n)^\vee$ such that h is positive on $C \setminus 0$, the set $\{h \leq r\} := \{ v \in C \mid h(v) \leq r \}$ is compact.*

Note that this implies in particular that the cross-sections $\{ v \in C \mid h(v) = r \}$ are compact sets.

Proof. The proof is from [KM98, Corollary 1.19]. Assume for a contradiction that $\{h \leq r\}$ is not compact. Since the set is closed, it cannot be bounded, so we can find elements $v_i \in \{h \leq r\}$ such that $\|v_i\| \to \infty$. But then, the sequence $v_i/\|v_i\|$ is bounded, which implies there is a convergent subsequence $v_{k(i)}$ with a limit $v \in \{h \leq r\}$. It follows that

$$h(v) = \lim_{i\to\infty} h\left(\frac{v_{k(i)}}{\|v_{k(i)}\|}\right) = \lim_{i\to\infty} \frac{h(v_{k(i)})}{\|v_{k(i)}\|} = 0.$$

But this contradicts the assumption that h is positive on $C \setminus 0$. $\square$

1.3 Consequences

Introducing the cone of curves gives us a new language to talk about so called "numerical properties". Ampleness is a numerical property of divisors, which has a nice criterion in terms of the cone of curves. The definition of ample line bundles is based on the notion that line bundles give rise to morphisms into projective spaces.

Proposition 1.16 ([Gat02, Lemma 7.5.14])**.** *Fix a scheme X over an algebraically closed field. Suppose $\mathscr{L}$ is a line bundle on X and $s_0, \dots, s_n$ are basepoint-free global sections, meaning that there is no point $x \in X$ where all of the sections vanish. Then, there is a unique morphism $X \to \mathbb{P}^n$ corresponding to the data $(\mathscr{L}, s_0, \dots, s_n)$. Similarly, every morphism $X \to \mathbb{P}^n$ corresponds to unique such tuple of data.*

Proof. Suppose $(\mathscr{L}, s_0, \dots, s_n)$ is as above. We can define a morphism $f\colon X \to \mathbb{P}^n$ as follows.

$$f\colon x \mapsto [s_0(x), \dots, s_n(x)]$$

Of course, the evaluating a section at a point is not well-defined. But since $\mathscr{L} \otimes \mathscr{L}^\vee \cong \mathscr{O}_X$, the quotients s_i/s_j are rational functions and have a well-defined value. Finally, if $f\colon X \to \mathbb{P}^n$ is any given morphism, then the data

$$(f^*\mathscr{O}_{\mathbb{P}^n}(1), f^*x_0, \dots, f^*x_n)$$

corresponds to f. $\square$

Definition 1.17 ([Vak17, Section 16.6])**.** Suppose X is a proper scheme over an algebraically closed field. A line bundle $\mathscr{L}$ on X is very ample, if there is a collection of basepont-free sections $s_0, \dots, s_n$ such that the morphism $X \to \mathbb{P}^n$ corresponding to $(\mathscr{L}, s_0, \dots, s_n)$ is a closed embedding. A line bundle $\mathscr{L}$ is called ample, if there is an $N > 0$ such that $\mathscr{L}^{\otimes N}$ is very ample.

Recall that for a divisor D, the sheaf $\mathscr{O}_X(D)$ is a line bundle. Therefore, we say D is ample when $\mathscr{O}_X(D)$ is. We now state the numerical criterion for ampleness of divisors.

Theorem 1.18 (Kleiman's Criterion, [Deb01, Theorem 1.27])**.** *Suppose X is a projective variety and D is a divisor on X. Then,*

$$D \textit{ is ample} \iff \overline{\mathrm{NE}}(X) \setminus \{0\} \subset \{D > 0\}$$

The proof and its prerequisites are laid out nicely by Debarre through [Deb01, Sections 1.5-1.7], but we omit them here. We see that ample divisors are precisely those that are positive on the cone of curves. We also give a name to the divisors that are non-negative on $\overline{\mathrm{NE}}(X)$.

Definition 1.19. Suppose X is a normal, proper variety. Then a divisor, D on X is said to be nef, if $\overline{\mathrm{NE}}(X) \subseteq \{D \geq 0\}$.

The criterion gives us important information about the convex geometry of the cone of curves, described in the proposition below, which we will need in the proof of the cone theorem. The proposition involves the concept of extremal rays, which we introduce now.

Definition 1.20. A ray R of a cone $C \subset \mathbb{R}^n$ is extremal, if for each pair of vectors $v, w \in C$,

$$v + w \in R \implies v, w \in R.$$

Proposition 1.21 ([Deb01, Lemma 6.7])**.** *If X is a normal, projective variety, then $\overline{\mathrm{NE}}(X)$ has compact cross-sections. Thus, the following results apply to it.*

a) If $C \subset \mathbb{R}^n$ is a closed convex cone with compact cross-sections, then for any point $v \in \partial C$, there is a functional $(\mathbb{R}^n)^\vee$ such that $f(v) = 0$ and f is non-negative on C,

b) C is the convex hull of its extremal rays, and

c) for any proper closed subcone $C' \subset C$, there is a functional $f \in (\mathbb{R}^n)^\vee$, which is positive on $C' \setminus 0$ and vanishes along some extremal ray of C.

Proof. Since X is projective, it admits an ample divisor. Thus, Kleiman's criterion directly implies that $\overline{\mathrm{NE}}(X)$ has compact cross-sections. Thus, let us now prove the general claims about cones.

a) Since v is a boundary point of C, there is a sequence (v_i) of vectors in the complement of C converging to v. Fix an index i. Since C has compact cross-sections, we can pass to a cross-section containing v_i. Let $\bar{v}_i$ be the vector in the cross-section that is the point of C closest to v_i. There is unique such points, since C is convex.

   Now, let $f_i \in (\mathbb{R}^n)^\vee$ be the functional such that $f_i(v_i) = 0$ and the restriction of $\{f_i = 0\}$ to the cross-section is orthogonal to $\bar{v}_i - v_i$. Then, f_i is positive on $C \setminus 0$. Indeed, if $f_i(w) \leq 0$ for some $w \in C \setminus 0$, then the line segment connecting w and $\bar{v}_i$ is contained in C by convexity. When we pass to the cross-section, this line intersects the open ball of radius $\|\bar{v}_i - v_i\|$ centred around v_i, but this contradicts the fact that $\|\bar{v}_i - v_i\|$ is minimal.

   Finally, we can scale the f_i so that they are bounded in $(\mathbb{R}^n)^\vee$. Thus, there is a convergent subsequence $f_{k(i)}$ converging to some $f \in (\mathbb{R}^n)^\vee$. Since the $f_{k(i)}$ are positive on $C \setminus 0$, the functional f is non-negative on C, and

$$f(v) = \lim_{i \to \infty} f_{k(i)}(v_{k(i)}) = 0.$$

b) The proof proceeds by induction on n. Note that the base case $n = 1$ is trivial. Now, fix a point $w \in \partial C$ and let f be a functional that is non-negative on C and $f(w) = 0$. By applying the induction hypothesis to the cone $\ker(f) \cap C$, we obtain an extremal ray $\mathbb{R}^+ r$ of $\ker(f) \cap C$. The ray $\mathbb{R}^+ r$ is extremal in C as well: if we have points $v_1, v_2 \in C$ such that $v_1 + v_2 \in \mathbb{R}^+ r$, then $f(v_1) + f(v_2) = 0$. Since $f(v_1), f(v_2) \geq 0$, this implies that $f(v_1) = f(v_2) = 0$.

   Next, fix an arbitrary point $v \in C$ and consider the set $\{\, \lambda \in \mathbb{R}^+ \mid v - \lambda r \in C \,\}$. The set is bounded by above, because C has compact cross-sections. Denote the maximum element by λ_0. Then, $v - \lambda_0 r \in \partial C$. Let g be a functional that is non-negative on C and $g(v - \lambda_0 r) = 0$.Then, $v - \lambda_0 r$ is in the convex hull of extremal rays of $\ker(g) \cap C$. Using the same argument as above, $v - \lambda_0 r$ is in the convex hull of extremal rays of C. Therefore, $v = (v - \lambda_0 r) + \lambda_0 r$ is in the convex hull of extremal rays of C.

c) Since C' is a proper subcone, there is a boundary point $w \in \partial C'$, which is contained in the interior of C. Then, let g be a functional that is non-negative on C' and $g(w)$. Furthermore, since C has compact cross-sections, there is a functional h, which is positive on $C \setminus 0$. Then, let

$$\lambda_0 = \inf \left\{ \, \lambda \in \mathbb{R}^+ \mid g + \lambda h \text{ is non-negative on } C \setminus 0 \, \right\}.$$

   Setting $f = g + \lambda_0 h$ gives a functional that is positive on C' and vanishes at some point of $C \setminus 0$. By part (b), the subcone $\ker(f) \cap C$ contains an extremal ray, which is also extremal in C. □

Now, let us turn to discussing contractions. Thus, fix a proper morphism $\pi : X \to Y$ of normal, projective varieties. Note first that if π contracts an irreducible curve $[C]$, then $\pi_*[rC] = 0$ for every $r \in \mathbb{R}^+$. Hence, we say π contracts the ray $\mathbb{R}^+[C]$. Furthermore, we can show that if π contracts only the ray $\mathbb{R}^+[C]$, then the ray is extremal.

Proposition 1.22 ([Deb01, Proposition 1.14]). *Suppose $\pi : X \to Y$ is a proper morphism of normal, projective varieties. If every irreducible curve contracted by π lies in the ray $R \subset \mathrm{NE}(X)$, then R is extremal.*

Proof. Let $z = \sum_{i\in I} z_i[C_i]$ and $w = \sum_{j\in J} w_j[C_j]$ be two elements of $\mathrm{NE}(X)$ with $z_i, w_j \geq 0$ for all $i \in I$ and $j \in J$ such that $z + w \in R$. Therefore, $z + w$ is contracted by π, so

$$\sum_{i\in I} z_i \pi_*[C_i] + \sum_{j\in J} w_j \pi_*[C_j] = 0.$$

This implies $\pi_*[C_k] = 0$ for all $k \in I \cup J$. Since Y is projective, this means the curves C_k are contracted by π. Hence, $z, w \in R$. □

Conversely, certain extremal rays R of $\overline{\mathrm{NE}}(X)$ have a corresponding map $\mathrm{contr}_R : X \to Y$ that contracts R. This theorem is difficult to prove and is the other key piece of the MMP puzzle along with the cone theorem.

Theorem 1.23 (Contraction Theorem, [Mat02]). *Let X be a normal, projective variety over a field of characteristic zero with only $\mathbb{Q}$-factorial and terminal singularities and R be an extremal ray in $\overline{\mathrm{NE}}(X)_{K_X<0}$. Then, there is a normal, projective variety Y and a morphism $\mathrm{contr}_R : X \to Y$ with connected fibres such that for any irreducible curve $C \subset X$,*

$$\mathrm{cont}_R \textit{ contracts } C \iff [C] \in R.$$

Deformations and Moduli 2

Deformation-theoretic methods form the back bone of Mori's technique; Mori obtains information about the cone of curves by deforming curves on the given variety. In addition to deformation theory, which is the local study of families of schemes, we will discuss moduli theory, which is the global counter-part of deformation theory. After laying out the foundations, the main goal will be to prove the dimension bound on the space of curves in Corollary 2.16. This will be a key ingredient in the proof of the bend-and-break theorem, which we will discuss in the following chapter. This chapter also prefaces much of the discussion in Chapter 5.

2.1 Moduli Spaces

When studying families of geometric objects, it is natural to ask how the members of the family are related to each other, exactly. The miracle of moduli theory is that sometimes a collection of families of algebro-geometric objects has a natural scheme structure, in which case it becomes susceptible to the methods of algebraic geometry. Broadly speaking, there are two steps in moduli theory: one first defines a *moduli problem* and then constructs the *moduli space* that solves the problem [HM98]. In modern treatments of the subject, moduli problems are described in terms of functors. Thus, we will begin by exploring the concept of the *functor of points* and how moduli problems are defined in this language. Then, we will mention *flatness*, which is the formalisation of the idea of schemes varying continuously in families. We will end this section by listing examples of moduli spaces.

The correspondence between spaces and functors is based on the famous Yoneda lemma from category theory [EH00]. I will denote by $\mathscr{D}^{\mathscr{C}}$ the category of functors $\mathscr{C} \to \mathscr{D}$ and by $\mathrm{Nat}(F, G)$ the set of natural transformations $F \Rightarrow G$.

Lemma 2.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be a category, X an object of $\mathscr{C}$, and $F\colon \mathscr{C} \to \mathbf{Set}$ a contravariant functor. Then there is a bijection*

$$\mathrm{Nat}(\mathrm{Mor}_{\mathscr{C}}(-, X), F) \cong F(X),$$

which is natural in both X and F.

A direct consequence of this lemma is that the functor

$$\begin{aligned} \mathscr{C} &\xrightarrow{\ h\ } \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}} \\ X &\longmapsto \mathrm{Mor}_{\mathscr{C}}(-, X) \end{aligned}$$

is fully faithfull. Indeed, if we set $F = \mathrm{Mor}_{\mathscr{C}}(-, Y)$, then the lemma implies

$$\mathrm{Nat}(\mathrm{Mor}_{\mathscr{C}}(-, X), \mathrm{Mor}_{\mathscr{C}}(-, Y)) \cong \mathrm{Mor}_{\mathscr{C}}(X, Y).$$

Thus, any category $\mathscr{C}$ embeds in the functor category $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$. In more concrete terms, the objects of any given category are uniquely determined by the morphisms into them. When we apply this in the case where $\mathscr{C}$ is the category **Sch** of schemes, we see that every scheme X has a corresponding functor $h_X: \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$, and X is uniquely determined by the functor h_X. But we can say more: since morphisms of schemes can be glued from morphisms on an affine open cover of the source, the scheme X can be determined from the restriction of h_X to the category $\mathbf{Aff}^{\mathrm{op}}$, where **Aff** is the category of affine scheme. Furthermore, $\mathbf{Aff}^{\mathrm{op}}$ is isomorphic to the category **CRing** of commutative rings by definition. Therefore, we make the following definition.

Definition 2.2. The functor of points of a scheme X is the functor

$$\begin{array}{ccc} \mathbf{CRing} & \xrightarrow{h_X} & \mathbf{Set} \\ R & \longmapsto & \mathrm{Mor}(\mathrm{Spec}(R), X) \end{array}$$

By the above discussion, a scheme X is completely determined by its functor of points. The explanation behind the name for these functors is that for a ring R, the elements of $h_X(R)$ are morphisms $\mathrm{Spec}\, R \to X$ and such morphisms are called R-points of X. Now, we will also want to use functors of points in the relative setting—for example, we may restrict attention to k-schemes, where k is a field. In this case we are only interested in k-morphisms. In this context, we take the functor of points of a k-scheme X to be the restriction to the category of k-algebras.

$$h_X^k : k\text{-}\mathbf{Alg} \to \mathbf{Set}$$

The k-scheme X is again uniquely determined by the functor h_X^k. When no confusion arises, we denote the functor of points of a scheme X by the same letter and the functor of points of a k-scheme by X_k. For a k-scheme X, the k-points in $X_k(k)$ correspond to the closed k-rational points of X. Indeed, for every k-point $\mathrm{Spec}\, k \to X$, there is a ring map $A \to k$, where $\mathrm{Spec}\, A$ is an affine open subset of X containing the image of $\mathrm{Spec}\, k \to X$. Then, the kernel of the map $A \to k$ is a maximal ideal. Conversely, any maximal ideal with residue field equal to k gives rise to a k-morphism $\mathrm{Spec}\, k \to X$.

Having laid out the framework of functors of points, we can come back to discussing moduli. A moduli problem is now simply a functor $F: k\text{-}\mathbf{Alg} \to \mathbf{Set}$ [HM98]. Then, the moduli space corresponding to the functor F is a k-scheme X which represents F, that is $F \cong h_X^k$. We define F so that the k-points of X—when the moduli space exists—correspond to the objects we wish to classify. Then, for example, a $k[x]$-point of X corresponds to a *family* of object above the affine line $\mathbb{A}_k^1 = \mathrm{Spec}\, k[x]$.

Here is the first example [Sta23, Tag 01ND]. Let k be an algebraically closed field and define the functor $F: k\text{-}\mathbf{Alg} \to \mathbf{Set}$ by

$$F(R) = \left\{ (\mathscr{L}, s_0, \dots, s_n) \;\middle|\; \begin{array}{c} \mathscr{L} \text{ a line bundle on Spec } R, \\ s_i \text{ basepoint-free sections} \end{array} \right\} \Big/ \sim,$$

where $(\mathscr{L}, s_0, \dots, s_n) \sim (\mathscr{N}, t_0, \dots, t_n)$ whenever there is an isomorphism $\varphi : \mathscr{L} \to \mathscr{N}$ sending $\varphi(s_i) = t_i$ for all i. Then, using Proposition 1.16, one can show that this functor is represented by $\mathbb{P}_k^n$. We know that $\mathbb{P}_k^n$ parametrises lines in the affine space $\mathbb{A}_k^{n+1}$, and indeed, every element of $F(k)$ is a line. The $n+1$ sections determine the "direction where the line points to" in $\mathbb{A}_k^{n+1}$. Also, the elements of $F(k[x])$, for example, are line bundles above $\mathbb{A}_k^1$, which is the natural way of representing a family of lines.

In order to define more moduli spaces, let us briefly discuss *flatness*. The notion of flatness encapsulates the intuitive notion that for some morphisms $\Phi : \mathscr{X} \to B$ of schemes, the fibres of Φ may vary continuously over B. This is of interest in moduli theory, because when we consider families of schemes over a base, we want to exclude morphisms, where the fibres do not vary "nicely".

Definition 2.3 ([EH00, Subsection II.3.4])**.** An R-module M is said to be flat, if for every injection $A \to B$ of R-modules, the map $M \otimes_R A \to M \otimes_R B$ is again injective. When X and Y are schemes, a morphism $\pi: X \to Y$ is flat, if for every $x \in X$, the ring $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{B,\pi(x)}$-module. The $\mathscr{O}_{B,\pi(x)}$-module structure is given by $\pi^{\#}$.

The definition is not very geometric, but intuitively speaking there are two consequences that flatness should have, which—among other properties of flatness—motivates the definition. Firstly, if B is a nonsingular curve, $\mathscr{X} \subset \mathbb{A}^n_B$, and Φ is the projection $\mathbb{A}^n_B \to B$, then any fibre of Φ should be the "limit" of the other fibres. This is true, if and only if Φ is flat; see [EH00] for the details. The other intuitive consequence, according to Mumford [Mum99], is that for *any* morphism of varieties $f\colon X \to Y$, almost all the fibres should vary continuously. More precisely, there should be a dense open set $U \subseteq Y$ such that $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is flat. Again, this is indeed the case.

Theorem 2.4 (Generic Flatness, [Vak17, Theorem 24.5.12])**.** *If $f\colon X \to Y$ is a morphism of finite type and Y is reduced, then there is a dense open subscheme $U \subset Y$ such that $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is flat.*

The definition of flatness guides us now to formulate the definition of families of curves, for example. For now, restrict to characteristic zero. We say a family of curves over a base B is a flat morphism $\pi\colon \mathscr{X} \to B$, the fibres of which are non-singular curves of a fixed genus g [EH00]. We fix the genus in the definition, because arithmetic genus of the fibres of a flat, projective morphism is always locally constant [Vak17, Corollary 24.7.2]. We can now define the moduli problem of curves with fixed genus.

Definition 2.5 ([EH00, Subsection VI.2.4])**.** Suppose $\operatorname{char} k = 0$ for the sake of simplicity, and fix a non-negative integer g. The moduli of curves is the functor $\mathscr{M}_g\colon \mathbf{Aff}_k^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$\mathscr{M}_g(B) = \{ \text{ families } \mathscr{X} \to B \text{ of curves of genus } g \ \} / \cong,$$

where $\cong$ denotes isomorphism of B-schemes.

It turns out that this functor is not representable by a scheme, but instead, it is representable by a *stack*. We will discuss this moduli space further in Chapter 5.

Next, we define the moduli spaces (more correctly, *parameter spaces*) that are relevant to Mori's technique. Given a k-scheme X, we wish to describe the space of all subschemes of X. Thus, define the following functor.

Definition 2.6 ([Kol96, Definition 1.3])**.** For a k-scheme X, the Hilbert functor $\mathrm{Hilb}_X\colon \mathbf{Aff}_k^{\mathrm{op}} \to \mathbf{Set}$ is defined as follows.

$$\mathrm{Hilb}_X(B) = \left\{ Y \subset X \times_k B \text{ subscheme} \;\middle|\; \begin{array}{c} Y \hookrightarrow X \times_k B \xrightarrow{\pi_B} B \\ \text{proper and flat} \end{array} \right\}$$

After fixing an ample line bundle on X one can define for each polynomial P the functor $\mathrm{Hilb}_{X,P}$ as the subfunctor consisting of subschemes with Hilbert polynomial P.

When X is projective, the schemes $\mathrm{Hilb}_{X,P}$ are representable by projective schemes, called the Hilbert schemes. Harris and Morrison [HM98] outline the main ideas of the proof, while Kollár [Kol96] provides a more detailed proof. Due to the generality of the Hilbert schemes, many moduli spaces are constructed from them. We are interested in the space of morphisms.

Definition 2.7 ([Kol96, Definition 1.9])**.** Suppose X and Y are k-schemes. Then, we can define the functor $\mathrm{Mor}(X, Y)\colon \mathbf{Aff}_k^{\mathrm{op}} \to \mathbf{Set}$ by

$$\mathrm{Mor}(X, Y)(T) = \left\{ k\text{-morphisms } X \times_k T \to Y \right\}$$

Note that the k-points of $\mathrm{Mor}(X, Y)$ are k-morphisms $X \to Y$. When we want to consider a k-morphism f as a point of $\mathrm{Mor}(X, Y)$, we denote it by $[f]$.

Since the graph of a morphism $X \to Y$ is a subscheme of $X \times_k Y$, the functor $\mathrm{Mor}(X, Y)$ is represented by a subscheme of $\mathrm{Hilb}_{X \times_k Y}$, called the space of morphisms [Kol96, Theorem 1.10]. Now, fix a k-scheme T. Every T-point of $\mathrm{Mor}(X, Y)$ corresponds can be thought of as a family f_t of morphisms $X \to Y$ parametrised by the points t of T [Deb01]. The T-point is called an evaluation map:

$$\mathrm{ev}\colon X \times T \to Y \ : (x, t) \mapsto f_t(x).$$

Let us make one more definition

Definition 2.8. Suppose B is a subscheme of X and $g: B \to Y$ is a morphism. We define the subscheme $\mathrm{Mor}(X, Y; g)$ of $\mathrm{Mor}(X, Y)$ as the fibre of the restriction map $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(B, Y)$ above $[g]$.

In the next section we will use deformation theory to study the structure of $\mathrm{Mor}(X, Y)$ locally at a fixed k-point $[f]$.

2.2 Deformations

The utility of the categorical way of taking schemes as functors is demonstrated, when we wish to study tangent spaces. This is immediately useful to us, since we wish to understand the local structure of moduli spaces, which are defined by the corresponding moduli problems. The following results are from [EH00, Subsection VI.1.3].

Proposition 2.9. *If X is a k-scheme, then the elements of $X_k\big(k[\varepsilon]/(\varepsilon^2)\big)$ correspond to pairs (p, v), where p is a k-point of X and $v \in T_pX$.*

Proof. Suppose first that we are given a $k[\varepsilon]/(\varepsilon^2)$-point $\varphi: \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to X$ in $X_k\big(k[\varepsilon]/(\varepsilon^2)\big)$. Let

$$\iota: \operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$$

be the inclusion corresponding to the quotient $\pi: k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ by (ε). Then, the composition

$$p = \varphi \circ \iota: \operatorname{Spec} k \to X$$

is a k-point on X. It remains to show that φ defines an element of T_pX.

After restricting to an affine open subset of X containing p, we may assume $X = \operatorname{Spec} A$. Furthermore, let $\mathfrak{m} \subset A$ be the maximal ideal corresponding to p. Recall that the Zariski tangent space T_pX is defined as the dual vector space of $\mathfrak{m}/\mathfrak{m}^2$. Thus, we wish to produce a functional on $\mathfrak{m}/\mathfrak{m}^2$ corresponding to φ. Now, consider the following triangle of k-algebra homomorphisms.

$$\begin{tikzcd} & k[\varepsilon]/(\varepsilon^2) \arrow[d, "\pi"] \\ A \arrow[ur, "\varphi^\#"] \arrow[r, "p^\#"'] & k \end{tikzcd}$$

Since $\mathfrak{m}$ is the kernel of $p^\#$, the image of $\mathfrak{m}$ under $\varphi^\#$ is contained in $\ker \pi = (\varepsilon)$. Thus, $\varphi^\#$ restricts to a k-module homomorphism $\mathfrak{m} \to (\varepsilon)$. Finally, since $\mathfrak{m}^2$ is contained in the kernel of this homomorphism, it factors through $\mathfrak{m}/\mathfrak{m}^2$.

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\varphi^\#|_{\mathfrak{m}}"] \arrow[d] & (\varepsilon) \cong k \\ \mathfrak{m}/\mathfrak{m}^2 \arrow[ur, "v"'] & \end{tikzcd}$$

The functional $v: \mathfrak{m}/\mathfrak{m}^2 \to k$ is the tangent vector corresponding to φ.

Next, suppose we are given a pair (p, v), where $p: \operatorname{Spec} k \to X$ is a k-point and $v \in T_pX$. As above, we may assume $X = \operatorname{Spec} A$ and $\mathfrak{m} \subset A$ is the maximal ideal corresponding to p. Then, p defines a k-algebra homomorphism $p^\#: A \twoheadrightarrow k$. Since $\mathfrak{m}^2$ is contained in the kernel of this morphism, it induces a surjection $A/\mathfrak{m}^2 \twoheadrightarrow k$. Then, we obtain the following short exact sequence of k-vector spaces.

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow A/\mathfrak{m}^2 \longrightarrow k \longrightarrow 0$$

As short exact sequences of vector spaces split, we have the decomposition $A/\mathfrak{m}^2 \cong k \oplus \mathfrak{m}/\mathfrak{m}^2$. We can then define

$$\bar{\varphi} \colon\ k \oplus \mathfrak{m}/\mathfrak{m}^2 \to k[\varepsilon]/(\varepsilon^2) \,:\, (s, m) \mapsto s + v(m)\varepsilon,$$

where $v \colon\ \mathfrak{m}/\mathfrak{m}^2 \to k$ is the functional defined by $v \in T_p X$. Finally, the following composition defines an element of $X_k\big(k[\varepsilon]/(\varepsilon^2)\big)$.

$$\begin{tikzcd}
A \arrow[d, two heads] \arrow[dr, "\varphi"] & \\
A/\mathfrak{m}^2 \arrow[r, "\bar{\varphi}"'] & k[\varepsilon]/(\varepsilon^2)
\end{tikzcd}$$

It is straight-forward to check that these two constructions are inverses to each other. □

Corollary 2.10. *Denote $\pi \colon\ k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ for the natural projection as above. Then, for a k-point p of X, the tangent space $T_p X$ is the fibre of the map $X_k(\pi) \colon\ X_k\big(k[\varepsilon]/(\varepsilon^2)\big) \to X_k(k)$ over p.*

Proof. Recall that $\varphi \in X_k\big(k[\varepsilon]/(\varepsilon^2)\big)$ corresponds to a vector at the k-point $\varphi \circ \iota$. Thus, it corresponds to a vector at $p \colon\ \operatorname{Spec} k \to X$, if and only if $p = \varphi \circ \iota = X_k(\pi)(\varphi)$. □

Remark. If $F \colon\ k\text{-}\mathbf{Alg} \to \mathbf{Set}$ is any functor that respects fibre products, one can actually define the tangent space of a F and equip it with a vector space structure that agrees with the structure of the Zariski tangent space in the case that F is the functor of points of a k-scheme [EH00].

We can now use this corollary to find an expression for the tangent spaces of the space of morphisms.

Proposition 2.11 ([Deb01, Proposition 2.4])**.** *If $f \colon\ X \to Y$ is a morphism of k-varieties, where X is projective and Y is quasi-projective, then*

$$T_{[f]} \operatorname{Mor}(X, Y) \cong H^0\big(X, \mathscr{H}om\big(f^*\Omega^1_Y, \mathscr{O}_X\big)\big).$$

If Y is furthermore smooth, then

$$T_{[f]} \operatorname{Mor}(X, Y) \cong H^0\big(X, f^*T_Y\big).$$

Proof. Using the corollary, we identify the tangent vectors at $[f]$ with $k[\varepsilon]/(\varepsilon^2)$-points lying in the fibre of $\operatorname{Mor}(X, Y)(\pi)$ above $[f]$. By the definition of $\operatorname{Mor}(X, Y)$, a $k[\varepsilon]/(\varepsilon^2)$-point is a morphism

$$\varphi \colon\ X \times k[\varepsilon]/(\varepsilon^2) \to Y$$

and the equality $\operatorname{Mor}(X, Y)(\pi)(\varphi) = [f]$ is equivalent to the commutativity of the following diagram.

$$\begin{tikzcd}
 & X \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[ddr, "\varphi"] & \\
X \times \operatorname{Spec} k \arrow[ur, "\mathrm{id}_X \times \iota"] & & \\
 & X \arrow[ul, "\cong"'] \arrow[uu] \arrow[r, "f"] & Y
\end{tikzcd}$$

Thus, the tangent vectors at $[f]$ correspond to such extensions $\varphi \colon\ X \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to Y$ of f. We wish to show that these are in correspondence with global sections of $\mathscr{H}om(f^*\Omega^1_Y, \mathscr{O}_X)$. It suffices to show this affine-locally: Cover X and Y with affine open sets U_i and V_i respectively such that $f(U_i) \subseteq V_i$. Then, φ can be constructed by gluing extensions $\varphi_i \colon\ U_i \times \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to V_i$ of $f|_{U_i}$. Similarly, a global section of $\mathscr{H}om(f^*\Omega^1_Y, \mathscr{O}_X)$ can be constructed by gluing sections over the U_i.

Thus, assume $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then, the above triangle corresponds to the following commutative triangle of k-algebras.

$$\begin{tikzcd} & B \otimes_k k[\varepsilon]/(\varepsilon^2) \arrow[d, two heads] \\ A \arrow[ur, "\varphi^\#"] \arrow[r, "f^\#"'] & B \end{tikzcd}$$

We can therefore write $\varphi^\# = f^\# + \varepsilon v$ for some map $v\colon A \to B$. One can now check that since $\varphi^\#$ respects multiplication, we have that

$$v(aa') = f^\#(a)v(a') + f^\#(a')v(a).$$

If we consider B as an A-module via the map $f^\#\colon A \to B$, then the above equation can be read as the Leibniz rule. Since v is also k-linear, it is a derivation of A into B. By the universal property of the module of differentials [Vak17], the map v factorises through $d\colon A \to \Omega_{k/A}$ as $A \xrightarrow{d} \Omega_{A/k} \longrightarrow B$. In conclusion, we see that the extensions of f are in correspondence with A-module homomorphisms $\Omega_{A/k} \to B$. These are the elements of

$$\operatorname{Hom}_A\big(\Omega_{A/k}, B\big) \cong \operatorname{Hom}_B\big(\Omega_{A/k} \otimes_A B, B\big) \cong H^0\big(X, \mathscr{H}om(f^*\Omega^1_Y, \mathscr{O}_X)\big).$$

Finally, if Y is smooth, then f^*T_Y and $f^*\Omega^1_Y$ are vector bundles by [Vak17, Exercise 21.2.Q]. Hence, we have that

$$\mathscr{H}om\big(f^*\Omega^1_Y, \mathscr{O}_X\big) \cong \big(f^*\Omega^1_Y\big)^\vee \otimes \mathscr{O}_X \cong f^*T_Y. \qquad \square$$

2.3 Dimension of the Space of Morphisms

Proposition 2.11 gives us an upper bound on the the dimensions of the irreducible components of $\operatorname{Mor}(X, Y)$ at a given point. This section is concerned with finding a lower bound. The lower bound is obtained from the dimension of the tangent space by subtracting by the dimension of a certain space of obstructions. The following lemma describes the obstructions in terms of cohomology.

Lemma 2.12 ([Deb01, Lemma 2.7])**.** *Suppose $(R, \mathfrak{m})$ is a Noetherian local k-algebra with residue field k and $I \subset R$ is an ideal such that $I \subseteq \mathfrak{m}$ and $\mathfrak{m}I = 0$. Fix a morphism $f\colon X \to Y$ of k-varieties, where Y is smooth, and suppose f has an extension $f_{R/I}\colon X \times \operatorname{Spec}(R/I) \to Y$. The obstruction to having an extension of $f_{R/I}$ as in the below diagram lies in $H^1\big(X, f^*T_Y\big) \otimes_k I$.*

$$\begin{tikzcd} X \times \operatorname{Spec} R \arrow[dr, "f_R"] & \\ X \times \operatorname{Spec}(R/I) \arrow[u] \arrow[r, "f_{R/I}"'] & Y \end{tikzcd}$$

Proof. Let us first suppose $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then, an extension f_R corresponds to the following commutative triangle of k-algebras.

$$\begin{tikzcd} & B \otimes_k R \arrow[d, two heads] \\ A \arrow[ur, "f_R^\#"] \arrow[r, "f_{R/I}^\#"'] & B \otimes_k R/I \end{tikzcd}$$

The existence of the lift $f_R^{\#}$ is always guaranteed by the Infinitesimal Lifting Property [Har97, Exercise II.8.6] as Y is smooth and $I^2 = 0$. Moreover, two such lifts differ by a k-derivation of A into $B \otimes_k I$. After applying the universal property of $\Omega_{A/k}$—as in the proof of Proposition 2.11—we see that these k-derivations are in correspondence with the elements of

$$\operatorname{Hom}_A\big(\Omega_{A/k}, B \otimes_k I\big) \cong H^0\big(X, \mathscr{H}\!om(f^*\Omega^1_Y, \mathscr{O}_X)\big) \otimes_k I \cong H^0\big(X, f^*T_Y\big) \otimes_k I.$$

Now, suppose X and Y are not necessarily affine. Cover $X \times \operatorname{Spec}(R/I)$ with affine open sets U_i and Y with affine open sets V_i such that $f_{R/I}(U_i) \subseteq V_i$. We can then construct lifts $f_i \colon U_i \times \operatorname{Spec} R \to V_i$. The difference of two lifts f_i and f_j on $U_i \cap U_j$ defines an element of $H^0\big(U_i \cap U_j, f^*T_Y\big) \otimes_k I$. These elements form a Čech 1-cocycle, which in turn defines an element of $H^1\big(X, f^*T_Y\big) \otimes_k I$, since the Čech cohomology of a quasi-coherent sheaf agrees with its sheaf cohomology on a quasi-compact separated scheme [Vak17]. The 1-cycle vanishes, if and only if the local extensions can be glued to a global one. $\square$

The idea now is to first use Proposition 2.11 to show that near $[f]$, $\operatorname{Mor}(X, Y)$ is a quasi-projective variety sitting in an affine variety of dimension $h^0(X, f^*T_Y)$, and then use Lemma 2.12 to show that it can be cut out by $h^1(X, f^*T_Y)$ equations. Thus, we prove the following theorem.

Theorem 2.13 ([Deb01, Theorem 2.6])**.** *Suppose X is a quasi-projective variety and Y is a smooth projective variety. Then, for any morphism $f \colon X \to Y$, we have*

$$\dim_{[f]} \operatorname{Mor}(X, Y) \geq h^0\big(X, f^*T_Y\big) - h^1\big(X, f^*T_Y\big).$$

Before we prove the theorem, we need two more lemmas.

Lemma 2.14. *Suppose $(R, \mathfrak{m})$ is a local ring and $I \subseteq R$ is an ideal. Denote by $\overline{\mathfrak{m}}$ the image of $\mathfrak{m}$ in R/I. Then,*

$$I \subseteq \mathfrak{m}^2 \iff T_{[\mathfrak{m}]}(\operatorname{Spec} R) = T_{[\overline{\mathfrak{m}}]}(\operatorname{Spec} R/I).$$

Proof. First, note that if $I \nsubseteq \mathfrak{m}$, then $I = (1)$ and the statement follows by direct verification in this case. Thus, suppose $I \subseteq \mathfrak{m}$. Then, the following sequence of k-vector spaces is exact.

$$0 \longrightarrow I/\mathfrak{m}^2 \longrightarrow T_{[\mathfrak{m}]}(\operatorname{Spec} R) \xrightarrow{\ \varphi\ } T_{[\overline{\mathfrak{m}}]}(\operatorname{Spec} R/I) \longrightarrow 0$$

Therefore, φ is an isomorphism, if and only if $I/\mathfrak{m}^2 = 0$. $\square$

Lemma 2.15. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring and $I \subseteq R$ is an ideal such that $I \subseteq \mathfrak{m}^2$. Furthermore, suppose $J \subseteq R$ is an ideal such that $\mathfrak{m}I \subseteq J \subseteq I$. If the projection $\pi \colon R/J \twoheadrightarrow R/I$ splits, then $I = J$.*

Proof. Since $J \subseteq I$, it suffices to show $I + J \subseteq J$. Thus, fix an element $x \in I$. As $I \subseteq \mathfrak{m}^2$, there are elements $a, b \in \mathfrak{m}$ such that $x = ab$. Now, denote the section of π by $\sigma \colon R/I \to R/J$. If we chase $a + I$ and $b + I$ around the commutative triangle

$$\begin{tikzcd} & R/J \arrow[d, two heads, "\pi"] \\ R/I \arrow[ur, "\sigma"] \arrow[r, "\mathrm{id}"'] & R/I \end{tikzcd}$$

then, we see that $\sigma(a + I) = a + a' + J$ and $\sigma(b + I) = b + b' + J$, for some $a', b' \in I$. Next, we have

$$\begin{aligned} J = \sigma(I) &= (\sigma \circ \pi)(x + J) \\ &= (\sigma \circ \pi)(ab + J) = \sigma(a + I)\sigma(b + I) \\ &= (a + a' + J)(b + b' + J) \\ &= x + ab' + a'b + a'b' + J. \end{aligned}$$

This implies that $x + J \in \mathfrak{m}I + I^2 + J$. Furthermore, $I^2 \subseteq \mathfrak{m}I$ and $\mathfrak{m}I \subseteq J$ imply that $\mathfrak{m}I + I^2 + J \subseteq J$. Since x was arbitrary, we have shown that $I + J \subseteq J$. $\square$

Proof of Theorem 2.13. As we have fixed the point $[f]$, we can restrict to the quasi-projective component of $\operatorname{Mor}(X, Y)$ parametrising morphisms with Hilbert polynomial $P(m) = \chi(Y, mf^*H)$. Therefore, near $[f]$ the space $\operatorname{Mor}(X, Y)$ is an algebraic set in some affine space $\mathbb{A}^n_k$, so there is an affine open neighbourhood

$$U = \operatorname{Spec} \frac{k[x_1, \dots, x_n]}{(g_1, \dots, g_m)} \hookrightarrow \operatorname{Mor}(X, Y)$$

of $[f]$. Now, consider the Jacobian matrix

$$J = \left(\frac{\partial g_i}{\partial x_j}([f]) \right)_{1 \le i \le m, 1 \le j \le n}$$

and denote $r := \operatorname{rank} J$. Suppose WLOG that the $r \times n$-submatrix consisting of the first r rows of J has rank r. Then U embeds in the *smooth* variety $V := \operatorname{Spec} \frac{k[x_1,\dots,x_n]}{(g_1,\dots,g_r)}$. Let us restrict further to the local picture by setting $R := \mathcal{O}_{V,[f]}$ and let $I \subseteq R$ be the ideal of functions vanishing on U. In summary, we have defined the following spaces.

$$\begin{tikzcd}
& \operatorname{Spec} R \arrow[r] & V & \\
\operatorname{Spec} k \arrow[r, "{[f]}"] & \operatorname{Spec} R/I \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & U \arrow[u] \arrow[r] & \operatorname{Mor}(X, Y)
\end{tikzcd}$$

Thus, the dimension of an irreducible component of $\operatorname{Mor}(X, Y)$ through $[f]$ is the same as the dimension of the corresponding component of $\operatorname{Spec} R/I$. Now, by Krull's height theorem [Vak17, Theorem 11.3.7], the codimension in $\operatorname{Spec} R$ of an irreducible component of $\operatorname{Spec} R/I$ is bounded from above by the number of generators of I. By Nakayama's lemma [Vak17, Exercise 7.2.H], the number of generators is bounded by the dimension of $I/\mathfrak{m}I$, where $\mathfrak{m}$ is the maximal ideal of R.

Note that $\operatorname{Spec} R$ has the same tangent space at $[f]$ as $\operatorname{Mor}(X, Y)$. Moreover, since $\operatorname{Spec} R$ is smooth, we have

$$\dim(\operatorname{Spec} R) = \dim T_{[f]} \operatorname{Mor}(X, Y) = h^0\big(X, f^*T_Y\big)$$

by Proposition 2.11. Thus, we are done after bounding the dimension of $I/\mathfrak{m}I$ by $h^1\big(X, f^*T_Y\big)$. To do this, we try to find an ideal $J \subseteq R$ such that $\mathfrak{m}I \subseteq J \subseteq I$, which we understand more explicitly than I and for which $R/J \twoheadrightarrow R/I$ splits, because then $\dim_k I/\mathfrak{m}I = \dim_k J/\mathfrak{m}I$ by the Lemma 2.15. We can apply the lemma, because the tangent spaces of $\operatorname{Spec} R$ and $\operatorname{Spec} R/I$ agree, which implies $I \subseteq \mathfrak{m}^2$ by Lemma 2.14.

Let us first try to find a section of $R/\mathfrak{m}I \twoheadrightarrow R/I$, or equivalently, an extension

$$\begin{tikzcd}
\operatorname{Spec} R/\mathfrak{m}I \arrow[dr] & \\
\operatorname{Spec} R/I \arrow[u] \arrow[r] & \operatorname{Mor}(X, Y)
\end{tikzcd}$$

By Lemma 2.12, the obstruction to the existence of such a lift lies in $H^1\big(X, f^*T_Y\big) \otimes_k (I/\mathfrak{m}I)$. Denote the obstruction by

$$\sum_{i=1}^{h^1} \sigma_i \otimes r_i, \text{ where } h^1 = h^1\big(X, f^*T_Y\big).$$

Thus, if we set $J = \mathfrak{m}I + (r_1, \dots, r_{h^1})$, then the obstruction to extending $\operatorname{Spec} R/I \to \operatorname{Mor}(X, Y)$ to $\operatorname{Spec} R/J$ vanishes. If we localise the resulting triangle at $[f]$, we obtain the following commutative triangle.

$$\begin{tikzcd} \operatorname{Spec} R/J \arrow[dr] & \\ \operatorname{Spec} R/I \arrow[u] \arrow[r] & \operatorname{Spec} R/I \end{tikzcd}$$

This corresponds in the algebra side to a section of $R/J \twoheadrightarrow R/I$. By Lemma 2.15, we have

$$I = J = \mathfrak{m}I + (r_1, \ldots, r_{h^1}),$$

which implies $I/\mathfrak{m}I$ is spanned by the classes of $r_1, \ldots, r_{h^1}$. □

We can summarise the key ideas in the proof.

1. We spend some time setting up the local picture, where the localisation $\operatorname{Spec} R/I$ of $\operatorname{Mor}(X, Y)$ at $[f]$ is embedded in some smooth local scheme $\operatorname{Spec} R$ with the same tangent space.
2. We reduce the problem of bounding $\dim_{[f]} \operatorname{Mor}(X, Y)$ from below to the problem of bounding $\dim_k I/\mathfrak{m}I$ from above.
3. We know that if $\mathfrak{m}I \subseteq J \subseteq I$ and $R/J \twoheadrightarrow R/I$ splits, then we can replace I by J. Finding a section of the projection is equivalent to extending $\operatorname{Spec} R/I \to \operatorname{Mor}(X, Y)$. Adjoining to $\mathfrak{m}I$ the $h^1(X, f^*T_Y)$ elements of I that kill the obstruction to the existence of the extension yields an ideal J such that the projection splits.
4. Hence, $I/\mathfrak{m}I = J/\mathfrak{m}I$ can be generated by $h^1(X, f^*T_Y)$ elements.

To end our study of the spaces of morphisms, note that these dimension bounds have a particularly nice form in the case when the source of f is a projective curve.

Corollary 2.16. *If X is a smooth quasi-projective variety and $f : C \to X$ is a morphism from a projective curve, then*

$$\dim_{[f]} \operatorname{Mor}(C, X) \geq -K_X \cdot f_*C + (1 - g(C)) \dim(X), \text{ where } g(C) = 1 - \chi(C, \mathcal{O}_C).$$

Moreover, if B is a finite subscheme of C, then

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_B) \geq -K_X \cdot f_*C + (1 - g(C) - \operatorname{length}(B)) \dim(X).$$

Proof. The proof is based on [Deb01, Section 2.3]. Since C is a curve, we can re-write Theorem 2.13 as $\dim_{[f]} \operatorname{Mor}(C, X) \geq \chi(C, f^*T_X)$. By the Hirzebruch-Riemann-Roch theorem (see [Har97, Appendix A]), we have that

$$\begin{aligned} \chi(C, f^*T_X) &= \deg\big(\operatorname{ch}(f^*T_X) \cdot \operatorname{td}(T_C)\big)_1 \\ &= \deg\Big(\big(\dim(X) - f^*K_X\big) \cdot \Big(1 - \frac{1}{2}K_C\Big)\Big)_1 \\ &= -\deg f^*K_X - \deg\Big(\frac{1}{2}K_C\Big) \dim(X) \\ &= -K_X \cdot f_*C + (1 - g(C)) \dim(X). \end{aligned}$$

Now, suppose B is a finite subscheme of C. Recall that $\operatorname{Mor}(C, X; f|_B)$ is defined as the fibre of the map $\operatorname{Mor}(C, X) \to \operatorname{Mor}(B, X)$ above $[f|_B]$. By [Har97, Exercise 3.22b], we have that

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_B) \geq \dim_{[f]} \operatorname{Mor}(C, X) - \dim_{[f|_B]} \operatorname{Mor}(B, X).$$

Since $\dim_{[f|_B]} \operatorname{Mor}(B, X) = \operatorname{length}(B) \dim(X)$, we have that

$$\begin{aligned} \dim_{[f]} \operatorname{Mor}(C, X; f|_B) &\geq \chi(C, f^*T_X) - \operatorname{length}(B) \dim(X) \\ &\geq -K_X \cdot f_*C + (1 - g(C) - \operatorname{length}(B)) \dim(X). \end{aligned}$$

□

3 Bend-and-Break

Using the theory developed in the previous chapter, we can now prove Mori's bend-and-break theorem (Theorem 3.8). The theorem produces K_X-negative rational curves on a smooth variety X, and it is used in the proof of the cone theorem, which describes $\overline{\mathrm{NE}}(X)_{K_X<0}$ in terms of K_X-negative rational curves. The proof of the bend-and-break theorem relies on the observation that when a curve is bent (read: deformed) fixing one point, it will break into multiple components, one of which is a rational curve. To complete the proof, we need to show that a given curve can be deformed to begin with. This is done using the dimension bound in Corollary 2.16 along with an argument from reduction to positive characteristic.

3.1 Rigidity and Indeterminacies

Before we can discuss the proofs, there are two concepts we need to understand first: the rigidity lemma and resolution of indeterminacies.

Proposition 3.1 (Rigidity Lemma, [Deb01, Lemma 1.15])**.** *Suppose X, Y, Z are varieties and*

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \arrow[r, "g"] & Z \end{tikzcd}$$

are proper morphism such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. If a fibre $f^{-1}(y)$ is contracted by g, there is an open neighbourhood $Y_0 \subseteq Y$ of y and a commutative diagram

$$\begin{tikzcd} & f^{-1}(Y_0) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y_0 \arrow[rr, "\exists"] & & Z \end{tikzcd}$$

Proof. Let $p\colon X \to Y \times Z$ be the unique morphism making the following diagram commute.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "p"] \arrow[dr, "g"] & \\ Y & Y \times Z \arrow[l, "\pi_Y"] \arrow[r, "\pi_Z"'] & Z \end{tikzcd}$$

Denote $\overline{X} = \mathrm{im}(p)$, so that the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "p"] \arrow[dr, "g"] & \\ Y & \overline{X} \arrow[l, "\pi_Y|_{\overline{X}}"] \arrow[r, "\pi_Z|_{\overline{X}}"'] & Z \end{tikzcd}$$

Now, suppose that g contracts the fibre of f above $y \in Y$. Since both f and g contract $f^{-1}(y)$, the product map p also contracts $f^{-1}(y)$. Thus,

$$\{\mathrm{pt}\} = p(f^{-1}(y)) = (p \circ p^{-1} \circ \pi_Y|_{\overline{X}}^{-1})(y) = \pi_Y|_{\overline{X}}^{-1}(y).$$

Then, by upper semi-continuity of fibre dimension [Vak17, Theorem 11.4.2], there is an open neighbourhood $Y_0 \subset Y$ of Y over which $\pi_Y|_{\overline{X}}$ has finite fibres. Denote $X_0 = f^{-1}(Y_0)$, $f_0 = f|_{X_0}$, $p_0 = p|_{X_0}$, $\overline{X}_0 = \pi_Y|_{\overline{X}}^{-1}(Y_0)$, and $\pi_0 = \pi_Y|_{\overline{X}_0}$. Next, we show that π_0 is finite and $\mathscr{O}_{Y_0} = \pi_{0,*}\mathscr{O}_{\overline{X}_0}$.

Since f_0 is proper and $f_{0,*}\mathscr{O}_{X_0} \cong \mathscr{O}_{Y_0}$, it is surjective by [Vak17, Exercise 28.1.G]. Hence, $\pi_Y|_{\overline{X}}$ is also proper by [Vak17, Proposition 10.3.4]. Then, by [Vak17, Theorem 29.6.2], the morphism $\pi_Y|_{\overline{X}}$ is finite over Y_0. Thus, we have the following two commutative triangles.

$$\begin{tikzcd} & X_0 \arrow[dl, two heads, "f_0"'] \arrow[d, two heads, "p_0"] \\ Y_0 & \overline{X}_0 \arrow[l, two heads, "\pi_0"] \end{tikzcd} \qquad \begin{tikzcd} & f_{0,*}\mathscr{O}_{X_0} \\ \mathscr{O}_{Y_0} \arrow[ur, "\cong"] \arrow[r, hook] & \pi_{0,*}\mathscr{O}_{\overline{X}_0} \arrow[u, hook] \end{tikzcd}$$

The inclusion $\pi_{0,*}\mathscr{O}_{\overline{X}_0} \hookrightarrow f_{0,*}\mathscr{O}_{X_0}$ is obtained by

$$\mathscr{O}_{\overline{X}_0} \subseteq p_{0,*}\mathscr{O}_{X_0} \implies \pi_{0,*}\mathscr{O}_{\overline{X}_0} \subseteq \pi_{0,*}p_{0,*}\mathscr{O}_{X_0}.$$

We conclude from the commutative triangle on the right that $\mathscr{O}_{Y_0} = \pi_{0,*}\mathscr{O}_{\overline{X}_0}$. Finally, this implies π_0 has connected fibres by Zariski's Connectedness Lemma [Vak17, Lemma 29.5.1], so π_0 is an isomorphism, as it is finite. Therefore, we obtain the following commutative diagram.

$$\begin{tikzcd} & X_0 \arrow[dl, "f_0"'] \arrow[d, "p_0"] \arrow[dr, "g_0"] & \\ Y_0 \arrow[r, "\pi_0^{-1}"'] & \overline{X}_0 \arrow[r, "\pi_Z|_{\overline{X}_0}"'] & Z \end{tikzcd}$$

□

Proposition 3.2 (Resolution of Indeterminacy, [Bea96, Theorem II.7])**.** *Let S be a smooth surface and X a projective variety. For any rational map $\varphi : S \dashrightarrow X$, there is a finite sequence of blow-ups of points $\rho : \widehat{S} \to S$ and a **morphism** $\hat{\varphi} : \widehat{S} \to X$ making the following triangle commute.*

$$\begin{tikzcd} \widehat{S} \arrow[d, "\rho"'] \arrow[dr, "\hat{\varphi}"] & \\ S \arrow[r, dashed, "\varphi"'] & X \end{tikzcd}$$

Proof. Since X is projective, we may assume that $X = \mathbb{P}^n$ for some n. We can further assume that $\varphi(S)$ is not contained in any hyperplane of $\mathbb{P}^n$. Now, let H be an ample divisor on X and consider the pullback $\varphi^* H$. A point $P \in X$ is a base-point of $\varphi^* H$, if and only if φ is not defined at P. If there are no base-points, φ is defined everywhere. Thus, suppose there is a base-point $P \in X$.

Next, construct the blow-up $\pi_1 : \widehat{S}_1 \to S$ at P. Then, the base locus of $\pi_1^*\varphi^* H$ contains the exceptional divisor E of the blow-up. Now, let k be the greatest number such that global sections of $\pi_1^*\varphi^* H$ vanish along E with multiplicity k. Then, the divisor $D_1 := \pi_1^*\varphi^* H - kE$ has a finite base locus. Indeed, as π_1 is an isomorphism away from E and $\varphi^* H$ has a finite base locus, so does $\pi_1^*\varphi^*$ away from E, and once we subtract a high enough multiple of E, the base locus is finite also E as well. Therefore, the divisor D_1 defines a rational map $\varphi_1 : \widehat{S}_1 \dashrightarrow \mathbb{P}^n$.

Note that

$$(D_1 \cdot D_1) = (\varphi^* H, \varphi^* H) - 2(\varphi^* H, k\pi_{1,*}E) - k^2 E^2 < (\varphi^* H, \varphi^* H).$$

We see that if we continue this process, we get a chain of blow-ups

$$\widehat{S}_n \xrightarrow{\pi_n} \widehat{S}_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} \widehat{S}_1 \xrightarrow{\pi} S$$

and a corresponding divisors D_i in $\widehat{S}_i$. Furthermore, the process stops, because we have

$$D_n^2 < D_{n-1}^2 < \cdots < D_2^2 < D_1^2$$

and $D_n^2 \geq 0$ since the base locus of D_n is finite, so there is a linearly equivalent divisor intersecting it at finitely many points. Take $\rho = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n$ and $\widehat{S} = \widehat{S}_n$. □

3.2 Producing Rational Curves

Proposition 3.3 (Bend-and-Break Lemma 1, [Deb01, Proposition 3.1]). *Suppose X is a smooth projective variety and $f\colon C \to X$ is a smooth curve with a distinguished point $c \in C$. If*

$$\dim_{[f]} \operatorname{Mor}(C, X; f|_{\{c\}}) \geq 1,$$

*then there is a **rational** curve on X through $f(c)$.*

Proof. The proof proceeds in four steps.

Step 1. Begin by choosing a 1-dimensional subvariety of $\operatorname{Mor}(C, X; f_{\{c\}})$ that passes through $[f]$. Suppose T is the normalisation of such a subvariety and $\overline{T}$ is a smooth compactification of T.

Step 2. The subvariety T is a T-point of $\operatorname{Mor}(C, X)$, and thus, defines an evaluation map $\operatorname{ev}\colon C \times T \to X$. The evaluation map ev can be extended to a rational map on $C \times \overline{T}$:

$$\begin{tikzcd} C \times T \arrow[d, hook] \arrow[dr, "\operatorname{ev}"] & \\ C \times \overline{T} \arrow[r, dashed, "\overline{\operatorname{ev}}"'] & X \end{tikzcd}$$

Claim. The map $\overline{\operatorname{ev}}\colon C \times \overline{T} \dashrightarrow X$ is not defined at some point of $\{c\} \times \overline{T}$.

Proof of Claim. Assume for a contradiction that $\overline{\operatorname{ev}}$ is defined on $\{c\} \times \overline{T}$. Then, there is an open neighbourhood $U \subseteq C$ of c such that $\overline{\operatorname{ev}}$ is defined on all of $U \times \overline{T}$. Now, we have two proper morphisms

$$U \xleftarrow{\pi_U} U \times \overline{T} \xrightarrow{\overline{\operatorname{ev}}} X$$

with $\pi_{U,*}\mathcal{O}_{U\times\overline{T}} \cong \mathcal{O}_U$. Thus, by the rigidity lemma, there is an open neighbourhood $V \subseteq C$ of c and a commutative diagram

$$\begin{tikzcd} & V \times \overline{T} \arrow[dl, "\pi_V"'] \arrow[dr, "\overline{\operatorname{ev}}"] & \\ V \arrow[rr] & & X \end{tikzcd}$$

Since the restriction of $\overline{\operatorname{ev}}$ to $V \times \{[f]\}$ is $f|_V$, we see from the commutativity of the diagram that $\overline{\operatorname{ev}}$ agrees with $f|_V \times \pi_V$ on $V \times \overline{T}$. Therefore ev agrees with $f \times \pi_C$ on $C \times T$. But this implies $T = \{[f]\}$, which is a contradiction as T is 1-dimensional. ■

Choose a point $t_0 \in \overline{T}$, such that ev is not defined at (c, t_0).

Step 3. Denote $S := C \times \overline{T}$ and resolve the indeterminacies of ev using blow-ups:

$$\begin{tikzcd}
\widehat{S} \arrow[d, "\rho"'] \arrow[dr, "\widehat{\mathrm{ev}}"] & \\
S \arrow[r, dashed, "\mathrm{ev}"'] & X
\end{tikzcd}$$

Step 4. The pre-image of $C \times \{t_0\}$ under ρ is the union of the strict transform of $C \times \{t_0\}$ and an exceptional set E. Now, E is not contracted by $\widehat{\mathrm{ev}}$, because otherwise ev would be defined at (c, t_0). A rational component of E must intersect the strict transform of $\{c\} \times \hat{T}$, which implies that the image of the rational component under $\widehat{\mathrm{ev}}$ passes through $f(c)$. □

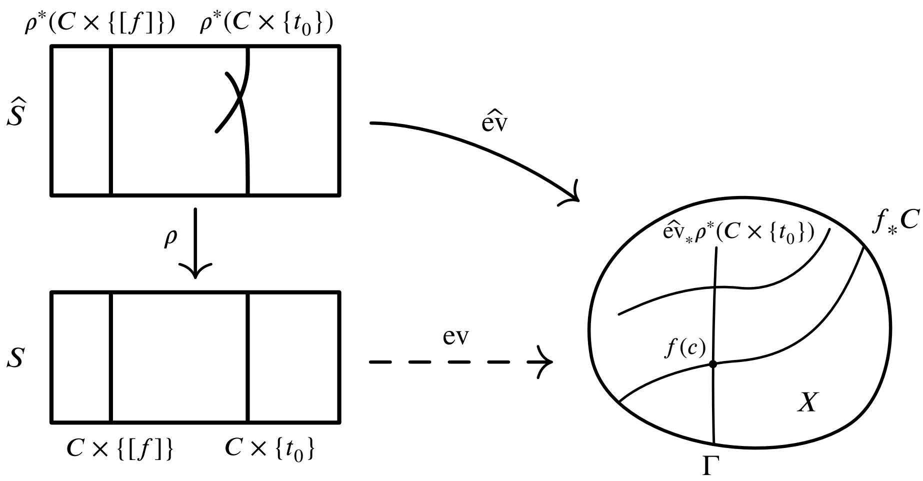


Figure 3.1 Illustration of the proof of Proposition 3.3.

Let us summarize what we did in each step:

1. Construct a 1-dimensional smooth, proper scheme $\overline{T}$, where a dense subset of points corresponds to deformations of $f : C \to X$
2. Show that the evaluation map ev : $C \times \overline{T} \dashrightarrow X$ has indeterminacies
3. Resolve the indeterminacies using blow-ups
4. Show that the resolution produces an exceptional 1-cycle, which is not contracted by the resolved morphism and a rational component of which is mapped to a curve that intersects $f(c)$

By extending this argument, we can even control the degree of the resulting rational curve.

Proposition 3.4 ([Deb01, Proposition 3.5])**.** *Suppose X is a smooth projective variety and $f : C \to X$ is a smooth curve with a distinguished subset $B \subset C$ of points. If $\dim_{[f]} \mathrm{Mor}(C, X; f|_B) \geq 1$, then there is a rational curve Γ on X intersecting $f(B)$ such that*

$$(H \cdot \Gamma) \leq \frac{2(H \cdot f_*C)}{|B|}.$$

Proof. Denote $B = \{ c_1, \dots, c_b \}$. As in the proof of Proposition 3.3, we want to fix a 1-dimensional subvariety of $\mathrm{Mor}(C, X; f|_B)$, but we also want to avoid deformations that do not move $f(C)$. Thus, let $\widetilde{C}$ be the normalisation of $f(C)$. By Proposition 2.13, the space of morphisms $C \to f(C)$ sending B to $f(B)$ has dimension at most $h^0(C, f^*T_{\widetilde{C}} \otimes I_B)$. We have two cases:

$\widetilde{C}$ is not rational When $\widetilde{C}$ is not rational, the tangent bundle $T_{\widetilde{C}}$ has non-positive degree. Since I_B has negative degree, so does $f^*T_{\widetilde{C}} \otimes I_B$. Therefore, the bundle has no global sections.

$\widetilde{C}$ is rational If $\deg f \geq b/2$, then we can simply set $\Gamma = \widetilde{C}$ and this finishes the proof. Hence, suppose $\deg f < b/2$. Since $\widetilde{C}$ is rational, we have

$$\deg(f^*T_{\widetilde{C}} \otimes I_B) = (\deg f) \cdot 2 + (-b) < \frac{b}{2}2 - b = 0.$$

We can again conclude that the bundle has no global sections.

We have now seen that these deformations form a 0-dimensional subvariety, and we can therefore choose a 1-dimensional subvariety through $[f]$ that corresponds to non-trivial deformations.

Following the argument in Proposition 3.3, we take a smooth compactification $\overline{T}$ of the normalisation of such a subvariety and we resolve the indeterminacies of the evaluation map $\mathrm{ev}\colon C \times \overline{T} \dashrightarrow X$:

$$\begin{tikzcd} \widehat{S} \arrow[d, "\rho"'] \arrow[dr, "\widehat{\mathrm{ev}}"] & \\ S \arrow[r, dashed, "\mathrm{ev}"'] & X \end{tikzcd}$$

This time we produce more exceptional 1-cycles. Denote by $E_{i,1}, \ldots, E_{i,n_i}$ the exceptional 1-cycles obtained from blowing-up points above $\{c_i\} \times \overline{T}$ for $1 \leq i \leq n$. As before, each 1-cycle $E_{i,j}$ has a component that is mapped to a rational curve on X through $f(C)$. It remains to find a 1-cycle that results in a rational curve satisfying the bound in the statement of the Proposition. This turns out to mostly be a somewhat lengthy computation in linear algebra. The reader is invited to find this series of calculations in the proof of [Deb01, Proposition 3.5]. □

3.3 Bending and Breaking Rational Curves

Now that we can produce rational curves on our variety—given there are enough deformations—we can bend and break the rational curve further to decrease its K_X-degree.

Proposition 3.5 (Bend-and-Break Lemma 2, [Deb01, Proposition 3.2])**.** *Suppose X is a smooth projective variety and $f\colon \mathbb{P}^1 \to X$ a rational curve. If $\dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}}) \geq 2$, then $f_*\mathbb{P}^1$ is numerically equivalent to an effective 1-cycle that intersects $f(0)$ and $f(\infty)$ and the components of which are rational and there are more than one of them (counting with multiplicity).*

Using the bound in Theorem 2.16, we see that $\dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}}) \geq 2$, when

$$\begin{aligned} &-(K_X \cdot f_*\mathbb{P}^1) - \dim(X) \geq 2 \\ &\qquad \iff -(K_X \cdot f_*\mathbb{P}^1) \geq 2 + \dim(X). \end{aligned}$$

In Theorem 3.8, we use this bend-and-break lemma repeatedly to produce a rational curve Γ with

$$-(K_X \cdot \Gamma) \leq 1 + \dim(X).$$

Proof. As in the previous proof, we wish to fix a 1-dimensional subvariety of $\mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}})$ while avoiding deformations that do not move $f(C)$. Such deformations correspond to automorphisms of $\mathbb{P}^1$ fixing 0 and ∞. These automorphisms are given by the multiplicative action of $\mathbb{G}_m$ on $\mathbb{P}^1$. We see that the $\mathbb{G}_m$-orbit of $[f]$ is 1-dimensional, so $\dim_{[f]} \mathrm{Mor}(\mathbb{P}^1, X; f|_{\{0,\infty\}}) \geq 2$ implies we can take a 1-dimensional subvariety passing through $[f]$, but which is not contained in the $\mathbb{G}_m$-orbit of $[f]$. Let T be the normalisation of such a 1-dimensional subvariety and $\overline{T}$ a smooth compactification of T.

The rest of this proof follows mostly [KM98]. Let us extend the $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times T \to T$ to a $\mathbb{P}^1$-bundle $S \to \overline{T}$ and extend the evaluation map on $\mathbb{P}^1 \times T$ to S:

$$\begin{tikzcd}
\mathbb{P}^1 \times T \arrow[r, hook] \arrow[d, "\pi_T"'] \arrow[rr, bend left, "\mathrm{ev}"] & S \arrow[r, dashed, "\mathrm{ev}"] \arrow[d, "\pi"] & X \\
T \arrow[r, hook] & \overline{T} &
\end{tikzcd}$$

As before, we resolve the indeterminacies of $\mathrm{ev} \colon S \dashrightarrow X$ using blow-ups:

$$\begin{tikzcd}
\widehat{S} \arrow[d, "\rho"'] \arrow[dr, "\widehat{\mathrm{ev}}"] & \\
S \arrow[r, dashed, "\mathrm{ev}"'] & X
\end{tikzcd}$$

Now, fix a fibre F of $\pi \colon S \to \overline{T}$. Since the fibres of a bundle are numerically equivalent, the fibre F is equivalent in particular to the fibre over $[f]$. Therefore,

$$\widehat{\mathrm{ev}}_* \rho^* F \equiv \widehat{\mathrm{ev}}_* \rho^* \pi^* [f] = \mathrm{ev}_* \pi^* [f] = f_* \mathbb{P}^1.$$

Since π is a $\mathbb{P}^1$-bundle and blow-ups create only rational curves, we see that $\widehat{\mathrm{ev}}_* \rho^* F$ is a sum of classes of rational curves. We still need to do some work to find a fibre for which the resulting curve breaks up to multiple components, and we will do this by induction on the number of blow-ups required to construct $\rho \colon \widehat{S} \to S$.

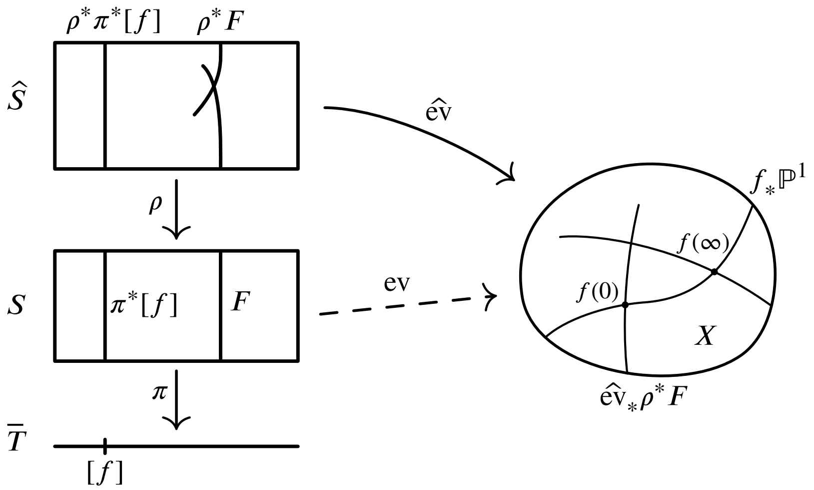


Figure 3.2 Illustration of the proof of Proposition 3.5.

Base Case: In the base case, the evaluation map $\mathrm{ev} \colon \mathbb{P}^1 \times T \to X$ extends to a morphism $\mathrm{ev} \colon S \to X$. Let T_0 and T_∞ be the sections of $\pi \colon S \to \overline{T}$ that extend the sections $\{0\} \times T$ and $\{\infty\} \times T$ of $\mathbb{P}^1 \times T \to T$. We will see that since $\mathrm{ev} \colon S \to \overline{T}$ is now a morphism, the fact that it contracts these two sections leads to a contradiction.

Let H be an ample divisor on X. Since C_0 and C_∞ are contracted by ev, the projection formula (Proposition 1.7) implies

$$(\mathrm{ev}^* H \cdot C_0) = 0 \text{ and } (\mathrm{ev}^* H \cdot C_\infty) = 0.$$

We can now apply Hodge Index Theorem ([Har97, Theorem V.1.9]) to see that

$$(C_0 \cdot C_0) < 0 \text{ and } (C_\infty \cdot C_\infty) < 0.$$

By [Har97, Proposition V.2.3], $N^1(W)$ is spanned by C_0 and a fibre F of $\pi\colon S \to \overline{T}$, and they satisfy $(C_0 \cdot F) = 1$ and $(F \cdot F) = 0$. Thus, write $C_\infty \equiv \alpha C_0 + \beta F$, so

$$1 = (C_\infty \cdot F) \equiv \alpha(C_0 \cdot F) + \beta(F \cdot F) = \alpha \implies C_\infty \equiv C_0 + \beta F.$$

But this leads to a contradiction:

$$0 = (\beta F)^2 = (C_\infty - C_0)^2 = C_0^2 + C_\infty^2 - 2(C_0 \cdot C_\infty) < 0.$$

Induction Step: We can factor ρ as $\widehat{S} \xrightarrow{\rho'} S' \xrightarrow{\tau} S$, where τ is the blow-up of a point $P \in S$ needed for the elimination of indeterminacy. As τ is a birational morphism, we can extend $\mathrm{ev}\colon S \dashrightarrow X$ to S':

$$\begin{tikzcd}
\widehat{S} \arrow[d, "\rho'"'] \arrow[dr, "\widehat{\mathrm{ev}}"] & \\
S' \arrow[r, dashed, "\mathrm{ev}'"] \arrow[d, "\tau"'] & X \\
S \arrow[ur, dashed, "\mathrm{ev}"'] &
\end{tikzcd}$$

Now, the point P lies in some fibre F of $\pi\colon S \to \overline{T}$. Then, the pullback $\tau^* F$ can be written as $\widetilde{F} + E$, where E is the exceptional divisor and $\widetilde{F}$ is a smooth rational curve intersecting E once. There are now three cases depending on the indeterminacies of ev' along $\widetilde{F}$.

Suppose first that there is a point $P' \in \widetilde{F} \setminus E$, where ev' is not defined. Then,

$$f_*\mathbb{P}^1 \equiv \widehat{\mathrm{ev}}_*\rho^* F = \widehat{\mathrm{ev}}_* \operatorname{red}(\rho^{-1}(P)) + \widehat{\mathrm{ev}}_* \operatorname{red}(\rho^{-1}(P')) + (\text{effective cycle}),$$

and we are done.

Now, suppose that ev' is not defined at the intersection point Q of $\widetilde{F}$ and E. As Q lies on both E and $\widetilde{F}$, each irreducible component of $\operatorname{red}((\rho')^{-1}(Q))$ appears with multiplicity at least two in $\rho^* F = (\rho')^*(\widetilde{F} + E)$. If $\widehat{\mathrm{ev}}$ contracts the components of $\operatorname{red}((\rho')^{-1}(Q))$, there was no need to blow-up Q to begin with. Therefore, $\widehat{\mathrm{ev}}_*\rho^* F$ consists of multiple components, and we are again done.

In the remaining case, the map ev' is defined along the whole of $\widetilde{F}$. Now, we know that E is a (-1)-curve, so

$$(\widetilde{F} \cdot \widetilde{F}) = \widetilde{F} \cdot (\tau^* F - E) = (\widetilde{F} \cdot \tau^* F) - (\widetilde{F} \cdot E) = (F \cdot F) - 1 = -1.$$

Thus, there is a contraction $\varphi_{\widetilde{F}}\colon S' \to Z$ of $\widetilde{F}$ by Castelnuovo's criterion.

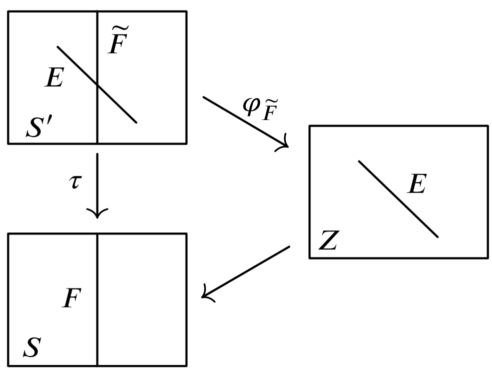


Figure 3.3 The curves contracted by τ and $\varphi_{\widetilde{F}}$.

Hence, we have the following commutative diagram.

$$\begin{tikzcd}
\widehat{S} \arrow[d, "\rho"] \arrow[dr] \arrow[drr, dashed, "\widehat{\mathrm{ev}}"] \arrow[dd, bend right=60, "\rho"'] & & \\
S' \arrow[r, "\varphi_{\widetilde{F}}"] \arrow[d, "\tau"] & Z \arrow[r, dashed, "\overline{\mathrm{ev}}"] \arrow[dl] & X \\
S \arrow[urr, dashed, "\mathrm{ev}"'] & &
\end{tikzcd}$$

Now, the map $\overline{\mathrm{ev}} : Z \dashrightarrow X$ requires one less blow-up to resolve its indeterminacies. Furthermore, the composition $Z \longrightarrow S \xrightarrow{\pi} \overline{T}$ defines a $\mathbb{P}^1$-bundle extending $\mathbb{P}^1 \times T$. Therefore, we can conclude by applying the induction hypothesis. $\square$

3.4 Gathering the Threads Together

This section is devoted to proving Theorem 3.8, in which we tie together the bend-and-break lemmas in this chapter and the results from last section to prove that on a smooth variety X, any K_X-negative curve $C \subset X$ can be deformed to produce a rational curve $\Gamma \subset X$. Furthermore, we can control the H-degree of Γ, where H is any amply divisor.

Lemma 3.6. *Suppose R is a finitely generated $\mathbb{Z}$-algebra. For any maximal ideal $\mathfrak{m} \subset R$, the quotient $R/\mathfrak{m}$ is a finite field.*

Proof. Let us first show that this statement reduces to showing that $\mathbb{Z}/(\mathbb{Z} \cap \mathfrak{m})$ is a finite field. Indeed, if R is finitely generated over $\mathbb{Z}$, then $R/\mathfrak{m}$ is finitely generated over $\mathrm{Frac}\,(\mathbb{Z}/\mathbb{Z} \cap \mathfrak{m})$

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr] \arrow[dr] \arrow[dd] & & R \arrow[dr] \arrow[dd] & \\
& \frac{\mathbb{Z}}{\mathbb{Z}\cap\mathfrak{m}} \arrow[rr] \arrow[dl] & & R/\mathfrak{m} \arrow[dl] \\
\mathrm{Frac}\left(\frac{\mathbb{Z}}{\mathbb{Z}\cap\mathfrak{m}}\right) \arrow[rr] & & \mathrm{Frac}(R/\mathfrak{m}) = R/\mathfrak{m} &
\end{tikzcd}$$

Since, the field extension $\mathrm{Frac}\,(\mathbb{Z}/\mathbb{Z} \cap \mathfrak{m}) \to R/\mathfrak{m}$ is finitely generated as rings, it is a finite field extension by Zariski's lemma [Vak17, Lemma 3.2.5]. Therefore, if $\mathrm{Frac}\,(\mathbb{Z}/\mathbb{Z} \cap \mathfrak{m})$ is a finite field, so is $R/\mathfrak{m}$.

The field $\mathrm{Frac}\,(\mathbb{Z}/\mathbb{Z} \cap \mathfrak{m})$ is finite, if and only if $\mathbb{Z}/(\mathbb{Z} \cap \mathfrak{m})$ is and this is the case, if and only if $\mathbb{Z} \cap \mathfrak{m}$ is not the zero ideal. Thus, assume $\mathbb{Z} \cap \mathfrak{m} = 0$. Then, $R/\mathfrak{m}$ is a finite field extension of $\mathbb{Q}$, while also being a finitely generate $\mathbb{Z}$-algebra. But this leads directly to a contradiction, because multiplication by some $n \in \mathbb{Z} \setminus \{-1, 0, +1\}$ induces an automorphism on any $\mathbb{Q}$-vector space but not on any finitely generated $\mathbb{Z}$-algebra. $\square$

Lemma 3.7. *Suppose R is a finitely generated $\mathbb{Z}$-algebra. Then, the closed points of* $\mathrm{Spec}(R)$ *form a dense set.*

Proof. The set of closed points is dense, if and only if its complement does not contain an open set. Therefore, it suffices to show that every distinguished open set contains a closed point. Thus, fix some $a \in R$ and consider the distinguished open set $U = \mathrm{Spec}(R_a)$. Then, let $\mathfrak{n} \subset R_a$ be any maximal ideal. Since $\mathrm{Frac}(R_a/\mathfrak{n}) = \mathrm{Frac}(R/R \cap \mathfrak{n}) = R/R \cap \mathfrak{n}$, we can use the same argument as in the above proof to see that $R_a/\mathfrak{n}$ is a finite field. Thus, so is $R/(R \cap \mathfrak{n})$. In particular, $\mathrm{Spec}\, \frac{R}{R\cap\mathfrak{n}}$ is a closed point of $\mathrm{Spec}\, R$, which is contained in U as can be seen from the below pullback diagram.

$$\begin{tikzcd}
U \arrow[r, hook] & \operatorname{Spec} R \\
\operatorname{Spec} \frac{R_a}{\mathfrak{n}} \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner", very near start] & \operatorname{Spec} \frac{R}{R\cap\mathfrak{n}} \arrow[u]
\end{tikzcd}$$

□

The proof of the bend-and-break theorem is done by two cases. First, we prove the theorem in positive characteristic with the help of the Fröbenius morphism, as we can control its degree. Then, we prove the characteristic zero case by forming a family of varieties over the spectrum of a finitely generated $\mathbb{Z}$-algebra such that our variety X of interest is the generic fibre of this family. Then, we look at the set of points, over which the theorem holds. Using the two lemmas above, we show that the set contains a dense set of closed points. Therefore, the set contains the generic point.

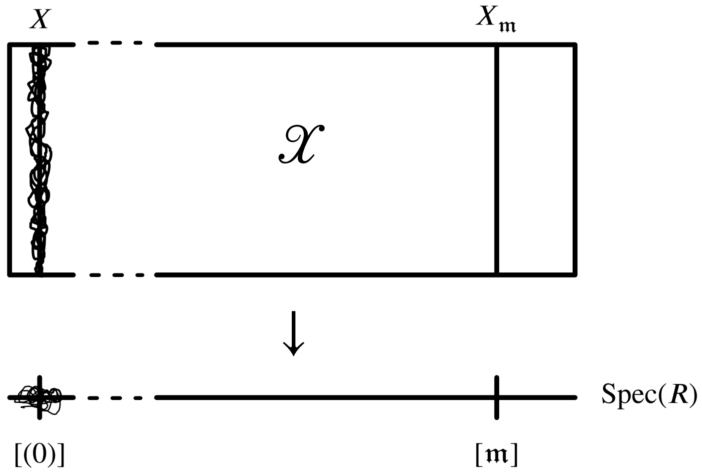


Figure 3.4 $\mathscr{X}$ projecting to Spec(R).

Theorem 3.8 (Bend-and-Break Theorem, [Deb01, Theorem 3.6]). *Let X be a smooth projective variety and fix an ample divisor H on X. If $f: C \to X$ is a smooth K_X-negative curve on X, then through every point of $f(C)$, there is a rational curve $\Gamma \subset X$ such that*

$$-(K_X \cdot \Gamma) \leq \dim(X) + 1 \text{ and } (H \cdot \Gamma) \leq 2\dim(X)\frac{H \cdot f_*C}{-K_X \cdot f_*C}.$$

Proof.

Positive characteristic. Let us suppose X is defined over a field of characteristic $p > 0$. For an integer $m > 0$, let $B_m \subset C$ be a set of size b_m and $f_m := f \circ F_m : C_m \to X$, where F_m is the m-fold composition of the Fröbenius morphism. Then, by Corollary 2.16, we have

$$\dim_{[f_m]} \operatorname{Mor}(C_m, X; f_m|_{B_m}) \geq -p^m K_X \cdot f_*C + (1 - g(C) - b_m)\dim(X).$$

To obtain enough deformations to produce rational curves, we want

$$\begin{aligned} &-p^m K_X \cdot f_*C + (1 - g(C) - b_m)\dim(X) \geq 1 \\ \iff\quad & 1 - \frac{1}{\dim X} + \frac{-p^m K_X \cdot f_*C}{\dim X} - g(C) \geq b_m \end{aligned}$$

Since we also want to make b_m as large as possible, we set

$$b_m = \left\lfloor \frac{-p^m K_X \cdot f_* C}{\dim(X)} - g(C) \right\rfloor .$$

Now, by the bend-and-break lemma (Proposition 3.4), we find a rational curve $\Gamma_m \subset X$ such that

$$(H \cdot \Gamma_m) \leq \frac{2(H \cdot f_{m,*}C)}{b_m} = \frac{2p^m(H \cdot f_* C)}{b_m}.$$

Finally, if we let m go to infinity, then the RHS approaches $2\dim(X)\frac{H \cdot f_* C}{-K_X \cdot f_* C}$, and since the LHS is an integer, the bound

$$(H \cdot \Gamma_m) \leq 2\dim(X)\frac{H \cdot f_* C}{-K_X \cdot f_* C}$$

is satisfied for a sufficiently large m. Finally, if $-K_X \cdot \Gamma_m > \dim(X) + 1$, then Theorem 2.16 implies that there are enough deformations of Γ_m to apply the second bend-and-break lemma (Proposition 3.5) to break Γ_m into pieces of lower H-degree. We can repeat this until we obtain a rational curve Γ satisfying both $-(K_X \cdot \Gamma) \leq \dim(X) + 1$ and $(H \cdot \Gamma) \leq 2\dim(X)\frac{H \cdot f_* C}{-K_X \cdot f_* C}$.

It remains to show that through every point of $f(C)$, such a rational curve passes through it. Let $V \subseteq f(C)$ be the set of points for which there is a rational curve passing through it satisfying the bound on its H-degree.

Claim. The set V is closed in $f(C)$.

Proof of Claim. Let M_d be the parameter space of rational curves on X of degree at most d. Note that this space is quasi-projective, as it is a subscheme of a suitable Hilbert scheme of curves of degree at most d. Such a Hilbert scheme consists of finitely many quasi-projective components. We also have the associated evaluation map

$$\mathrm{ev}_d : \mathbb{P}^1 \times M_d \to X.$$

Note that V is the intersection of the image of this map with $f(C)$. Thus, it suffices to show that $\mathrm{im}(\mathrm{ev}_d)$ is closed in X.

Assume for the contrary that the set $\overline{\mathrm{im}(\mathrm{ev}_d)} \setminus \mathrm{im}(\mathrm{ev}_d)$ is non-empty, and fix a point x contained in it. Choose a curve in $\overline{\mathrm{im}(\mathrm{ev}_d)}$ that passes through both x and $\mathrm{im}(\mathrm{ev}_d)$ such that it is dominated by a 1-dimensional subvariety of $\mathbb{P}^1 \times M_d$. Let T be the normalisation of the subvariety of $\mathbb{P}^1 \times M_d$ and $\overline{T}$ a smooth compactification of T.

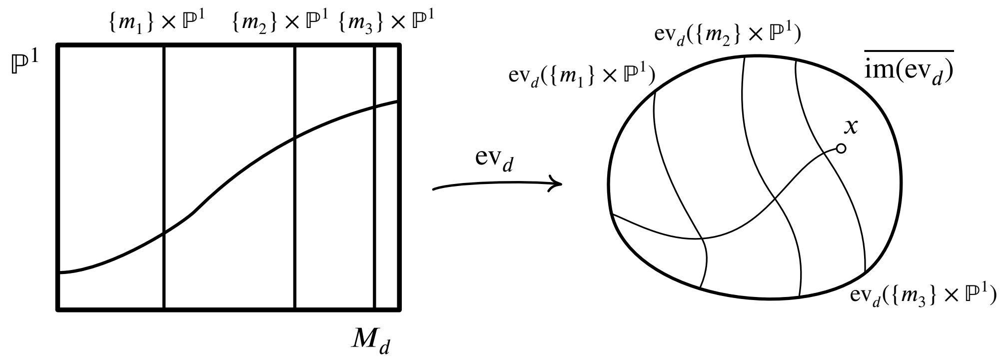


Figure 3.5 The set-up.

Consider the composition $T \to \mathbb{P}^1 \times M_d \to M_d$. Note that this is not constant, because otherwise the curve on X that T is dominating would be a rational curve of degree $\leq d$, contradicting the fact that x is not in $\mathrm{im}(\mathrm{ev}_d)$. Thus, it induces a corresponding evaluation map $\mathrm{ev} : \mathbb{P}^1 \times T \to X$, which in turn can be used to define the rational map

$$\mathrm{ev} : \mathbb{P}^1 \times \overline{T} \dashrightarrow X.$$

We can resolve the indeterminacies of this map to get

$$\begin{tikzcd} \widehat{S} \arrow[d, "\rho"'] \arrow[dr, "\widehat{\mathrm{ev}}"] & \\ \mathbb{P}^1 \times \overline{T} \arrow[r, dashed, "\mathrm{ev}"'] & X \end{tikzcd}$$

Finally, note that $\mathrm{im}(\widehat{\mathrm{ev}})$ must contain x But $\widehat{S}$ is covered by the fibres of the projection $\widehat{S} \to \overline{T}$, which are unions of rational curves of degree $\leq d$. Therefore, such a curve must pass through x. ■

Since V is closed, it is either finite or all of $f(C)$. If it was finite, we could modify B_m so that $f(B_m)$ does not intersect V. But this is a contradiction, as the bend-and-break lemma produces a rational curve through a point of $f(B_m)$. Hence, we conclude that through every point of $f(C)$, there is a rational curve satisfying the desired bounds.

Zero characteristic. Suppose now that X is defined over a field k of characteristic 0. Let us fix a point $x \in f(C)$. We will construct a rational curve through x by reducing to positive characteristic. Since X is a projective variety, it is cut out by finitely many polynomials in some projective space, and so is $f(C)$. Also, the morphism f is defined by finitely many polynomials, as are the transition functions of H. Now, let R be the ring obtained by adjoining to $\mathbb{Z}$ the coefficients of all these polynomials along with the coordinates of x. Then, the equations of X define a projective R-scheme $\mathcal{X} \to \mathrm{Spec}(R)$ along with an R-point x_R such that X is obtained from the generic fibre of the map by base change to k and x is obtained from x_R:

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "x_r"] & \mathcal{X} \arrow[r] & \mathrm{Spec}(R) \\ \mathrm{Spec}(k) \arrow[u] \arrow[r, "x"] \arrow[ur, phantom, "\urcorner"] & X \arrow[u] \arrow[r] \arrow[ur, phantom, "\urcorner"] & \mathrm{Spec}(k) \arrow[u] \end{tikzcd}$$

Now, by generic flatness (Theorem 2.4), there is a dense open set $U \subseteq \mathrm{Spec}(R)$ such that $\mathcal{X} \to \mathrm{Spec}(R)$ is flat over U. After removing a finitely many points from U, we may assume

1. $\mathcal{X}$ is smooth of dimension n over U,

2. the transition functions of H define ample line bundles on the fibres above U, and

3. the 1-cycles in the fibres over U corresponding to f_*C are K_X-negative 1-cycles in the fibres.

With this choice of U, whenever we take a closed point $\mathrm{Spec}(R/\mathfrak{m})$ in U, the fibre $X_\mathfrak{m}$ above it is a smooth variety with a curve $f_\mathfrak{m} : C_\mathfrak{m} \to X_\mathfrak{m}$ and an ample line bundle $H_\mathfrak{m}$.

$$\begin{tikzcd} X_\mathfrak{m} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(R/\mathfrak{m}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

By Lemma 3.6, we know that the quotient ring $R/\mathfrak{m}$ is finite, which implies $X_\mathfrak{m}$ is a variety over a field of positive characteristic. Therefore, we know that through every point of $f_\mathfrak{m}(C)$, there is a rational curve in $X_\mathfrak{m}$ satisfying the bounds of the bend-and-break theorem.

Now, let $d = 2\dim(X)\frac{H \cdot f_*C}{-K_X \cdot C}$ and $M_d \to \mathrm{Spec}(R)$ be the parameter space of non-constant morphisms $\mathbb{P}^1_R \to \mathcal{X}$ of H-degree at most d sending $0 \mapsto x_R$. As before, the bound on the degree ensures that M_d consists of finitely many quasi-projective components. By what we observed above, we know that the geometric fibre above every closed point of U is nonempty. Thus, the image of ρ contains all closed points of U. By Chevalley's theorem [Vak17, Theorem 7.4.2], the image of ρ is constructible (a finite union of locally closed subsets), and by Lemma 3.7, the closed points form a dense subset of U. These two facts imply that the image of ρ is dense in $\operatorname{Spec}(R)$. Therefore, it contains the generic point of $\operatorname{Spec}(R)$. Therefore, the generic fibre is nonempty, and an element of the fibre defines a rational curve in X satisfying the bounds of the bend-and-break theorem. $\square$

It is useful to picture the H-degree bound on Γ as a slice of $\overline{\mathrm{NE}}(X)$, where the Γ must be located in and which is controlled by $f\colon C \to X$.

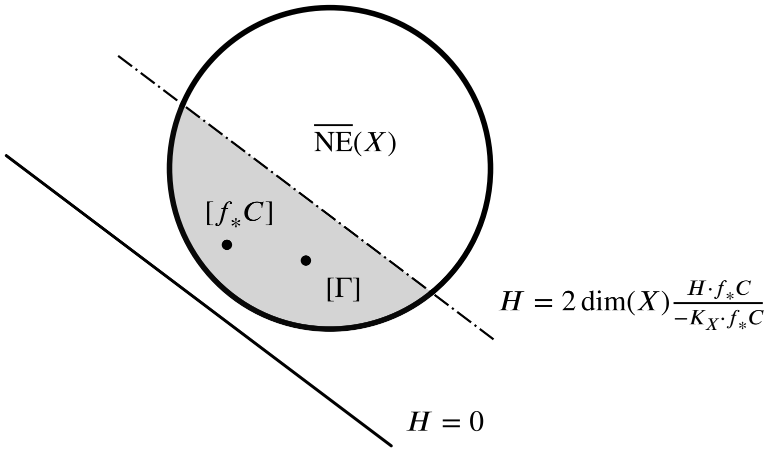


Figure 3.6 The H-degree bound.

4 Cone Theorem

WITH Mori's bend-and-break lemma at hand, we can finally prove the cone theorem, which describes the structure of the K_X-negative part of the cone of curves.

Theorem 4.1. *Let X be a smooth projective variety. There exists a countable set I and a corresponding collection $(\Gamma_i)_{i\in I}$ of curves on X such that*

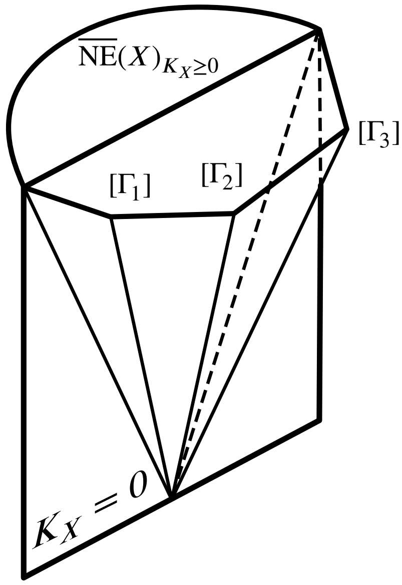
$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \sum_{i\in I} \mathbb{R}_{\geq 0}[\Gamma_i].$$

The curves Γ_i

1. *are rational,*
2. *span extremal rays $\mathbb{R}_{\geq 0}[\Gamma_i]$, and*
3. *satisfy a bound on their K_X-degrees: $0 < -K_X \cdot \Gamma_i \leq \dim(X)+1$.*

Furthermore, these extremal rays are locally finite in $\overline{\mathrm{NE}}(X)_{K_X<0}$, that is for every ample divisor H and $\delta > 0$, there is a finite subset $J \subseteq I$ such that

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\delta H\geq 0} + \sum_{j\in J} \mathbb{R}_{\geq 0}[\Gamma_j].$$



Note that since Mori's proof is based on his bend-and-break lemma, the statement holds in all characteristics! Note also the relevance of this theorem to the MMP: The cone theorem guarantees that whenever K_X is not nef, we can find a curve on X which generates a K_X-negative extremal ray. Then, the contraction theorem (Theorem 1.23) immediately yields a contraction of X. This grants the following definition.

Definition 4.2. A projective variety X is said to be *minimal*, if K_X is nef. We say a variety representing the birational equivalence class of X a *minimal model* of X.

4.1 Proof of the Theorem

We will deviate from our main source [Deb01] in the proof of the theorem and instead combine ideas from several different sources [Mat02; KM98; Kol92]. Begin by fixing an ample divisor H and a real number $\delta > 0$. Let $(\Gamma_j)_{j\in J_\delta}$ be the collection of *all* rational curves on X such that $\delta H \cdot \Gamma_j < -K_X \cdot \Gamma_j \leq \dim(X)+1$ for all $j \in J_\delta$. For brevity, denote

$$V_\delta := \overline{\mathrm{NE}}(X)_{K_X+\delta H\geq 0} + \sum_{j\in J_\delta} \mathbb{R}_{\geq 0}[\Gamma_j].$$

Lemma 4.3. *The set J_δ is finite and V_δ is closed in $N_1(X)_{\mathbb{R}}$.*

Proof. Recall that $N_1(X)_{\mathbb{R}}$ is constructed from the $\mathbb{Z}$-module $N_1(X)$. Thus, classes of integral 1-cycles correspond to integer lattice points in the $\mathbb{R}$-vector space. In particular, they form a discrete subset. Now, if we let $r = (\dim(X) + 1)/\delta$, then $\{ [\Gamma_j] \mid j \in J_\delta \}$ is a discrete subset of $\overline{\mathrm{NE}}(X)_{H \le r}$. Since $\overline{\mathrm{NE}}(X)$ has finite cross-sections (Proposition 1.21, Proposition 1.15), the set $\overline{\mathrm{NE}}(X)_{H \le r}$ is compact, so J_δ is finite. This directly implies that V_δ is closed. Indeed, each term in the finite sum

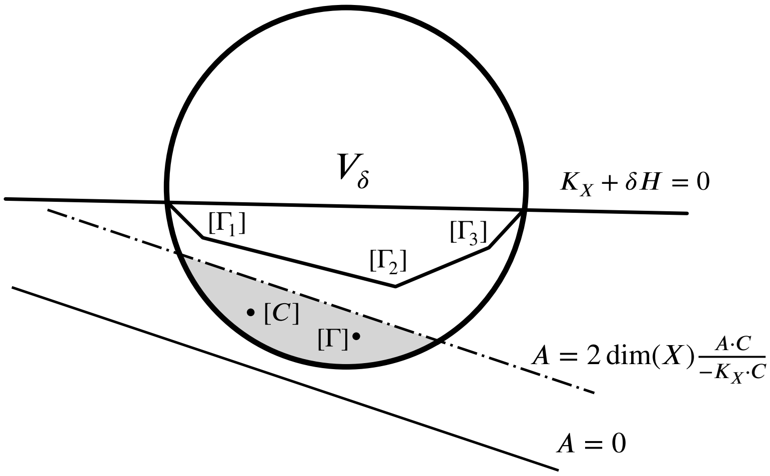
$$\overline{\mathrm{NE}}(X)_{K_X + \delta H \ge 0} + \sum_{j \in J_\delta} \mathbb{R}_{\ge 0}[\Gamma_j]$$

is closed, and it is easy to see that two closed convex cones generate a closed convex cone. $\square$

The idea now is to first assume for a contradiction that $V_\delta \subsetneq \overline{\mathrm{NE}}(X)$, then try to find a curve $C \subset X$ and an ample divisor A such that the intersection

$$\left\{ z \in \overline{\mathrm{NE}}(X) \mid A \cdot z \le 2 \dim(X) \frac{A \cdot C}{-K_X \cdot C} \right\} \cap V_\delta$$

does not contain any classes of integral 1-cycles. Hence, when we apply the bend-and-break lemma on C, we obtain a rational curve Γ such that its class $[\Gamma]$ is inside the slice $\{ z \in \overline{\mathrm{NE}}(X) \mid A \cdot z \le 2 \dim(X) \frac{A \cdot C}{-K_X \cdot C} \}$ but satisfies $-K_X \cdot \Gamma \le \dim(X) + 1$, which leads to a contradiction.



Proof of the Cone Theorem. Fix an ample divisor H, a real number $\delta > 0$, and define J_δ and V_δ as above. It suffices to show that $\overline{\mathrm{NE}}(X) = V_\delta$. Indeed, the union $\bigcup_{\delta > 0} J_\delta$ is countable by Lemma 4.3, and we can pick out the subset I of this union such that the rays $\mathbb{R}_{\ge 0}[\Gamma_i]$ are extremal for all $i \in I$ while

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \ge 0} + \sum_{i \in I} \mathbb{R}_{\ge 0}[\Gamma_i].$$

The last part of the theorem also follows from Lemma 4.3.

Thus, assume for a contradiction that $V_\delta \subsetneq \overline{\mathrm{NE}}(X)$. Then, we can fix a point $z \in \overline{\mathrm{NE}}(X)$ at the boundary of $\overline{\mathrm{NE}}(X)$. Moreover, since the intersection product $N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ is a perfect pairing, we can find a nef $\mathbb{R}$-divisor M such that $(M \cdot z) = 0$ and M is positive on V_δ.

Recall we showed in the proof of Lemma 4.3 that the set $\overline{\mathrm{NE}}(X)_{H \le 1}$ is compact and contains finitely many classes of integral 1-cycles. Let $\tilde{z}$ be the non-zero integral 1-cycle, where the map

$$\left(\overline{\mathrm{NE}}(X)_{H \le 1} \cap N_1(X)_{\mathbb{Z}} \right) \setminus \{0\} \to \mathbb{R} : z \mapsto (H \cdot z)$$

attains its minimum and denote by $\alpha := (H \cdot \tilde{z})$ the minimum value. Next, since $\overline{\mathrm{NE}}(X)_{H \le \alpha}$ is compact, so is the section $\overline{\mathrm{NE}}(X)_{H = \alpha}$ and $V_\delta \cap \overline{\mathrm{NE}}(X)_{H = \alpha}$ as V_δ is closed. Hence, the map

$$V_\delta \cap \overline{\mathrm{NE}}(X)_{H = \alpha} \to \mathbb{R} : z \mapsto (M \cdot z)$$

has a non-zero minimum value. We can therefore scale M so that $M > 1$ on $V_\delta \cap \{ H = \alpha \}$. Note that for any $z \in Z_\delta \cap \{ H \geq \alpha \}$, we have $(M \cdot z) > 1$:

$$\left(H \cdot \frac{\alpha z}{H \cdot z}\right) = \alpha \implies \left(M \cdot \frac{\alpha z}{H \cdot z}\right) > 1 \implies (M \cdot z) > \alpha^{-1}(H \cdot z) \geq 1.$$

Therefore, the slice $\overline{\mathrm{NE}}(X) \cap \{ M \leq 1 \}$ does not intersect $V_\delta \cap \{ H \geq \alpha \}$, and $V_\delta \cap \{ H < \alpha \}$ contains no non-zero classes of integral 1-cycles.

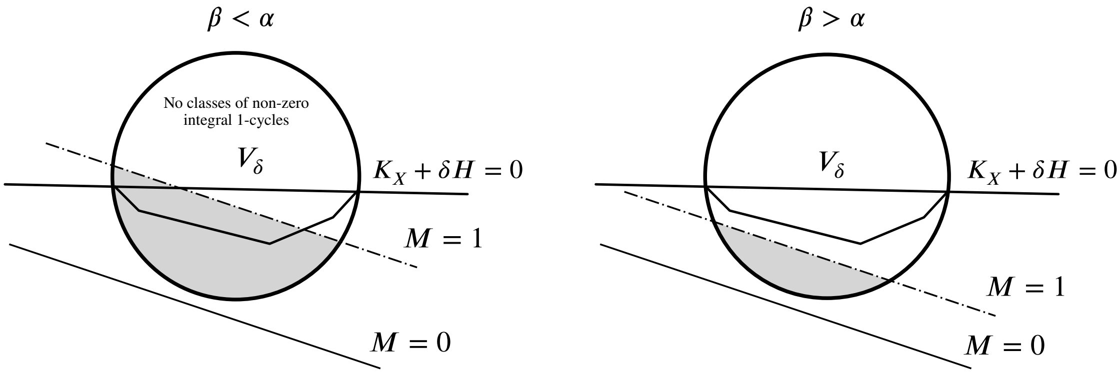


Figure 4.1 Two sections of $\overline{\mathrm{NE}}(X)$ at $H = \beta$.

Next, take a sequence M_i of ample $\mathbb{Q}$-divisors converging to M and a sequence z_i of effective 1-cycles converging to z. For sufficiently large i, the slice $\overline{\mathrm{NE}}(X) \cap \{ M_i \leq 1 \}$ does not intersect $V_\delta \cap \{ H \geq 1 \}$ and $\{ M_i \leq 1 \}$ does not contain any non-zero classes of integral 1-cycles in V_δ. Now, as

$$\lim_{i \to \infty} 2 \dim(X) \frac{M_i \cdot z_i}{-K_X \cdot z_i} = 2 \dim(X) \frac{0}{-K_X \cdot z_i} = 0,$$

taking larger i still gives

$$2 \dim(X) \frac{M_i \cdot z_i}{-K_X \cdot z_i} \leq 1.$$

Now, write $z_i = \sum \alpha_j [C_j]$, where the C_j are irreducible curves. Note that we can discard any K_X-nonnegative curves from this sum and the above inequality still holds.

Claim ([KM98]). If $c, d > 0$, then $\frac{a+b}{c+d} \geq \min \{ \frac{a}{c}, \frac{b}{d} \}$.

Proof of Claim. Assume WLOG that $\frac{a}{c} \leq \frac{b}{d}$. Then,

$$bc \geq ad, \text{ and } \frac{a+b}{c+d} = \frac{1}{c} \frac{ac+bc}{c+d} \geq \frac{1}{c} \frac{ac+ad}{c+d} = \frac{a}{c} \frac{c+d}{c+d} = \frac{a}{c}. \qquad \blacksquare$$

Using this result and induction, we see that

$$\min_j 2 \dim(X) \frac{M_i \cdot C_j}{-K_X \cdot C_j} \leq 2 \dim(X) \frac{M_i \cdot z_i}{-K_X \cdot z_i} \leq 1.$$

Fix the index j where the minimum is attained. Then, apply bend-and-break (Theorem 3.8) to the normalisation of the curve C_j to obtain a rational curve Γ such that

$$M_i \cdot \Gamma \leq 2 \dim(X) \frac{M_i \cdot C_j}{-K_X \cdot C_j} \leq 1 \text{ and } 0 < -(K_X \cdot \Gamma) \leq \dim(X) + 1.$$

The first inequality shows that $[\Gamma]$ is in the slice $\overline{\mathrm{NE}}(X) \cap \{ M \leq 1 \}$ and the second inequality shows it is in V_δ, but since $[\Gamma]$ is the class of a curve, this is a contradiction. Therefore, we must have $V_\delta = \overline{\mathrm{NE}}(X)$. $\square$

4.2 Kollár's Generalisation

Kollár [Kol92] formulates a slight generalisation of the theorem, which allows us to obtain information about $\overline{\mathrm{NE}}(X)_{K_X<0}$, when we know that X has *enough deformations* on the complement of some closed subvariety. This theorem will be used in the next chapter.

Definition 4.4. For a variety X over k and a closed subvariety Z, we say that $X \setminus Z$ has enough deformations, if

- char $k = p > 0$ and for every proper, irreducible K_X-negative curve $f : C \to X$ not contained in Z, there is a surjection $g : D \to C$ from a smooth, proper, irreducible curve D such that

$$\lim_{m\to\infty} \frac{\dim_{[f\circ g_m]} \mathrm{Mor}(D_m, X)}{p^m(-D \cdot K_X)} \geq 1,$$

  where $g_m : D_m \to D \to C$ is the composition with the Fröbenius morphism F_m, or

- char $k = 0$ and for every finitely generated $\mathbb{Z}$-algebra $R \subset k$, there is a dense subset $U \subset \mathrm{Spec}\, R$ such that for every maximal ideal $\mathfrak{m}$ in U, the reduction of $X \setminus Z$ modulo $\mathfrak{m}$ has enough deformations.

Theorem 4.5 ([Kol92, Theorem 3.3])**.** *Suppose X is a projective variety over k and Z is a closed subvariety. If $X \setminus Z$ has enough deformations, then there are countably many rational curves $(\Gamma_i)_{i\in I}$ on X such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X\geq 0} + \mathrm{im}\left[\overline{\mathrm{NE}}(Z) \to \overline{\mathrm{NE}}(X)\right] + \sum_{i\in I} \mathbb{R}^+[\Gamma_i].$$

Furthermore, the rational curves satisfy

$$0 < K_X \cdot \Gamma_i \leq 2 + 2\dim(X).$$

As before, the rays $\mathbb{R}^+[\Gamma_i]$ are locally finite in $\overline{\mathrm{NE}}(X)_{K_X<0}$.

The theorem is proved in much the same way as the original cone theorem.

Extending the Technique 5

ALTHOUGH Mori's technique is a beautiful gem of mathematics, the MMP demands stronger results on the structure of the cone of curves. As mentioned in the introduction, the cone theorem has been proven for a class of singular varieties that is sufficiently large for the MMP. However, it is still fascinating to try to extend Mori's deformation-theoretic approach. The upside of Mori's technique is that it works in all characteristics. Another advantage over the cohomological method is that the deformation-theoretic approach results in a tighter bound on the K_X-degree of the rational curves that generate the extremal rays in $\overline{\mathrm{NE}}(X)_{K_X<0}$. The starting point of extending Mori's technique is in a result of Kollár [Kol92], which generalises the dimension bound of Corollary 2.16 to LCI schemes.

Definition 5.1 ([CT09, Definition 1.5])**.** We say a scheme X is LCI (locally complete intersection), if for every point $x \in X$, one can find an étale neighbourhood U of x such that U is a complete intersection.

Theorem 5.2 ([Kol92, Theorem 2.10])**.** *Suppose X is an* LCI *scheme and $f\colon C \to X$ is a proper, connected curve such that no component of C is contained in the singular locus of X. Then,*

$$\dim_{[f]} \mathrm{Mor}(C, X) \geq -K_X \cdot f_*C + (1 - g(C)) \dim(X).$$

The proof is mostly an argument in homological algebra. One could now use this theorem in conjunction with Kollár's version of the cone theorem (Theorem 4.5), but Kollár went further by introducing what he calls *bug-eyed covers* to prove the cone theorem for LCIQ varieties.

Definition 5.3 ([CT09, Definition 1.5])**.** We say a scheme X is LCIQ, if for every point $x \in X$, one can find an étale neighbourhood U of x such that $U \cong V/G$, where V is an LCI scheme and G is a group acting on V with action that is étale in codimension 1.

In this chapter we discuss the approach laid out by Chen and Tseng [CT09], which proves the theorem using Deligne-Mumford stacks instead of bug-eyed covers. Below is a table comparing the different versions of the cone theorem, where we have listed the bounds that the theorems give on the K_X-degrees of the curves generating the extremal rays.

Reference	Approach	Singularities	Characteristic	Bound
[Mor82]	deformations	none	all	$\dim X + 1$
[Kaw84]	cohomology	terminal, $\mathbb{Q}$-factorial	0	$2 \dim X$
[Kol92]	deformations + bug-eyed covers	LCIQ	all	$2 \dim X$
[CT09]	deformations + DM stacks	LCIQ	all	$\dim X + 1$

Table 5.1 Different cone theorems.

5.1 Deligne-Mumford Stacks

Our goal in this section is to get an idea of what stacks are and why introducing them here would help in proving the cone theorem for LCIQ varieties. The precise definition of stacks is rather lengthy and technical, so we will settle for an overview of the topic; for a nice introduction, see [Fan01]. Everything in this section is based on the wonderful article [Góm99]. Stacks are a class of spaces that generalises schemes, and for every scheme, there is a stack associated to it. The main difference between schemes and stacks is the following. Recall that a scheme X is completely determined by the functor of points

$$h_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$

Now, the "functor of points" of a stack $\mathcal{X}$ is of the form

$$\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Grpd},$$

where **Grpd** is the category of groupoids. Recall that a groupoid is a category where every morphism is an isomorphism and that every group can be seen as a groupoid with one object. The definition of stacks is a general category-theoretic construction, and stacks can be defined on any *sites*—categories with a generalised notion of topology. The category of schemes is a site with the Zariski topology, but it has other topologies as well. When we define stacks over **Sch**, we want to use the étale topology or some other topology that is finer than the Zariski topology.

A Deligne-Mumford stack is a stack $\mathcal{X}$ over **Sch** that has an *atlas* of schemes. Formally speaking, the atlas is an étale surjective morphism $U \to \mathcal{X}$ from a fixed scheme U. Compare this to the definition of manifolds, where a manifold X has a smooth atlas $\{U_i\}_{i\in I}$ along with transition functions. The data of the atlas can equivalently be encoded in the surjection $\bigsqcup U_i \to X$. Using this definition, we can tie the abstract notion of a stack back to the concrete theory of schemes. Using the atlas, we can translate the scheme-theoretic language to stacks. Indeed, when P is a local property of schemes in the étale topology, then we say $\mathcal{X}$ has P if the atlas U has P. Similarly, for morphisms of Deligne-Mumford stacks, we can define scheme-theoretic properties of morphisms that are local on the source and target in the étale topology. Finally, we can also define quasi-coherent sheaves on Deligne-Mumford stacks. In particular, we can define differentials and the cotangent bundle $\Omega^1_{\mathcal{X}}$.

Remark. There is a small technical twist in the definition of Deligne-Mumford stacks. Namely, we need to know what it means for the atlas $U \to \mathcal{X}$ to be an étale surjection. Thus, we put an extra condition on $\mathcal{X}$ that allows us to decide whether or not a given morphism $V \to \mathcal{X}$ from a scheme is an étale surjection.

One of the main reasons for introducing stacks is that we can define quotients stacks of schemes by group actions. For example, the moduli of curves (Definition 2.5) can be constructed, if we can construct a certain quotient space. Indeed, on every smooth curve C, the divisor $3K_C$ is very ample [HM98]. Thus, curves of fixed genus can be embedded in a projective space of sufficiently high dimension. Then, non-singular curves of genus g form a subscheme of the Hilbert scheme of curves of genus g in this projective space. The moduli space $\mathcal{M}_g$ could in theory be constructed by taking the quotient of this subscheme by the equivalence relation that identifies isomorphic curves. Deligne and Mumford [DM69] described this quotient, and this was in fact the first appearance of Deligne-Mumford stacks. The problem with taking the quotient is caused by curves having too many automorphisms, and stacks solve the problem by retaining the information of the automorphisms in the data of the quotient stack. For example, if X is a k-variety and G is an affine group variety over k acting on X, then we define the quotient stack $[X/G]$ in such a way that the groupoid of k-points of $[X/G]$ are G-orbits of k-points of X and the morphisms describe how these orbits are related together [Góm99].

Now, the reason why stacks are used to prove the cone theorem for LCIQ varieties is the fact that for any normal LCIQ variety X, there is a Deligne-Mumford stack $\mathcal{X}$, which is LCI and "approximates" X—in fact X and $\mathcal{X}$ are isomorphic in codimension 1 [CT09]. Thus, we can prove the cone theorem for LCIQ varieties, if we can prove it for LCI stacks that approximate the LCIQ varieties.

5.2 Twisted Stable Maps

The key to proving the cone theorem in the smooth case is that we can deform morphisms $C \to X$ of smooth projective varieties, where C is a curve. To prove the cone theorem for projective LCIQ varieties X, instead of deforming curves on X, we deform curves on the LCI stacks $\mathcal{X}$ that approximate the varieties X. Thus, we would like to lift morphisms $C \to X$ from curves to morphisms $C \to \mathcal{X}$. However, this is in general not possible [CT09]. If $C \to X$ is a morphism from a smooth curve such that the image of C intersects the smooth locus of X, then we can actually obtain a lift $\mathscr{C} \to \mathcal{X}$, where $\mathscr{C}$ is obtained by adjoining additional stacky structure to finitely many points of C. We say $\mathscr{C}$ is a twisted curve.

Now, as in the proof of the bend-and-break lemmas in the smooth case, we would like to construct a moduli space of morphisms $\mathscr{C} \to \mathcal{X}$ and study the deformations of a fixed such morphism. Recall that in the proofs of the bend-and-break lemmas, we fixed a 1-dimensional subvariety T of $\mathrm{Mor}(C, X)$ and compactified it. Then, we used resolution of indeterminacy by blow-ups to construct rational curves over the points where the evaluation map $\mathrm{ev}: C \times \overline{T} \dashrightarrow X$ is not defined. We will prove the bend-and-break lemmas in the context of LCI stacks using a similar argument, but here we compactify the whole moduli space of twisted maps $\mathscr{C} \to \mathcal{X}$. Abramovich and Vistoli [AV02] proved in their phenomenal paper that the compactification is obtained by adding maps from so-called twisted stable curves, which are twisted curves obtained from nodal curves satisfying some stability condition that ensures that the compactification is separated. The moduli stack of twisted stable maps to $\mathcal{X}$ is denoted by $\mathscr{K}_{g,n}(\mathcal{X}, d)$.

5.3 Conclusion

After setting up the correct context, the rest of [CT09] follows Mori's technique of proving the bend-and-break theorem and the cone theorem in addition to Kollár's results [Kol92], but instead using Deligne-Mumford stacks and deformations of twisted stable maps. Thus, we prove the bend-and-break theorem for a normal, projective variety X with "tame" LCIQ singularities and a smooth K_X-negative curve $f: C \to X$, such that $f(C)$ is not contained in the singular locus of X, by lifting $f: C \to X$ to a twisted stable map $\bar{f}: \mathscr{C} \to \mathcal{X}$ into the LCI stack $\mathcal{X}$ that approximates X. Then we follow the proof of Theorem 5.2 along with using the Fröbenius morphism and reduction to positive characteristic to show that $\bar{f}: \mathscr{C} \to \mathcal{X}$ can be deformed. Thus, we fix a smooth curve T along with a morphism $T \to \mathscr{K}_{g,n}(\mathcal{X}, d)$ and compactifying it to get a morphism $\overline{T} \to \mathscr{K}_{g,n}(\mathcal{X}, d)$ that extends $T \to \mathscr{K}_{g,n}(\mathcal{X}, d)$. Then, one can find a point $t_0 \in \overline{T} \setminus T$ such that the twisted stable map $\mathscr{C}_0 \to \mathcal{X}$ corresponding to t_0 is such that $\mathscr{C}_0$ has a rational component that is not contracted by $\mathscr{C}_0 \to \mathcal{X}$, with the desired properties.

Finally, we simply use Theorem 4.5 along with this new bend-and-break theorem to prove the cone theorem for LCIQ varieties.

Theorem 5.4. *For a projective variety X with tame* LCIQ *singularities, there is a countable collection $(\Gamma_i)_{i \in I}$ of rational K_X-negative curves such that*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \mathrm{im}\left[\overline{\mathrm{NE}}(X_{\mathrm{sing}}) \to \overline{\mathrm{NE}}(X)\right] + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

Furthermore,

$$0 < -K_X \cdot \Gamma_i \leq 2 + 2\dim(X).$$

Chen and Tseng [CT09] manage to improve the bound to $-K_X \cdot \Gamma_i \leq 1 + \dim(X)$ by using the Fröbenius morphism in positive characteristic and passing to zero characteristic with the help of the existence of contractions in zero characteristic.

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