A MESH DEPENDENT NORM ANALYSIS OF LOW ORDER MIXED FINITE ELEMENT FOR ELASTICITY WITH WEAKLY SYMMETRIC STRESS

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Abstract. We consider mixed finite elements for linear elasticity with weakly symmetric stress. We propose a low order three dimensional rectangular element with optimal $O(h)$ rate of convergence for all the unknowns. The element is a rectangular analogue of the simplified Arnold–Falk–Winther element. Instead of the elasticity complex approach, our stability analysis is based on new mesh dependent norms.

1. Introduction

In the Hellinger–Reissner formulation of linear elasticity the symmetry of stress tensor is an obstacle for using mixed methods. A successful alternative is to impose the symmetry of stress weakly, dating back to Ref. [13, 14]. In this paper we construct a new low order three dimensional rectangular element with weakly symmetric stress and show its stability using new mesh dependent norms.

For triangular meshes, numerous mixed finite elements with weakly symmetric stress have been constructed [2, 3, 5, 9, 11, 12, 15, 18, 19, 20, 21]. For rectangular meshes, the first mixed finite elements with weakly symmetric stress were studied in Ref. [4] where the proposed elements are rectangular analogues of the Arnold–Falk–Winther (AFW) elements and the simplified AFW elements in two dimensions. Our motivation is to continue the work started in Ref. [4] and construct a rectangular analogue of the simplified AFW element in three dimensions. For optimal second order rectangular elements see Ref. [10].

In our element the numbers of local degrees of freedom (DOFs) are 36 for the stress, 3 for the displacement and 3 for the rotation unknowns. To the best of our knowledge, the number of DOFs of our element is smaller than any previously known three dimensional rectangular element for elasticity with weakly symmetric stress. The error analysis shows $O(h)$ convergence of errors for all the unknowns, which is optimal for the shape functions. Moreover, there are no vertex DOFs, so hybridization technique is available for implementation.

For the stability analysis, instead of the elasticity complex approach of Ref. [3] and [4], we use the mesh dependent norm idea of Ref. [17] and [20] combined with discrete Korn’s inequality proved in Ref. [6]. The stability analysis using mesh dependent norms explicitly shows necessary conditions for stable stress spaces and our element is constructed to fulfill these conditions by enriching the lowest order rectangular Raviart–Thomas element.

The paper is organized as follows. In section 2 we summarize the notations and review the Hellinger–Reissner formulation of linear elasticity with weakly symmetric stress. In section 3 we formally describe the proposed mixed finite element and
prove unisolvency of the degrees of freedom. In section 4 we introduce the mesh dependent norms and show the stability of the method. In section 5 we show the a priori error analysis and finally, in section 6 we present the explicit construction of our element.

2. Preliminaries

Let \( \mathbb{R}^{3 \times 3} \), \( \mathbb{R}_{\text{sym}}^{3 \times 3} \) and \( \mathbb{R}_{\text{skw}}^{3 \times 3} \) denote the spaces of all, symmetric and skew-symmetric \( 3 \times 3 \) matrices, respectively. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with a Lipschitz boundary and let \( G \subset \Omega \). For \( \sigma, \tau : \Omega \to \mathbb{R}^{3 \times 3} \) and \( v, w : \Omega \to \mathbb{R}^3 \), the inner products on Hilbert spaces \( L^2(G; \mathbb{R}^{3 \times 3}) \) and \( L^2(G; \mathbb{R}^3) \) are

\[
(\sigma, \tau)_G = \int_G \sigma : \tau \, dx \quad \text{and} \quad (v, w)_G = \int_G v \cdot w \, dx,
\]

where \( \sigma : \tau \) is the Frobenius inner product of matrices. The induced \( L^2 \) norms are denoted by \(| \cdot |_G \) and, if \( G = \Omega \), by \(| \cdot |_1 \). We use \( H^1(\Omega) \) to denote standard Sobolev spaces \( \mathbb{X} \) and for a finite dimensional inner product space \( \mathbb{X} \), \( H^1(\Omega; \mathbb{X}) \) is the space of \( \mathbb{X} \)-valued functions such that each component is in \( H^1(\Omega) \). The associated norm is denoted by \(| \cdot |_1 \). In addition, for a subspace \( \mathbb{X} \) of \( \mathbb{R}^{3 \times 3} \), we define

\[
|\sigma|_{\text{div}}^2 = |\sigma|^2 + |\text{div} \sigma|^2 \quad \text{and} \quad H(\text{div}, \Omega; \mathbb{X}) = \{ \sigma \in L^2(\Omega; \mathbb{X}) \mid |\sigma|_{\text{div}} < \infty \}.
\]

For \( \sigma : \Omega \to \mathbb{R}^{3 \times 3} \), the symmetric and skew-symmetric parts are denoted by

\[
\text{sym} \sigma = \frac{\sigma + \sigma^T}{2} \quad \text{and} \quad \text{skw} \sigma = \frac{\sigma - \sigma^T}{2},
\]

where \( \sigma^T \) is the transpose of \( \sigma \). For \( \sigma : \Omega \to \mathbb{R}^{3 \times 3} \) and \( u : \Omega \to \mathbb{R}^3 \) their components are denoted by \( \sigma_{ij} \) and \( u_i \), \( i, j = 1, 2, 3 \). The operators \( \text{div} \), \( \text{grad} \) and \( \text{curl} \) are defined row-wise, for example,

\[
(\text{div} \sigma)_i = \partial_j \sigma_{ij} \quad \text{and} \quad (\text{grad} u)_i = \partial_j u_i.
\]

For a given displacement \( u \in H^1(\Omega; \mathbb{R}^3) \), the strain tensor \( \epsilon = \epsilon(u) : \Omega \to \mathbb{R}_{\text{sym}}^{3 \times 3} \) is \( \epsilon(u) = \text{sym} \text{ grad} u \). In linear elasticity, for given external body force \( f \in L^2(\Omega; \mathbb{R}^3) \), the balance law of linear momentum leads to

\[
-\text{div} C(\epsilon(u)) = f \quad \text{in} \ \Omega.
\]

From the generalized Hooke’s law the stress tensor is \( \sigma = C\epsilon(u) \) where \( C : \mathbb{R}_{\text{sym}}^{3 \times 3} \to \mathbb{R}_{\text{sym}}^{3 \times 3} \) is a symmetric, bounded, positive definite stiffness tensor and for each \( x \in \Omega \), \( C(x)^{-1} \) is also symmetric, bounded and positive definite. For an isotropic elastic medium

\[
C^{-1} \tau = \frac{1}{2\mu} \left( \tau - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\tau) I \right),
\]

where \( \mu(x), \lambda(x) \) are positive scalar functions, called the Lamé parameters, and \( \text{tr}(\tau) \) is the trace of \( \tau \in H(\Omega, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \).

The Hellinger–Reissner functional \( \mathcal{J} : H(\Omega, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times L^2(\Omega; \mathbb{R}^3) \to \mathbb{R} \)

\[
\mathcal{J}(\tau, v) = \int_{\Omega} \left( \frac{1}{2} C^{-1} \tau : \tau + \text{div} \tau \cdot v - f \cdot v \right) \, dx,
\]

and it is known that the functional \( \mathcal{J} \) has a unique critical point

\[
(\sigma, u) \in H(\Omega, \text{div}; \mathbb{R}_{\text{sym}}^{3 \times 3}) \times L^2(\Omega; \mathbb{R}^3),
\]
which is the solution of equations
\[ \sigma = C \varepsilon(u), \quad -\text{div} \sigma = f \quad \text{in } \Omega, \]
with the homogeneous displacement boundary condition \( u = 0 \) on \( \partial \Omega \). Unless otherwise stated, we assume this boundary condition in the rest of the paper.

For the weakly symmetric stress approach, we need to extend the domain of \( C^{-1} \) to \( \mathbb{R}^{3 \times 3} \)-valued functions on \( \Omega \). The extended operator, denoted by \( \tilde{A} \), is defined by
\[ \tilde{A} \tau = C^{-1} \text{sym} \tau + \text{skw} \tau \quad \text{for } \tau : \Omega \rightarrow \mathbb{R}^{3 \times 3}. \]

We define an extended functional \( \tilde{J} \) as
\[ \tilde{J} : H(\text{div}, \mathbb{R}^{3 \times 3}) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3 \times 3}_{\text{skw}}) \rightarrow \mathbb{R}, \]
\[ (\tau, v, \eta) \mapsto \int_{\Omega} \left( \frac{1}{2} \tilde{A} \tau : \tau + \text{div} \tau \cdot v - f \cdot v + \text{skw} \tau : \eta \right) \, dx. \]

This functional has a unique critical point \((\sigma, u, \gamma)\) and the first two components of the critical point coincide with the critical point of \( J \) in (3).

For \( \psi, \tau \in H(\text{div}, \mathbb{R}^{3 \times 3}) \), \( v \in L^2(\Omega; \mathbb{R}^3) \) and \( \eta \in L^2(\Omega; \mathbb{R}^{3 \times 3}_{\text{skw}}) \) we define bilinear forms
\[ \mathcal{A}(\psi, \tau) := (A\psi, \tau) \quad \text{and} \quad \mathcal{B}(v, \eta; \tau) := (v, \text{div} \tau) + (\eta, \tau). \]

The critical point \((\sigma, u, \gamma)\) of \( \tilde{J} \) satisfies
\[ A(\sigma, \tau) + \mathcal{B}(u, \gamma; \tau) = 0, \quad \tau \in H(\text{div}, \mathbb{R}^{3 \times 3}), \]
\[ -\mathcal{B}(v, \eta; \sigma) = (f, v), \quad (v, \eta) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3 \times 3}_{\text{skw}}). \]

For numerical solutions we consider the associated discrete problem: Find \((\sigma_h, u_h, \gamma_h) \in \Sigma_h \times U_h \times \Gamma_h \subset H(\text{div}, \mathbb{R}^{3 \times 3}) \times L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^{3 \times 3}_{\text{skw}})\) such that
\[ \mathcal{A}(\sigma_h, \tau) + \mathcal{B}(u_h, \gamma_h; \tau) = 0, \quad \tau \in \Sigma_h, \]
\[ -\mathcal{B}(v, \eta; \sigma_h) = (f, v), \quad (v, \eta) \in U_h \times \Gamma_h. \]

\section{Definition of the New Finite Element}

Let \( T_h \) be a rectangular mesh of \( \Omega \) and \( K \) be a cube in \( T_h \). By \( P_k(G; \mathbb{X}) \) for \( G \subset \Omega \), we denote \( \mathbb{X} \)-valued polynomials of degree \( \leq k \) on \( G \). Let \( RM(K) \subset P_1(K; \mathbb{R}^3) \) denote the space of rigid body motions on \( K \in T_h \). For a face \( F \) of \( K \) we define
\[ \text{Tr}(RM; F) = \{ \xi |_F \mid \xi \in RM(K) \}, \]
\[ \tilde{\text{Tr}}(RM; F) = \{ \xi \in \text{Tr}(RM; F) \mid (\xi, q)_F = 0, \quad q \in P_0(F; \mathbb{R}^3) \}. \]

In addition, we define
\[ \tilde{\text{Tr}}_h(RM; \partial K) = \{ \xi : \partial K \rightarrow \mathbb{R}^3 \mid \xi|_F \in \tilde{\text{Tr}}(RM; F) \text{ for a face } F \text{ of } K \}. \]

For \( \xi \in \tilde{\text{Tr}}_h(RM; \partial K) \), \( \xi|_F \) and \( \xi|_{F'} \) can be traces of different rigid body motions on \( K \) when \( F \) and \( F' \) are different faces of \( K \).

\begin{lemma}
Let \( H \) be a plane in \( \mathbb{R}^3 \). The trace space of three dimensional rigid body motions on \( H \) is a vector space of dimension 6.
\end{lemma}
Proof. A rigid body motion in $\mathbb{R}^3$ is a function of $x \in \mathbb{R}^3$ of the form $Sx + d$ with $S \in \mathbb{R}^{3 \times 3}_{\text{skw}}$ and $d \in \mathbb{R}^3$. By direct computations, one can see that the set of rigid body motions $\{Sx + d | S \in \mathbb{R}^{3 \times 3}_{\text{skw}}, d \in \mathbb{R}^3\}$ is invariant under the change of coordinates with rotations and translations. Thus, without loss of generality, we may assume that $H = \{x \in \mathbb{R}^3 | x_1 = 0\}$.

A general form of rigid body motion is
\[
\begin{pmatrix}
0 & s_1 & s_2 \\
-s_1 & 0 & s_3 \\
-s_2 & -s_3 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
+ \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix}
\]
and the trace of this on $H$ is
\[
\begin{pmatrix}
s_1 x_2 + s_2 x_3 \\
s_3 x_3 \\
-s_3 x_2
\end{pmatrix}
+ \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix}.
\]
This has 6 free coefficients $s_i$, $d_i$, $i = 1, 2, 3$, so the trace space is of dimension 6.

As a corollary of Lemma 3.1, $\dim \widetilde{\text{Tr}}(RM; \partial K) = 3$ because $\dim \text{Tr}(RM; F) = 6$ and $\dim P_0(F; \mathbb{R}^3) = 3$. This implies that
\[
\dim \widetilde{\text{Tr}}_h(RM; \partial K) = 18.
\]

Lemma 3.2. Suppose that $K \in T_h$ and $\nu$ is the unit outward normal vector field on $\partial K$. Then there exists a vector space $B(K) \subset P_2(K; \mathbb{R}^{3 \times 3})$ such that
\[
\text{div } B(K) = 0,
\]
and the map $\tau \mapsto \tau \nu |_{\partial K}$ from $B(K)$ to $\widetilde{\text{Tr}}_h(RM; \partial K)$ is bijective. This bijection implies that $\dim B(K) = \dim \widetilde{\text{Tr}}_h(RM; \partial K) = 18$.

We postpone the proof of this lemma and the explicit construction of $B(K)$ to section 6.

Let $RT_0(K)$ be the space of lowest order rectangular Raviart–Thomas shape functions on $K$. We define $\Sigma_K$ as
\[
\Sigma_K = RT_0(K) + B(K) \quad \text{where} \quad RT_0(K) := \begin{pmatrix} RT_0(\{K\}) \\ RT_0(\{K\}) \end{pmatrix} \subset \mathcal{P}_1(K; \mathbb{R}^{3 \times 3}).
\]

The degrees of freedom of $\tau \in \Sigma_K$ are
\[
\tau \mapsto \int_F \tau \nu \cdot q \, ds, \quad q \in P_0(F; \mathbb{R}^3), \quad (6 \times 3 = 18 \text{ DOFs})
\]
\[
\tau \mapsto \int_F \tau \nu \cdot q \, ds, \quad q \in \widetilde{\text{Tr}}(RM; F), \quad (6 \times 3 = 18 \text{ DOFs})
\]
for a face $F$ of $K$ and the unit outward normal vector $\nu$ on $F$. To see that Eq. (15) gives 18 DOFs, we refer to Eq. (10).

Theorem 3.3. The space of shape functions $\Sigma_K$ is unisolvent for the DOFs (14) and (15).
Proof. It is well known [8] that $\mathcal{R}T_0(K)$ is unisolvent for DOFs (14) and $B(K)$ is unisolvent for DOFs (15) by Lemma 3.2. Since $\dim \mathcal{R}T_0(K) = 18$ and $\dim B(K) = 18$, $\dim \Sigma_K = 36$. Suppose that $\tau \in \mathcal{R}T_0(K) \cap B(K)$, then $\tau \nu|_K \in \mathcal{P}_0(\partial K; \mathbb{R}^3) \cap \widetilde{\mathcal{T}}_h(RM; \partial K) = \{0\}$. By the unisolvency of $\mathcal{R}T_0(K)$, $\tau = 0$ and thus $\dim \Sigma_K = 36$.

Now we need to show that $\tau \in \Sigma_K$ vanishes if DOFs (14) and (15) vanish. Write $\tau = \tau_0 + \tau_b$ where $\tau_0 \in \mathcal{R}T_0(K)$ and $\tau_b \in B(K)$. By definition $\widetilde{\mathcal{T}}(RM; F)$ is orthogonal to $\mathcal{P}_0(F; \mathbb{R}^3)$. Therefore, $\int_F \tau \nu \cdot q \, ds = \int_F \tau_0 \nu \cdot q \, ds$, $q \in \mathcal{P}_0(F; \mathbb{R}^3)$,

$$\int_F \tau \nu \cdot q \, ds = \int_F \tau_b \nu \cdot q \, ds, \quad q \in \widetilde{\mathcal{T}}(RM; F).$$

The unisolvency of $\mathcal{R}T_0(K)$ for DOFs (14) and of $B(K)$ for DOFs (15) completes the proof.

The finite element spaces of our proposed method are

\begin{align*}
(16) & \quad \Sigma_h = \{ \tau \in H(\text{div}, \Omega; \mathbb{R}^{3 \times 3}) \mid \tau|_K \in \Sigma_K, \ K \in T_h \}, \\
(17) & \quad U_h = \{ v \in L^2(\Omega; \mathbb{R}^3) \mid v|_K \in \mathcal{P}_0(K; \mathbb{R}^3), K \in T_h \}, \\
(18) & \quad \Gamma_h = \{ \eta \in L^2(\Omega; \mathbb{R}^{3 \times 3}_{\text{skw}}) \mid \eta|_K \in \mathcal{P}_0(K; \mathbb{R}^{3 \times 3}_{\text{skw}}), K \in T_h \}.
\end{align*}

It is well known [8] that $\text{div} \mathcal{R}T_0(K) = \mathcal{P}_0(K; \mathbb{R}^3)$. Combining this with Eq. (11), it is easy to see that

$$\text{div} \Sigma_h = U_h.$$ 

4. Stability

In this section we show the Babuška–Brezzi stability conditions [8] for $\Sigma_h$ and $U_h \times \Gamma_h$ using mesh dependent norms. The proof uses ideas of Ref. [17] and a discrete Korn’s inequality in Ref. [6].

Define the space of element-wise rigid body motions $RM_h = \{ v \in L^2(\Omega; \mathbb{R}^3) \mid v|_K \in RM(K), K \in T_h \}$. For $\eta \in \Gamma_h$, $\eta|_K$ is a constant skew-symmetric matrix on every $K \in T_h$, so we can find a unique $r_\eta \in RM_h$ such that

$$\text{grad}_h r_\eta = \eta,$$

$$\int_K r_\eta \, dx = 0, \quad K \in T_h,$$

where $\text{grad}_h$ denotes the element-wise gradient.

Let $F^o$ be the set of faces of $T_h$ in the interior of $\Omega$, $F^\partial$ be the set of faces on $\partial \Omega$, and define $F = F^o \cup F^\partial$. For an element-wise polynomial $g$ on $T_h$, we define

$$\|g\|_F = \begin{cases} 
\text{the jump of } g \text{ over } F, & \text{if } F \in F^o, \\
g, & \text{if } F \in F^\partial.
\end{cases}$$
The mesh dependent norms are
\begin{align}
|\tau|^2_{0,h} &= \sum_{F \in \mathcal{F}} h_F |\tau|_{TF}^2, \\
|(v, \eta)|^2_h &= \sum_{F \in \mathcal{F}} h_F^{-1} \|v + r_\eta\|_F^2,
\end{align}
for \( \tau \in \Sigma_h \) and \((v, \eta) \in U_h \times \Gamma_h\), where \( h_F \) denotes the size of the face \( F \) and \( \nu_F \) denotes a unit normal vector on \( F \).

From the definition of DOFs \( \{14\} \) and \( \{15\} \), it is clear that \(| \cdot |_{0,h}\) is a norm on \( \Sigma_h \). Furthermore, by a standard scaling argument \( \{7\} \), one can see that
\begin{equation}
|\tau|_{0,h} \leq |\tau| \leq c_1 |\tau|_{0,h}, \quad \tau \in \Sigma_h,
\end{equation}
for some \( c_0, c_1 > 0 \) independent of mesh sizes.

To see that \(|(\cdot, \cdot)|_h\) is a norm on \( U_h \times \Gamma_h \), we need a discrete Korn’s inequality given in Lemma 4.1 below, cf. estimate (1.22) in Ref. \[1\].

**Lemma 4.1.** Suppose that \( g \in L^2(\Omega; \mathbb{R}^3) \) is an element-wise polynomial function on \( \mathcal{T}_h \) and \( \Gamma_D \) is a subset of \( \partial \Omega \) with a positive measure. Let \( \epsilon_h(g) \) denote the element-wise symmetric gradient of \( g \). Then there exists \( c > 0 \), independent of \( g \) and mesh sizes, such that
\begin{equation}
|g|^2 + \|\text{grad}_h g\|^2 \leq c \left( |\epsilon_h(g)|^2 + |g|^2_{L^2_D} + \sum_{F \in \mathcal{F}} h_F^{-1} \|g\|_F^2 \right).
\end{equation}

**Lemma 4.2.** Suppose that \( h_F \leq 1 \) for all \( F \in \mathcal{F} \). For \((v, \eta) \in U_h \times \Gamma_h\) there exists \( c > 0 \) independent of mesh sizes such that
\begin{equation}
|v| + |\eta| \leq c |(v, \eta)|_h.
\end{equation}

**Proof.** Since \( v \in U_h \) is piecewise constant and \( r_\eta \) is orthogonal to \( v \) by property \( \{21\} \), we have \(|v|^2 \leq |v + r_\eta|^2\). By \( \{20\} \), \( \text{grad}_h(v + r_\eta) = \eta \). These together give
\begin{equation}
|v|^2 + |\eta|^2 \leq |v + r_\eta|^2 + \|\text{grad}_h(v + r_\eta)\|^2.
\end{equation}

Applying the discrete Korn’s inequality \( \{26\} \) to \( v + r_\eta \) with \( \Gamma_D = \partial \Omega \), we get
\begin{equation}
|v|^2 + |\eta|^2 \leq c \left( |\epsilon_h(v + r_\eta)|^2 + |v + r_\eta|^2_{L^2_D} + \sum_{F \in \mathcal{F}} h_F^{-1} \|v + r_\eta\|_F^2 \right).
\end{equation}

Clearly \( \epsilon_h(v + r_\eta) = 0 \). By the definition of the jump \( \{22\} \) and the assumption \( h_F \leq 1 \) we have
\begin{equation}
|v|^2 + |\eta|^2 \leq c \left( |v + r_\eta|^2_{L^2_D} + \sum_{F \in \mathcal{F}} h_F^{-1} \|v + r_\eta\|_F^2 \right) \leq c \sum_{F \in \mathcal{F}} h_F^{-1} \|v + r_\eta\|_F^2.
\end{equation}
Recalling the definition of \(|(\cdot, \cdot)|_h\) completes the proof. \( \square \)

**Corollary 4.3.** For \((v, \eta) \in U_h \times \Gamma_h\), \(|(v, \eta)|_h = 0\) implies \((v, \eta) = (0, 0)\) and hence \(|(\cdot, \cdot)|_h\) is a norm on \( U_h \times \Gamma_h \).

**Theorem 4.4** (Stability). \( \mathcal{A}(\cdot, \cdot) \) in \( \{5\} \) is continuous and coercive on \( \Sigma_h \) with the norm \(|\cdot|_{0,h}\).

\( \mathcal{B}(\cdot, \cdot) \) in \( \{5\} \) is continuous on \( U_h \times \Gamma_h \times \Sigma_h \) with the norms \(|\cdot|_h\) and \(|\cdot|_{0,h}\). In addition, for any \( 0 \neq (v, \eta) \in U_h \times \Gamma_h \), there exists \( \tau \in \Sigma_h \) satisfying
\begin{equation}
\mathcal{B}(v, \eta; \tau) \geq |(v, \eta)|_h^2 \quad \text{and} \quad |\tau|_{0,h} \leq |(v, \eta)|_h.
\end{equation}
Proof. Recall the equivalence of norms (25). Since $A$ is bounded, it holds that

$$|A(\psi, \tau)| = |(A\psi, \tau)| \leq c|\psi||\tau| \leq c|\psi|_{0,h}||\tau||_{0,h},$$

and since $A$ is positive definite, it holds

$$|A(\tau, \tau)| = |(A\tau, \tau)| \geq c||\tau||^2 \geq c||\tau||_{0,h}^2,$$

for $\psi, \tau \in \Sigma_h$. Thus $A(\cdot, \cdot)$ is continuous and coercive.

Let $\tau \in \Sigma_h$, $v \in U_h$ and $\eta \in \Gamma_h$. The property (20) and element-wise integration by parts give

$$B(v, \eta; \tau) = (\text{div} \, \tau, v) + (\tau, \eta) = (\text{div} \, \tau, v) + (\tau, \text{grad}_h \eta)$$

$$= \sum_{K \in T_h} ((\tau \eta), \text{grad}_K v) - (\tau, \text{div}_h v)K + ((\tau \nu), \tau)K + (\text{div} \, \tau, \tau)K.$$ 

Since $v$ is piecewise constant, $\text{grad}_h v = 0$. By property (21) the last term also vanishes because $\text{div} \, \tau$ is piecewise constant. Thus we get

$$B(v, \eta; \tau) = \sum_{F \in \mathcal{F}} (\tau \nu, |v + \eta|)_F.$$ 

The continuity

$$|B(v, \eta; \tau)| \leq c||v||_{0,h}||\eta||_h$$

follows combining (29) with the definition of the norms (23) and (24).

To show (28), suppose that $0 \neq (v, \eta) \in U_h \times \Gamma_h$ is given. Note that, on $F$, $|v + \eta| \in \text{Tr}(RM; F)$ because $v + \eta$ is a piecewise rigid body motion on each cube $K$. By the DOFs (14) and (15), one can find $\tau \in \Sigma_h$ such that

$$\tau \nu|_F = h_F^{-1}|v + \eta|_F, \quad \forall F \in \mathcal{F}.$$ 

Using the definition of norms (23) and (24), and the equality (29), we get

$$|\tau||^2_{0,h} = \sum_{F \in \mathcal{F}} h_F |\tau \nu|^2_F = \sum_{F \in \mathcal{F}} h_F^{-1}||v + \eta||^2_F = ||(v, \eta)||^2_h,$$

$$B(v, \eta; \tau) = \sum_{F \in \mathcal{F}} (\tau \nu, |v + \eta|)_F = ||(v, \eta)||^2_h,$$

which completes the proof.

The Babuška–Brezzi theory and Theorem 4.4 ensure that the problem (8)–(9) is well-posed and that there exists a unique solution $(\sigma_h, u_h, \gamma_h) \in (\Sigma_h, U_h, \Gamma_h)$.

A similar argument works for mixed boundary conditions too as long as the domain $\Gamma_D \subset \partial \Omega$ for Dirichlet boundary conditions has a positive measure.

**Pure traction boundary conditions.** The pure traction boundary conditions are covered with small modifications. In this case the global rigid body motions $(v, \eta) \in \mathcal{P}_0(\Omega; \mathbb{R}^3) \times \mathcal{P}_0(\Omega; \mathbb{R}^{3 \times 3})$ are in the kernel of the problem. Therefore we modify the corresponding discrete spaces to

$$\tilde{U}_h = \left\{ v \in U_h \mid \int_{\Omega} v \, dx = 0 \right\},$$

and

$$\tilde{\Gamma}_h = \left\{ \eta \in \Gamma_h \mid \int_{\Omega} \eta \, dx = 0 \right\}.$$ 

Suppose for convenience that the problem has homogeneous pure traction boundary conditions. Then the stress space is

$$\tilde{\Sigma}_h = \{ \tau \in \Sigma_h \mid \tau \nu|_{\partial \Omega} = 0 \}.$$
The definition of mesh dependent norm $|\cdot|_{h}$ in \[24\] is modified as well. The sum over all faces $F$ is replaced by the sum over interior faces $F^\circ$, that is,

$$
\| (v, \eta) \|_h^2 = \sum_{F \in F^\circ} h_F^{-1} \| v + r_\eta \|_F^2.
$$

To see that $\| (\cdot, \cdot) \|_h$ is a norm on $\hat{U}_h \times \hat{V}_h$, we use, instead of Lemma 4.1, the estimate (1.23) in Ref. [6]. It gives

$$
|v|^2 + |\eta|^2 \leq c \sum_{F \in F^\circ} h_F^{-1} \| v + r_\eta \|_F^2 = c \| (v, \eta) \|_h^2, \quad \forall (v, \eta) \in \hat{U}_h \times \hat{V}_h,
$$

and the proof of Lemma 4.2 shows that $\| (\cdot, \cdot) \|_h$ is a norm on $\hat{U}_h \times \hat{V}_h$.

With these modifications, showing stability as in Theorem 4.4 is straightforward, so we omit the details.

5. Error Analysis

Let $\Pi_h : H^1(\Omega; \mathbb{R}^{3\times 3}) \to \Sigma_h$ be the Raviart–Thomas interpolation operator defined by \[13\] and let $P_h : L^2(\Omega; \mathbb{R}^3) \to U_h$ and $Q_h : L^2(\Omega; \mathbb{R}^{3\times 3}) \to \Gamma_h$ denote the orthogonal $L^2$ projections. It holds that

$$
(31) \quad |\tau - \Pi_h \tau| + |v - P_h v| + |\eta - Q_h \eta| \leq c( |\tau|_1 + |v|_1 + |\eta|_1)
$$

for $(\tau, v, \eta) \in H^1(\Omega; \mathbb{R}^{3\times 3}) \times H^1(\Omega; \mathbb{R}^3) \times H^1(\Omega; \mathbb{R}^{3\times 3})$.

**Lemma 5.1.** Suppose $(\sigma, u, \gamma)$ is the exact solution of \[6\]-\[7\] and $(\sigma_h, u_h, \gamma_h)$ is the corresponding discrete solution of \[8\]-\[9\]. Then there exists $c > 0$ independent of mesh sizes such that

$$
(32) \quad |\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h \leq c( |\sigma|_1 + |\gamma|_1).
$$

**Proof.** By Theorem 4.4 and the Babuška–Brezzi theory there exists $(\tau, v, \eta) \in \Sigma_h \times U_h \times \Gamma_h$ such that

$$
(33) \quad |\tau|_{0,h} + |(v, \eta)|_h \leq c
$$

and

$$
(34) \quad |\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h \leq A(\Pi_h \sigma - \sigma_h, \tau) + B(P_h u - u_h, Q_h \gamma - \gamma_h; \tau) + B(v, \eta; \Pi_h \sigma - \sigma_h).
$$

Subtracting Eqs. \[6\]-\[7\] from Eqs. \[8\]-\[9\] gives

$$
A(\Pi_h \sigma - \sigma_h, \tau) + B(P_h u - u_h, Q_h \gamma - \gamma_h; \tau) + B(v, \eta; \Pi_h \sigma - \sigma_h) = A(\Pi_h \sigma - \sigma, \tau) + B(P_h u - u, Q_h \gamma - \gamma; \tau) + B(v, \eta; \Pi_h \sigma - \sigma).
$$

By Eq. (34), the above equality, and the definitions in \[5\], we obtain

$$
|\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h \leq A(\Pi_h \sigma - \sigma, \tau) + B(P_h u - u, Q_h \gamma - \gamma; \tau) + B(v, \eta; \Pi_h \sigma - \sigma) = (A(\Pi_h \sigma - \sigma), \tau) + (P_h u - u, \div \tau) + (Q_h \gamma - \gamma, \tau) + (v, \eta; \Pi_h \sigma - \sigma).
$$

By the property $\div \Pi_h = P_h \div$ and the fact $v \in U_h$, we have

$$
(v, \div(\Pi_h \sigma - \sigma)) = (v, P_h \div \sigma - \div \sigma) = 0.
$$
Moreover, \((P_h u - u, \text{div} \tau) = 0\) because \(\text{div} \tau \in U_h\) by \((19)\). Thus we have
\[
|\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h \\
\leq (A|\Pi_h \sigma - \sigma|, |\tau|) + (Q_h \gamma - \gamma, |\tau|) + (\eta, |\Pi_h \sigma - \sigma|)
\]
\[
\leq c|\Pi_h \sigma - \sigma| |\tau| + |Q_h \gamma - \gamma| |\tau| + |\eta| |\Pi_h \sigma - \sigma|.
\]

The norm equivalence \((25)\), Lemma \(4.2\) and Eq. \((33)\) show that
\[
|\tau| + |v| + |\eta| \leq c (|\tau|_{0,h} + |(v, \eta)|_h) \leq c
\]
and therefore
\[
|\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h \leq c (|\Pi_h \sigma - \sigma| + |Q_h \gamma - \gamma|).
\]
The interpolation estimate \((31)\) completes the proof. \(\square\)

**Theorem 5.2.** Suppose \((\sigma, u, \gamma)\) is the exact solution of \((3)\)–\((7)\) and \((\sigma_h, u_h, \gamma_h)\) is the corresponding discrete solution of \((8)\)–\((9)\). Then the following error estimate holds:
\[
|\sigma - \sigma_h| + |u - u_h| + |\gamma - \gamma_h| \leq ch(|\sigma|_1 + |\gamma|_1).
\]

**Proof.** By the triangle inequality, norm equivalence \((25)\) and Lemma \(4.2\) we have
\[
|\sigma - \sigma_h| + |u - u_h| + |\gamma - \gamma_h|
\]
\[
\leq |\sigma - \Pi_h \sigma| + |u - P_h u| + |\gamma - Q_h \gamma| + |\Pi_h \sigma - \sigma_h| + |P_h u - u_h| + |Q_h \gamma - \gamma_h|
\]
\[
\leq |\sigma - \Pi_h \sigma| + |u - P_h u| + |\gamma - Q_h \gamma|
\]
\[
+ c(|\Pi_h \sigma - \sigma_h|_{0,h} + |(P_h u - u_h, Q_h \gamma - \gamma_h)|_h).
\]
Using the interpolation estimate \((31)\) and Lemma \(5.1\) we get
\[
|\sigma - \sigma_h| + |u - u_h| + |\gamma - \gamma_h| \leq ch(|\sigma|_1 + |u|_1 + |\gamma|_1)
\]
and the inequality \(|u|_1 \leq c (|\sigma| + |\gamma|)\) completes the proof. \(\square\)

6. **Construction of \(B(K)\)**

For simplicity we use a convention that \(i + 3 = i\) for subscripts \(i = 1, 2, 3\). Since we only consider rectangular meshes, without loss of generality, we may take the cube \(K\) as
\[
K = [-h_1, h_1] \times [-h_2, h_2] \times [-h_3, h_3],
\]
with \(h_1, h_2, h_3 > 0\). We denote faces of the cube \(K\) by
\[
F_i^+ = \partial K \cap \{x_i = h_1\} \quad \text{and} \quad F_i^- = \partial K \cap \{x_i = -h_1\}, \quad i = 1, 2, 3.
\]
Denoting by \(S_3\) the set of permutations of \(\{1, 2, 3\}\), we define
\[
p_{ij}^+ = \text{curl}(w_{ij}^+ e_k^1), \quad p_{ij}^- = \text{curl}(w_{ij}^- e_k^1), \quad \text{for } (ijk) \in S_3,
\]
in which the curl is a row-wise operator, \(e_k\) is the unit column vector pointing in the positive direction of \(x_k\)-axis and
\[
w_{ij}^+ := \begin{cases} -(h_i + x_i)(h_j^2 - x_j^2), & \text{if } j = i + 1, \\
(h_i + x_i)(h_j^2 - x_j^2), & \text{if } j = i + 2,
\end{cases}
\]
\[
w_{ij}^- := \begin{cases} -(h_i - x_i)(h_j^2 - x_j^2), & \text{if } j = i + 1, \\
(h_i - x_i)(h_j^2 - x_j^2), & \text{if } j = i + 2.
\end{cases}
\]
By direct calculations, we have

$$p_{12}^+ = (2x_2(h_1 + x_1) - h_2^2 - x_2^2, 0), \quad p_{12}^- = (2x_2(h_1 - x_1) - h_2^2 + x_2^2, 0),$$
$$p_{13}^+ = (2x_3(h_1 + x_1) - h_3^2 - x_3^2, 0), \quad p_{13}^- = (2x_3(h_1 - x_1) - h_3^2 + x_3^2, 0),$$
$$p_{23}^+ = (0, 2x_3(h_2 + x_2) - h_3^2 - x_3^2, 0), \quad p_{23}^- = (0, 2x_3(h_2 - x_2) - h_3^2 + x_3^2, 0),$$
$$p_{21}^+ = (h_1^2 - x_1^2, 2x_1(h_2 + x_2) - h_1^2 + x_1^2, 2x_1(h_2 - x_2) - h_1^2 - x_1^2, 0),$$
$$p_{31}^+ = (h_1^2 - x_1^2, 0, 2x_1(h_3 + x_3), p_{31}^- = (-h_1^2 + x_1^2, 0, 2x_1(h_3 - x_3)),$$

Note that these are row vectors and $e_i$ is a column vector, so $p_{ij}^+ e_i, p_{ij}^- e_i$ are scalar valued. Again by direct computations, for $i, j, l = 1, 2, 3$ with $i \neq j$, one can see that

$$p_{ij}^+ e_i |_{F_t^+} = p_{ij}^- e_i |_{F_t^-} = 0,$$

and that

$$p_{ij}^+ e_i |_{F_t^+} = p_{ij}^- e_i |_{F_t^-} = \begin{cases} 4h_i x_j, & \text{if } i = l, \\ 0, & \text{if } i \neq l. \end{cases}$$

We define

$$\phi_{ij}^+ = \begin{pmatrix} e_{i+1} p_{i,i+2}^+ - e_{i+2} p_{i,i+1}^+ \\ e_i, p_{ij}^+ \end{pmatrix}, \quad \phi_{ij}^- = \begin{pmatrix} p_{ij}^- \end{pmatrix},$$

and similarly $\phi_{ij}^-$. Note that $\phi_{ij}^+$ and $\phi_{ij}^-$ are $\mathbb{R}^{3 \times 3}$-valued because $e_i$ is a column vector and both $p_{ij}^+$ and $p_{ij}^-$ are row vectors. For example,

$$\phi_{12}^+ = \begin{pmatrix} 0 \\ p_{12}^- \end{pmatrix}, \quad \phi_{12}^+ = \begin{pmatrix} p_{12}^+ \\ 0 \end{pmatrix}, \quad \phi_{13}^+ = \begin{pmatrix} p_{13}^+ \\ 0 \end{pmatrix}.$$

By the properties (42) and (43) we have

$$\phi_{ij}^+ e_i |_{F_t^+} = \phi_{ij}^- e_i |_{F_t^-} = 0,$$

$$\phi_{ij}^+ e_i |_{F_t^+} = \phi_{ij}^- e_i |_{F_t^-} = \begin{cases} 4h_i(x_{i+2} e_{i+1} - x_{i+1} e_{i+2}), & \text{if } i = j = l, \\ 4h_i x_j e_i, & \text{if } i = l, i \neq j, \\ 0, & \text{if } i \neq l, \end{cases}$$

and by property (40)

$$\text{div } \phi_{ij}^+ = \text{div } \phi_{ij}^- = 0,$$

for $i, j, l = 1, 2, 3$.

We define spaces $\Phi_i^+, \Phi_i^-$ by

$$\Phi_i^+ = \text{span}\{\phi_{ij}^+ : j = 1, 2, 3\}, \quad \Phi_i^- = \text{span}\{\phi_{ij}^- : j = 1, 2, 3\},$$

and $B(K) \subset P_2(K; \mathbb{R}^{3 \times 3})$ as

$$B(K) := \sum_{i=1}^3 (\Phi_i^+ + \Phi_i^-) = \text{span}\{\phi_{ij}^+, \phi_{ij}^- : i, j = 1, 2, 3\}.$$
To show that Lemma 3.2 holds for $B(K)$ we need the following results.

**Lemma 6.1.** For the faces $F^+_i$, $F^-_i$, $i = 1, 2, 3$ of the cube $K$, defined in (36), it holds that

$$
\tilde{\text{Tr}}(RM; F^+_i) = \tilde{\text{Tr}}(RM; F^-_i) = \text{span}\{x_{i+1}e_i, x_{i+2}e_i, x_{i+2}e_{i+1} - x_{i+1}e_{i+2}\}.
$$

**Proof.** For simplicity, we check the claim in detail only for $(51)$ with constants $(52)$ and $(53)$.

A general form of rigid body motions in \(\mathbb{R}^3\) is

$$
\begin{pmatrix}
0 & s_1 & s_2 \\
-s_1 & 0 & s_3 \\
-s_2 & -s_3 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
+ \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix},
$$

with constants \(s_i, d_i \in \mathbb{R}, i = 1, 2, 3\), and the trace of this on \(F^+_1\) is

$$
\begin{pmatrix}
s_1x_2 + s_2x_3 + d_1 \\
s_3x_3 - h_1s_1 + d_2 \\
-s_3x_2 - h_1s_2 + d_3
\end{pmatrix}.
$$

Since \(\tilde{\text{Tr}}(RM; F^+_1)\) is orthogonal to \(P_0(F^+_1; \mathbb{R}^3)\), an element in \(\tilde{\text{Tr}}(RM; F^+_1)\) has a form

$$
\begin{pmatrix}
s_1x_2 + s_2x_3 \\
s_3x_3 \\
-s_3x_2
\end{pmatrix} = s_1x_2e_1 + s_2x_3e_1 + s_3(x_3e_2 - x_2e_3).
$$

Checking the claim for the rest of the faces is straightforward. \(\Box\)

**Corollary 6.2.** For \(\Phi^+_i\), \(i = 1, 2, 3\) defined in (49), we have

$$
\tau \nu|_F = 0, \text{ for all } \tau \in \Phi^+_i \text{ if } F \neq F^+_i,
$$

$$
\{\tau \nu|_{F^+_i} \mid \tau \in \Phi^+_i\} = \tilde{\text{Tr}}(RM; F^+_i).
$$

A similar result holds for \(\Phi^-_i\) with \(F^-_i\).

**Proof.** Eq. (54) is a direct consequence of (46) and (47). Eq. (55) follows from (47)–(51). \(\Box\)

**Proof of Lemma 3.2.** It is clear from the definition (50) that \(B(K) \subset P_2(K; \mathbb{R}^3)\) and (11) follows from (48).

By Corollary 6.2 it is clear that Eq. (12) holds. By the same corollary the map \(\tau \mapsto \tau \nu|_{K}\) is a bijection between \(\Phi^+_i\) and \(\tilde{\text{Tr}}(RM; F^+_i)\), and hence the inverse map from \(\tilde{\text{Tr}}(RM; F^+_i)\) to \(\Phi^+_i\) is well-defined. Same holds between \(\Phi^-_i\) and \(\tilde{\text{Tr}}(RM; F^-_i)\), respectively. Moreover, by (54) we can extend these inverse maps from \(\tilde{\text{Tr}}_h(RM; \partial K)\) to \(B(K)\), which shows that the map \(\tau \mapsto \tau \nu|_{K}\) from \(B(K)\) to \(\tilde{\text{Tr}}_h(RM; \partial K)\) is bijective. \(\Box\)

**Practical implementation.** The basis functions corresponding to the lowest order Raviart–Thomas space are well known [8].

Using Corollary 6.2 and definition (50) we see that the \(\phi^+_i\) and \(\phi^-_i\) in Eq. (44) are the actual basis functions for the additional degrees of freedom in the cube \(K\). Furthermore, using the explicit forms of \(\mathcal{p}^+_i\) and \(\mathcal{p}^-_i\) in Eq. (41), it is easy to write
explicit forms of $\phi_{ij}^+$ and $\phi_{ij}^-$. For example, $\phi_{11}^+$, $\phi_{12}^+$, and $\phi_{13}^+$ are the three basis functions corresponding to face $F_1^+$ and using (45) we immediately see that

$$
\phi_{11}^+ = \begin{pmatrix}
2x_3(h_1 + x_1) & 0 & 0 \\
-2x_2(h_1 + x_1) & -h_2^2 + x_2^2 & h_3^2 - x_3^2
\end{pmatrix},
$$

$$
\phi_{12}^+ = \begin{pmatrix}
-2x_2(h_1 + x_1) & h_2^2 - x_2^2 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

$$
\phi_{13}^+ = \begin{pmatrix}
2x_3(h_1 + x_1) & 0 & h_3^2 - x_3^2 \\
0 & 0 & 0
\end{pmatrix}.
$$

7. Conclusion

We showed a mesh dependent norm analysis of mixed finite elements for elasticity with weakly symmetric stress and applied it to develop a new rectangular element in three dimensions. The stability analysis explicitly shows necessary conditions for stable stress spaces and our element is constructed to fulfill these requirements. It shows optimal $O(h)$ convergence of all the unknowns and has the smallest number of DOFs up to date among the stable three dimensional rectangular elements for elasticity with weakly symmetric stress.

Our mesh dependent norm analysis can be applied to general meshes with planar faces including tetrahedral meshes. However, the construction of the stress space fulfilling all the necessary conditions for general hexahedral meshes can be difficult.

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REFERENCES


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