Gluing Constructions: A Central Thread in Geometry

Miika Rankaviita

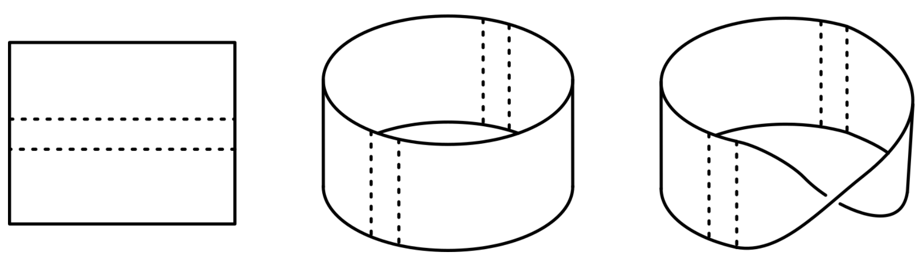
1 Introduction

Like any branch of mathematics, geometry is a complex web of interconnected ideas. Running through this web is a thread that seems to touch every branch of geometry, called *gluing*. In this article, I wish to explore with you the pervasiveness of this mathematical idea. We will see that there are two key consequences of gluing: it allows us to pass from the extrinsic to intrinsic and from the local to global. We will discuss how geometric shapes can be glued together, followed by gluing "things-on-a-shape".

2 Gluing shapes

Gluing geometric shapes[1] abstractly works very much the same way as in real life. To glue two pieces of paper together, you need a glue stick and gluing instructions. To glue two geometric shapes together, you do not need a glue stick, but you do need some sort of "gluing instructions". Two common ways of formulating these mathematical gluing instructions are by defining either an equivalence relation between points of the spaces, or an isomorphism between parts of the spaces. Both the equivalence relation and the isomorphism specifies, which points of the spaces should be glued together.

As an example, suppose we start with two rectangles that we wish to glue together. A simple way of gluing these would be along an edge to form a larger rectangle. A more interesting way would be to glue opposite pairs of edges to form a cylinder. We can do something even more interesting: If we do one half-twist before gluing, we obtain a *Möbius strip*. Interestingly, doing two half-twists before gluing results in something that looks similar to a Möbius strip, but in fact the resulting shape is *topologically equivalent*[2] to a cylinder! Once we know how to glue things-on-a-shape we will be able to prove this.



Exercise (hard)
Can you find a way to transform the strip with two half-twists into the cylinder by flipping one of the half-twists in the fourth dimension without tearing it?

[1]By "a shape" I mean a geometric object living in some space. I allow for broad interpretations of the word, as I am trying to make a general statement about geometry. If you would like, you can interpret a "shape" to mean a polygon, a subset of $\mathbb{R}^n$, a topological space, a manifold, etc.

[2]Two shapes are said to be topologically equivalent, if one can be deformed to another without tearing the shape. For example, a square is topologically equivalent to a disk, but a circle is not topologically equivalent to a line.

2.1 The intrinsic point of view

Even though gluing is a rather simple procedure, it has great implications to how mathematicians study geometry today, because it lets us move from *extrinsic* definitions to *intrinsic* ones. A shape is defined extrinsically, when it is seen as a part of some *ambient space*. As an example, we can define the circle as a set of points satisfying some equation. Formally,

$$S^1 = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} .$$

The "ambient space" in this situation is the Euclidean plane $\mathbb{R}^2$. One consequence of this extrinsic definition is that the circle has a defined centre point.

On the other hand, an intrinsic definition of the circle would describe it as the shape obtained by gluing the endpoints of a line segment. In this definition, there is no ambient space or a centre point. One can also note that the intrinsic definition is more minimal. It tries to encapsulate only the essential features of the shape without describing how it relates to some surrounding space.

Thinking back to the construction of the Möbius strip, we realise its definition was intrinsic. It would have been possible to give an extrinsic definition in terms of equations.

Exercise
Can you write down equations that describe the Möbius strip as a subset of $\mathbb{R}^3$?

Understanding the notions of extrinsic and intrinsic properties changes our perspective on geometry. So far we have been making a distinction between shapes and spaces. For example, we call the plane $\mathbb{R}^2$ a "space", thinking of it as floating in some abstract realm of ideas, not being dependent on any larger space. In contrast, we say that the sphere is a "shape" living in three-dimensional space. If we think of the sphere intrinsically instead of extrinsically, we can view it as a two-dimensional *space* itself, just like the plane. Hence this shift in our perspective puts "shapes" and "spaces" on an equal footing. In fact, modern geometers do not make a distinction between the two and instead call everything a "space". From now on, we will use this conventional terminology.

2.2 The local point of view

In addition to allowing us to shift to the intrinsic viewpoint, one of the purposes of gluing is to construct complicated spaces from simple building blocks. Then, all the facts we know about the building blocks apply *locally* to the complicated space as well. For instance, if we fix a point p on the Möbius strip, then p is contained in at least one of the rectangles that we glued together. Therefore, the properties of the Möbius strip *near* p are exactly the same as the properties of the rectangle near p.

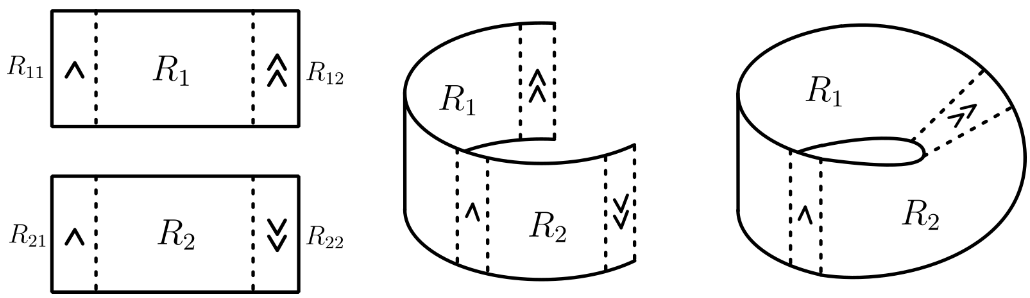
Because of the intrinsic and local viewpoints, many classes of spaces in modern geometry are defined by gluing together spaces that we understand well. The most widely used class is that of *manifolds*. These are spaces obtained from gluing (open) subsets of $\mathbb{R}^n$.

3 Gluing things-on-a-space

When mathematicians study some complicated object, it is useful to enrich the object with as much additional structure as possible. In the context of geometric spaces, I call these additional objects associated to a given space "things-on-a-space". In this section we will explore some examples of things-on-a-space and how gluing constructions extend to them.

3.1 Gluing functions

As the first example of a thing-on-a-space, let us consider functions. Recall that the Möbius strip was constructed from two rectangles by gluing. Label the rectangles[3] as R_1 and R_2. Furthermore, denote by R_{11} a small, vertical strip along the left edge of R_1 and by R_{12} a strip along the right edge. Define R_{21} and R_{22} similarly on R_2. Then, to form the Möbius strip M, one can first glue R_{11} to R_{21} and then twist the rectangles before gluing R_{12} to R_{22}.



Now, suppose $f_1 : R_1 \to \mathbb{R}$ and $f_2 : R_2 \to \mathbb{R}$ are two functions, which take the same values on the points that are glued together. Then we can define a new function $f : M \to \mathbb{R}$ on the Möbius strip as the function, whose value at a point $p \in M$ is determined by f_1 or f_2 depending on which rectangle the point is on. Note that on the intersection of the rectangles, we can still define the value of f without ambiguity, since the two functions have the same value there. We say that the function f is formed by *gluing* the functions f_1 and f_2. In general, we can glue functions on a space, whenever they agree on the intersections of the domains of definitions.

Recall the problem of showing that the strip with two half-twists is topologically equivalent to a cylinder. Firstly, two topological spaces are equivalent, if there is a *homeomorphism* between them.

Definition
A continuous function $f : X \to Y$ is a homeomorphism, if there is another continuous function $g : Y \to X$ such that

$$g \circ f = \mathrm{id}_X \text{ and } f \circ g = \mathrm{id}_Y.$$

Now, suppose we form both the cylinder and the doubly twisted strip by gluing the rectangles R_1 and R_2. Then, one can check that the identity functions

$$\mathrm{id}_{R_1} : R_1 \to R_1 \text{ and } \mathrm{id}_{R_2} : R_2 \to R_2$$

agree on the intersections in this case, and hence, can be glued to form a homeomorphism from the cylinder to the doubly twisted strip.

3.2 Gluing properties

Consider the situation where we are given a space X that is formed by gluing spaces U and V and we want to check if X has some property P. In some cases, it is possible to deduce that X has the property P by showing U and V have the property, along with verifying some condition on the

[3]Note that one could for example define $R_1 = (0, 2) \times (0, 1)$ and $R_2 = (2, 4) \times (0, 1)$, and write down precise formulas for the constructions. But for the sake of clarity, I omit the details and encourage the reader to verify them.

intersection $U \cap V$. I will demonstrate this in the case where the property is *simply connectedness*. Intuitively, a space is simply connected when it does not have "holes". Let us also define *path-connectedness*.

Definition (path-connected)
A (topological) space X is path-connected, if between every pair of points, there is a continuous path contained in X starting from the first point and ending on the second point.

Definition (simply connected)
A path-connected space X is simply connected, if every continuous loop in X can be continuously deformed onto a point, without breaking the loop.

For example, the plane $\mathbb{R}^2$ is simply connected, because every loop can be contracted to a point. But the plane without the origin $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected. Indeed, if we consider a loop circling around the origin and try to contract it to a point, it will always "get stuck" on the hole, where the origin used to be.

Now, suppose X is obtained by gluing two simply connected spaces U and V. With a bit of knowledge from topology, one can prove that X is simply connected, if and only if $U \cap V$ is path-connected. Therefore, we manage to extend the property of being simply-connected to the whole space, provided some condition is satisfied over the intersection[4]. Although the analogy might be distant, one can still appreciate the similarity of the argument to that of gluing functions on a space. Note that we can immediately see that the cylinder and Möbius strip are not simply connected.

3.3 Passing from local to global

A part of the local study of spaces is to define things-on-a-space locally. Then, the natural question is:

When do locally defined things-on-a-space extend to globally defined ones?

In this section we have seen that functions and simply connectedness can be extended globally by gluing, provided that some condition is satisfied on the intersections. I will mention one more example: local solutions to a differential equation can sometimes be glued together to form a global solution over the entire space. Hence, we have seen examples of gluing local functions, properties, and solutions to form global ones. The benefit of these local-to-global principles is that it is often easy to construct solutions, say, on a simple space. Then, if our space is formed by gluing simple spaces, we can hope that the solutions can be glued to obtain a solution on the entire space.

4 Conclusion

Following this thread through the web of geometry took us from constructing spaces by gluing, studying them locally, to finally gluing local structures to form a global picture of the spaces. This is a template for a process that repeats throughout geometry, since it allows us to break problems into tractable pieces, placing the gluing construction at the centre of modern geometry.

I will end on a philosophical note: As gluing constructions are furthermore exclusive to geometry, they are a distinguishing feature of the very nature of geometry. I believe understanding the significance of gluing will give us insight into fundamentally *what space is*.

[4] A note to the advanced reader: These kinds of gluing constructions can be *automated*—in a sense—by an algebraic machine, called *homology*. The relation between gluing and homology is expressed in the *Mayer-Vietoris sequence*. A deeper link is provided by *Čech cohomology*.