Robust finite element methods for Biot’s consolidation model

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Summary. We propose new locking-free finite element methods for Biot’s consolidation model by coupling nonconforming and mixed finite elements. We show a priori error estimates of semidiscrete and fully discrete solutions. The main advantage of our method is that a uniform-in-time pressure error estimate is provided with an analytic proof. In our error analysis, we do not use Gronwall’s inequality, so the exponential growing factors in time do not appear in the error bounds.

Key words: Biot’s consolidation model, nonconforming finite elements, poroelasticity, locking-free

Introduction

The Biot’s consolidation model describes deformation of saturated elastic porous media and viscous fluid flow in the porous media, simultaneously. In studies of numerical solutions for Biot’s model with continuous Galerkin finite elements, nonphysical pressure oscillations of numerical solutions are observed [1, 2, 3]. This nonphysical pressure oscillation phenomenon is called poroelasticity locking and various numerical methods have been suggested to resolve it [4, 7, 5]. Most of these numerical methods use nonconforming elements or discontinuous Galerkin methods with stabilization techniques. Although some numerical results of the methods illustrate that they are locking-free, the complete a priori error analysis is achieved with the assumption that the constrained specific storage coefficient, denoted by $c_0$, is uniformly positive. However, in an heuristic analysis [12] the locking phenomena occur when $c_0$ is vanishing or very close to 0, so the analysis is not enough to guarantee that the methods are locking-free.

In our work we propose a new locking-free finite element scheme for Biot’s consolidation model. In our numerical scheme we obtain an a priori error estimate of the pressure in $L^\infty([0,T];L^2)$ norm without assuming uniformly positive $c_0$, so the error analysis confirms that our method is locking-free.

Biot’s consolidation model

Consider a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with piecewise smooth boundary. We use $u$, $p$, $f$, $g$ to denote the displacement of porous media, fluid pressure, body force, source/sink density function of fluid, respectively. The governing equations of quasistatic Biot’s consolidation model are

$$-\text{div} \mathcal{C} \varepsilon(u) + \alpha \nabla p = f,$$  \hspace{1cm} (1)

$$c_0 \dot{p} + \alpha \text{div} \dot{u} - \text{div}(\kappa \nabla p) = g.$$  \hspace{1cm} (2)
in which $C, c_0 \geq 0, \kappa, \alpha > 0$ denote the elastic stiffness tensor, the constrained specific storage
coefficient, the hydraulic conductivity tensor which is positive definite, and the Biot–Willis
constant, respectively. Here $\dot{p}, \dot{u}$ stand for the time derivatives of $p, u$.

To define boundary conditions we consider two partitions of $\partial \Omega$,

$$\partial \Omega = \Gamma_p \cup \Gamma_f, \quad \partial \Omega = \Gamma_d \cup \Gamma_t,$$

and assume that $\Gamma_d$ is of positive measure. We assume that boundary conditions are

$$p = 0 \text{ on } \Gamma_p, \quad -\kappa \nabla p \cdot n = 0 \text{ on } \Gamma_f, \quad u = 0 \text{ on } \Gamma_d, \quad \sigma n = 0 \text{ on } \Gamma_t,$$

(3)

for all time, where $n$ is the outward unit normal vector field on $\partial \Omega$ and $\sigma := C \epsilon(u) + \alpha p I$ is the
Cauchy stress tensor. We also assume that the initial data of $u, p$ and $f$ satisfy the compatibility
condition (1).

**Variational formulation**

For simplicity we set $\alpha = 1$ and $\kappa$ is the identity tensor. By introducing a new unknown $z := \nabla p
in (1)$ and (2), we have a new system

$$-\text{div} C \epsilon(u) + \nabla p = f,$$

(4)

$$z - \nabla p = 0,$$

(5)

$$c_0 \dot{p} + \text{div} \dot{u} - \text{div} z = g,$$

(6)

Defining

$$\Sigma = \{ u \in H^1(\Omega; \mathbb{R}^n) | u|_{\Gamma_d} = 0 \},$$

$$V = \{ z \in H(\text{div}, \Omega) | z \cdot n|_{\Gamma_t} = 0 \},$$

$$W = L^2(\Omega),$$

the variational formulation of (4)-(6) is to seek $(u, z, p) \in \Sigma \times V \times W$ such that

$$a(u, v) - (p, \text{div} v) = (f, v), \quad v \in \Sigma,$$

(7)

$$(z, w) + (p, \text{div} w) = 0, \quad w \in V,$$

(8)

$$(c_0 \dot{p}, q) + (\text{div} \dot{u}, q) - (\text{div} z, q) = (g, q), \quad q \in W,$$

(9)

in which $a(u, v) = (C\epsilon(u), \epsilon(v))$.

**Finite element spaces**

We assume that our triangular mesh is shape regular. The finite element space $V_h \subset V$ the
lowest order Raviart–Thomas element and $W_h \subset W$ is the piecewise constant element. For $\Sigma_h
we use vector-valued nonconforming $H^1$ finite elements, say Mardal–Tai–Winther type elements,
which are originally developed for Stokes–Darcy flow problems [8, 9]. The discrete $H^1$ norm for
$\Sigma_h$ is defined by

$$\|v\|_{1,h}^2 = \sum_{T \in T_h} \|
\nabla u\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h} \frac{1}{h_E} \|[u]\|_{0,E}^2,$$

(10)

in which $T_h, \mathcal{E}_h$ are the set of triangles/tetrahedra and the set of edges/faces, respectively. Here,
if $E \subset \partial \Omega$, then the jump $[u]$ on $E$ means the trace of $u$.

The Mardal–Tai–Winther type elements satisfy a discrete Korn’s inequality, and the pair
$(\Sigma_h, W_h)$ satisfies the inf-sup condition with the above discrete $H^1$ norm and the $L^2$ norm. It
turns out that these features are very useful to develop robust numerical schemes for Biot’s consolidation model.

Rectangular Mardal–Tai–Winther type elements [5] and higher order triangular Mardal–Tai–Winther type elements [10] are developed. Therefore our approach can be extended to rectangular meshes and triangular meshes with higher order elements.

**Differential algebraic equations and compatibility conditions**

The semidiscrete problem corresponding to the system (7)-(9), is not a system of ordinary differential equations, so existence and uniqueness of its solutions should be treated carefully. In particular, the algebraic equations (7) and (8) give a compatibility condition on initial data. We say that numerical initial data is compatible if it satisfies the compatibility conditions given by the corresponding discrete equations of (7) and (8). We prove that the semidiscrete problem of (7)-(9) has a unique solution if its initial data is compatible.

The compatibility of numerical initial data is also important for robustness of time discretization schemes. If the initial data of differential algebraic equation is not compatible, then the Crank–Nicolson scheme, which is absolutely stable, may give a spurious numerical solution because numerical solutions do not satisfy the algebraic equations in all time steps. To avoid this problem, we show that we can always find a compatible numerical initial data which is close to the original initial data.

**Semidiscrete and fully discrete error estimates**

We use \( \|u\|_{L^\infty([0,T];H^1_h)} \) to denote the standard space-time norm with the discrete \( H^1 \) norm in (10). Let \((u, z, p)\) and \((u_h, z_h, p_h)\) be the exact and semidiscrete solutions. If the exact solution is sufficient regular, then we have

\[
\|u - u_h\|_{L^\infty([0,T];H^1_h)} + \|z - z_h\|_{L^2([0,T];L^2)} + \|p - p_h\|_{L^\infty([0,T];L^2)} = O(h).
\]

In our error analysis, to avoid Grönwall’s inequality, we adapt the energy estimates of linear evolutionary partial differential equations to our differential algebraic equation.

For the fully discrete solutions we use the Crank–Nicolson scheme with time step \( \Delta t \). Let \( N \) be the last time step. Defining \( t_j = j \Delta t \) for integer \( 0 \leq j \leq N \) and denoting the fully discrete solution at \( j \)-th time step by \((U^j, Z^j, P^j)\), we have

\[
\sup_{0 \leq j \leq N} \|u(t_j) - U^j\|_{1,h} + \sum_{0 \leq j \leq N} \|z(t_j) - Z^j\|_0^2 + \sup_{0 \leq j \leq N} \|p(t_j) - P^j\|_0 = O(h + (\Delta t)^2).
\]

In the error analysis of fully discrete solutions there is an additional technical difficulty arising from time discretization. It is mainly due to the fact that the \( c_0 \)-weighted \( L^2 \) norm of \( p \) is not an upper bound of \( \|p\|_0 \) because \( c_0 \) is not uniformly positive. We are able to handle this problem in the help of the inf-sup condition of \((\Sigma_h, W_h)\) and the error estimate of \( u \).

**References**


