Global $\mathbb{R}$-linear GMRES for solving a class of $\mathbb{R}$-linear matrix equations

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Abstract

We present a new minimal residual method, called global $\mathbb{R}$-linear GMRES, to solve the $\mathbb{R}$-linear matrix equations $X + A\overline{X}B = C$ and $X + AX^*B = C$, where $C, X \in \mathbb{C}^{m \times n}$, $\overline{X}$ denotes the complex conjugate of $X$, $X^*$ its complex conjugate transpose, and $A, B$ are complex matrices with appropriate dimensions. We show that the new method requires fewer matrix-matrix products than the global GMRES method applied to the related generalized Sylvester matrix equations $W - AAWB = C$ with $X = W - AWB$ and $W - AB^*WA^*B = C$ with $X = W - AW^*B$. A numerical example is given to illustrate our theoretical results.

Keywords: $\mathbb{R}$-linear matrix equation, global GMRES, global $\mathbb{R}$-linear GMRES, matrix-matrix product

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1. Introduction

We consider the $\mathbb{R}$-linear matrix equations

\begin{align*}
X + A\overline{X}B &= C \quad (1.1) \\
X + AX^*B &= C, \quad (1.2)
\end{align*}

where $C, X \in \mathbb{C}^{m \times n}$, $\overline{X}$ denotes the complex conjugate of $X$, $X^*$ its complex conjugate transpose, and $A, B$ are complex matrices with appropriate dimensions. Such matrix equations, or more general, the matrix equations $EXF + A\overline{X}B = C$ and $EXF + AX^*B = C$ with appropriately chosen matrices $A, B, E$ and $F$, have been extensively investigated; see, for example, [1–12]. In this paper, we present a new minimal residual method, called global $\mathbb{R}$-linear GMRES (GI-RL-GMRES), to solve (1.1) and (1.2) by the global Arnoldi algorithm (see Algorithm 2), which extends the $\mathbb{R}$-linear GMRES (RL-GMRES) method [13] for $\mathbb{R}$-linear systems to $\mathbb{R}$-linear matrix equations.

Replacing $X$ by $W - AWB$, we obtain the generalized Sylvester matrix equation related to (1.1)

\begin{equation}
W - A\overline{W}B = C. \quad (1.3)
\end{equation}

Replacing $X$ by $W - AW^*B$, we obtain the generalized Sylvester matrix equation related to (1.2)

\begin{equation}
W - AB^*WA^*B = C. \quad (1.4)
\end{equation}
The global GMRES (Gl-GMRES) method [14] can be used to solve (1.3) and (1.4). In this paper, we show that Gl-RL-GMRES applied to (1.1) and (1.2) is faster than Gl-GMRES applied to (1.3) and (1.4) in terms of matrix-matrix products; see Remark 4.3.

The paper is organized as follows. In section 2, we review the R-linear GMRES method. In section 3 we present the global R-linear GMRES method. In section 4 we investigate the global R-linear GMRES method through the equivalent R-linear systems. In section 5 we report numerical experiments illustrating our theoretical results. We present some concluding remarks in the last section.

2. R-linear GMRES

The R-linear GMRES method [13] was proposed for solving the R-linear system

\[ \kappa z + M \bar{z} = b, \]  

(2.1)

where \( \kappa \in \mathbb{C}, z, b \in \mathbb{C}^n, M \in \mathbb{C}^{n \times n} \). Such R-linear systems arise in the inverse problem of reconstructing an unknown electric conductivity in the unit disc from boundary measurements [15–17], especially in the numerical discretization of the \( \mathbb{R} \)-linear Beltrami equation [18] and the \( \bar{\partial} \)-equation [19]. Let \( \tau \) denote the complex conjugation operator \( \tau x = \bar{x} \) on \( \mathbb{C}^n \) and \( I \) the identity matrix whose dimension is clear from the context. For notational simplicity, denote \( \kappa I + M \tau \) by \( M \kappa \).

Let \( z_0 \) be the initial guess and \( r_0 = b - M \kappa z_0 \) the corresponding residual. Let \( \| \cdot \| \) denote the 2-norm. The \( i \)th iterate, \( z_i \), determined by RL-GMRES satisfies

\[ \| b - M \kappa z_i \| = \min_{w \in z_0 + K_i(M_k, r_0)} \| b - M_k w \|, \quad z_i \in z_0 + K_i(M_k, r_0). \]

Here \( K_i(M_k, r_0) \) denotes the \( i \)th Krylov subspace generated by \( M_k \) and \( r_0 \in \mathbb{C}^n \),

\[ K_i(M_k, r_0) := \text{span}\{r_0, M_k r_0, \ldots, M_k^{i-1} r_0\} \subset \mathbb{C}^n. \]

Let \( \tilde{I} \) denote the \( i \times i \) identity matrix augmented with the row of zeros as the last row and \( e_1 \) the first column of the identity matrix with appropriate dimension. Let \( H_{i+1, j} \) be the upper Hessenberg matrix generated in the Arnoldi process (see step 2 of Algorithm 1 below). The \( i \)th iterate \( z_i \) satisfies

\[ \| b - M_k z_i \| = \min_{s \in \mathbb{C}^n} \| r_0 \|_2 e_1 - \kappa \tilde{I} s - H_{i+1, i} \bar{s} \|_2. \]

The above minimal problem

\[ \min_{s \in \mathbb{C}^n} \| r_0 \|_2 e_1 - \kappa \tilde{I} s - H_{i+1, i} \bar{s} \|_2 \]

can be solved by employing the \( \mathbb{R} \)-linear QR decomposition [13]. The work and storage of RL-GMRES (as a function of the number of iterations) are comparable to those of GMRES [23]. We give the details of RL-GMRES in Algorithm 1.
Algorithm 1: \( \mathbb{R} \)-linear GMRES

1. Compute \( r_0 = b - M_0z_0, z_0 \) is the initial guess
2. Generate the Arnoldi basis and the matrix \( H_{i+1,i} \):
   \[
   v_1 = r_0/\|r_0\|; \\
   \text{for } j = 1, 2, \ldots, \text{ do} \\
   \quad w = \tilde{M}v_j \\
   \quad \text{for } i = 1 \text{ to } j \text{ do} \\
   \quad \quad h_{ij} = v_i^* w \\
   \quad \quad w = w - h_{ij}v_i \\
   \quad \text{end for} \]
   \[
   h_{j+1,j} = \|w\| \\
   v_{j+1} = w/h_{j+1,j} \\
   \text{Solve the minimal problem } \min_{s \in \mathbb{C}} \|r_0\|e_1 - \kappa_1 s - H_{i+1,i}\tilde{s} \text{ for } s \\
   \text{Set } z_i = z_0 + V_is \text{ and } r_i = b - M_0z_i \\
   \text{Exit if satisfied} \]

By the right preconditioner \( \kappa I - M\tilde{\tau} \), one obtains the \( \mathbb{C} \)-linear system
\[
|\kappa|^2w - \overline{M}w = b. \tag{2.2}
\]
If (2.2) is solved, then \( z = \overline{\kappa}w - M\tilde{\tau}w \). Through the equivalent real formulations of (2.1), it was shown in [24] that RL-GMRES applied to (2.1) is faster than GMRES applied to (2.2) in terms of matrix-vector products; see Theorem 3.9 and Remark 3.10 of [24]. Here, we give a different proof [25].

**Theorem 2.1.** Let \( r_i \) and \( r_i^G \) be the \( i \)th residual of RL-GMRES applied to (2.1) and the \( i \)th residual of GMRES applied to (2.2), respectively. If further assume that \( r_0 = r_0^G \), then we have
\[
\|r_2\| \leq \|r_i^G\|.
\]

**Proof.** By the shift-invariance property [26] of Krylov subspaces, we have
\[
\|r_i^G\| = \min_{u \in K(|\kappa|^2 I - \overline{M})} \|r_0 - (|\kappa|^2 I - \overline{M})u\| \\
= \min_{u \in K(M\tilde{\tau})} \|r_0 - (|\kappa|^2 I - M\overline{\tau})u\| \\
= \min_{u \in K(M\tilde{\tau})} \|r_0 - M\kappa u\| \\
= \min_{u \in (\kappa I - M\tau)K(M\overline{\tau})} \|r_0 - M\kappa u\| = \|r_2\|.
\]
The inequality follows from
\[
(\kappa I - M\tau)K(\overline{M}, r_0) = (\kappa I - M\tau)K((\tau)^2, r_0) \subseteq K_2(M, r_0) = K_2(M, r_0).
\]

**Remark 2.2.** The assumption \( r_0 = r_0^G \) in Theorem 2.1 is attainable by setting the zero vector as the initial guess. For this case we have \( r_0 = r_0^G = b \).
3. Global $\mathbb{R}$-linear GMRES

For two matrices $A, B \in \mathbb{C}^{p \times q}$, we define the inner product $(A, B)_F = \text{trace}(A^*B)$. The associated norm is the well-known Frobenius norm denoted by $\| \cdot \|_F$. In the sequel, we mainly present our results for (1.1) and its related systems. The results for (1.2) and its related systems are similar. We define the $\mathbb{R}$-linear operator $\mathcal{M}$ by

$$\mathcal{M} : X \rightarrow X + A\bar{X}B.$$ 

We call $\{V_1, V_2, \ldots, V_i\}$ an $F$-orthonormal basis of the matrix Krylov subspace

$$\mathcal{K}_i(M, V) = \text{span}\{V, MV, \ldots, M^{i-1}V\},$$

if for $j, k = 1, \ldots, i$,

$$\langle V_j, V_k \rangle_F = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

and

$$\text{span}\{V_1, V_2, \ldots, V_i\} = \mathcal{K}_i(M, V).$$

Here $M^iV$ is defined recursively as $M(M^{i-1}V)$. We describe the global Arnoldi algorithm as follows.

**Algorithm 2: Global Arnoldi algorithm**

1. Compute $\|V\|_F$, and let $V_1 = V/\|V\|_F$
2. for $j = 1, 2, \ldots, i$
   
   $W = AV_jB$
   
   for $k = 1$ to $j$
   
   $h_{kj} = \text{trace}(V_k^*W)$
   
   $W = W - h_{kj}V_k$
   
   end for
   
   $h_{j+1} = \|W\|_F$
   
   $v_{j+1} = W/h_{j+1}$
   
end for

**Proposition 3.1.** The global Arnoldi algorithm constructs an $F$-orthonormal basis $V_1, V_2, \ldots, V_i$ of the Krylov subspace $\mathcal{K}_i(M, V)$

**Proof.** Note that the shift-invariance property of Krylov subspace. The proof follows the same arguments of the proof of Theorem 3.1 of [13].

Let $X_0$ be the initial guess and $R_0 = C - MX_0$ the corresponding residual. Gl-RL-GMRES constructs the approximate solution $X_i \in X_0 + \mathcal{K}_i(M, R_0)$ at step $i$ such that

$$\|R_i\|_F := \|C - MX_i\|_F = \min_{Z \in \mathcal{K}_i(M, R_0)} \|R_0 - MZ\|_F.$$ (3.1)

**Theorem 3.2.** Let $\bar{I}_i$ denote the $i \times i$ identity matrix augmented with the row of zeros as the last row and $e_1$ the first column of the identity matrix with appropriate dimension. Let $H_{i+1,j}$ be the upper Hessenberg matrix generated in Algorithm 1 with $V = R_0$. We have

$$\|R_i\|_F = \min_{s \in \mathcal{C}^i} \|R_0\|_F e_1 - \bar{I}_i s - H_{i+1,j} \bar{s}\|.$$
Proof. Let \( s = (s_1, \ldots, s_i)^T \). From (3.1)
\[
\|R_i\|_F = \min_{s \in \mathbb{C}_i} \left\| R_0 - \sum_{j=1}^i V_j s_j - \sum_{j=1}^i s_j A V_j B \right\|_F
\]
\[
= \min_{s \in \mathbb{C}_i} \left\| \|R_0\|_F V_1 - \sum_{j=1}^i V_j s_j - \sum_{j=1}^i \sum_{k=1}^{j+1} s_j h_{kj} V_k \right\|_F
\]
\[
= \min_{s \in \mathbb{C}_i} \left\| \|R_0\|_F e_1 - \tilde{I}_i s - H_{i+1, i} \| \right\|.
\]
The last equality holds because \( V_1, \ldots, V_{i+1} \) are \( F \)-orthonormal.

The minimal problem
\[
\min_{s \in \mathbb{C}_i} \left\| \|R_0\|_F e_1 - \tilde{I}_i s - H_{i+1, i} \| \right\|
\]
can be solved by employing the \( \mathbb{R} \)-linear QR decomposition [13]. We present the details of Gl-RL-GMRES as follows.

**Algorithm 3: Global \( \mathbb{R} \)-linear GMRES**

1. Compute \( R_0 = C - M X_0 \), \( X_0 \) is the initial guess
2. for \( j = 1, 2, \ldots \),
   1. Generate \( \{V_1, V_2, \ldots\} \) and \( H_{i+1, i} \) by Algorithm 2 with \( V = R_0 \)
   2. Solve the problem \( \min_{s \in \mathbb{C}_i} \left\| \|R_0\|_F e_1 - \tilde{I}_i s - H_{i+1, i} \| \right\| \) for \( s = (s_1, \ldots, s_i)^T \)
   3. Set \( X_i = X_0 + \sum_{j=1}^i V_j s_j \) and \( R_i = C - M X_i \)
   4. Exit if satisfied
end for

4. Equivalent \( \mathbb{R} \)-linear systems

Let \( X^T \) denote the transpose of \( X \). Define \( \text{vec}(X) \in \mathbb{C}^{mn} \) by \( \text{vec}(X) = [x_1^T, \ldots, x_n^T]^T \), where \( X = [x_1, \ldots, x_n] \) with \( x_i \in \mathbb{C}^m, i = 1, \ldots, n \). Taking the vec operator on both sides of (1.1) and (1.2) we obtain the equivalent \( \mathbb{R} \)-linear systems
\[
(I + (B^T \otimes A)\tau)\text{vec}(X) = \text{vec}(C), \tag{4.1}
\]
and
\[
(I + (B^T \otimes A)P\tau)\text{vec}(X) = \text{vec}(C). \tag{4.2}
\]
Here \( \otimes \) denotes the Kronecker product [27, Chapter 4] and \( P \) the permutation described in [27, Theorem 4.3.8]. The related \( \mathbb{C} \)-linear systems to (4.1) and (4.2) are
\[
(I - (BB^T) \otimes (AA^T))\text{vec}(W) = \text{vec}(C), \tag{4.3}
\]
and
\[
(I - (A^* B)^T \otimes (AB^*))\text{vec}(W) = \text{vec}(C). \tag{4.4}
\]
Let $\sigma(M)$ denote the spectrum of a matrix $M \in \mathbb{C}^{m \times n}$. The $\mathbb{R}$-linear matrix equation (1.1) has a unique solution if and only if $\lambda_i \mu_j \neq 1$ for $\lambda_i \in \sigma(\overline{A}A)$, $\mu_j \in \sigma(\overline{B}B)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The $\mathbb{R}$-linear matrix equation (1.2) has a unique solution if and only if $\lambda_i \lambda_j \neq 1$ for $\lambda_i, \lambda_j \in \sigma(AB^*)$, $i, j = 1, \ldots, m$.

Note that for any $X \in \mathbb{C}^{m \times n}$, $\|X\|_F = \|\text{vec}(X)\|$. It is easy to prove the following proposition.

**Proposition 4.1.** Let $R_i, R_i^G$, $r_i$ and $r_i^G$ be the $i$th residual of GI-RL-GMRES applied to (1.1), the $i$th residual of GI-GMRES applied to (1.3), the $i$th residual of RL-GMRES applied to (4.1) and the $i$th residual of GMRES applied to (4.3), respectively.

1. GI-RL-GMRES applied to (1.1) is mathematically equivalent to RL-GMRES applied to (4.1), i.e., if $r_0 = \text{vec}(R_0)$ then $r_i = \text{vec}(R_i)$.

2. GI-GMRES applied to (1.3) is mathematically equivalent to GMRES applied to (4.3), i.e., if $r_0^G = \text{vec}(R_0^G)$ then $r_i^G = \text{vec}(R_i^G)$.

By Theorem 2.1 and Proposition 4.1, we immediately obtain the following corollary.

**Corollary 4.2.** Let $R_i$ and $R_i^G$ be the $i$th residual of GI-RL-GMRES applied to (1.1) and the $i$th residual of GI-GMRES applied to (1.3), respectively. If further assume that $R_0 = R_0^G$, then we have

$$\|R_2\|_F \leq \|R_i^G\|_F.$$

**Remark 4.3.** By Corollary 4.2, GI-RL-GMRES applied to (1.1) requires fewer matrix-matrix products than GI-GMRES applied to (1.3) (Note that GI-RL-GMRES applied to (1.1) requires two matrix-matrix products every iteration, and GI-GMRES applied to (1.3) requires four matrix-matrix products every iteration and for $X$ two extra matrix-matrix products are required). Roughly speaking, the cost of every iteration of GI-GMRES is twice as much as that of GI-RL-GMRES.

**Remark 4.4.** If $A$ and $B$ in (1.1) are not large, then by saving the matrices $A\overline{A}$ and $B\overline{B}$, GI-GMRES applied to (1.3) only requires two matrix-matrix products every iteration. When $A$ and $B$ are large and have some special structure, for example, sparsity, Toeplitz, $A\overline{A}$ and $B\overline{B}$ do not preserve such structure in general. Therefore, storing $A\overline{A}$ and $B\overline{B}$ is inadvisable.

5. **Numerical experiments**

We compare GI-RL-GMRES with GI-GMRES. The initial guess is set to be the zero matrix and the iteration stops if $\|r_i\|_F/\|C\|_F \leq 10^{-12}$ (GI-RL-GMRES) or $\|R_i^G\|_F/\|C\|_F \leq 10^{-12}$ (GI-GMRES). Throughout, the computation is performed in MATLAB 2008a on a laptop with 2.26G CPU and 4GB memory.

Consider the $\mathbb{R}$-linear matrix equation

$$X - A\overline{A}F = C$$

where

$$A = \begin{bmatrix}
1 & -2 - i & -1 + i \\
0 & i & 0 \\
0 & -1 & 1 - i
\end{bmatrix}, \quad F = \begin{bmatrix}
2i & i \\
1 & -1 + i \\
-1 + i & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
-1 + i & 1 \\
0 & i \\
-1 - i & 1 - 2i
\end{bmatrix}.$$

This equation has been studied in [28]. Both GI-RL-GMRES and GI-GMRES require 6 iterations to obtain the satisfactory solutions. GI-RL-GMRES requires 12 matrix-matrix products and GI-GMRES requires 26 matrix-matrix products (see Remark 4.3). We plot the convergence history of GI-RL-GMRES and GI-GMRES in Figures 1-2.
Figure 1: Convergence history of Gl-RL-GMRES and Gl-GMRES in terms of iterations

Figure 2: Convergence history of Gl-RL-GMRES and Gl-GMRES in terms of matrix-matrix products
6. Concluding remarks

We have presented the global $\mathbb{R}$-linear GMRES method, which is an extension of the $\mathbb{R}$-linear GMRES method for solving the $\mathbb{R}$-linear system (2.1) to solve the $\mathbb{R}$-linear matrix equations (1.1) and (1.2). We have proved that the new method requires fewer matrix-matrix products than the global GMRES method applied to the related generalized Sylvester matrix equations (1.3) and (1.4).

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References


