1.1 *Elementary properties of probability measures.* Prove that any probability measure P on a countable sample space satisfies:

(a) $P(\emptyset) = 0$,

(b) $P(A^c) = 1 - P(A)$,

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

(d) $0 \leq P(A) \leq 1$,

(e) $A \subset B \implies P(A) \leq P(B)$,

(f) $P(B \setminus A) = P(B) - P(A \cap B)$.

1.2 *Uniform distribution of a finite and infinite set.*

(a) Show that the uniform distribution $P(\omega) = \frac{1}{|\Omega|}$ of a finite set Ω is a probability function (probability mass function).

(b) Let us model a 5-card poker hand using the uniform distribution of the set $\Omega = K^{(5)}$ where $K^{(5)} = \{A \subset K : |A| = 5\}$ is set of 5-element subset of the deck $K = \{1, 2, \ldots, 52\}$. Compute the probability $P(\omega)$ of the sample $\omega = \{1, 2, 3, 4, 5\}$.

(c) We wish to pick a random number from the set $\mathbb{N} = \{1, 2, \ldots\}$ so that the probability of every outcome is equally big. Is it possible to perform such a choice? What is then the probability of obtaining number 7?

1.3 *Product of uniform distributions.* Let the probability function μ_1 be the uniform distribution of the finite set S_1, and let the probability function μ_2 be the uniform distribution of the finite set S_2. Is $\mu_1 \times \mu_2$ then also a uniform distribution? Prove the statement true or provide a counterexample to show it false.

Continues on the next page...

1.4 *Taxonomy of randomness.* Browse through David Aldous's article at

`http://www.stat.berkeley.edu/~aldous/Real-World/100.html`

on examples of perceiving randomness in the real world.

(a) Propose 3–5 categorizes in which different manifestations of randomness can be characterized. You are free to set up your own categorization criteria.

(b) Do you think it possible to categorize Aldous's list into 3–5 categories?

(c) If not, explain why.

1.5 *Gibbs distribution.* Let H be a positive (i.e. $H \geq 0$) function on a finite set $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $\beta \geq 0$. Define

$$P(\omega) = Z_\beta^{-1} e^{-\beta H(\omega)}, \quad \omega \in \Omega,$$

where $Z_\beta = \sum_{\omega \in \Omega} e^{-\beta H(\omega)}$. The probability function P on is the *Gibbs distribution* induced by the energy function H. The number $1/\beta$ corresponds to temperature in many models of statistical physics.

(a) Prove that P if a probability function on Ω.

(b) What kind of Gibbs distribution is obtained by taking $\beta = 0$?

(c) Assume that the function H has a unique minimum at the point ω_1 and a unique maximum at the point ω_n. Examine how the probability function P as $\beta \to \infty$.