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Optimal second order rectangular elasticity elements with weakly symmetric stress

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Abstract We present new second order rectangular mixed finite elements for linear elasticity where the symmetry condition on the stress is imposed weakly with a Lagrange multiplier. The key idea in constructing the new finite elements is enhancing the stress space of the Awanou's rectangular elements (rectangular Arnold–Falk–Winther elements) using bubble functions. The proposed elements have only 18 and 63 degrees of freedom for the stress in two and three dimensions, respectively, and they achieve the optimal second order convergence of errors for all the unknowns. We also present a new simple a priori error analysis and provide numerical results illustrating our analysis.

Keywords mixed finite element · elasticity · weakly imposed symmetry · rectangular finite element

Mathematics Subject Classification (2000) 65N30

1 Introduction

We consider rectangular mixed finite elements for linear elasticity. In the mixed form of linear elasticity, for given external body force and boundary conditions, the stress and displacement are sought as a saddle point of the Hellinger–Reissner functional. In this saddle point approach, stress, which is of primary interest in mechanics, is directly obtained and numerical solutions do not suffer from locking in nearly incompressible materials [4].

The stress tensor in linear elasticity is symmetric due to the conservation of angular momentum, so it is reasonable to use a symmetric finite element space for numerical solutions of the stress tensor. However, the construction of mixed finite elements

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with symmetric finite element stress space is very difficult and requires a relatively large number of degrees of freedom (DOFs) [2,4,7,20,25]. There is an alternative way to deal with the symmetry of the numerical stress by imposing it weakly, that is, by requiring it to be orthogonal to a certain space of skew-symmetric tensors. This weak symmetry idea dates back to 1970's [1,26,27] and there are various triangular mixed finite elements based on this approach [3,6,10,13,16–18,22–24].

To our best knowledge, rectangular finite elements for elasticity with weakly imposed symmetry of stress were first studied by Awanou in [8]. He constructed a rectangular element version of the elasticity complex developed in [6]. Then he described two low order mixed finite elements for the two dimensional problem. The first achieves the optimal $O(h)$ order of convergence in all the unknowns and he pointed out that the second one is suboptimal. For the three dimensional problem he only described an analogue of the suboptimal element. Those suboptimal elements can be viewed as low order rectangular Arnold–Falk–Winther (AFW) elements and the optimal one is an analogue of the reduced AFW element in [6]. The purpose of this work is to show that we can modify the suboptimal elements in [8] with a few additional DOFs and achieve the optimal $O(h^2)$ convergence in all the unknowns.

The idea in developing our new elements is to enrich the shape functions of the stress space using rectangular bubble functions. These functions were originally proposed in [13] and used in [13,18] to improve the AFW elements. Our elements can be viewed as rectangular analogues of the lowest order elements in [18]. Because the proposed stress space is not a subspace of the rectangular BDM_2-based element, instead of following the stability proof in [18], we show a different proof relying on the features of the additional shape functions. We also present a new proof of a priori error estimates based on the stronger stability conditions that first appeared in [6]. This proof is different from [10] and [19], and applies, for example, to the elements in [6,13,18].

The same rectangular bubble functions are used to construct discontinuous Galerkin methods for linear elasticity in [14]. We point out that our elements have remarkably fewer DOFs than the methods proposed in [14] if they have the same order of convergence for stress. For example, the three dimensional RT_1 method in [14] has at least 108 local DOFs for stress whereas our three dimensional element has only 63 local DOFs for stress.

The paper is organized as follows. In section 2, we summarize the notations of the paper and review the Hellinger–Reissner formulation of linear elasticity. In section 3, we describe the new two and three dimensional mixed finite elements and prove that they are stable. In section 4, we give a simple proof of a priori error estimates and an analysis of local post-processing. In section 5, we show numerical results illustrating the presented analysis.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2,3$ with a Lipschitz boundary. We use $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, and $\mathbb{R}^{n\times n}_{\mathrm{skw}}$ to denote the spaces of all, symmetric, and skew-symmetric $n \times n$

matrices, respectively. For $\mathbb{R}^{n\times n}$ and $\mathbb{R}^n$ valued functions σ and u, their components are denoted by σ_{ij} and u_i.

On a set $G \subset \Omega$, for $\sigma, \tau : G \to \mathbb{R}^{n\times n}$, and $u, v : G \to \mathbb{R}^n$, we define inner products

$$(\sigma,\tau)_G := \int_G \sigma : \tau \, dx, \quad (u,v)_G := \int_G u \cdot v \, dx,$$

where $\sigma : \tau$ is the Frobenius inner product of matrices. The Hilbert spaces $L^2(G;\mathbb{R}^{n\times n})$ and $L^2(G;\mathbb{R}^n)$ are defined by these inner products, and $\|\cdot\|_G$ denotes the corresponding norms. In the case $G = \Omega$ we use (σ,τ) and $\|\cdot\|$ instead of $(\sigma,\tau)_\Omega$ and $\|\cdot\|_\Omega$ for simplicity. The $\operatorname{div}\sigma$ and $\operatorname{grad} u$ are defined by the row-wise divergence and the row-wise gradient

$$(\operatorname{div}\sigma)_i = \sum_{j=1}^n \partial_j \sigma_{ij}, \qquad (\operatorname{grad} u)_{ij} = \partial_j u_i,$$

respectively, where ∂_j is the partial derivative with respect to x_j. For a scalar function ϕ and a $\mathbb{R}^2$ (row) valued function $u = (u_1, u_2)$ on a two dimensional domain, $\overrightarrow{\operatorname{curl}}\,\phi$ and $\operatorname{rot} u$ are defined by

$$\overrightarrow{\operatorname{curl}}\,\phi = (-\partial_2\phi, \partial_1\phi), \qquad \operatorname{rot} u = -\partial_2 u_1 + \partial_1 u_2.$$

We use curl to denote the standard curl operator for three dimensional vector fields. Similar to the div operator, curl and rot are row-wise operators for matrix-valued functions. For a subspace $\mathbb{X}$ of $\mathbb{R}^{n\times n}$, we define

$$\|\sigma\|_{\operatorname{div}}^2 = \|\sigma\|^2 + \|\operatorname{div}\sigma\|^2, \quad H(\operatorname{div},\Omega;\mathbb{X}) = \{\sigma \in L^2(\Omega;\mathbb{X}) \,|\, \|\sigma\|_{\operatorname{div}} < \infty\}.$$

For an integer $0 \le m < \infty$, we use $H^m(\Omega)$ to denote the standard Sobolev spaces (see e.g., [15]). For a finite dimensional inner product space $\mathbb{X}$, $H^m(\Omega;\mathbb{X})$ is the space of $\mathbb{X}$-valued functions such that each component of a function is in $H^m(\Omega)$.

Throughout this paper we simply use h_K to denote the mesh size of a rectangle K up to a uniform constant.

For given displacement $u : \Omega \to \mathbb{R}^n$, the linear strain tensor $\varepsilon(u)$ is defined by

$$\varepsilon(u) = \frac{1}{2}(\operatorname{grad} u + (\operatorname{grad} u)^T).$$

From the generalized Hooke's law the stress tensor is $\sigma = C\varepsilon(u)$, in which C is the stiffness tensor such that $C(x) : \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}^{n\times n}_{\text{sym}}$ for all $x \in \Omega$ and

$$c_0 \tau : \tau \le C(x)\tau : \tau \le c_1 \tau : \tau, \quad \tau \in \mathbb{R}^{n\times n}_{\text{sym}},$$

with positive constants c_0, c_1 independent of $x \in \Omega$.

For each $x \in \Omega$, $C(x)^{-1}$ is also bounded and positive definite. If an elastic medium is isotropic, then $C^{-1}\tau$ has the form

$$C^{-1}\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda}\operatorname{tr}(\tau) I\right), \tag{2.1}$$

where $\mu(x), \lambda(x) > 0$ are the Lamé parameters, and $\mathrm{tr}(\tau)$ is the trace of τ.

Throughout this paper we assume homogeneous displacement boundary condition $u = 0$ on $\partial\Omega$ for simplicity. For a given $f \in L^2(\Omega;\mathbb{R}^n)$, the Hellinger–Reissner functional $\mathscr{J} : H(\Omega,\mathrm{div};\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{R}^n) \to \mathbb{R}$ is defined by

$$\mathscr{J}(\tau,v) = \int_\Omega \left(\frac{1}{2}C^{-1}\tau : \tau + \mathrm{div}\,\tau\cdot v - f\cdot v\right)dx, \tag{2.2}$$

and it is known that $\mathscr{J}$ has a unique critical point

$$(\sigma,u) \in H(\Omega,\mathrm{div};\mathbb{R}^{n\times n}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{R}^n),$$

which is the solution of linear elasticity problem with homogeneous displacement boundary condition.

For the approach with weakly imposed symmetry of stress we extend C^{-1} to be defined on $\mathbb{R}^{n\times n}$ by taking the identity map for skew-symmetric matrices. We denote this extended operator by A. We define function spaces Σ, U, and Γ by

$$\Sigma = H(\mathrm{div},\Omega;\mathbb{R}^{n\times n}), \quad U = L^2(\Omega;\mathbb{R}^n), \quad \Gamma = L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}}),$$

and a functional $\tilde{\mathscr{J}} : \Sigma\times U\times\Gamma \to \mathbb{R}$ by

$$\tilde{\mathscr{J}}(\tau,v,\eta) = \int_\Omega \left(\frac{1}{2}A\tau : \tau + \mathrm{div}\,\tau\cdot v + \tau : \eta - f\cdot v\right)dx. \tag{2.3}$$

The functional $\tilde{\mathscr{J}}$ has a unique critical point (σ,u,γ) (see [5]) and the first two components coincide with the critical point of $\mathscr{J}$ in (2.2). By variational methods, the critical point (σ,u,γ) of $\tilde{\mathscr{J}}$ satisfies

$$(A\sigma,\tau) + (u,\mathrm{div}\,\tau) + (\gamma,\tau) = 0, \qquad \tau\in\Sigma, \tag{2.4}$$

$$-(\mathrm{div}\,\sigma,v) = (f,v), \qquad v\in U, \tag{2.5}$$

$$(\sigma,\eta) = 0, \qquad \eta\in\Gamma. \tag{2.6}$$

The associated discrete problem with finite element spaces $\Sigma_h\times U_h\times\Gamma_h \subset \Sigma\times U\times\Gamma$ is seeking $(\sigma_h,u_h,\gamma_h)\in\Sigma_h\times U_h\times\Gamma_h$ such that

$$(A\sigma_h,\tau) + (u_h,\mathrm{div}\,\tau) + (\gamma_h,\tau) = 0, \qquad \tau\in\Sigma_h, \tag{2.7}$$

$$-(\mathrm{div}\,\sigma_h,v) = (f,v), \qquad v\in U_h, \tag{2.8}$$

$$(\sigma_h,\eta) = 0, \qquad \eta\in\Gamma_h. \tag{2.9}$$

In mixed finite element methods [12], the Babuška–Brezzi conditions for stability are:

(S1) There is a constant $c > 0$ independent of mesh sizes such that

$$c\|\tau\|^2_{\mathrm{div}} \le (A\tau,\tau),$$

for $\tau\in\Sigma_h$ satisfying $(\mathrm{div}\,\tau,v) + (\tau,\eta) = 0$ for all $(v,\eta)\in U_h\times\Gamma_h$.

(S2) There is a constant $c > 0$ independent of mesh sizes such that

$$\inf_{0\neq(v,\eta)\in U_h\times\Gamma_h} \sup_{0\neq\tau\in\Sigma_h} \frac{(\operatorname{div}\tau, v) + (\tau, \eta)}{\|\tau\|_{\operatorname{div}}(\|v\| + \|\eta\|)} \geq c > 0.$$

Throughout the paper c is a generic positive constant independent of mesh sizes. One can see that these conditions are fulfilled if

(A1) $\operatorname{div}\Sigma_h = U_h$
(A2) There is a constant $c > 0$ independent of mesh sizes such that for any $(v, \eta) \in U_h \times \Gamma_h$, there exists $\tau \in \Sigma_h$ satisfying $\operatorname{div}\tau = v$ and

$$(\tau, \eta') = (\eta, \eta'), \quad \forall \eta' \in \Gamma_h, \qquad \|\tau\|_{\operatorname{div}} \leq c(\|v\| + \|\eta\|).$$

The condition (**A1**) implies (**S1**) because A is positive definite and it is not difficult to see that (**A2**) implies (**S2**).

3 New finite elements

In this section we define our elements and show that they satisfy the stability conditions **(A1)** and **(A2)**. To give a unified stability proof for the two and three dimensions, we first describe the proposed elements in an abstract form and show that the abstract construction satisfies **(A1)** and **(A2)**. Then we provide explicit descriptions of the two and three dimensional elements which fit the abstract framework.

For a finite dimensional space $\mathbb{X}$ and an element $K \in \mathscr{T}_h$, $\mathscr{P}_k(K;\mathbb{X})$ denotes the space of $\mathbb{X}$ valued polynomials of degree $\leq k$, and $\tilde{\mathscr{P}}_1(K;\mathbb{X})$ denotes the L^2 orthogonal complement of $\mathscr{P}_0(K;\mathbb{X})$ in $\mathscr{P}_1(K;\mathbb{X})$. We define the piecewise polynomial spaces

$$\mathscr{P}_k(\mathscr{T}_h;\mathbb{X}) := \{p \in L^2(\Omega;\mathbb{X}) : p|_K \in \mathscr{P}_k(K;\mathbb{X}) \text{ for } K \in \mathscr{T}_h\},$$
$$\tilde{\mathscr{P}}_1(\mathscr{T}_h;\mathbb{X}) := \{p \in L^2(\Omega;\mathbb{X}) : p|_K \in \tilde{\mathscr{P}}_1(K;\mathbb{X}) \text{ for } K \in \mathscr{T}_h\}.$$

We use $X \sim Y$ to denote that $cX \leq Y \leq c'X$ holds for positive constants c, c' independent of mesh sizes.

3.1 Abstract description

Suppose there is a vector space $B_h \subset H(\Omega, \operatorname{div}; \mathbb{R}^{n\times n}) \cap \mathscr{P}_3(\mathscr{T}_h; \mathbb{R}^{n\times n})$ satisfying the following conditions:

(B1) $\operatorname{div} B_h = 0$
(B2) $B_h \perp \mathscr{P}_0(\mathscr{T}_h; \mathbb{R}^{n\times n}_{\text{skw}})$ (orthogonal in $L^2(\Omega; \mathbb{R}^{n\times n})$)
(B3) The L^2 projection $\tilde{Q}_h : B_h \to \tilde{\mathscr{P}}_1(\mathscr{T}_h; \mathbb{R}^{n\times n}_{\text{skw}})$ is an isomorphism and $\tilde{Q}_h^{-1}$ is uniformly bounded with respect to mesh sizes.

The rectangular AFW elements in [8] are $(\Sigma_h, U_h, \Gamma_h)$ where

$$\Sigma_h = n \text{ tuple of the rectangular } BDM_1 \text{ elements},$$
$$U_h = \mathscr{P}_0(\mathscr{T}_h; \mathbb{R}^n),$$
$$\Gamma_h = \mathscr{P}_0(\mathscr{T}_h; \mathbb{R}^{n\times n}_{\text{skw}}),$$

and the triple satisfies **(A1)** and **(A2)** [8].

Our new elements are $(\Sigma'_h, U_h, \Gamma'_h)$ where

$$\Sigma'_h = \Sigma_h + B_h, \tag{3.1}$$
$$\Gamma'_h = \mathscr{P}_1(\mathscr{T}_h; \mathbb{R}^{n\times n}_{\text{skw}}). \tag{3.2}$$

The degrees of freedom for Σ'_h will be discussed later with explicit forms of B_h spaces in two and three dimensions.

Theorem 3.1 *Suppose that $(\Sigma_h, U_h, \Gamma_h)$ is the rectangular AFW element. If Σ'_h and Γ'_h are defined as in* (3.1–3.2) with B_h satisfying **(B1)**, **(B2)** and **(B3)**, then $(\Sigma'_h, U_h, \Gamma'_h)$ satisfies **(A1)** and **(A2)**.

Proof Note that $\operatorname{div}\Sigma'_h = \operatorname{div}\Sigma_h = U_h$ because $\operatorname{div} B_h = 0$ by **(B1)**, so **(A1)** holds. Let $\tilde{\Gamma}_h = \tilde{\mathscr{P}}_1(\mathscr{T}_h; \mathbb{R}^{n\times n}_{\text{skw}})$ and note that

$$\Gamma'_h = \Gamma_h \oplus \tilde{\Gamma}_h, \qquad \Gamma_h \perp \tilde{\Gamma}_h.$$

To prove **(A2)**, suppose that $0 \neq (v, \eta') \in U_h \times \Gamma'_h$ is given and rewrite $\eta' = \eta + \tilde{\eta}$ with $\eta \in \Gamma_h$, $\tilde{\eta} \in \tilde{\Gamma}_h$. By **(A2)** for the rectangular AFW elements, there exists $\tau \in \Sigma_h$ such that $\operatorname{div}\tau = v$, $\|\tau\|_{\text{div}} \leq c(\|v\| + \|\eta\|)$ and

$$(\tau, \xi) = (\eta, \xi), \quad \forall \xi \in \Gamma_h. \tag{3.3}$$

By **(B3)** $\tilde{Q}_h : B_h \to \tilde{\Gamma}_h$ is an isomorphism, so by the Riesz representation theorem, there is a unique $\tau_b \in B_h$ such that

$$(\tau_b, \tilde{\xi}) = (\tilde{Q}_h \tau_b, \tilde{\xi}) = (\tilde{\eta} - \tau, \tilde{\xi}), \quad \forall \tilde{\xi} \in \tilde{\Gamma}_h. \tag{3.4}$$

We set $\tau' = \tau + \tau_b$ and will check that τ' satisfies the conditions in **(A2)**. We first see that $\operatorname{div}\tau' = \operatorname{div}\tau = v$. For $\xi' = \xi + \tilde{\xi} \in \Gamma_h \oplus \tilde{\Gamma}_h$,

$$\begin{aligned}
(\tau', \xi') &= (\tau, \xi + \tilde{\xi}) + (\tau_b, \xi + \tilde{\xi}), && \text{(definition of } \tau')\\
&= (\tau, \xi + \tilde{\xi}) + (\tau_b, \tilde{\xi}) && \text{by } \mathbf{(B2)}\\
&= (\tau, \xi) + (\tau, \tilde{\xi}) + (\tilde{\eta} - \tau, \tilde{\xi}) && \text{by (3.4)}\\
&= (\eta, \xi) + (\tilde{\eta}, \tilde{\xi}) && \text{by (3.3)}\\
&= (\eta', \xi'). && \text{by } \Gamma_h \perp \tilde{\Gamma}_h
\end{aligned}$$

To complete the proof of **(A2)**, we need to show $\|\tau'\|_{\text{div}} \leq c(\|v\| + \|\eta'\|)$. We know that $\|\tau\|_{\text{div}} \leq c(\|v\| + \|\eta\|)$ and by (3.4), $\|\tau_b\|_{\text{div}} = \|\tau_b\| \leq c(\|v\| + \|\tilde{\eta}\|)$. Hence

$$\|\tau'\|_{\text{div}} \leq \|\tau\|_{\text{div}} + \|\tau_b\|_{\text{div}} \leq c(\|v\| + \|\eta\| + \|\tilde{\eta}\|).$$

We have $\|\eta\| \leq \|\eta'\|$ and $\|\tilde{\eta}\| \leq \|\eta'\|$ due to the orthogonality $\Gamma_h \perp \tilde{\Gamma}_h$, so **(A2)** holds. □

3.2 Two dimensional element

The goal of this section is to construct B_h space satisfying $(\mathbf{B1}),(\mathbf{B2}),(\mathbf{B3})$ in two dimensions. Let $K \in \mathscr{T}_h$ be a rectangle with edges E_i, $i = 1,2,3,4$ such that E_i, E_{i+2} $(i = 1,2)$ are parallel. Define ϕ_1^K, ϕ_2^K to be the linear polynomials on K such that

$$\phi_i^K|_{E_i} \equiv 0, \quad \phi_i^K|_{E_{i+2}} \equiv 1, \qquad i = 1,2.$$

Let $b_K = \phi_1^K(1-\phi_1^K)\phi_2^K(1-\phi_2^K)$. We define B_h by

$$B_h := \{p \in L^2(\Omega;\mathbb{R}^{2\times 2}) \,:\, p|_K \in B_K\}, \tag{3.5}$$

where

$$B_K = \operatorname{span}\left\{ \begin{pmatrix} \overrightarrow{\operatorname{curl}}\, b_K \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \overrightarrow{\operatorname{curl}}\, b_K \end{pmatrix} \right\}.$$

Since b_K vanishes on ∂K, the gradient of b_K is normal to each edge $E \subset \partial K$. Hence the $\overrightarrow{\operatorname{curl}}\, b_K$, which is a $\pi/2$ rotation of the gradient of b_K, has vanishing normal components on $E \subset \partial K$. Thus it is clear that $B_h \subset H(\operatorname{div};\mathbb{R}^{2\times 2})$ and that $\operatorname{div} B_h = 0$. Now **(B1)** holds and next we check that also **(B2)** and **(B3)** hold.

Lemma 3.1 *B_h in* (3.5) *satisfies* **(B3)**.

Proof Let $\tilde{Q}_K$ denote the restriction of $\tilde{Q}_h$ to an element $K \in \mathscr{T}_h$, that is, the L^2 projection of B_K onto $\tilde{\mathscr{P}}_1(K;\mathbb{R}^{2\times 2}_{\text{skw}})$. In order to prove **(B3)** it suffices to show that $\tilde{Q}_K$ is an isomorphism and that $\tilde{Q}_K^{-1}$ is uniformly bounded with respect to mesh sizes.

Note that B_K and $\tilde{\mathscr{P}}_1(K;\mathbb{R}^{2\times 2}_{\text{skw}})$ have bases $\{\tau_1,\tau_2\}$ and $\{\eta_1,\eta_2\}$ such that

$$\tau_1 = \begin{pmatrix} \overrightarrow{\operatorname{curl}}\, b_K \\ 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 \\ \overrightarrow{\operatorname{curl}}\, b_K \end{pmatrix}, \tag{3.6}$$

$$\eta_i = \begin{pmatrix} 0 & x_i - a_i \\ -x_i + a_i & 0 \end{pmatrix}, \quad \int_K (x_i - a_i)\, dx = 0, \qquad a_i \in \mathbb{R}, \quad i = 1,2. \tag{3.7}$$

Integrating by parts gives

$$(\tilde{Q}_K \tau_i, \eta_j) = (\tau_i, \eta_j)_K = \begin{cases} (b_K, 1)_K, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{3.8}$$

From this identity one can see that $\tilde{Q}_K$ is injective, and by counting dimensions, it is an isomorphism. To show that $\tilde{Q}_K^{-1}$ is uniformly bounded with respect to mesh sizes, it is enough to show that

$$\frac{|(\tilde{Q}_K \tau_i, \eta_i)_K|}{\|\tau_i\|_K \|\eta_i\|_K} \geq c, \quad i = 1,2. \tag{3.9}$$

By the identity (3.8),

$$(\tilde{Q}_K \tau_i, \eta_i)_K = (b_K, 1)_K \geq c h_K^2.$$

By Friedrichs' inequality and the inverse inequality [11, chapter 4]

$$\|\eta_i\|_K \le ch_K \|\operatorname{rot}\eta_i\|_K \le ch_K^2,$$
$$\|\tau_i\|_K = \|\overrightarrow{\operatorname{curl}}\, b_K\|_K \le \frac{c}{h_K}\|b_K\|_K \le c.$$

From these results one can see that (3.9) holds. □

Lemma 3.2 *B_h in* (3.5) *satisfies* **(B2)**.

Proof For $\eta \in \mathscr{P}_0(K;\mathbb{R}^{2\times 2}_{\mathrm{skw}})$,

$$(\tau_1,\eta)_K = \left(\begin{pmatrix} b_K \\ 0 \end{pmatrix}, \operatorname{rot}\eta \right)_K = 0,$$

by the integration by parts. The same holds for τ_2. □

Now we define the DOFs for Σ_h' in (3.1) and prove the unisolvency. Let Σ_K be the space of local shape functions of Σ_h of the rectangular AFW element. The DOFs for Σ_K are

$$\tau \mapsto \int_E \tau\nu \cdot q\, ds, \quad q \in \mathscr{P}_1(E;\mathbb{R}^2), \quad (4\times 2\times 2 = 16 \text{ DOFs}) \tag{3.10}$$

where ν is the unit normal vector for each edge E of K. Unisolvency of Σ_K for these DOFs is well-known from the unisolvency of BDM_1 element [12]. We define the local shape functions of Σ_h' as $\Sigma_K' = \Sigma_K + B_K$. The DOFs for Σ_K' are given by the DOFs in (3.10) and

$$\tau \mapsto \int_K \tau : \eta\, dx, \quad \eta \in \tilde{\mathscr{P}}_1(K;\mathbb{R}^{2\times 2}_{\mathrm{skw}}). \quad (2 \text{ DOFs}) \tag{3.11}$$

Theorem 3.2 *The space Σ_K' is unisolvent for the DOFs in* (3.10) *and* (3.11).

Proof Note that $\dim \Sigma_K' = 18$ which is same as the number of DOFs given by (3.10) and (3.11). For the unisolvency suppose that $\tau \in \Sigma_K'$ and all DOFs of τ given by (3.10) and (3.11) vanish. We consider a decomposition

$$\tau = \tau_0 + c_1\tau_1 + c_2\tau_2, \quad \tau_0 \in \Sigma_K,$$

for $c_1, c_2 \in \mathbb{R}$ and τ_1, τ_2 in (3.6). The DOFs given by (3.10) vanish for τ_1 and τ_2 because they have vanishing normal components on ∂K. Therefore

$$\int_E \tau\nu \cdot p\, ds = \int_E \tau_0\nu \cdot p\, ds, \quad p \in \mathscr{P}_1(E;\mathbb{R}^2),$$

and the unisolvency of Σ_K implies $\tau = c_1\tau_1 + c_2\tau_2$. By (3.8) and the assumption that all DOFs of τ given by (3.11) vanish, we have $c_1 = c_2 = 0$. □

3.3 Three dimensional element

To define B_h in three dimensions we need the following matrix valued bubble function $\mathbf{b}_K$. For a cube $K \in \mathscr{T}_h$, let F_i, $i = 1, \cdots, 6$ be the faces of K such that F_i and F_{i+3} are parallel for $i = 1, 2, 3$. Let $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^3$ be the outward unit normal (column) vectors on F_1, F_2, F_3, respectively. Define ψ_i by the linear polynomial on K which is 1 on F_i and vanishes on F_{i+3}, and define

$$b_i = \psi_i(1-\psi_i), \qquad i = 1, 2, 3.$$

We adopt a convention that $b_{i+3} = b_i$, $i \in \mathbb{N}$ and set

$$\mathbf{b}_K = \sum_{i=1}^{3} \nu_i \nu_i^T b_{i+1} b_{i+2}.$$

Observe that $\mathbf{b}_K$ is an $\mathbb{R}^{3\times 3}$ valued function on K. Note that the choice of F_i and F_{i+3} between two parallel faces does not change the definition of $\mathbf{b}_K$ because $\nu_i \nu_i^T = (-\nu_i)(-\nu_i)^T$, so $\mathbf{b}_K$ is uniquely defined for the given cube K. From the definition it is easy to see that $0 < c \le \|\mathbf{b}_K\|_{L^\infty(K)} \le 1$.

As an example, we consider a case that $K = [0, h_1] \times [0, h_2] \times [0, h_3]$, and ν_i is the unit vector in the positive direction of x_i-axis. Then

$$\mathbf{b}_K = \begin{pmatrix} b_2 b_3 & 0 & 0 \\ 0 & b_1 b_3 & 0 \\ 0 & 0 & b_1 b_2 \end{pmatrix}. \tag{3.12}$$

Lemma 3.3 *Let $\overrightarrow{b_K}$ be a row vector of $\mathbf{b}_K$. Then $\overrightarrow{b_K} \times \nu = 0$ on any face F of K for the outward unit normal vector ν on F.*

Proof From the definition of $\mathbf{b}_K$ one can see that $\overrightarrow{b_K}$ is a linear combination of $\nu_i^T b_{i+1} b_{i+2}$, $i = 1, 2, 3$, so it suffices to check the claim for $\nu_i^T b_{i+1} b_{i+2}$.

If the normal vector ν on a face F is parallel to ν_i, then $\nu_i^T b_{i+1} b_{i+2}$ is parallel to ν on F, so $\nu_i^T b_{i+1} b_{i+2} \times \nu = 0$. If the ν on F is parallel to ν_{i+1} or ν_{i+2}, then $\nu_i^T b_{i+1} b_{i+2} \equiv 0$ on F because $b_{i+1} \equiv 0$ or $b_{i+2} \equiv 0$ on F. □

Let $\tilde{\Gamma}_K = \tilde{\mathscr{P}}_1(K; \mathbb{R}^{3\times 3}_{\text{skw}})$. By direct computations, one can check that $\operatorname{curl} : \tilde{\Gamma}_K \to \mathscr{P}_0(K; \mathbb{R}^{3\times 3})$ is injective. Define B_h by

$$B_h = \{ p \in L^2(\Omega; \mathbb{R}^{3\times 3}) \, : \, p|_K \in B_K \}, \tag{3.13}$$

where

$$B_K = \operatorname{span}\{ \operatorname{curl}((\operatorname{curl} \xi)\mathbf{b}_K) \,|\, \xi \in \tilde{\Gamma}_K \}. \tag{3.14}$$

Lemma 3.4 *If $\tau_b \in B_K$, then all normal components of τ_b vanish on all faces of K.*

Proof Let $\tau_b = \mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K)$ for some $\xi \in \tilde{\Gamma}_K$. Since $\mathrm{curl}\,\xi \in \mathscr{P}_0(K;\mathbb{R}^{3\times 3})$, each row of the matrix $(\mathrm{curl}\,\xi)\mathbf{b}_K$ is a linear combination of row vectors of $\mathbf{b}_K$ in (3.12). From this, it is sufficient to show that $\mathrm{curl}\,\overrightarrow{b_K} \cdot \nu \equiv 0$ for any row vector $\overrightarrow{b_K}$ of $\mathbf{b}_K$ on ∂K. By the Stokes' theorem, for $\phi \in C^1(\overline{K})$,

$$
\begin{aligned}
0 &= \int_K \mathrm{div}(\mathrm{curl}\,\overrightarrow{b_K})\phi\,dx = \int_{\partial K} (\mathrm{curl}\,\overrightarrow{b_K}\cdot\nu)\phi\,ds - \int_K (\mathrm{curl}\,\overrightarrow{b_K})\cdot \mathrm{grad}\,\phi\,dx \\
&= \int_{\partial K} (\mathrm{curl}\,\overrightarrow{b_K}\cdot\nu)\phi\,ds - \int_{\partial K} (\overrightarrow{b_K}\times\nu)\cdot \mathrm{grad}\,\phi\,ds - \int_K \overrightarrow{b_K}\cdot \mathrm{curl}(\mathrm{grad}\,\phi)\,dx \\
&= \int_{\partial K} (\mathrm{curl}\,\overrightarrow{b_K}\cdot\nu)\phi\,ds - \int_{\partial K} (\overrightarrow{b_K}\times\nu)\cdot \mathrm{grad}\,\phi\,ds.
\end{aligned}
$$

By Lemma 3.3, $\int_{\partial K}(\overrightarrow{b_K}\times\nu)\cdot\mathrm{grad}\,\phi\,ds = 0$. Since $\phi \in C^1(\overline{K})$ is arbitrary, $\mathrm{curl}\,\overrightarrow{b_K}\cdot\nu \equiv 0$ on ∂K. □

By Lemma 3.4, $B_h \subset H(\Omega,\mathrm{div};\mathbb{R}^{3\times 3})$ and $\mathrm{div}\,B_h = 0$. Now **(B1)** holds and next we check that also **(B2)** and **(B3)** hold.

Lemma 3.5 *B_h in* (3.13) *satisfies* **(B3)**.

Proof Let $\tilde{Q}_K$ be the restriction of $\tilde{Q}_h$ to an element $K \in \mathscr{T}_h$. In order to prove **(B3)** we show that $\tilde{Q}_K$ is an isomorphism and that $\tilde{Q}_K^{-1}$ has a bound independent of mesh sizes.

To show that $\tilde{Q}_K$ is an isomorphism, we first show that it is injective. Suppose that $\tilde{Q}_K\,\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K) = 0$ for $\xi \in \tilde{\Gamma}_K$. By Lemma 3.3,

$$
(\tilde{Q}_K\,\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K),\xi)_K = (\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K),\xi)_K = ((\mathrm{curl}\,\xi)\mathbf{b}_K,\mathrm{curl}\,\xi)_K. \tag{3.15}
$$

Recall that $\mathbf{b}_K$ is positive definite and $\mathrm{curl} : \tilde{\Gamma}_K \to \mathscr{P}_0(K;\mathbb{R}^{3\times 3})$ is injective. Therefore $\xi = 0$ and $\tilde{Q}_K$ is injective.

To show that $\tilde{Q}_K$ is surjective, it is enough to prove $\dim B_K \ge \dim \tilde{\Gamma}_K$. By the definition of B_K we only need to show that the linear map from $\tilde{\Gamma}_K$ to B_K defined by

$$
\xi \mapsto \mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K), \tag{3.16}
$$

is injective. Suppose that there is $0 \ne \xi \in \tilde{\Gamma}_K$ such that $\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K) = 0$. However, by (3.15) we have $((\mathrm{curl}\,\xi)\mathbf{b}_K,\mathrm{curl}\,\xi)_K = 0$, which is a contradiction. Thus the map in (3.16) is injective and $\dim B_K \ge \dim \tilde{\Gamma}_K$.

Finally, we show that $\tilde{Q}_K^{-1}$ is uniformly bounded with respect to the mesh size of K. We observe a preliminary result

$$
\|\xi\|_K \sim h_K \|\mathrm{curl}\,\xi\|_K, \qquad \xi \in \tilde{\Gamma}_K, \tag{3.17}
$$

which is obtained by Friedrichs' inequality, the inverse inequality, and standard scaling argument. By the definitions of the L^2 norm and $\tilde{Q}_K$, we have

$$
\begin{aligned}
\|\tilde{Q}_K\,\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K)\|_K &= \sup_{\eta\in\tilde{\Gamma}_K,\|\eta\|_K=1} (\mathrm{curl}((\mathrm{curl}\,\xi)\mathbf{b}_K),\eta)_K \qquad (3.18)\\
&= \sup_{\eta\in\tilde{\Gamma}_K,\|\eta\|_K=1} ((\mathrm{curl}\,\xi)\mathbf{b}_K,\mathrm{curl}\,\eta)_K. \quad \text{(by Lemma 3.3)}
\end{aligned}
$$

Hence the supremum is attained with $\eta = \xi/\|\xi\|_K$ and we have

$$\|\tilde{Q}_K \operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)\|_K = \frac{((\operatorname{curl}\xi)\mathbf{b}_K, \operatorname{curl}\xi)_K}{\|\xi\|_K} \geq c\frac{\|\operatorname{curl}\xi\|_K^2}{\|\xi\|_K} \geq ch_K^{-2}\|\xi\|_K,$$

where the last step is due to (3.17). Furthermore, by the inverse inequality and (3.17), we have

$$\|\operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)\|_K \leq ch_K^{-1}\|(\operatorname{curl}\xi)\mathbf{b}_K\|_K \leq ch_K^{-1}\|\operatorname{curl}\xi\|_K \leq ch_K^{-2}\|\xi\|_K.$$

Hence we have, for $\tau = \operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)$ with $0 \neq \xi \in \tilde{\Gamma}_K$,

$$\frac{\|\tilde{Q}_K \tau\|_K}{\|\tau\|_K} = \frac{\|\tilde{Q}_K \operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)\|_K}{\|\operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)\|_K} \geq c > 0.$$

□

Lemma 3.6 B_h *in* (3.13) *satisfies* **(B2)**.

Proof For $\xi \in \tilde{\Gamma}_K$ and $\eta \in \mathscr{P}_0(K;\mathbb{R}^{3\times 3}_{\text{skw}})$,

$$(\operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K), \eta)_K = ((\operatorname{curl}\xi)\mathbf{b}_K, \operatorname{curl}\eta)_K = 0,$$

by Lemma 3.3. □

Now we define the DOFs for Σ_h' and prove the unisolvency. Let Σ_K be the space of shape functions of Σ_h on K. For $\tau \in \Sigma_K$, degrees of freedom are given by

$$\tau \mapsto \int_F \tau\nu \cdot q\,ds, \quad q \in \mathscr{P}_1(F;\mathbb{R}^3), \qquad (6\times 9 = 54 \text{ DOFs}), \tag{3.19}$$

where F is a face of K and ν is the outward unit normal vector on F. The unisolvency of Σ_K comes from the unisolvency of the rectangular BDM_1 element [12]. From the definition of Σ_h', the space of local shape functions is $\Sigma_K' = \Sigma_K + B_K$. The DOFs of $\tau \in \Sigma_K'$ are defined by (3.19) and

$$\tau \mapsto \int_K \tau : \eta\,dx, \quad \eta \in \tilde{\Gamma}_K. \qquad (9 \text{ DOFs}) \tag{3.20}$$

Theorem 3.3 *The space* Σ_K' *is unisolvent for the DOFs* (3.19) *and* (3.20).

Proof Note that $\dim\Sigma_K' = \dim\Sigma_K + \dim\mathscr{P}_0(K;\mathbb{R}^{3\times 3}) = 54 + 9 = 63$. To show the unisolvency of Σ_K' for the DOFs (3.19) and (3.20), we follow the argument similar to the two dimensional case.

We suppose that $\tau = \tau_0 + \tau_b$ for $\tau_0 \in \Sigma_K$, $\tau_b \in B_K$ and all DOFs of τ given by (3.19) and (3.20) vanish. By Lemma 3.4

$$\int_F \tau\nu \cdot q\,ds = \int_F \tau_0\nu \cdot q\,ds, \quad q \in \mathscr{P}_1(F;\mathbb{R}^3)$$

holds and $\tau_0 = 0$ by the unisolvency of Σ_K. By the definition of B_K, for some $\xi \in \tilde{\Gamma}_K$, $\tau_b = \operatorname{curl}((\operatorname{curl}\xi)\mathbf{b}_K)$. Combining the definition of DOFs (3.20) with **(B3)**, we have $\xi = 0$, and thus $\tau_b = 0$. □

4 Error analysis and local post-processing

4.1 Improved error estimates

In this section we discuss a priori error analysis of the elements $(\Sigma_h', U_h, \Gamma_h')$. By the Babuška–Brezzi theory, the stability conditions give an a priori error estimate

$$\begin{aligned} &\|\sigma - \sigma_h\|_{\mathrm{div}} + \|u - u_h\| + \|\gamma - \gamma_h\| \\ &\quad \le c \inf_{(\tau,v,\eta)\in \Sigma_h' \times U_h \times \Gamma_h'} (\|\sigma - \tau\|_{\mathrm{div}} + \|u - v\| + \|\gamma - \eta\|). \end{aligned}$$

However, considering the approximability of $\Sigma_h \subset \Sigma_h'$, this estimate does not give an optimal error bound of $\|\sigma - \sigma_h\|$, so we show an improved error analysis which gives an optimal error bound of $\|\sigma - \sigma_h\|$. The same error bounds are obtained in previous literature [6, 13, 18, 19] but we show an alternative proof based on **(A1)** and **(A2)**.

Throughout this section, P_h and Q_h' denote the orthogonal L^2 projections onto U_h and Γ_h', respectively, and Π_h denotes the standard BDM_1 interpolation operator onto Σ_h [12]. This interpolation operator satisfies a commuting diagram property $\mathrm{div}\,\Pi_h \tau = P_h \,\mathrm{div}\,\tau$ for $\tau \in H^1(\Omega;\mathbb{R}^{n\times n})$.

Theorem 4.1 *Let (σ, u, γ) be a solution of* (2.4–2.6) *such that $\sigma \in H^m(\Omega;\mathbb{R}^{n\times n})$ and $\gamma \in H^m(\Omega;\mathbb{R}^{n\times n}_{\mathrm{skw}})$ for $1 \le m \le 2$. Suppose that $(\Sigma_h', U_h, \Gamma_h')$ are the elements defined in previous sections and let $(\sigma_h, u_h, \gamma_h)$ be a solution of the discrete problem corresponding to* (2.4–2.6) *with $(\Sigma_h', U_h, \Gamma_h')$. Then the following error estimates hold:*

$$\|\mathrm{div}\,\sigma - \mathrm{div}\,\sigma_h\| = \|\mathrm{div}\,\sigma - P_h \,\mathrm{div}\,\sigma\|, \tag{4.1}$$

$$\begin{aligned} \|\sigma - \sigma_h\| + \|u_h - P_h u\| + \|\gamma - \gamma_h\| &\le c(\|\sigma - \Pi_h \sigma\| + \|\gamma - Q_h' \gamma\|) \\ &\le c h^m (\|\sigma\|_m + \|\gamma\|_m), \qquad 1 \le m \le 2. \end{aligned} \tag{4.2}$$

Proof The error equations are

$$(A(\sigma - \sigma_h), \tau) + (u - u_h, \mathrm{div}\,\tau) + (\gamma - \gamma_h, \tau) = 0, \qquad \tau \in \Sigma_h', \tag{4.3}$$

$$(\mathrm{div}(\sigma - \sigma_h), v) = 0, \qquad v \in U_h, \tag{4.4}$$

$$(\sigma - \sigma_h, \eta) = 0, \qquad \eta \in \Gamma_h'. \tag{4.5}$$

Since $\mathrm{div}\,\sigma_h \in U_h$, (4.4) implies that $\mathrm{div}\,\sigma_h = P_h \,\mathrm{div}\,\sigma$, so (4.1) holds.

For the proof of (4.2), let

$$\Sigma_{h,0}' = \{\tau \in \Sigma_h' : \mathrm{div}\,\tau = 0\},$$

and consider an auxiliary system seeking $(\sigma, \gamma) \in \Sigma_{h,0}' \times \Gamma_h'$ for

$$(A\sigma, \tau) + (\gamma, \tau) + (\sigma, \eta) = F(\tau) + G(\eta), \qquad (\tau, \eta) \in \Sigma_{h,0}' \times \Gamma_h', \tag{4.6}$$

with bounded linear functionals (F, G) on $\Sigma_{h,0}' \times \Gamma_h'$. As a special case of **(A2)**, for $v = 0$ and any given $\eta \in \Gamma_h'$ there exists $\tau \in \Sigma_{h,0}'$ such that $(\tau, \eta') = (\eta, \eta')$ for all $\eta' \in \Gamma_h'$ and $\|\tau\| \le c\|\eta\|$. From this observation and **(A1)**, $\Sigma_{h,0}' \times \Gamma_h'$ is a stable mixed finite element for the system (4.6) with the L^2 norms. By restricting $\tau \in \Sigma'_{h,0}$, the sum of (4.3) and (4.5) becomes

$$(A(\sigma - \sigma_h), \tau) + (\gamma - \gamma_h, \tau) + (\sigma - \sigma_h, \eta) = 0,$$

which gives

$$(A(\sigma_h - \Pi_h \sigma), \tau) + (\gamma_h - Q'_h \gamma, \tau) + (\sigma_h - \Pi_h \sigma, \eta) \\ = (A(\sigma - \Pi_h \sigma), \tau) + (\gamma - Q'_h \gamma, \tau) + (\sigma - \Pi_h \sigma, \eta). \quad (4.7)$$

Note that $\sigma_h - \Pi_h \sigma \in \Sigma'_{h,0}$ because $\operatorname{div} \sigma_h = P_h \operatorname{div} \sigma = \operatorname{div} \Pi_h \sigma$. From the Babuška–Brezzi stability, there is $(\tau, \eta) \in \Sigma'_{h,0} \times \Gamma'_h$ such that $\|\tau\| + \|\eta\| \le c$ and

$$\|\sigma_h - \Pi_h \sigma\| + \|\gamma_h - Q'_h \gamma\| \le (A(\sigma_h - \Pi_h \sigma), \tau) + (\gamma_h - Q'_h \gamma, \tau) + (\sigma_h - \Pi_h \sigma, \eta).$$

Combining this, (4.7), and the Cauchy–Schwarz inequality with $\|\tau\| + \|\eta\| \le c$, we get

$$\|\sigma_h - \Pi_h \sigma\| + \|\gamma_h - Q'_h \gamma\| \le (A(\sigma - \Pi_h \sigma), \tau) + (\gamma - Q'_h \gamma, \tau) + (\sigma - \Pi_h \sigma, \eta) \\ \le c(\|\sigma - \Pi_h \sigma\| + \|\gamma - Q'_h \gamma\|).$$

By the triangle inequality and the above,

$$\|\sigma - \sigma_h\| + \|\gamma - \gamma_h\| \le \|\sigma - \Pi_h \sigma\| + \|\Pi_h \sigma - \sigma_h\| + \|\gamma - Q'_h \gamma\| + \|Q'_h \gamma - \gamma_h\| \\ \le c(\|\sigma - \Pi_h \sigma\| + \|\gamma - Q'_h \gamma\|) \\ \le ch^m(\|\sigma\|_m + \|\gamma\|_m), \qquad 1 \le m \le 2,$$

so (4.2) for $\|\sigma - \sigma_h\|$, $\|\gamma - \gamma_h\|$ is proved. To estimate $\|u_h - P_h u\|$, we first observe that (4.3) gives

$$(A(\sigma - \sigma_h), \tau) + (P_h u - u_h, \operatorname{div} \tau) + (\gamma - \gamma_h, \tau) = 0, \qquad \tau \in \Sigma'_h, \quad (4.8)$$

because $\operatorname{div} \tau \in U_h$ and $\operatorname{div} \tau \perp u - P_h u$. By **(A2)** we take τ in (4.8) such that $\operatorname{div} \tau = P_h u - u_h$ and $\|\tau\|_{\operatorname{div}} \le c\|P_h u - u_h\|$. Then we have

$$\|P_h u - u_h\|^2 = -(A(\sigma - \sigma_h), \tau) - (\gamma - \gamma_h, \tau) \\ \le c(\|\sigma - \sigma_h\| + \|\gamma - \gamma_h\|)\|P_h u - u_h\|.$$

Dividing by $\|P_h u - u_h\|$ and combining the result with the estimates of $\|\sigma - \sigma_h\|$, $\|\gamma - \gamma_h\|$, we have

$$\|P_h u - u_h\| \le ch^m(\|\sigma\|_m + \|\gamma\|_m), \qquad 1 \le m \le 2,$$

as desired. □

Remark 4.1 In the above proof, only **(A1)**, **(A2)**, and the commuting diagram and approximation properties of Π_h are used, so the proof applies to other mixed finite elements for linear elasticity with weakly imposed symmetry, for example, [6, 13, 18].

4.2 Local post-processing

A local post-processing can be used to obtain a better numerical solution of displacement for some mixed finite elements for elasticity [18,24] including the proposed elements. We use the post-processing method suggested in [18], which is slightly different from the one suggested by Stenberg [24]. Here we follow the proof presented in [21] for triangular elements.

Recall that $U_h = \mathscr{P}_0(\mathscr{T}_h;\mathbb{R}^n)$. We define

$$\begin{aligned} U_h^* &= \mathscr{P}_1(\mathscr{T}_h;\mathbb{R}^n), \\ \tilde{U}_h &= \{v \in U_h^* \,|\, (v,w) = 0, \quad \forall w \in U_h\}, \end{aligned}$$

and use P_h^* and $\tilde{P}_h$ to denote the orthogonal L^2 projections from $L^2(\Omega;\mathbb{R}^n)$ onto U_h^* and $\tilde{U}_h$, respectively. It is obvious that $P_h^* = \tilde{P}_h + P_h$. Let $(\sigma_h, u_h, \gamma_h)$ be a solution of the discrete problem, corresponding to (2.4–2.6) with $(\Sigma_h', U_h, \Gamma_h')$. On each $K \in \mathscr{T}_h$, we define $u_{h,K}^* \in U_h^*|_K$ by

$$(\operatorname{grad} u_{h,K}^*, \operatorname{grad} w) = (A\sigma_h + \gamma_h, \operatorname{grad} w), \qquad w \in \tilde{U}_h|_K, \tag{4.9}$$

$$(u_{h,K}^*, v) = (u_h, v), \qquad v \in U_h|_K. \tag{4.10}$$

We first show that $u_{h,K}^*$ in (4.9–4.10) is well-defined. Note that (4.9–4.10) is a system of linear equations with same number of equations and unknowns, so we only need to show that $u_{h,K}^*$ vanishes if the right-hand sides in (4.9–4.10) vanish. Suppose all the right-hand sides vanish, then from (4.10) we see that $u_{h,K}^*$ is in $\tilde{U}_h$. By taking $w = u_{h,K}^*$ in (4.9), we have $\operatorname{grad} u_{h,K}^* = 0$, which implies $u_{h,K}^* = 0$ because $\tilde{U}_h$ does not include constant functions and therefore grad is an injective operator on $\tilde{U}_h$.

Theorem 4.2 *Suppose that* (σ, u, γ), $(\sigma_h, u_h, \gamma_h)$ *are defined as in Theorem* 4.1 *and assume that* $\|u\|_3 < \infty$. *We define* u_h^* *by* $u_h^*|_K = u_{h,K}^*$ *with* $u_{h,K}^*$ *defined in* (4.9–4.10). *Then we have*

$$\|u - u_h^*\| \le ch^2 \|u\|_3. \tag{4.11}$$

Remark 4.2 Note that $\|\sigma\|_2 + \|\gamma\|_2 \le c\|u\|_3$ from the definitions of σ and γ, therefore (4.2) with $m = 2$ leads to

$$\|u_h - P_h u\| \le ch^2 \|u\|_3. \tag{4.12}$$

Proof Note that $\|u - P_h^* u\| \le ch^2 \|u\|_2$ by the Bramble–Hilbert lemma. Using the triangle inequality, we get

$$\|u - u_h^*\| \le \|u - P_h^* u\| + \|P_h^* u - u_h^*\| \le ch^2 \|u\|_2 + \|P_h^* u - u_h^*\|,$$

thus we only need to show

$$\|P_h^* u - u_h^*\| \le ch^2 \|u\|_3. \tag{4.13}$$

Recall that $P_h u_h^* = u_h$ by the definition of u_h^*. Since $P_h^* = P_h + \tilde{P}_h$, we have $u_h^* = u_h + \tilde{P}_h u_h^*$. Using this fact and $P_h^* = P_h + \tilde{P}_h$, one can see

$$P_h^* u - u_h^* = (P_h u + \tilde{P}_h u) - (P_h u_h^* + \tilde{P}_h u_h^*) = (P_h u - u_h) + (\tilde{P}_h u - \tilde{P}_h u_h^*). \tag{4.14}$$

By the triangle inequality, $\|P_h^* u - u_h^*\| \le \|P_h u - u_h\| + \|\tilde{P}_h u - \tilde{P}_h u_h^*\|$. Since $\|P_h u - u_h\| \le ch^2 \|u\|_3$ by (4.12), to prove (4.13), it suffices to show

$$\|\tilde{P}_h u - \tilde{P}_h u_h^*\| \le ch^2 \|u\|_3.$$

In order to prove the above estimate, we use the relation $\operatorname{grad} u = A\sigma + \gamma$. Let grad_h be the element-wise gradient operator and observe that

$$(\operatorname{grad} u, \operatorname{grad}_h w) = (A\sigma + \gamma, \operatorname{grad}_h w), \quad w \in \tilde{U}_h.$$

The difference of this equation and the sum of element-wise equations in (4.9) gives

$$(\operatorname{grad}_h (u - u_h^*), \operatorname{grad}_h w) = (A(\sigma - \sigma_h) + \gamma - \gamma_h, \operatorname{grad}_h w), \quad w \in \tilde{U}_h. \tag{4.15}$$

By (4.14), we have

$$u - u_h^* = (u - P_h^* u) + (P_h^* u - u_h^*) = (u - P_h^* u) + (P_h u - u_h) + (\tilde{P}_h u - \tilde{P}_h u_h^*).$$

Using this to rewrite (4.15), with $\operatorname{grad}_h (P_h u - u_h) = 0$, we have

$$\begin{aligned}
&(\operatorname{grad}_h (\tilde{P}_h u - \tilde{P}_h u_h^*), \operatorname{grad}_h w) \\
&\qquad = -(\operatorname{grad}_h (u - P_h^* u), \operatorname{grad}_h w) + (A(\sigma - \sigma_h) + \gamma - \gamma_h, \operatorname{grad}_h w).
\end{aligned}$$

Taking $w = \tilde{P}_h u - \tilde{P}_h u_h^*$ and using the Cauchy–Schwarz inequality, we have

$$\|\operatorname{grad}_h (\tilde{P}_h u - \tilde{P}_h u_h^*)\| \le c(\|\operatorname{grad}_h (u - P_h^* u)\| + \|\sigma - \sigma_h\| + \|\gamma - \gamma_h\|). \tag{4.16}$$

Now we note that

$$h\|\operatorname{grad}_h w\| \le c\|w\|, \qquad w \in U_h^*, \tag{4.17}$$
$$\|w\| \le ch\|\operatorname{grad}_h w\|, \qquad w \in \tilde{U}_h, \tag{4.18}$$

where (4.17) is an inverse estimate (see [11], p. 110) and (4.18) is a result of the fact that $\tilde{U}_h$ is orthogonal to piecewise constants, and Friedrichs' inequality with scaling. Using these inequalities and (4.16), we get

$$\begin{aligned}
\|\tilde{P}_h u - \tilde{P}_h u_h^*\| &\le ch\|\operatorname{grad}_h (\tilde{P}_h u - \tilde{P}_h u_h^*)\| \\
&\le ch(\|\operatorname{grad}_h (P_h^* u - u)\| + \|\sigma - \sigma_h\| + \|\gamma - \gamma_h\|) \\
&\le c(\|P_h u - u_h\| + h\|\operatorname{grad}_h (P_h^* u - u)\| + h\|\sigma - \sigma_h\| + h\|\gamma - \gamma_h\|),
\end{aligned}$$

where the last inequality is due to (4.17). By (4.12), the Bramble–Hilbert lemma, and the fact $(\|\sigma\|_2 + \|\gamma\|_2) \le c\|u\|_3$, we obtain $\|\tilde{P}_h u - \tilde{P}_h u_h^*\| \le ch^2 \|u\|_3$. □

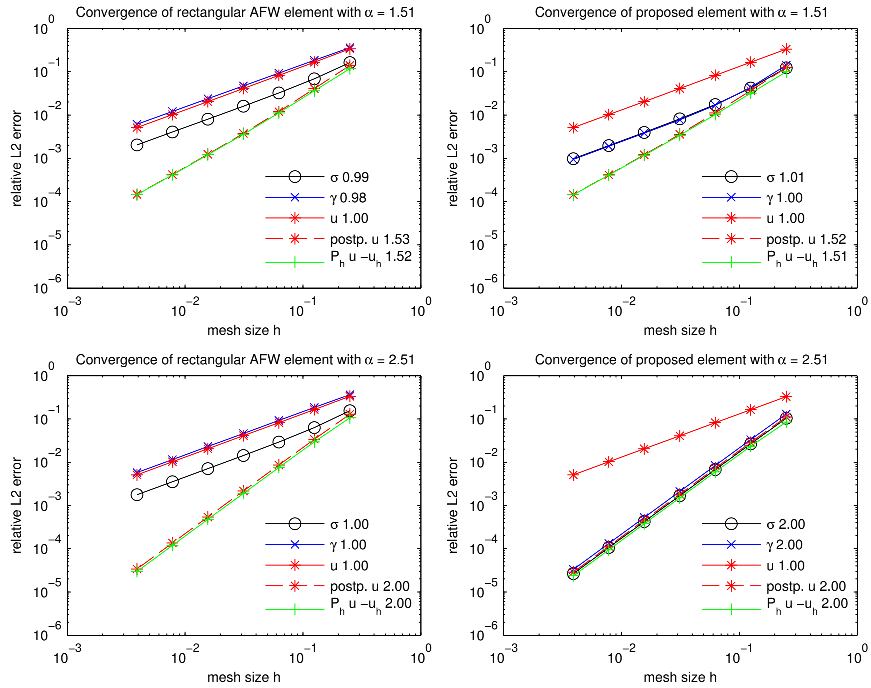

Fig. 5.1 Convergence of the relative L^2-error; rectangular AFW element [8] on the left and the proposed element on the right. On the top the parameter $\alpha = 1.51$ and on the bottom $\alpha = 2.51$. The rate of convergence is in the legend.

5 Numerical results

We measure the convergence of the errors $\|u-u_h\|$, $\|\sigma-\sigma_h\|$, $\|\gamma-\gamma_h\|$ and the convergence of the post-processed solution $\|u-u_h^*\|$. The proof of Theorem 4.2 shows that the critical part of the estimate for $\|u-u_h^*\|$ is the convergence of $\|P_h u - u_h\|$. Thus we also measure the convergence of $\|P_h u - u_h\|$.

Consider the linear elasticity problem in the unit square $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ with the exact solution set to

$$u(x,y) = \begin{pmatrix} 10x^\alpha(1-x)y^\alpha(1-y) \\ \frac{1}{2}\sin(\pi x)\sin(\pi y) \end{pmatrix},$$

in which the parameter $\alpha \in \mathbb{R}$, and the Lamé parameters are $\lambda = 1$ and $\mu = 1$. The exact solution fulfills homogenous displacement boundary condition and the load function is set to $f = -\operatorname{div} C\varepsilon(u)$. The parameter α controls the smoothness of the solution:

$$\begin{aligned} u &\in H^{\alpha+1/2-s}(\Omega;\mathbb{R}^2), & \sigma &\in H^{\alpha-1/2-s}(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}}), \\ f &\in H^{\alpha-3/2-s}(\Omega;\mathbb{R}^2), & \gamma &\in H^{\alpha-1/2-s}(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{skw}}), \end{aligned}$$

for $\alpha > 3/2$ and an arbitrarily small $s > 0$.

Figure 5.1 shows the rate of convergence of the relative L^2-errors for both the rectangular AFW and the proposed elements. The results are shown for the parameter values $\alpha = 1.51$ and $\alpha = 2.51$. The rates of convergence follow Theorem 4.1. For example, the rate of convergence of $\|u - u_h\|$ is linear for both elements and for both values of α. For $\alpha = 2.51$ the proposed element achieves quadratic rate of convergence for $\|\sigma - \sigma_h\|$ and $\|\gamma - \gamma_h\|$ whereas the rectangular AFW element has linear rates.

For $\alpha = 1.51$, $\|u - u_h^*\|$ and $\|P_h u - u_h\|$ show approximately $\mathscr{O}(h^{3/2})$ rate of convergence numerically. This is higher than the linear rate predicted by Theorem 4.2. However, the proof of this theorem shows that $\|P_h u - u_h\|$ and $h(\|\sigma - \sigma_h\| + \|\gamma - \gamma_h\|)$ limit the convergence of $\|u - u_h^*\|$, and for this example the estimate of $\|P_h u - u_h\|$ can be improved with the duality argument presented below.

For the duality argument suppose that $(\hat{\sigma}, \hat{u}, \hat{\gamma})$ is the solution of

$$(A\hat{\sigma}, \tau) + (\hat{u}, \operatorname{div} \tau) + (\hat{\gamma}, \tau) = 0, \qquad \tau \in \Sigma, \tag{5.1}$$

$$(\operatorname{div} \hat{\sigma}, v) = (P_h u - u_h, v), \qquad v \in U, \tag{5.2}$$

$$(\hat{\sigma}, \eta) = 0, \qquad \eta \in \Gamma. \tag{5.3}$$

Set $\tau = \sigma - \sigma_h$, $v = u - u_h$, $\eta = \gamma - \gamma_h$ and sum the equations to get

$$\begin{aligned}(A\hat{\sigma}, \sigma - \sigma_h) + (\hat{u}, \operatorname{div}(\sigma - \sigma_h)) + (\hat{\gamma}, \sigma - \sigma_h) + (\operatorname{div} \hat{\sigma}, u - u_h) + (\hat{\sigma}, \gamma - \gamma_h) \\ = (P_h u - u_h, u - u_h).\end{aligned}$$

Note that $(P_h u - u_h, u - u_h) = \|P_h u - u_h\|^2$. By the Galerkin orthogonality we can add $\Pi_h \hat{\sigma}$, $P_h \hat{u}$, $Q_h' \hat{\gamma}$ and arrive to

$$\begin{aligned}(A(\hat{\sigma} - \Pi_h \hat{\sigma}), \sigma - \sigma_h) + (\hat{u} - P_h \hat{u}, \operatorname{div}(\sigma - \sigma_h)) + (\hat{\gamma} - Q_h' \hat{\gamma}, \sigma - \sigma_h) \\ + (\operatorname{div}(\hat{\sigma} - \Pi_h \hat{\sigma}), u - u_h) + (\hat{\sigma} - \Pi_h \hat{\sigma}, \gamma - \gamma_h) = \|P_h u - u_h\|^2.\end{aligned}$$

By (5.2), $\operatorname{div} \hat{\sigma} = P_h \operatorname{div} \hat{\sigma} = \operatorname{div} \Pi_h \hat{\sigma}$, so $(\operatorname{div}(\hat{\sigma} - \Pi_h \hat{\sigma}), u - u_h) = 0$. In addition, use that $\operatorname{div} \sigma = f$, $\operatorname{div} \sigma_h = P_h f$ and the standard interpolation to have

$$\|P_h u - u_h\|^2 \leq ch(\|\sigma - \sigma_h\| \|\hat{\sigma}\|_1 + \|\gamma - \gamma_h\| \|\hat{\gamma}\|_1) + |(\hat{u} - P_h \hat{u}, f - P_h f)|.$$

Recall that in the unit square the following elliptic regularity estimate holds [9].

$$\|\hat{\sigma}\|_1 + \|\hat{u}\|_2 + \|\hat{\gamma}\|_1 \leq c\|P_h u - u_h\|.$$

Therefore the duality argument gives

$$\|P_h u - u_h\|^2 \leq ch\big(\|\sigma - \sigma_h\| + \|\gamma - \gamma_h\|\big) \|P_h u - u_h\| + |(\hat{u} - P_h \hat{u}, f - P_h f)|. \tag{5.4}$$

Let $W^{s,p}(\Omega; \mathbb{R}^n)$ denote the Sobolev space based on L^p norm with s weak derivates [11]. Since $H^2(\Omega; \mathbb{R}^2)$ is continuously embedded in $W^{1,1+1/\delta}(\Omega; \mathbb{R}^2)$ for any $\delta > 0$,

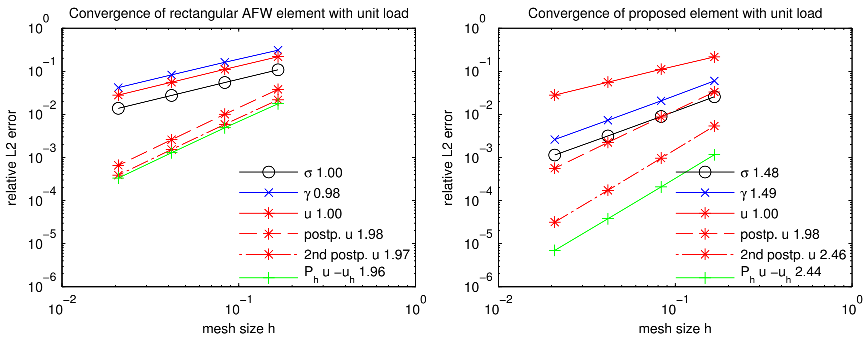

Fig. 5.2 Convergence of the relative L^2-error with unit load; rectangular AFW element on the left and the proposed element on the right. The rate of convergence is in the legend.

we get

$$
\begin{aligned}
&|(\hat{u}-P_h\hat{u}, f-P_h f)| \\
&\leq c\|\hat{u}-P_h\hat{u}\|_{L^{1+1/\delta}(\Omega;\mathbb{R}^2)}\|f-P_h f\|_{L^{1+\delta}(\Omega;\mathbb{R}^2)} && \text{(Hölder inequality)}\\
&\leq ch\|\hat{u}\|_{W^{1,1+1/\delta}(\Omega;\mathbb{R}^2)}\, h^{1/2}\|f\|_{W^{\frac{1}{2},1+\delta}(\Omega;\mathbb{R}^2)} && \text{(Bramble–Hilbert)}\\
&\leq ch^{3/2}\|\hat{u}\|_{H^2(\Omega;\mathbb{R}^2)}\|f\|_{W^{\frac{1}{2},1+\delta}(\Omega;\mathbb{R}^2)} && \text{(Sobolev embedding)}\\
&\leq ch^{3/2}\|P_h u-u_h\|\|f\|_{W^{\frac{1}{2},1+\delta}(\Omega;\mathbb{R}^2)}. && \text{(elliptic regularity)}
\end{aligned}
$$

Combining this with (5.4) gives

$$\|P_h u-u_h\| \leq c\Big[h\big(\|\sigma-\sigma_h\|+\|\gamma-\gamma_h\|\big)+h^{3/2}\|f\|_{W^{\frac{1}{2},1+\delta}(\Omega;\mathbb{R}^2)}\Big]. \tag{5.5}$$

Now the results are explained observing that for $\alpha = 1.51$ and sufficiently small $\delta > 0$, the norm $\|f\|_{W^{\frac{1}{2},1+\delta}(\Omega;\mathbb{R}^2)}$ is finite.

The duality result (5.4) also gives a more familiar result

$$\|P_h u-u_h\| \leq ch\Big(\|\sigma-\sigma_h\|+\|\gamma-\gamma_h\|+\|P_h f-f\|\Big). \tag{5.6}$$

Suppose that $f \in U_h$, that is, $\|P_h f - f\| = 0$. Then for the proposed element the estimate (5.6) predicts cubic convergence of $\|P_h u-u_h\|$ for smooth enough solutions. This suggests that it is possible to post-process the displacement variable up to piecewise quadratic. The procedure is as explained in section 4.2 except that U_h^* is replaced by

$$U_h^{**} = \{v \in L^2(\Omega;\mathbb{R}^n) \,:\, v|_K \in \mathscr{P}_2(K;\mathbb{R}^n), \quad K \in \mathscr{T}_h\},$$

and we look for $u_h^{**} \in U_h^{**}$. To show that the second post-processing works in practice we study the linear elasticity problem in the unit square Ω with the unit load $f = \begin{pmatrix}1\\1\end{pmatrix}$ and Lamé parameters $\lambda = 1$ and $\mu = 1$. Clearly for this problem $f \in U_h$.

Figure 5.2 shows the relative error in the L^2-norm for both the rectangular AFW and the proposed element for the unit load example. As expected, the rate of convergence of $\|u - u_h\|$ is linear for both elements. The convergence of $\|\sigma - \sigma_h\|$ and $\|\gamma - \gamma_h\|$ is linear for the rectangular AFW element and roughly $\mathscr{O}(h^{3/2})$ for the proposed element. This information with estimate (5.6) predicts quadratic rate of $\|P_h u - u_h\|$ convergence for the rectangular AFW element and $\mathscr{O}(h^{5/2})$ rate for the proposed element. Figure 5.2 shows exactly these predicted convergence rates. For the rectangular AFW element the quadratic rate of $\|P_h u - u_h\|$ explains the observed quadratic rate of $\|u - u_h^*\|$ and $\|u - u_h^{**}\|$. This confirms that the rectangular AFW element is unable to benefit from the second post-processing. For the proposed element $\|P_h u - u_h\|$ has $\mathscr{O}(h^{5/2})$ rate of convergence which enables $\|u - u_h^*\|$ to achieve $O(h^2)$ rate and $\|u - u_h^{**}\|$ to converge with $\mathscr{O}(h^{5/2})$ rate. This shows that the second post-processing can indeed improve the convergence beyond the quadratic rate for the proposed element if $f \in U_h$ and the solution is smooth enough.

6 Concluding remarks

We proposed new optimal second order rectangular mixed finite elements for elasticity and showed the a priori error analysis. The elements have a few additional DOFs compared to the ones in [8] and have higher orders of convergence for all errors. We point out that the current theory does not guarantee the same order of convergence for general quadrilateral meshes. However, we believe that the idea of our construction can be extended to higher order rectangular elements, which will be studied in our future work.

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