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NUMERICAL SOLUTION OF THE $\mathbb{R}$-LINEAR BELTRAMI EQUATION

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Abstract. The $\mathbb{R}$-linear Beltrami equation appears in applications, such as in the inverse problem of recovering the electrical conductivity distribution in the plane. In this paper, a new way to discretize the $\mathbb{R}$-linear Beltrami equation is considered. This gives rise to large and dense $\mathbb{R}$-linear systems of equations with structure. For their iterative solution, norm minimizing Krylov subspace methods are devised. In the numerical experiments, these improvements combined are shown to lead to speed-ups of almost two orders of magnitude in the electrical conductivity problem.

1. Introduction

The $\mathbb{R}$-linear Beltrami equation

$$\overline{\partial}\omega + \nu\overline{\partial\omega} + \alpha\overline{\omega} + \beta = 0 \tag{1.1}$$

in the plane is a first order partial differential equation that can be converted, under certain assumptions, into an $\mathbb{R}$-linear integral equation. Here the asymptotic condition $\omega(z) = \mathcal{O}(1/z)$ as $z \to \infty$ for the solution is imposed. Moreover, let $\Omega \subset \mathbb{C}$ be a bounded domain with connected complement. (We may let, for simplicity, Ω to be the open unit disc.) Then suppose $\nu, \alpha, \beta \in L^\infty(\mathbb{C})$ with $|\nu(z)| \leq \kappa\chi_\Omega(z)$, where $\kappa < 1$ is a constant, for almost every $z \in \mathbb{C}$ and $\mathrm{supp}(\alpha) \subset \Omega, \mathrm{supp}(\beta) \subset \Omega$. These assumptions are realistic in several applications; see [8, 6, 2] and references therein. For the Beltrami equation and its applications, see [1, 5]. Motivated by the inverse problem of recovering the electrical conductivity distribution in the plane considered in [2], in this paper a new way to discretize the $\mathbb{R}$-linear Beltrami equation is suggested.

Using the properties of the Cauchy and Beurling transforms, an $\mathbb{R}$-linear integral equation formulation of (1.1) is treated directly. This gives rise to an integral operator which, from the numerical analysis point of view, is not as appealing as that considered in [2]. The reason is that the latter splits nicely as "the identity + compact", the numerical treatment of which is classical. However, the respective discretized problems can be shown to have the same solution, making the potential numerical inconvenience illusory. This is of importance since the discretized version considered in this paper turns out to be better suited for fast solving.

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In practice, the appearing $\mathbb{R}$-linear systems of equations are too large and dense to be solved with direct methods. However, their structure involves explicitly products of diagonal matrices and block Toeplitz matrices with Toeplitz blocks. Hence, by invoking the FFT, matrix-vector products can be performed rapidly, making the use of iterative methods attractive. A short but exhaustive treament of norm minimizing Krylov subspace methods for solving such linear systems is given. One of the methods considered is new. When executed with the discretization proposed in this paper, significant computational savings result. For example, in the electrical conductivity problem treated in [2], speed-ups of almost two orders of magnitude are obtained.

The paper is organized as follows. In Section 2 integral equation formulations of the $\mathbb{R}$-linear Beltrami equation (1.1) are considered. In Section 3 these integral equation formulations are discretized. The discretizations are shown to be equivalent. In Section 4 iterative methods (and preconditioning) are considered for solving the arising $\mathbb{R}$-linear systems of equations. Numerical experiments are given in Section 5.

2. The $\mathbb{R}$-linear Beltrami equation

In analyzing the $\mathbb{R}$-linear Beltrami equation, the Cauchy and Beurling transforms are of central relevance. First we give their required properties concisely. Then integral equation formulations of the Beltrami equation (1.1) are considered.

2.1. **Preliminaries.**

The Cauchy transform is a weakly singular integral operator defined, initially for $f \in C_0^\infty(\mathbb{C})$, by

$$Pf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\, dm(\zeta), \tag{2.1}$$

where m is the Lebesgue measure on the plane $\mathbb{C}$. The Beurling transform is a singular integral operator defined, again initially for $f \in C_0^\infty(\mathbb{C})$, by the principal value integral

$$Sf(z) = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\zeta - z|>\varepsilon} \frac{f(\zeta)}{(\zeta - z)^2}\, dm(\zeta). \tag{2.2}$$

These operators satisfy $Sf = \partial Pf$ and $S\overline{\partial} f = \partial f$ for all $f \in C_0^\infty(\mathbb{C})$.

In the following, $L^p(\mathbb{C})$ denotes the standard Lebesgue space on the plane $\mathbb{C}$ and $W^{1,p}(\mathbb{C})$ the Sobolev space. For a domain $\Omega \subset \mathbb{C}$, we define

$$L^p(\Omega) = \left\{ f \in L^p(\mathbb{C}) \,\middle|\, f|_{\mathbb{C}\setminus\Omega} \equiv 0 \right\}.$$

The Cauchy and Beurling transforms can be extended beyond $C_0^\infty(\mathbb{C})$. We need the following facts in particular:

- $P : L^p(\Omega) \to W^{1,p}(\mathbb{C})$ is bounded for $2 < p < \infty$,
- $P : L^p(\Omega) \to L^p(\mathbb{C})$ is compact for $2 < p < \infty$,
- $S : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.

Furthermore, we have, in the sense of weak derivatives, $\partial Pf = Sf$ and $\overline{\partial} Pf = f$ for all $f \in L^p(\mathbb{C})$, $1 < p < \infty$. For these and further properties of P and S, we refer to [1, Chapter 4].

2.2. **Two integral equation formulations.** We now describe two integral equation formulations of the Beltrami equation (1.1). The first of these was considered in [2] and the second one is our suggested replacement.

In the following, $\tau(f) = \overline{f}$ denotes the conjugation operator. Assume now $2 < p < 1 + 1/\kappa$ and define an $\mathbb{R}$-linear operator $K : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ by

$$Kg = P(I + \nu\tau S)^{-1}(\alpha\overline{g}).$$

It is shown in [3, Proposition 4.1] that K is compact, $I + K$ is invertible in $L^p(\mathbb{C})$ and maps $L^p(\mathbb{C})$ into $W^{1,p}(\mathbb{C})$. Let $\Omega \subset \mathbb{C}$ be the open unit disk. The integral equation

$$\omega + P(I + \nu\tau S)^{-1}(\alpha\overline{\omega}) = -K(\chi_\Omega) \tag{2.3}$$

can be used to establish a unique solution $\omega \in W^{1,p}(\mathbb{C})$ to the equation (1.1), when $\beta = \alpha$ [3, Section 4]. In [2], the equation (2.3) is used as a starting point for numerical computations.

For another integral equation formulation, we need the following (nonstandard) definition. For $1 < p < \infty$, define

$$W^{1,p}(\mathbb{C}, \Omega) = \left\{ f \in W^{1,p}(\mathbb{C}) \,\middle|\, f \text{ is analytic in } \mathbb{C} \setminus \overline{\Omega} \right\}.$$

$W^{1,p}(\mathbb{C}, \Omega)$ is a closed subset of $W^{1,p}(\mathbb{C})$ with the following property.

Proposition 2.1. *Assume $2 < p < \infty$. The Cauchy transform is a bounded linear operator $P : L^p(\Omega) \to W^{1,p}(\mathbb{C}, \Omega)$ with a bounded inverse $P^{-1} = \overline{\partial}$.*

Proof. For $f \in L^p(\Omega)$, it follows easily that Pf is analytic in $\mathbb{C} \setminus \overline{\Omega}$. Also, as mentioned above, $\overline{\partial} Pf = f$. Next, let $f \in W^{1,p}(\mathbb{C}, \Omega)$ and $g = P\overline{\partial} f$. Applying $\overline{\partial}$ to the both sides, we get $\overline{\partial} g = \overline{\partial} f$. Denoting $h = f - g$ we have $\overline{\partial} h = 0$ so that h satisfies the Cauchy-Riemann equations on the entire plane in the sense of weak derivatives. Hence h is equal to an analytic function almost everywhere on $\mathbb{C}$. Since $h \in L^p(\mathbb{C})$, $h \equiv 0$ by an L^p-version of Liouville's theorem. Therefore, $P : L^p(\Omega) \to W^{1,p}(\mathbb{C}, \Omega)$ is a bijection and $P = \overline{\partial}^{-1}$. □

Now suppose again $2 < p < 1 + 1/\kappa$. As just mentioned, the equation (1.1) has a unique solution $\omega \in W^{1,p}(\mathbb{C})$. Actually $\omega \in W^{1,p}(\mathbb{C}, \Omega)$ by the fact that (1.1) reduces to the Cauchy-Riemann equations outside $\overline{\Omega}$. Now introduce $u \in L^p(\Omega)$ by $\overline{u} = -\overline{\partial}\omega$. Then $\omega = -P\overline{u}$ and $\partial\omega = -S\overline{u}$. Hence (1.1) transforms into

$$u + (\overline{\nu} S + \overline{\alpha} P)\overline{u} = \overline{\beta}, \tag{2.4}$$

which has a unique solution $u \in L^p(\Omega)$ due to Proposition 2.1. This is an $\mathbb{R}$-linear singular integral equation. Here $\overline{\nu} S\tau$ is a singular integral operator and $\overline{\alpha} P\tau$ is a compact operator. See [7] for the theory of $\mathbb{C}$-linear equations of this type.

The operator $I + \overline{\nu} S\tau + \overline{\alpha} P\tau$ is invertible in $L^p(\Omega)$. To see this, note that $I + \overline{\nu} S\tau$ is invertible in $L^p(\Omega)$ according to [3, Theorem 3.2] (multiply by τ both left and right to use the theorem.) Hence the equation

$$\big(I + (I + \overline{\nu} S\tau)^{-1}(\overline{\alpha} P\tau)\big)u = (I + \overline{\nu} S\tau)^{-1}(\overline{\beta})$$

is equivalent to (2.4). The operator on the left-hand side is invertible by the fact that its null space is trivial and $(I + \overline{\nu} S\tau)^{-1}(\overline{\alpha} P\tau)$ is compact. Therefore, $I + \overline{\nu} S\tau + \overline{\alpha} P\tau$ is invertible as well.

Regarding the regularity of the solution of the equation (2.4), it follows from [1, Theorem 15.6.2] that for locally Hölder-continuous functions ν, α and β, the

solution u of (2.4) is locally Hölder-continuous as well. In applications, the functions ν, α and β may have discontinuities. Therefore we assume Ω to be decomposed into a finite number of subdomains each with a piecewise smooth boundary curve. If ν, α and β are locally Hölder-continuous in each subdomain, the solution u is locally Hölder-continuous in each subdomain as well. The solution is, in general, discontinuous on the boundaries of the subdomains.

3. Discretizing the $\mathbb{R}$-linear Beltrami equation

Next, both the equations (2.3) and (2.4) are discretized by collocating at a uniform grid of points. The purpose is to preserve the convolution structure of S and P so that the FFT can be readily employed. The discretizations are then shown to be equivalent. We take $\Omega \subset \mathbb{C}$ as the open unit disk.

3.1. **Discretization of** (2.4)**.**

To discretize the equation (2.4), take the rectangle $R = [-1, 1)^2$ and choose a positive integer m. Let $N = 2^m$ and $h = 2/N$. Then form a uniform grid of points jh in R, where $j \in \mathbb{Z}_N^2$ and

$$\mathbb{Z}_N^2 = \left\{ (j_1, j_2) \in \mathbb{Z}^2 \,\middle|\, -\frac{N}{2} \leq j_1, j_2 < \frac{N}{2} \right\}. \tag{3.1}$$

Our discrete form of the equation (2.4) is

$$u(jh) - \overline{\nu(jh)} h^2 \sum_{\substack{k \in \mathbb{Z}_N^2 \\ k \neq j}} \frac{\overline{u(kh)}}{\pi(jh - kh)^2} + \overline{\alpha(jh)} h^2 \sum_{\substack{k \in \mathbb{Z}_N^2 \\ k \neq j}} \frac{\overline{u(kh)}}{\pi(jh - kh)} = \overline{\beta(jh)} \quad (j \in \mathbb{Z}_N^2),$$

where we identify $jh = (j_1 h, j_2 h)$ and $(j_1 + i j_2)h \in \mathbb{C}$. This can be written in terms of matrices by packing the values $u(jh)$ into a vector $x \in \mathbb{C}^{N^2}$ by setting $x_{(j_1+N/2)+N(j_2+N/2)+1} = u(jh)$. As a result, we obtain an $\mathbb{R}$-linear system which can be given in a matrix form as

$$x + (\overline{D_1} T_1 + \overline{D_2} T_2)\overline{x} = \overline{D_3} b. \tag{3.2}$$

The matrices and the vector b in these equations have the following structure:

- b is an all-ones vector,
- D_1, D_2 and D_3 are diagonal matrices formed from the values $\nu(jh), \alpha(jh)$ and $\beta(jh)$, respectively,
- T_1 is a block-Toeplitz with Toeplitz blocks (BTTB) matrix formed from the values $-h^2/(\pi(jh - kh)^2)$ $(j \neq k)$. It is the discretization of the Beurling transform S. T_1 is complex symmetric.
- T_2 is a BTTB matrix formed from the values $h^2/(\pi(jh - kh))$ $(j \neq k)$. It is the discretization of the Cauchy transform P. T_2 is complex skew-symmetric.

By using the FFT, matrix-vector products with T_1 and T_2 cost $\mathcal{O}(N^2 \log(N))$ floating point operations.

Finally, an approximate solution $\omega_h \approx \omega$ to (1.1) in the rectangle R is obtained by solving x from the equation (3.2) and then computing

$$\omega_h = -T_2 \overline{x}. \tag{3.3}$$

An approximation to ω outside R can be obtained by discretizing P in a larger rectangle or by direct numerical integration of (2.1) at a single point $z \in \mathbb{C}$.

3.2. **Discretization of** (2.3)**.** For comparison, we describe the discretization of the equation (2.3) suggested in [2]. To our mind, the periodization proposed there is unnecessary. This is shown below. We also make some remarks on the use of the Neumann series in [2].

To describe the discretization of [2], first a periodic integral equation is derived on the square $Q = [-s, s)^2$, where $s = 2 + 3\varepsilon$ and $\varepsilon > 0$. To this end, define $\widetilde{g}(z) = \eta(z)/(\pi z)$ and $\widetilde{\gamma}(z) = -\eta(z)/(\pi z^2)$ for $z \in Q$, where η is a smooth cut-off function taking value 1 in the disk $\overline{B}(0, 2)$ and 0 outside the disk $\overline{B}(0, s)$. The functions $\widetilde{g}$ and $\widetilde{\gamma}$ are then $2s$-periodically extended over the entire plane $\mathbb{C}$. The respective periodic Cauchy and Beurling transforms are then defined by

$$\widetilde{P}f(z) = \int_Q \widetilde{g}(z-\zeta)f(\zeta)\,dm(\zeta),$$
$$\widetilde{S}f(z) = \int_Q \widetilde{\gamma}(z-\zeta)f(\zeta)\,dm(\zeta).$$

The periodic integral equation is then

$$(I + \widetilde{K})\omega = -\widetilde{K}(\chi_\Omega), \tag{3.4}$$

where

$$\widetilde{K}\omega = \widetilde{P}(I + \nu\tau\partial\widetilde{P})^{-1}(\alpha\overline{\omega}). \tag{3.5}$$

The functions ν, α, ω and χ_Ω are now regarded as being $2s$-periodic. It is then shown that this periodic equation has a unique solution that agrees with the solution of the original integral equation in the unit disk [2, Theorem 2]. Once this solution is obtained, it is possible to extend it to have the full solution of the original equation on the whole plane [2, Corollary 1]. To clarify the following description of the discretization, we rewrite the equation (3.4) as the system

$$v + \nu\tau\partial\widetilde{P}v = \alpha\overline{\omega}, \tag{3.6}$$

$$\omega + \widetilde{P}v = -\widetilde{K}(\chi_\Omega). \tag{3.7}$$

To discretize the equation (3.4), choose a positive integer m and let $M = 2^m$, $h = 2s/M$. A uniform grid in Q is formed by the points jh, where $j \in \mathbb{Z}_M^2$ as in (3.1). Consider the following system of equations

$$v(jh) + \nu(jh)\tau h^2 \sum_{\substack{k\in\mathbb{Z}_M^2\\ k\neq j}} \widetilde{\gamma}(jh-kh)v(kh) = \alpha(jh)\overline{\omega(jh)}, \qquad (j \in \mathbb{Z}_M^2) \tag{3.8}$$

$$\omega(jh) + h^2 \sum_{\substack{k\in\mathbb{Z}_M^2\\ k\neq j}} \widetilde{g}(jh-kh)v(kh) = f(jh), \qquad (j \in \mathbb{Z}_M^2) \tag{3.9}$$

obtained from discretizing (3.6) and (3.7), where $\omega(jh)$ for all $j \in \mathbb{Z}_M^2$ are to be solved with given values $f(jh)$. Here $f(jh) = 0$ when $|jh| \geq 1$. In [2], the equation (3.8) is solved only approximately. In fact, the operator $(I + \nu\tau\partial\widetilde{P})^{-1}$ is approximated by truncating its Neumann series and then discretizing the resulting sum in the grid in Q. (Solving the equation (3.8) would amount to summing their full Neumann series to avoid the approximation error.) To express this in a matrix form, we write the equations (3.8) and (3.9) as

$$x + C_2(I + D_1\tau C_1)^{-1}D_2\overline{x} = c,$$

where D_1 and D_2 are diagonal matrices and C_1 and C_2 are block-circulant matrices with circulant blocks (BCCB). The vector c is formed from the values $f(jh)$. Specifically, $c = -C_2(I + D_1\tau C_1)^{-1}D_2 b$, where b is an all-ones vector.

In [2], for a given vector y, it is then the vector $(I + D_1\tau C_1)^{-1}y$ that is is approximated by truncating the Neumann series.

Since $\nu(jh) = \alpha(jh) = 0$ for $|jh| \geq 1$, the solution of (3.8) satisfies $v(jh) = 0$ for $|jh| \geq 1$. Therefore, in (3.8) and (3.9) we may sum over $k \in \mathbb{Z}^2_{M/2}$ instead of $k \in \mathbb{Z}^2_M$. Moreover, we are only interested in the values $\omega(jh)$ for $|jh| < 1$, so that the equations reduce to

$$v(jh) + \nu(jh)\tau h^2 \sum_{\substack{k\in\mathbb{Z}^2_{M/2}\\ k\neq j}} \widetilde{\gamma}(jh-kh)v(kh) = \alpha(jh)\overline{\omega(jh)}, \qquad (j \in \mathbb{Z}^2_{M/2}) \tag{3.10}$$

$$\omega(jh) + h^2 \sum_{\substack{k\in\mathbb{Z}^2_{M/2}\\ k\neq j}} \widetilde{g}(jh-kh)v(kh) = f(jh). \qquad (j \in \mathbb{Z}^2_{M/2}) \tag{3.11}$$

Further, note that when $|jh|, |kh| < 1$, then $|jh-kh| < 2$ so that $\gamma(jh-kh) = \widetilde{\gamma}(jh-kh)$ and $g(jh-kh) = \widetilde{g}(jh-kh)$, where $g(z) = 1/(\pi z)$ and $\gamma(z) = -1/(\pi z^2)$ (defined for $z \in \mathbb{C}$). Since we are only interested in the values of $\omega(jh)$ for $|jh| < 1$, we can conclude that the smooth cut-off function η serves no purpose. Furthermore, these are the equations that we would have got had we discretized directly in the square $[-1, 1)^2$ without first periodizing the equation (2.3). Thereby, the periodization is unnecessary.

3.3. **Equivalence of the discretizations of** (2.4) **and** (2.3)**.** Let us return to solving (3.2). To compare it with the discretization of (2.3) just described, the equations (3.10) and (3.11) may be written in a matrix form as

$$x + T_2(I + D_1\tau T_1)^{-1}D_2\overline{x} = c, \tag{3.12}$$

where the matrices D_1, D_2, T_1 and T_2 are as in (3.2) and $N = M/2$. Here N is as in §3.1 and M as in §3.2. The vector c is formed from the values $f(jh)$, specifically

$$c = -T_2(I + D_1\tau T_1)^{-1}D_2 b.$$

We apply the (invertible) change of variables $\widetilde{x} = x + b$, to obtain the equation

$$x + T_2(I + D_1\tau T_1)^{-1}D_2\overline{x} = b, \tag{3.13}$$

where we have omitted the tilde above x.

The problem here is that the inverse of $I + D_1\tau T_1$ is not readily available. To circumvent this obstacle, multiply both sides of (3.13) by $\overline{D_2}$ from the left and set $y = \overline{D_2}x$ to get

$$y + \overline{D_2}T_2(I + D_1\overline{T_1}\tau)^{-1}\overline{y} = \overline{D_2}b. \tag{3.14}$$

Define $\overline{z} = (I + D_1\overline{T_1}\tau)^{-1}\overline{y}$, so that $y = z + \overline{D_1}T_1\overline{z}$. Then (3.14) converts into

$$z + (\overline{D_1}T_1 + \overline{D_2}T_2)\overline{z} = \overline{D_2}b. \tag{3.15}$$

This is the same equation as (3.2). (Here $D_3 = D_2$, because for (2.3) we took $\beta = \alpha$.)

The change of variables $y = \overline{D_2}x$ may not be invertible. This, however, is not a problem. To see this, suppose we solve (3.15) for z and then compute

$$x = b - T_2\overline{z}.$$

This x solves (3.13) as the following computation shows. We have $D_2\overline{(b-T_2\overline{z})} = \overline{D_2}(b-T_2\overline{z}) = \overline{z+D_1T_1\overline{z}} = \overline{z}+D_1\overline{T_1}z$. Therefore

$$\begin{aligned} b-&T_2\overline{z}+T_2(I+D_1\overline{T_1}\tau)^{-1}D_2\overline{(b-T_2\overline{z})} \\ &= b-T_2\overline{z}+T_2(I+D_1\overline{T_1}\tau)^{-1}(\overline{z}+D_1\overline{T_1}z) \\ &= b-T_2\overline{z}+T_2\overline{z} = b. \end{aligned}$$

Hence the discretization (3.2) is equivalent to the one used in [2] without the approximation error resulting from the Neumann series truncation.

4. Iterative methods for $\mathbb{R}$-linear systems of equations

To solve the $\mathbb{R}$-linear system (3.2), denote the matrix multiplying the unknown conjugated vector by $A_\# = \overline{D_1}T_1 + \overline{D_2}T_2$. For iterative methods, consider the product

$$(M+M_\#\tau)(I+A_\#\tau) \tag{4.1}$$

with $M, M_\# \in \mathbb{C}^{n\times n}$. This gives rise to the conditions

$$\begin{cases} M+M_\#\overline{A_\#} &= I \\ MA_\#+M_\# &= 0 \end{cases} \tag{4.2}$$

for the inverse. Relaxing these conditions corresponds to different (preconditioned) Krylov subspace methods for solving

$$x+A_\#\overline{x} = b \tag{4.3}$$

for $b\in\mathbb{C}^n$. Of course, considering the real and imaginary parts of (4.3) separately yields a standard (real) linear system of doubled size. However, this approach is not advisable [4].

4.1. Conditions for the standard GMRES.

If the second equation in (4.2) holds exactly, i.e., $M_\# = -MA_\#$, then a $\mathbb{C}$-linear system is obtained on which standard iterative methods can be executed. The first equation then reads

$$M(I-A_\#\overline{A_\#}) = I, \tag{4.4}$$

so that any approximation to the inverse of $I-A_\#\overline{A_\#}$ is advisable in choosing M.

The following is readily shown.

Proposition 4.1. *The operator $I+A_\#\tau$ is invertible if and only if the matrix $I-A_\#\overline{A_\#}$ is invertible.*

Simplest options for choosing M are diagonal matrices. In particular, with

$$M = I \quad\text{and}\quad M_\# = -A_\# \tag{4.5}$$

we end up solving

$$(I-A_\#\overline{A_\#})y = b \tag{4.6}$$

with $x = (I-A_\#\tau)y$. This alternative has not been considered before. The standard GMRES [9] can be executed here.

In assesing the speed of convergence of GMRES, observe that the spectrum of $A_\#\overline{A_\#}$ is best understood in terms of the Youla decomposition of $A_\#$ [11]. Whenever $A_\#^T = \pm A_\#$ the convergence is easy to analyze since then $\overline{A_\#} = \pm A_\#^*$. In the skew-symmetric case $I-A_\#\overline{A_\#} = I+A_\#A_\#^*$, i.e., we have a positive definite coefficient

matrix whose eigenvalues are larger than one. Then it is advisable to apply the conjugate gradient method for solving (4.6).

4.2. **Conditions for the $\mathbb{R}$-linear GMRES.** In the $\mathbb{R}$-linear system (4.3) the identity multiplies the unknown vector. There are good reasons to preserve this structure since then there exists an $\mathbb{R}$-linear GMRES algorithm for solving the problem [4]. For preservation, the first equation in (4.2) must hold exactly yielding

$$M = I - M_{\#}\overline{A_{\#}}. \tag{4.7}$$

Substituting this into the second equation gives

$$M_{\#}(\overline{A_{\#}}A_{\#} - I) = A_{\#}. \tag{4.8}$$

If the matrix $A_{\#}$ is invertible, then this is equivalent to

$$M_{\#}(\overline{A_{\#}} - A_{\#}^{-1}) = I \tag{4.9}$$

which may be more accessible. (Observe that if $A_{\#}$ is Hermitian, then $\overline{A_{\#}}A_{\#} - I$ is complex symmetric whereas $\overline{A_{\#}} - A_{\#}^{-1}$ remains Hermitian.) Assuming invertibility of $A_{\#}$ appears to be unrealistic in discretizing the $\mathbb{R}$-linear Beltrami equation, though.

In preconditioning large scale problems, the matrix equation (4.8) can be solved only approximately. No preconditioning corresponds to the choices

$$M = I \quad \text{and} \quad M_{\#} = 0; \tag{4.10}$$

see [8, 4] for convergence results in an electrical conductivity problem. Then (4.8) is not very accurately solved. Thereby there should exist room for an improvement by first constructing a reasonable approximate solution $M_{\#}$. Thereafter M is determined by the formula (4.7), assuring that the $\mathbb{R}$-linear structure of (4.3) is conserved in the product (4.1).

To end this section, let us mention that with the electrical conductivity problem in the plane, various ideas were tested in choosing a preconditioner $M + M_{\#}\tau$. In this case both of the simple choices (4.5) and (4.10) turned out to perform so well that no significant improvement was produced with these trials; see the numerical experiments in the next section. However, we do not have an understanding on which properties of the operator the speed of convergence of these methods depend. Thereby, at present, we cannot say which method should be preferred.

5. Numerical experiments with the electrical conductivity problem

We compared the performance of three, to our mind, most attractive Krylov subspace methods to solve the system (3.2) which we denote by $x + A_{\#}\overline{x} = b$. The following were considered:

(i) Rewrite as a real system of doubled dimension ($x = u + iv$)

$$\begin{bmatrix} I + \mathrm{Re}(A_{\#}) & \mathrm{Im}(A_{\#}) \\ \mathrm{Im}(A_{\#}) & I - \mathrm{Re}(A_{\#}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(b) \\ \mathrm{Im}(b) \end{bmatrix}$$

and solve by executing the standard GMRES.

(ii) Substitute $x = (I - A_{\#}\tau)y$ and solve

$$(I - A_{\#}\overline{A_{\#}})y = b$$

by executing the standard GMRES.

(iii) Execute directly the $\mathbb{R}$-linear GMRES given in [4].

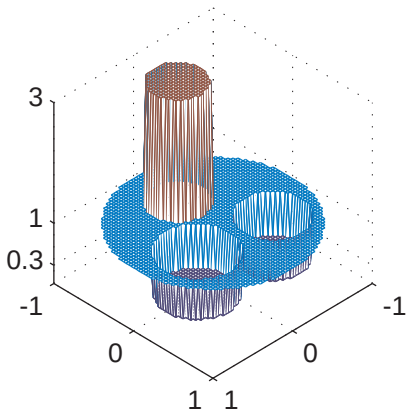

FIGURE 1. The conductivity in the unit disk.

We do not discuss the accuracy of the discretization given in §3.1, but refer to [2] in this regard. However, as shown in §3.2, the discretized equation (3.2) yields more accurate solutions than the method in [2].

We construct a piecewise continuous conductivity σ in the unit disk similar to the one used in [2]. This is shown in Figure 1. The base conductivity is 1, the high conductivity 3 and the low conductivity is 0.3. The values of σ outside the unit disk are taken to be 1. To compute the scattering transform associated with this σ, we need to solve equations of the form

$$\overline{\partial}\omega + \nu\overline{\partial\omega} + \alpha\overline{\omega} + \alpha = 0,$$

where $\nu(z,k) = -e_{-k}(z)\mu(z)$, $\alpha(z,k) = -i\overline{k}\nu(z,k)$ and $\mu(z) = (1-\sigma)/(1+\sigma)$, $e_k(z) = \exp\big(i(kz + \overline{k}\overline{z})\big)$. Here $k \in \mathbb{C}$ is a parameter and derivatives are taken with respect to z.

Referring to §3.1, we now choose $m = 8$ and thus use a uniform grid of size 256×256 on the rectangle $[-1,1)^2$. Note that the diagonal matrices D_1 and D_2 now satisfy $D_2 = -i\overline{k}D_1$, so that $A_\# = \overline{D_1}(T_1 + ikT_2)$ and therefore multiplication by $A_\#$ consists of a multiplication by a BTTB matrix followed by a diagonal matrix multiplication.

`MATLAB` version 7.9.0.529 was used in the computations. For the alternative (iii), the `MATLAB` code from [4] was used. For the alternatives (i) and (ii), this code was modified in a straightforward manner to get the standard GMRES so that the implementations of the $\mathbb{R}$-linear GMRES and standard GMRES were very similar. In addition, the alternatives (i) and (ii) were solved using the `MATLAB` provided `gmres` function. In all cases, iteration was started with zero initial guess and stopped once the relative residual satisfied $\|x + A_\#\overline{x} - b\|/\|b\| < 10^{-12}$. Note that this criterion results in the same accuracy for all cases.

Computations were done on a desktop machine with Core 2 Duo 2.4 GHz and 3 GB RAM. Table 1 shows the performance of the three methods when $k = 10$. Timing was done using `MATLAB`s profiler with 10 successive executions with averages (in seconds) recorded in the table. Total time is the time spent in the iterative solver as a whole. Multiplication time records the time spent in its matrix-vector

Method	Iterations	Total time (s)	Multiplication time (s)
(i) `MATLAB`	24	3.7	1.7
(ii) `MATLAB`	13	2.5	1.8
(i)	24	2.8	1.6
(ii)	13	2.0	1.6
(iii)	24	2.9	1.5

TABLE 1. Performance of the three methods on the 256×256 grid.
Method (i): the real system of doubled dimension solved by GMRES.
Method (ii): the system $(I - A_{\#}\overline{A_{\#}})y = b$ solved by GMRES.
Method (iii): the $\mathbb{R}$-linear system $x + A_{\#}\overline{x} = b$ solved by the $\mathbb{R}$-linear GMRES.

Method	Iterations	Total time (s)	Multiplication time (s)
(i) `MATLAB`	24	16.9	8.6
(ii) `MATLAB`	12	10.2	7.7
(i)	24	12.6	7.7
(ii)	12	7.8	6.5
(iii)	24	12.8	7.2

TABLE 2. Performance of the three methods on the 512×512 grid.
Method (i): the real system of doubled dimension solved by GMRES.
Method (ii): the system $(I - A_{\#}\overline{A_{\#}})y = b$ solved by GMRES.
Method (iii): the $\mathbb{R}$-linear system $x + A_{\#}\overline{x} = b$ solved by the $\mathbb{R}$-linear GMRES.

multiplications. The latter times are greater for `MATLAB`s `gmres` since it performs two unnecessary extra multiplications.

The computer was of similar processing capability as the laptop used in [2], where computations took about one minute on a 256×256 grid and 7 minutes on a 512×512 grid. Since we do not periodize the equation, the corresponding grid sizes in our case are 128×128 and 256×256. Since the solution time depends on the given conductivity σ, we cannot directly compare these times to ours. The number of iterations required increases with lowering the low-conductivity and raising the high-conductivity. Since our σ is reasonably similar to those used in [2], we believe our methods are faster due to the large solution time discrepancy and smaller multiplication cost per iteration step. Table 2 gives further measurements on a 512×512 grid ($m = 9$).

5.1. **Remark on computing the scattering transform.** In [2], the scattering transform τ (notation not to be confused with the conjugation operator) is computed from the formula

$$\overline{\tau(k)} = \frac{1}{2\pi} \int_{\mathbb{C}} \overline{\partial}_z \big(\omega(z,k) - \omega^{-}(z,k)\big)\, dz_1 dz_2,$$

where ω^- is computed using $-\mu$ and ω using μ. Given that we actually solve $u(z,k)$, where $\overline{u(z,k)} = -\overline{\partial}_z \omega(z,k)$, the scattering transform can be directly computed from

$$\tau(k) = \frac{1}{2\pi} \int_\Omega (u^- - u)\, dz_1 dz_2,$$

where u^- corresponds to $-\mu$ and u to μ. The integral can be approximated by e.g. the trapezoid rule in the rectangle $[-1,1]^2$.

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