Finite Difference Method in Sound Synthesis

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Abstract

This paper reviews the finite difference method in the sound synthesis of string instruments. The mathematical basis for the method and the evaluation of the recursion equations are considered. Some stability conditions are discussed. Initial and boundary conditions are reviewed for piano- and guitar-like strings.

1. INTRODUCTION

Physical modeling of musical instruments is nowadays one of the most active fields in sound synthesis, musical acoustics and computer music research. The basis of its attractivity is that it gives better tools for controlling and producing both traditional and new synthesized sounds. Physical modeling simulates the fundamental physical behaviour of an actual instrument by employing the knowledge of the physical laws that govern the motions and interactions within the instrument and expressing them as mathematical formulae and equations.

One method for physical modeling is the numerical solving of the mathematical equations that describe a given phenomenon. Finite difference method, in fact, is not one method but a field of schemes for numerical solving of partial differential equations. The basis of these schemes is approximating derivatives with differences. This can be done in numerous ways. The "best" method to choose depends always on the problem and the computational resources available. The method presented in this work is particularly suitable for wave equations.

This approach in physical modeling fo sound was first taken in (Hiller & Ruiz 1971*a*) and (Hiller & Ruiz 1971*b*). It has been used in sound synthesis especially for string instruments.

2. THE IDEAL STRING EQUATION

Using finite difference method in the sound synthesis of string instruments is based on modeling the physical properties of a vibrating string. Therefore, we start with the wave equation for the ideal string, which is the simplest mathematical model of a vibrating string (Hiller & Ruiz 1971*a*). The essential properties of the ideal string are the following. The string vibrates in one plane only and does not vibrate longitudinally or stretch. The tension changes are negligible and the weight of the string is small compared to the tension. The string is perfectly flexible, has a uniform linear density and is rigidly supported at both ends. The amplitude of the oscillations is small compared to the length of the string and the effect of the surrounding medium is negligible. Therefore the ideal string equation is as simple as

$$y_{tt} = c^2 y_{xx}, \tag{1}$$

where y is the displacement of the string, x is the axis along the string, t is time and c is the transverse velocity of the string. We denote the partial derivatives $\frac{\partial}{\partial x} f(x,t) = f_x$ etc.

When solving equation (1) with finite difference method we first divide the string of length L into K intervals of equal lengths $\Delta x = \frac{L}{K}$. We denote the end points of each interval by x_k, $k = 1, \ldots, K$ and name the starting point of the string x_0. Then we divide also the time in intervals of equal length Δt and denote them by t_n, $n = 0, 1, \ldots$. We can gain insight on the problem by representing the spatial and time intervals on a grid of points in the (x,t)-plane. If we have a function defined for continuously varying x and t, namely $y(x,t)$, we use the notation $y(k,n) = y(k\Delta x, n\Delta t) = y(x_k, t_n)$ (Strikwerda 1989).

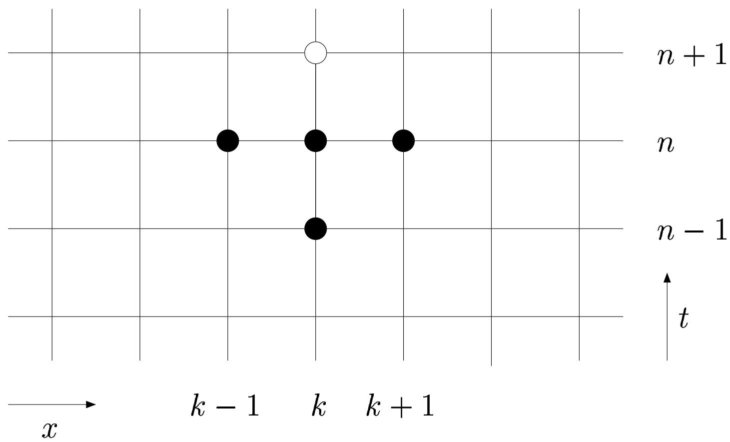

Figure 1: Dependence of displacement $y(k, n+1)$ (marked with ○) on previous values of the displacement (marked with ●) in the (x,t)-grid of an ideal string.

Figure 1 is an illustration of the information we need for each point of string when applying finite difference method to the ideal string equation. The displacement of the string at position k at time $n+1$ is computed from the displacements at the present time n at positions $k-1$, k and $k+1$ and at the preceding time $n-1$ at position k. In the following we shall see how this is achieved.

The basic idea of finite difference schemes is to replace derivatives by finite differences. Since the derivative of a function f of one variable can be defined as a limit of a difference,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \tag{2}$$

it is natural to approximate

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{or} \quad f'(x) \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \tag{3}$$

for small Δx. The latter formula is preferable, since it computes the difference without weighting information from one or the other side of the point in question. For a function

y of two variables we have then the partial derivatives

$$y_x(x,t) \approx \frac{y(x+\Delta x,t)-y(x-\Delta x,t)}{2\Delta x}, \tag{4}$$

$$y_t(x,t) \approx \frac{y(x,t+\Delta t)-y(x,t-\Delta t)}{2\Delta t}, \tag{5}$$

which can be written with the notations defined previously in this section

$$y_x(k,n) \approx \frac{y(k+1,n)-y(k-1,n)}{2\Delta x}, \tag{6}$$

$$y_t(k,n) \approx \frac{y(k,n+1)-y(k,n-1)}{2\Delta t}. \tag{7}$$

In the ideal string equation we have second-order derivatives. Let us approximate $f''(x)$ just like we did $f'(x)$,

$$\begin{aligned} f''(x) &\approx \frac{f'(x+\Delta x)-f'(x-\Delta x)}{2\Delta x} \\ &\approx \frac{\frac{f(x+2\Delta x)-f(x)}{2\Delta x}-\frac{f(x)-f(x-2\Delta x)}{2\Delta x}}{2\Delta x} \\ &\approx \frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}, \end{aligned} \tag{8}$$

by changing the notation $2\Delta x \rightarrow \Delta x$. Applying this to the displacement function yields

$$y_{xx}(k,n) \approx \frac{y(k+1,n)-2y(k,n)+y(k-1,n)}{(\Delta x)^2}, \tag{9}$$

$$y_{tt}(k,n) \approx \frac{y(k,n+1)-2y(k,n)+y(k,n-1)}{(\Delta t)^2}. \tag{10}$$

Now we have the approximate derivatives we need for the ideal string equation. Inserting (9) and (10) into equation (1) we get

$$\frac{y(k,n+1)-2y(k,n)+y(k,n-1)}{(\Delta t)^2} = c^2\frac{y(k+1,n)-2y(k,n)+y(k-1,n)}{(\Delta x)^2}. \tag{11}$$

Studying equation (11) we see that it would be convenient if the spatial intervals Δx and the time intervals Δt were somehow related. We can indeed write by the Von Neumann stability condition (Strikwerda 1989)

$$c\frac{\Delta t}{\Delta x} = r \leq 1. \tag{12}$$

Now we can write the displacement of string at x_k at the time t_{n+1} as

$$y(k,n+1) = 2(1-r^2)y(k,n) + r^2\left[y(k+1,n)+y(k-1,n)\right] - y(k,n-1), \tag{13}$$

and if $r=1$, as is the case in the ideal string,

$$y(k,n+1) = y(k+1,n)+y(k-1,n)-y(k,n-1). \tag{14}$$

The recursion formula (13) is illustrated in Figure 1. This equation can be directly used for computing the displacements of the chosen discrete points on the string.

The recursive equation (14) can be interpreted as a spatio-temporal digital waveguide filter (Karjalainen 2002). Equation (14) can be generalized as

$$y(k, n+1) = g_k^- y(k+1, n) + g_k^+ y(k-1, n) + a_k y(k, n-1), \qquad (15)$$

where coefficients g_k^-, g_k^+ and a_k can be used for simulating the losses, scattering etc. This leads to the digital waveguide filter shown in Figure 2 (Karjalainen 2002). However, in this work we examine the implementation of different modifications of the ideal string equation with the mathematical model as a starting point.

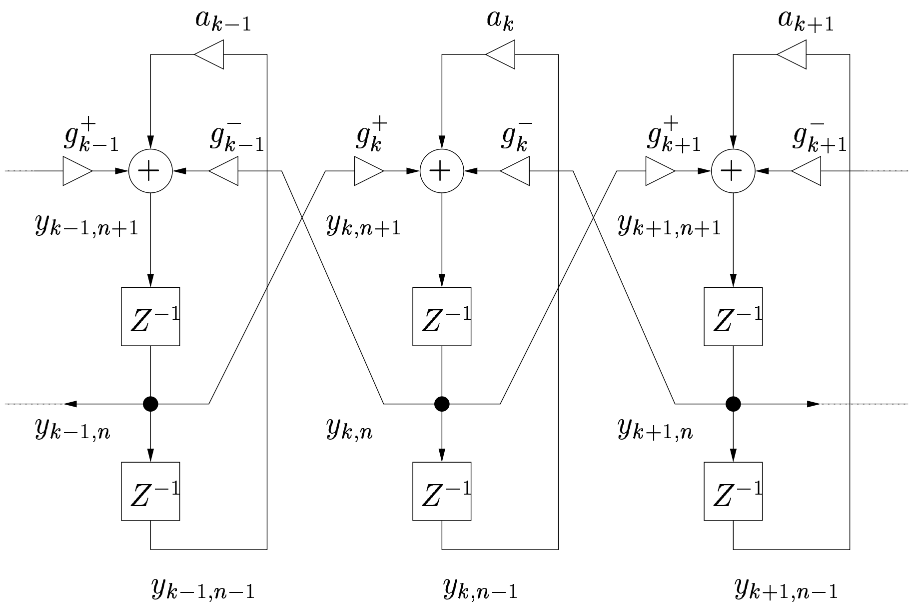

Figure 2: Digital Waveguide structure based on finite difference approximation of one-dimensional wave equation.

2.1. Stability and Numerical Dispersion

The choice $\frac{\Delta x}{\Delta t} = c$ gives the exact solution with no numerical dispersion (Chaigne 1992). However, $r = 1$ is valid only in the case of an ideal string.

It is convenient to write the stability condition (12) in terms of sampling frequency $f_e = \frac{1}{\Delta t}$ and fundamental frequency of the string $f_1 = \frac{c}{2L}$, yielding

$$K f_1 \leq \frac{f_e}{2}. \qquad (16)$$

The Nyquist theorem states that the upper frequency in the spectrum should be less than $\frac{f_e}{2}$ in order to avoid aliasing and to guarantee a unique continuous reconstruction. Therefore, in the ideal case when the eigenfrequencies of the string are equally spaced, that is, when $\Delta f = f_1$, condition (16) indicates that the maximum number of frequencies in the spectrum is K.

In practice, condition (16) can be used for selecting the appropriate number K of spatial points for synthesizing a sound with fundamental frequency f_1, if the sampling frequency f_e is given. However, since K is an integer, only discrete values of $f_1 = \frac{f_e}{2K}$ can be obtained without any truncation error, that is, using $r = 1$. Since these discrete series in general do not correspond to the musical scales of common use in western music, we must accept a reasonably small truncation error in order to get accurate adjustments of f_1. This can be obtained by spatial oversampling, that is, taking $r < 1$.

3. MODIFICATIONS OF THE STRING EQUATION

The ideal string equation is not a highly accurate approximation of the behaviour of real strings. Therefore we need to modify the equation by adding properties that resemble real strings. The stability conditions for the following modifications of the string equations can be found in (Chaigne 1992).

3.1. Stiff String

By adding a term that models the stiffness of the string (Hiller & Ruiz 1971*a*), we can write the wave equation for a stiff string as

$$y_{tt} = c^2 y_{xx} - \varepsilon c^2 L^2 y_{xxxx}, \tag{17}$$

where L is the string length and ε the stiffness parameter that is given by

$$\varepsilon = \kappa^2 \frac{ES}{TL^2}, \tag{18}$$

where κ is the radius of gyration of the string, E is Young's modulus, S is the area of the string cross section and T is the string tension.

For the third- and fourth-order derivatives of a function of one variable we have the approximations

$$f^{(3)}(x) \approx \frac{f(x+2\Delta x) - 2f(x+\Delta x) + 2f(x-\Delta x) - f(x-2\Delta x)}{2(\Delta x)^3},$$
$$f^{(4)}(x) \approx \frac{f(x+2\Delta x) - 4f(x+\Delta x) + 6f(x) - 4f(x-\Delta x) + f(x-2\Delta x)}{(\Delta x)^4},$$

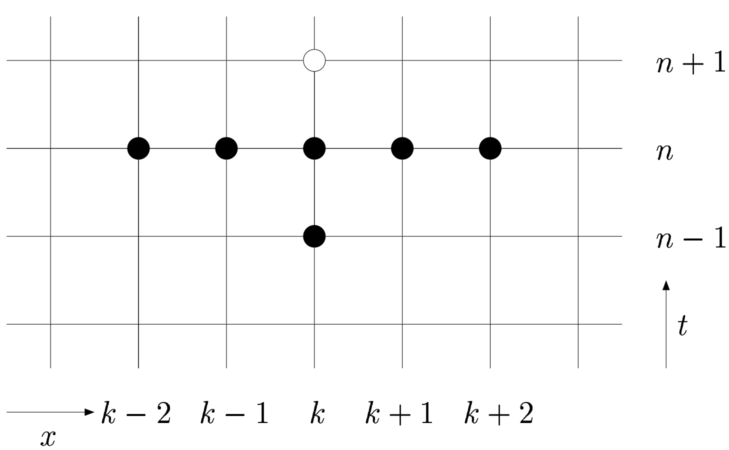

Figure 3: Dependence of displacement $y(k, n+1)$ (marked with ○) on previous values of the displacement (marked with ●) in the (x,t)-grid of a stiff string.

which are attained in the same way as the first- and second-order derivatives. Therefore, we have for the derivative in the stiffness term

$$y_{xxxx}(k,n) \approx \frac{y(k+2,n) - 4y(k+1,n) + 6y(k,n) - 4y(k-1,n) + y(k-2,n)}{(\Delta x)^4}. \tag{19}$$

Writing

$$a = \frac{\varepsilon L^2}{(\Delta x)^2}, \tag{20}$$

the stiff string equation can now be converted into the recursive formula

$$\begin{aligned} y(k,n+1) =& 2\left[1-(1+3a)r^2\right]y(k,n) + r^2(1+4a)\left[y(k+1,n)+y(k-1,n)\right] \\ &- ar^2\left[y(k+2,n)+y(k-2,n)\right] - y(k,n-1). \end{aligned} \tag{21}$$

Equation (21) can be illustrated in the same way as equation (21). As one can see in Figure 3, in a stiff string more distant values of the displacements affect the displacement at a future time, which is quite evident, when thinking of the meaning of stiffness.

3.2. Damped String

By including frequency-dependent losses due to the friction with air, viscosity and finite mass of the string (Hiller & Ruiz 1971*a*), we get the wave equation for a damped stiff string

$$y_{tt} = c^2 y_{xx} - \varepsilon c^2 L^2 y_{xxxx} - 2b_1 y_t + 2b_3 y_{ttt}, \tag{22}$$

where b_1 and b_3 are the loss parameters, which are obtained via the analysis of real instrument tones. The third-order time derivative can be approximated as the difference

$$y_{ttt}(k,n) \approx \frac{y(k,n+2) - 2y(k,n+1) + 2y(k,n-1) - y(k,n-2)}{(\Delta t)^3}, \tag{23}$$

which means that if we just insert differences in place of derivatives in the damped string equation (22), there will be a term $y(k,n+2)$ in the equation we can get for computating $y(k,n+1)$. Therefore, in order to compute the displacement at time $n+1$ similarly as in the preceding sections, we should already know the displacement at time $n+2$. This is of course impossible, and implicit methods are needed to solve the equation. However, since the perturbation term $2b_3 y_{ttt}$ is relatively small, we can approximate the term $y(k,n+2)$ by using twice the recursive equation (14) for an ideal string,

$$\begin{aligned} y(k,n+2) &= y(k+1,n+1) + y(k-1,n+1) - y(k,n) \\ &= y(k+2,n) - y(k+1,n-1) + y(k,n) + y(k-2,n) - y(k-1,n-1). \end{aligned}$$

Now we are able to write the recursion formula

$$\begin{aligned} y(k,n+1) =& \frac{2 - 2r^2 - 6a^2r^2 + \frac{b_3}{\Delta t}}{D} y(k,n) \\ &+ \frac{r^2(1+4a)}{D}\left[y(k+1,n)+y(k-1,n)\right] \\ &+ \frac{\frac{b_3}{\Delta t} - ar^2}{D}\left[y(k+2,n)+y(k-2,n)\right] \\ &+ \frac{-1 + b_1\Delta t + 2\frac{b_3}{\Delta t}}{D} y(k,n-1) \\ &- \frac{b_3}{D\Delta t}\left[y(k+1,n-1) + y(k-1,n-1) + y(k,n-2)\right], \end{aligned} \tag{24}$$

where

$$D = 1 + b_1\Delta t + 2\frac{b_3}{\Delta t}, \tag{25}$$

and a is given by formula (20). Although equation (24) may seem complicated, it is again straightforward to illustrate its form, which is shown in Figure 4 (Tolonen, Välimäki & Karjalainen 1998). The increase in time steps required in the formula is natural, since the damping effect is proportional to the time derivatives of the displacement.

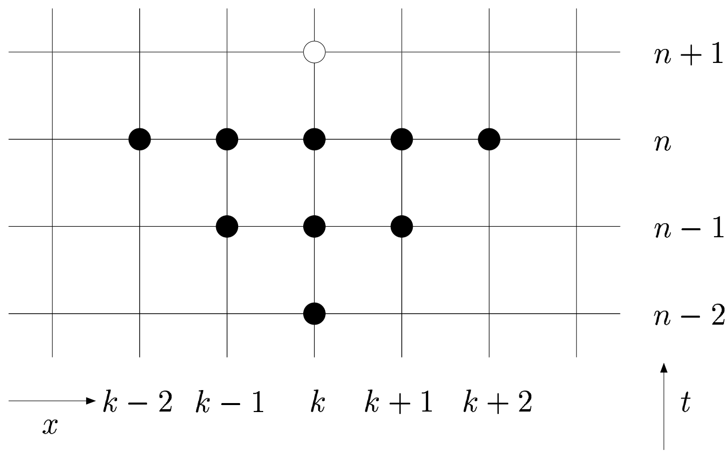

Figure 4: Dependence of displacement $y(k, n+1)$ (marked with ○) on previous values of the displacement (marked with ●) in the (x,t)-grid of a damped stiff string.

4. INITIAL CONDITIONS

Although the string equations carry much information of the behaviour of the string, additional conditions are needed in order to model the phenomenon of the emerging sound.

The initial excitation can be simulated by adding an acceleration term to the wave equation,

$$y_{tt} = c^2 y_{xx} - \varepsilon c^2 L^2 y_{xxxx} - 2b_1 y_t + 2b_3 y_{ttt} + f(x, x_0, t), \tag{26}$$

where $f(x, x_0, t)$ is the excitation acceleration applied at point x_0 (Tolonen et al. 1998). It is assumed here that the force density does not propagate along the string. Therefore, the time and space dependence can be separated, and we get

$$f(x, x_0, t) = f_0(t) g(x, x_0). \tag{27}$$

The term $g(x, x_0)$ can be understood as a spatial window that distributes the excitation energy to the string (Chaigne 1992). The force density term $f_0(t)$ is related to the force $F(t)$ exerted in the excitation by

$$f_0(t) = \frac{F(t)}{\mu \int_{x_0-\delta x}^{x_0+\delta x} g(x, x_0)\, dx}, \tag{28}$$

where μ is the linear mass density of the string and the effective length of the string section interacting with the exciter is $2\delta x$.

By applying the finite difference scheme to equation (26) and discretizing also the integral in formula (28) we get the recursion formula

$$\begin{aligned}
y(k,n+1) =& \frac{2-2r^2-6a^2r^2+\frac{b_3}{\Delta t}}{D} y(k,n) \\
&+ \frac{r^2(1+4a)}{D}\left[y(k+1,n)+y(k-1,n)\right] \\
&+ \frac{\frac{b_3}{\Delta t}-ar^2}{D}\left[y(k+2,n)+y(k-2,n)\right] \\
&+ \frac{-1+b_1\Delta t+2\frac{b_3}{\Delta t}}{D} y(k,n-1) \\
&- \frac{b_3}{D\Delta t}\left[y(k+1,n-1)+y(k-1,n-1)+y(k,n-2)\right] \\
&+ \frac{(\Delta t)^2 K}{D M_S} F(n) g(k,k_0),
\end{aligned} \tag{29}$$

with a and D given by formulae (20) and (25), respectively. Here M_S is the mass of the string, $F(n)$ is the force applied at time n and $g(k,k_0)$ is the value of the spatial window at position $x_0 = k_0\Delta x$ (Chaigne, Askenfelt & Jansson 1990).

4.1. Plucked String

For the guitar-like model of the string we assume that the string is initially at rest, that is,

$$y_t(x,0) = 0 \quad \Rightarrow \quad y(k,1) = y(k,-1). \tag{30}$$

The initial form of the string is given by

$$y(x,0) = h(x) \quad \Rightarrow \quad y(k,0) = h(k). \tag{31}$$

In the case of an ideal plucked string the initial shape can be approximated as triangular,

$$h(x) = \begin{cases} y(x_0)\frac{x}{x_0}, & 0 \le x \le x_0, \\ y(x_0)\frac{x-L}{x_0-L}, & x_0 < x \le L, \end{cases} \tag{32}$$

where x_0 represents the point of plucking (Hiller & Ruiz 1971*a*).

Further considerations of the plucked string can be found for example in (Chaigne 1992).

4.2. Struck String

The piano-like model for the excitation of the string is developed in (Chaigne & Askenfelt 1994*a*) and (Chaigne & Askenfelt 1994*b*). We assume that the string is initially at rest and has a zero initial displacement,

$$y(k,0) = 0. \tag{33}$$

In order to be able to use the recursive formula (29) with $F(n)$ replaced by the hammer force $F_H(n)$, we need to estimate somehow the displacement of the string at the first three time steps. The first one we already have in formula (33). By using approximated Taylor series we obtain

$$y(k,1) = \frac{y(k+1,0) + y(k-1,0)}{2}. \tag{34}$$

The hammer force is modeled as

$$F_H(t) = K_H \left|\eta(t) - y(x_0,t)\right|^p, \tag{35}$$

where K_H is the generalized stiffness of the hammer, p is the effective nonlinearity exponent (preferred range of values $2 < p < 3$ (Russell 1997)) and $\eta(t)$ is the displacement of the hammer defined by

$$M_H \frac{d^2\eta}{dt^2} = -F_H(t), \tag{36}$$

where M_H is the mass of the hammer. Now for the displacement of the hammer at time $n = 1$ we calculate

$$\eta(1) = V_{H0}\Delta t, \tag{37}$$

where V_{H0} is the hammer velocity at $t = 0$. For the force exerted by the hammer we get

$$F_H(1) = K_H \left|\eta(1) - y(k_0,1)\right|^p. \tag{38}$$

Now $y(k,2)$ can be computed using a simplified version of equation (29),

$$y(k,2) = y(k-1,1) + y(k+1,1) - y(k,0) + \frac{(\Delta t^2) K F_H(1)}{M_S}, \tag{39}$$

where the stiffness and damping terms are dropped out in order to limit the space and time dependence. The displacement of the hammer and the hammer force are computed by

$$\eta(2) = 2\eta(1) - \eta(0) - \frac{(\Delta t)^2 F_H(1)}{M_H}, \tag{40}$$

$$F_H(2) = K_H \left|\eta(2) - y(k_0,2)\right|^p. \tag{41}$$

With these estimations it is possible to use directly equation (29) for computing the displacements at future times. The force $F_H(n)$ is assumed to be known and its effect for the string is taken into account until time n when

$$\eta(n+1) < y(K_0, n+1). \tag{42}$$

After this the string is left free for vibrations unless recontact of the hammer is modeled.

5. BOUNDARY CONDITIONS

The boundary conditions are shaped to describe the contact of the string with the environment (Chaigne 1992). For example, the end points of a piano string can be considered as hinged,

$$y(0,t) = y(L,t) = 0, \tag{43}$$

$$y_{xx}(0,t) = y_{xx}(L,t) = 0. \tag{44}$$

By discretizing these for the ideal string equation (1) we obtain the boundary conditions

$$y(0,n) = y(K,n) = 0 \tag{45}$$
$$y(-1,n) = -y(1,n) \quad \text{and} \quad y(K+1,n) = -y(K-1,n). \tag{46}$$

The displacements at points $y(-1,n)$ and $y(N+1,n)$ are important since they are needed for computing the displacement at points $y(1,n)$ and $y(N-1,n)$ (as can be seen in Figures 3 and 4), even if they are not included in the string model itself.

If the wave equation describing the string includes damping and stiffness terms, conditions in formula (46) transform into somewhat more complicated boundary conditions

$$\begin{aligned}(1+4a^2)\,[y(K+1,n) + y(K-1,n)]& \\ + \left(\frac{b_3}{\Delta t} - a^2\right)[y(K+2,n) + y(K-2,n)]& \\ - \frac{b_3}{\Delta t}\,[y(K+1,n-1) + y(K-1,n-1)] = 0,&\end{aligned} \tag{47}$$

and

$$\begin{aligned}(1+4a^2)\,[y(1,n) + y(-1,n)]& \\ + \left(\frac{b_3}{\Delta t} - a^2\right)[y(2,n) + y(-2,n)]& \\ - \frac{b_3}{\Delta t}\,[y(1,n-1) + y(-1,n-1)] = 0&\end{aligned} \tag{48}$$

which are attained by writing the formula (24) with the conditions (45). The boundary conditions for piano-like strings are illustrated in Figure 5.

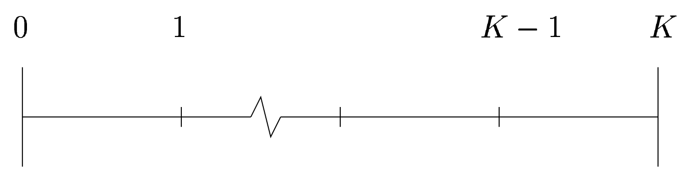

Figure 5: Model for piano-like boundary conditions.

In guitar, the terminations of strings are not completely rigid. The bridge has finite impedance and the finger terminating the string against the fingerboard is far from rigid. Therefore, in the guitar model it is assumed that the displacement $y(K,n)$ of the string is non-zero. Further, the string is clamped just behind the bridge, so that the distance between the bridge and the clamping position is above the audible wavelength range of human, and we can assume that

$$y(K+1,n) = y(K,n). \tag{49}$$

The boundary conditions for guitar-like strings are illustrated in Figure 6.

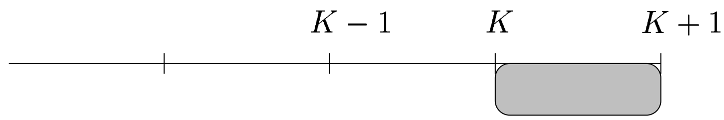

Figure 6: Model for a guitar-like boundary condition.

6. SUMMARY

The aim of this work was to go through the formulation of finite difference method in sound synthesis. This was done from the mathematical point of view and not considering very much the actual implementation of sound synthesis or the computational resources.

The models described in this work have been evaluated and compared to real instruments by (Chaigne & Askenfelt 1994*b*) and (Chaigne et al. 1990).

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