Appendix F

$\mathrm{L}^p_{\mathrm{strong}}$, $\mathrm{L}^p_{\mathrm{weak}}$ and Integration

Order and simplification are the first steps toward mastery of a subject — the actual enemy is the unknown.

— Thomas Mann (1875–1955)

For a function $f : Q \to \mathcal{B}(B, B_2)$, Bochner-measurability corresponds to the uniform (i.e., Banach space) topology of $\mathcal{B}(B, B_2)$. However, for several applications it suffices that, e.g., $fx : Q \to B_2$ is measurable for each $x \in B$ (i.e., that f is *strongly measurable*). We study this and the corresponding weak concept; in particular, we define and study $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ spaces (in Section F.1; we note that $\mathrm{L}^\infty_{\mathrm{strong}}$ is usually a Banach space (Theorem F.1.9) but $\mathrm{L}^p_{\mathrm{strong}}$ is often incomplete for $p < \infty$ (Example F.1.10)).

In Section F.2, we define and study integration and convolution for strongly or weakly measurable functions. In Section F.3, we treat $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ spaces and the Laplace transform of strongly or weakly measurable functions.

In this chapter, B, B_2 and B_3 denote Banach spaces over the scalar field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$ (we have $\mathbf{K} = \mathbf{C}$ in Section F.3), U, H, and Y denote Hilbert spaces, and μ is a complete positive measure on a set Q.

F.1 $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$

If you think you have the solution, the question was poorly phrased.

In this section, we study strong and weak (operator) measurability and corresponding L^p spaces. We start with the definitions of measurability:

Definition F.1.1 (Strong and weak measurability, L, $\mathrm{L}_{\mathrm{strong}}$, $\mathrm{L}_{\mathrm{weak}}$) *By* $\mathrm{L}(Q;*)$ *we denote the (equivalence classes of) Bochner measurable functions* $Q \to *$.

Let $F : Q \to \mathcal{B}(B, B_2)$. *Then* F *is* strongly measurable *(*$[F] \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$*) if* $Fx \in \mathrm{L}(Q; B_2)$ *for all* $x \in B$*, and* F *is* weakly measurable *(*$[F] \in \mathrm{L}_{\mathrm{weak}}(Q; \mathcal{B}(B, B_2))$*) if* $\Lambda F x \in \mathrm{L}(Q)$ *for all* $x \in B$, $\Lambda \in B_2^*$.

Elements $F, G \in \mathrm{L}$ *(resp* $\mathrm{L}_{\mathrm{strong}}$, $\mathrm{L}_{\mathrm{weak}}$*) are identified (*$[F] = [G]$*, or* $F \in [G]$*, where* $[F]$ *is the equivalence class of* F*) if* $F = G$ *a.e. (resp.* $Fx = Gx$ *a.e. for all* $x \in B$, $\Lambda F x = \Lambda G x$ *a.e. for all* $x \in B$, $\Lambda \in B_2^*$*).*

If $F : Q \to \mathcal{B}(B, B_2)$ *and* $G : Q \to \mathcal{B}(B_2^*, B^*)$ *are strongly (resp. weakly) measurable and* $\langle Fx, \Lambda\rangle_{\langle B_2, B_2^*\rangle} = \langle x, G\Lambda\rangle_{\langle B, B^*\rangle}$ *a.e. for all* $x \in B$, $\Lambda \in B_2^*$*, then* $[G]$ *is the* adjoint of $[F]$ in $\mathrm{L}_{\mathrm{strong}}$ *(resp.* $\mathrm{L}_{\mathrm{weak}}$*) and we write* $[F]^* = [G]$.

If $F \in \mathrm{L}(Q; \mathcal{B}(B, B_2))$ *and* $G \in \mathrm{L}(Q; \mathcal{B}(B_2, B_3))$*, then we define* $[G][F] := [GF] \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_3))$.

The above definition can be generalized to situations where F is B_3-valued and there is a continuous bilinear mapping $B \times B_3 \to B_2$, but these situations can be reduced to the above by considering B_3 as a subspace of $\mathcal{B}(B, B_2)$. The definition of $[G][F]$ will be justified in the proof of Lemma F.1.3(b).

We write $[F]_{\mathrm{L}}$, $[F]_{\mathrm{L}_{\mathrm{strong}}}$ or $[F]_{\mathrm{L}_{\mathrm{weak}}}$ when there may be confusion about the sense in which an equivalence class is defined. We use the standard convention to write F in place of $[F]$ when there is no risk of confusion.

The Bochner measurability of an operator-valued function is often called *uniform measurability* (in the literature, also the term "strong measurability" is used, but we shall use that term for $\mathrm{L}_{\mathrm{strong}}$ only).

We shall interpret the definition of $\mathrm{L}_{\mathrm{strong}}$ and $\mathrm{L}_{\mathrm{weak}}$ for vector-valued functions as follows: if $f : Q \to B$, then we consider f as a function $Q \to \mathcal{B}(\mathbf{K}; B)$, so that strong (operator) measurability reduces to Bochner measurability and weak (operator) measurability reduces to "weak vector measurability", i.e., to the condition that $\Lambda f \in \mathrm{L}$ for all $\Lambda \in B^*$ (for operator-valued functions, by weak measurability we refer to weak operator measurability, as defined in Definition F.1.1).

Lemma F.1.2

(a) $\mathrm{L}(Q; \mathcal{B}(B, B_2)) \subset \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2)) \subset \mathrm{L}_{\mathrm{weak}}(Q; \mathcal{B}(B, B_2))$.

(b1) *If* $[F] \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$ *has an adjoint in* $\mathrm{L}_{\mathrm{strong}}$*, then this adjoint is unique.*

(b2) *If* $F, F^* \in \mathrm{L}_{\mathrm{strong}}$*, then* $[F^*]_{\mathrm{L}_{\mathrm{strong}}} = [F]^*_{\mathrm{L}_{\mathrm{strong}}}$. *If* $F, F^* \in \mathrm{L}_{\mathrm{weak}}$*, then* $[F^*]_{\mathrm{L}_{\mathrm{weak}}} = [F]^*_{\mathrm{L}_{\mathrm{weak}}}$.

(c) *Let B be reflexive. If $[F] \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$, then $[F^*] \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B_2^*,B^*))$ and $[F^*] = [F]^*$.*

In contrast to (a) and (b), in Example 3.1.4 we construct $F \in \mathrm{L}_{\mathrm{strong}} \setminus \mathrm{L}$ s.t. $F^* \in \mathrm{L}_{\mathrm{weak}} \setminus \mathrm{L}_{\mathrm{strong}}$ and $\operatorname{ess\,sup} \|F^*\Lambda\|_{B^*} = \infty$ for certain $\Lambda \in B_2^*$, even though $F \in [0]_{\mathrm{L}_{\mathrm{strong}}}$ and hence $[F]^*_{\mathrm{L}_{\mathrm{strong}}} = [0]^*_{\mathrm{L}_{\mathrm{strong}}} = [0]_{\mathrm{L}_{\mathrm{strong}}}$. Thus, e.g., $[F] \in \mathrm{L}_{\mathrm{strong}}$ may have an adjoint in $\mathrm{L}_{\mathrm{strong}}$ even if $F^* \notin \mathrm{L}_{\mathrm{strong}}$.

Proof: (a) This follows from (B.18) (Naturally, the inclusions should be injective. By Lemma B.2.6, this is the case (but the equivalence classes may be enlarged with "less measurable" elements and there may appear new equivalence classes as we move from L to $\mathrm{L}_{\mathrm{strong}}$ or from $\mathrm{L}_{\mathrm{strong}}$ to $\mathrm{L}_{\mathrm{weak}}$).)

(b2)&(c) These follow directly from the definition.

(b1) For $[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ we have, by Lemma B.2.6, that $[F] = [0] \Leftrightarrow \Lambda F x = 0$ a.e. for all $x \in B$, $\Lambda \in B_2^*$.

If $[F]^* = [G]$ and $[F]^* = [H]$, then $\langle x, (G-H)\Lambda\rangle = 0$ a.e. for all $\Lambda \in B_2^*$, $x \in B$. Therefore, hence $(G-H)\Lambda = 0$ a.e. for all $\Lambda \in B_2^*$, by Lemma B.2.6 (because $\{x \in B \,\big|\, \|x\|_B \le 1\} \subset (B^*)^*$ is norming), i.e., $[G] = [H]$.

□

Now we go on with further properties of strongly and weakly measurable functions:

Lemma F.1.3 *Let $[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, $[G] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B_2,B_3))$, $[f] \in \mathrm{L}(Q;B)$, $[h] \in \mathrm{L}(Q)$, $H : Q \to \mathcal{B}(B,B_2)$ and $1 \le p \le \infty$. Then we have the following:*

(a) *$Ff \in \mathrm{L}(Q,B_2)$ and $hF \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$.*

(b) *We have $[GF] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_3))$. In particular, $\mathrm{L}_{\mathrm{strong}}(Q,\mathcal{B}(B))$ is an algebra.*

(c) *If $H_n \in \mathrm{L}_{\mathrm{strong}}$ for all $n \in \mathbf{N}$ and $H_n \to H$ a.e., then $H \in \mathrm{L}_{\mathrm{strong}}$.*

(d) *If $\dim B < \infty$, then $\mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}(Q;\mathcal{B}(B,B_2))$ (and $[F]_{\mathrm{L}_{\mathrm{strong}}} = [F]_{\mathrm{L}}$); if B_2 is separable, then $\mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ (and $[F]_{\mathrm{L}_{\mathrm{strong}}} = [F]_{\mathrm{L}_{\mathrm{weak}}}$).*

(e) *If also R is a measure space, then $((q,r) \mapsto F(q)) \in \mathrm{L}_{\mathrm{strong}}(Q \times R)$ and $(q,r) \mapsto F(r-q) \in \mathrm{L}_{\mathrm{strong}}(Q \times R)$.*

(f1) *Assume that B is separable. Then $\|F\|$ is measurable and $\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \operatorname{ess\,sup}\|F\|_{\mathcal{B}(B,B_2)} \le \infty$, in particular, $[F] = [0] \Leftrightarrow F = 0$ a.e.*

Moreover, $[F] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$ iff $F(q) \in \mathcal{G}\mathcal{B}$ for a.e. $q \in Q$ and $[F^{-1}] \in \mathrm{L}_{\mathrm{strong}}$. If, B is also reflexive, then $F^* \in \mathrm{L}_{\mathrm{strong}}$ and $[F^*] = [F]^*$.

(f2) *Assume that Q is separable and $\mu(\Omega) > 0$ for open $\Omega \subset Q$.*

Then $\mathcal{C}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$. Moreover, $\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^\infty(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, with equal norms.

Assume, in addition, that $F \in \mathcal{C}(Q;\mathcal{B}(B,B_2))$. *Then* $[F^*] = [F]^*$ *and* $\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_Q \|F\|_{\mathcal{B}(B,B_2)} \le \infty$*; in particular,* $[F] = [0] \Leftrightarrow F \equiv 0$.

If $F(q) \in \mathcal{GB}(B,B_2)$ *for a.e.* $q \in Q$*, then* $[F]^{-1} = [F^{-1}] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$*. Conversely, if* $[F] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$ *and* $[G] = [F]^{-1}$ *and* $\|G\|_{\mathcal{B}(B_2,B)} \le M$ *a.e., then* $F(q) \in \mathcal{GB}(B,B_2)$ *for all* $q \in Q$.

(g) Assume that $B_3 = B$*. Then any separable sets* $X_0 \subset B$ *and* $Y_0 \subset B_2$ *are contained, respectively, in closed separable subspaces* $X \subset B$ *and* $Y \subset B_2$*, s.t. there is a null set* $N \subset Q$ *satisfying* $F(q)x \in Y$ *and* $G(q)y \in X$ *for all* $x \in X$*,* $y \in Y$ *and* $q \in Q \setminus N$*. (Cf. Lemma 3.2.6.)*

(w) Replace $\mathrm{L}_{\mathrm{strong}}$ *by* $\mathrm{L}_{\mathrm{weak}}$ *everywhere above in this lemma. Then parts (c) and (e) above hold, we have* $hF \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$*, and* $g \in \mathrm{L}(Q;B_2^*)$ *implies that* $gFf \in \mathrm{L}(Q)$*. Parts (f1) and (f2) (provided that both* B *and* B_2 *are assumed to be separable in (f1)) also hold except possibly the claims concerning* $\mathcal{G}\mathrm{L}_{\mathrm{strong}}$ *and* $\mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$.

Moreover, if $f \in \mathrm{L}$*, then* $Ff \in \mathrm{L}_{\mathrm{weak}}$*; if* $F \in \mathrm{L}_{\mathrm{strong}}$*, then* $GF \in \mathrm{L}_{\mathrm{weak}}$*;*

Note that any measurable subset of $\mathbf{R}^n$ with the Lebesgue measure (or any countable set with the counting measure) satisfies the assumptions of (f2), by Lemma B.2.3(e).

Proof: (a) The claim on hF is a special case of (b). Now $g_j := \sum_{k=1}^{j} Fx_k\chi_{E_k} \in \mathrm{L}(Q;B_2)$ for all $j \in \mathbf{N}$, $\{x_k\} \subset B$ and disjoint, measurable E_k ($k \in \mathbf{N}$). Therefore, $Ff = \lim_{j\to+\infty} g_j \in \mathrm{L}$, when $f = \sum_{k=1}^{\infty} x_k\chi_{E_k}$. If $f \in \mathrm{L}$ is arbitrary, $f_n \to f$ a.e., and f_n is countably-valued ($n \in \mathbf{N}$), then $\mathrm{L} \ni Ff_n \to Ff$ a.e., as $n \to +\infty$, hence then $Ff \in \mathrm{L}$, by Lemma B.2.5(c).

(b) 1° Now $Fx \in \mathrm{L}$, hence $GFx \in \mathrm{L}$, by (a), for any $x \in B$. Thus, $[GF] \in \mathrm{L}_{\mathrm{strong}}$.

2° We shall now show that $[G][F] := [GF]$ is well defined, as promised below Definition F.1.1: Let $F' \in [F]$, $G' \in [G]$ and $x \in B$. By Lemma B.2.5(b1), there is a separable subset $B_0 \subset B$ s.t. $F(q)x \in B_0$ for a.e. $q \in Q$. Choose a null set $N \subset Q$ s.t. $Fy = F'y$ on N^c for all y in a dense, countable subset of B_0, hence for all $y \in B_0$. Then $FGx = F'G'x$ a.e. on N^c, hence a.e., hence $[FG] = [F'G']$, hence multiplication is well-defined.

3° Apply 1° to $B_2 = B = B_3$ to see that $\mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B))$ is an algebra.

(c) Now $Hx = \lim_n H_n x$ a.e. hence $Hx \in \mathrm{L}_{\mathrm{strong}}$, for any $x \in B$ (thus, it were sufficient if $H_n \to H$ strongly).

(d) We assume that $\dim B < \infty$ and prove that $F \in \mathrm{L}(Q;\mathcal{B}(B,B_2))$. Take a base $e_1,\ldots,e_n \subset B$, and set $f_j := Fe_j \in \mathrm{L}(Q;B_2)$ for all j. Then $F\sum_{j=1}^n \alpha_j e_j = \sum_{j=1}^n \alpha_j f_j$, hence $F = \sum_j f_j P_j \in \mathrm{L}$, where $P_j \in B^*$ is the mapping $\sum_{j=1}^n \alpha_j e_j \mapsto \alpha_j$. Obviously, $Fx = 0$ a.e. for all $x \in B$ iff $F = 0$ a.e., hence $[0]_{\mathrm{L}_{\mathrm{strong}}} = [0]_{\mathrm{L}}$.

If, instead B_2 is separable, then any $F \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ is strongly measurable, by Lemma B.2.5(b1) (and $\Lambda Fx = 0$ a.e. for all $\Lambda \in B_2^*$ iff $Fx = 0$ a.e., hence $[0]_{\mathrm{L}_{\mathrm{strong}}} = [0]_{\mathrm{L}_{\mathrm{weak}}}$).

(e) Now $F(\cdot)x \in \mathrm{L}(Q,B_2) \subset \mathrm{L}(Q \times R,B_2)$ for all $x \in B$, hence $q \mapsto F(q)$ is in $\mathrm{L}_{\mathrm{strong}}(Q \times R)$. The second claim follows analogously, by using Lemma B.2.9.

(f1) 1° Let $\{b_k\}_{k\in\mathbf{N}}$ be dense in the unit ball of B. Then $\|F\|_{\mathcal{B}(B,B_2)} = \sup_k \|Fb_k\|_{B_2}$ is measurable and $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \sup_k \|Fb_k\|_{\mathrm{L}^\infty} \le \operatorname{ess\,sup} \|F\|$. But if $\|F\| > M$ on E with $\mu(E) > 0$, then $\cup_k E_k = E$, where $E_k := \{q \in E \,\big|\, \|F(q)b_k\| > M\}$, hence then $\mu(E_k) > 0$ for some k, thus $\|Fb_k\|_\infty > M$. Consequently, $\sup_k \|Fb_k\|_{\mathrm{L}^\infty} \le \operatorname{ess\,sup}\|F\|$.

2° *Claim* $[F]^{-1} = [F^{-1}]$*:* If F^{-1} exists a.e. and $[F^{-1}] \in \mathrm{L}_{\text{strong}}$, then, obviously, $[F^{-1}][F] = I$ and $[F][F^{-1}] = I$ in $\mathrm{L}_{\text{strong}}$.

For the converse, assume that $B_3 = B$ and $[G] = [F]^{-1}$. Set $X_0 := B$ and apply (g) to obtain a null set $N \subset Q$ and a closed separable subspace $Y \subset B_2$ s.t. $F(q)[B] \subset Y$ for all $q \in Q \setminus N$.

Now $I_B - GF = 0$ a.e. on N^c and $(I_{B_2} - FG)_{|Y} = 0$ a.e. on N^c Therefore, $G_{|Y} = F^{-1}$ a.e. on N^c, say, on N_1^c, where N_1 is a null set. But for any $y \in B_2$ we have $FGy = y$ a.e. on N_1^c, hence $B_2 = Y$, i.e., $G = F^{-1}$ on N_1^c, hence a.e.

3° Assume now that B is also reflexive. Let $\Lambda \in B_2^*$. Then $\langle x, F^*\Lambda\rangle = \langle Fx, \Lambda\rangle$ is measurable for all $x \in B = B^{**}$, and $F^*\Lambda : Q \to B^*$ is separably-valued (B^* is separable, by Lemma A.3.4(R2)), hence $F^*\Lambda$ is measurable, by Lemma B.2.5(b1). Obviously, $[F^*] = [F]^*$.

(f2) (In fact, piecewise continuity suffices (or that $Q = \cup_{n\in\mathbf{N}} Q_n$, where, for each n, $Q_n \subset Q$ is a Borel set, $F \in \mathcal{C}(Q_n; \mathcal{B}(U,Y))$, and $\mu(\Omega) > 0$ for all open $\Omega \subset Q_n$).) Note that we implicitly assumed that Q is a topological space and that all Borel-sets are measurable.

1° By Lemma B.2.5(e), $\mathcal{C} \subset \mathrm{L}$. (Note that we have identified F and $[F]_{\mathrm{L}}$ for $F \in \mathcal{C}$; by 3°, this inclusion is injective.) Combine this with Lemma F.1.2(a) to obtain $\mathcal{C} \subset \mathrm{L} \subset \mathrm{L}_{\text{strong}}$.

2° *We have* $\|F\|_{\mathrm{L}^\infty_{\text{strong}}} = \sup_Q \|F\|$*:* If $F \in \mathcal{C}$ and $\|F(q)x\| > M := \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$ for some $x \in B$ s.t. $\|x\| \le 1$, then $\Omega := \{q \in Q \,\big|\, \|F(q)x\| > M\}$ has a positive measure, hence $\|Fx\|_\infty > M$, a contradiction, hence $\sup_Q \|F\|_{\mathcal{B}(B,B_2)} \le M$, hence $\sup_Q \|F\|_{\mathcal{B}(B,B_2)} = M$.

3° $\mathcal{C}_{\mathrm{b}} \subset \mathrm{L}^\infty \subset \mathrm{L}^\infty_{\text{strong}}$ *with equal norms:* this follows from 1° and 2°.

4° Now also F^* is continuous, hence $F^* \in \mathrm{L} \subset \mathrm{L}_{\text{strong}}$, by 1°. From the definition of $[F]^*$ we observe that $[F^*] = [F]^*$.

5° If $F(q) \in \mathcal{GB}(B,B_2)$ for $q \in N^c$, where N is a null set, then $F^{-1} \in \mathcal{C}(N^c; \mathcal{B}(B_2,B)) \subset \mathrm{L}_{\text{strong}}(N^c; \mathcal{B}(B_2,B)) = \mathrm{L}_{\text{strong}}(Q; \mathcal{B}(B_2,B))$.

6° Assume that $[F] \in \mathcal{G}\mathrm{L}_{\text{strong}}$, $[G] = [F]^{-1}$ and $\|G\|_{\mathcal{B}(B_2,B)} \le M$ on N_0^c, where N_0 is a null set.

Let $x_0 \in B$ and $y_0 \in B_2$ be arbitrary. Set $X_0 := \{x_0\}$, $Y_0 := \{y_0\}$, and apply (g) to obtain closed separable subspaces X, Y and a set N s.t. $F(q)X \subset Y$ and $G(q)Y \subset X$ for all $q \in N^c$, $x_0 \in X \subset B$ and $y_0 \in Y \subset B_2$. By continuity, $F(q)X \subset Y$ for all $q \in Q$.

Since $GFx = x$ and $FGy = y$ a.e. for all $x \in B$ and $y \in B_2$, hence for all $x \in X$ and $y \in Y$, we have $[F]^{-1} = [G]$ also in $\mathrm{L}^\infty_{\text{strong}}(Q; \mathcal{B}(Y,X))$. Thus, we can apply (f1) to obtain that $F(q)_{|X} \in \mathcal{GB}(X,Y)$ for a.e. $q \in Q$, say for $q \in N_{x_0,y_0}^c$, where N_{x_0,y_0} is a null set.

We now show that $F(q)_{|X} \in \mathcal{GB}(B,B_2)$ for all $q \in Q$: To obtain a contradiction, assume that $F(q_0)_{|X} \notin \mathcal{GB}(B,B_2)$ for some $q_0 \in Q$. Then there is an open $V \subset \mathcal{B}(X,Y)$ s.t. $F(q_0) \in V$ and $T \in V$ & $T \in \mathcal{GB}(X,Y) \Rightarrow \|T^{-1}\| > M$, by Lemma A.3.3(A4). It follows that $V' := \{q \in Q \,\big|\, F(q)_{|X} \in V\}$ is open and $V' \subset N_0 \cup N_{x_0,y_0}$, hence $\mu(V') = 0$, hence $V' = \emptyset$, a contradiction.

In particular, $F(q)x_0 \neq 0$ and $y_0 \in \mathrm{Ran}(F(q))$, for all $q \in Q$. Because x and y were arbitrary, we have $\mathrm{Ker}(F(q)) = \{0\}$ and $\mathrm{Ran}(F(q)) = B_2$, hence $F(q) \in \mathcal{GB}(B,B_2)$ for any $q \in Q$.

(g) Let $D_X \subset X_0$ and $D_Y \subset Y_0$ be dense and countable. For any $n \in \mathbf{N}$, we have $F(GF)^n, (GF)^n G, (GF)^n, (FG)^n \in \mathrm{L}_{\mathrm{strong}}$, by (b).

For each $x \in D_X$, there is a null set $N_0^x \subset Q$ s.t. $Y_0^x := F[Q \setminus N_0^x]x$ is separable. Set $Y_1 := \overline{\mathrm{span}(Y_0 \cup \cup_{x \in D_X} Y_0^x)}$, $N_0' := \cup_{x \in D_X} N_0^x$. It follows that $F[Q \setminus N_0']x \subset Y_1$ for all $x \in X_0$, by continuity. Moreover, $Y_1 \subset Y$ is separable, by Lemma B.2.3(a)&(c), and N_0' is a null set.

For each $k \in 1+\mathbf{N}$, given N_k' and Y_k, choose, analogously, a null set N_{k+1} and a separable subspace $X_{k+1} \subset B$ s.t. $X_k \subset X_{k+1}$ and $G[Q \setminus N_{k+1}]Y_k \subset X_{k+1}$. On the other hand, for each $k \in 1+\mathbf{N}$, given N_k and X_k, choose a null set N_k' and a separable subspace $Y_k \subset B_2$ s.t. $Y_{k-1} \subset Y_k$ and $F[Q \setminus N_k']X_k \subset Y_k$.

Set $N := \cup_k N_k \cup N_k'$, $X := \overline{\mathrm{span}(\cup_k X_k)}$, $Y := \overline{\mathrm{span}(\cup_k Y_k)}$. If $q \in Q \setminus N$, then $F(q)x \in Y$ for all $x \in \cup_k X_k$, hence for all $x \in X$, by linearity and continuity; analogously, $G(q)y \in X$ for all $y \in Y$.

(w) 1° $Ff, GF, hF, gFf \in \mathrm{L}_{\mathrm{weak}}$*:* Let $f \in \mathrm{L}$. A slight modification of the proof of (a) shows that $Ff \in \mathrm{L}_{\mathrm{weak}}$. Let now $G \in \mathrm{L}_{\mathrm{weak}}$ and $F \in \mathrm{L}_{\mathrm{strong}}$. Then $Fx \in \mathrm{L}$ for each $x \in B$, hence $GFx \in \mathrm{L}_{\mathrm{weak}}$, by the above; consequently, $GF \in \mathrm{L}_{\mathrm{weak}}$. The claims on hF and gFf follow.

2° *The other claims:* The above proofs of parts (c), (e) and (f) need only be slightly changed (in (f) we use a countable norming subset of B_2^* and a dense subset of B). □

Definition F.1.4 ($\mathbf{L^p_{strong}}(Q;\mathcal{B}(B,B_2))$) *Let* $1 \leq p \leq \infty$.

By $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *we denote the space of* $[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *having a finite norm*

$$\|F\|_{\mathrm{L}^p_{\mathrm{strong}}} := \sup_{\|x\|_B \leq 1} \|Fx\|_{\mathrm{L}^p(Q,B_2)}. \tag{F.1}$$

By $\mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2)))$ *we denote the space of* $[F] \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ *having a finite norm*

$$\|[F]\|_{\mathrm{L}^p_{\mathrm{weak}}} := \sup_{\|x\|_B, \|\Lambda\|_{B_2} \leq 1} \|\Lambda F x\|_{\mathrm{L}^p(Q)}. \tag{F.2}$$

It follows that $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p(Q;\mathcal{B}(B,B_2)) \cong (\mathrm{L}^p(Q;B_2))^n$ when $n := \dim B < \infty$ (cf. Lemma A.1.1(a4)). Note also that $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}} \leq \|F\|_{\mathrm{L}^p_{\mathrm{strong}}} \leq \|F\|_{\mathrm{L}^p}$ for $F \in \mathrm{L}^p$ and that $\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}$ (resp. $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}}$) is the norm of the operator $B \ni x \mapsto Fx \in \mathrm{L}^p$ (resp. the "bilinear norm" of $B \times B_2^* \ni (x,\Lambda) \mapsto \Lambda F x \in \mathrm{L}^p(Q)$; cf. Lemma A.3.4(J1)).

One could insist that $\mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ should consist of *all* linear $F : B \mapsto \mathrm{L}_{\mathrm{strong}}(Q;B_2)$ (and analogously for $\mathrm{L}_{\mathrm{weak}}$, $\mathrm{L}^p_{\mathrm{strong}}$, $\mathrm{L}^p_{\mathrm{weak}}$), not just for those that take the form of a function (a.e.). See Theorem F.2.1(g) etc. for details. However, that broader definition would cause problems in several applications.

The spaces $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ are normed spaces:

Lemma F.1.5 *Let $1 \le p \le \infty$. Then*

(a1) $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *is a subspace of* $\mathcal{B}(B,\mathrm{L}^p(Q;B_2))$ *with same norm.*

(a2) $\mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ *is a subspace of* $\mathcal{B}(B,\mathcal{B}(B_2,\mathrm{L}^p(Q)))$ *with same norm.*

(b) If $Q \subset \mathbf{R}^n$ and $\mu = m$, and B is a Hilbert space or $B_2 = \mathbf{K}$, then $\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$.

(c1) We have $\mathrm{L}^p(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$, *continuously.*

(c2) If $F \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, then $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$; if $F \in \mathrm{L}(Q;\mathcal{B}(B,B_2))$, then $\|F\|_{\mathrm{L}^\infty} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$.

(d) If $F \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ and $T \in \mathcal{B}(B_2,B_3)$, then $TF \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_3))$ and $\|TF\|_{\mathrm{L}^p_{\mathrm{strong}}} \le \|T\|_{\mathcal{B}}\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}$. Also the analogous "weak" claim holds.

(e) If $F \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ and B is reflexive, then $F^* \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B_2^*,B^*))$ and $\|F^*\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|F\|_{\mathrm{L}^p_{\mathrm{weak}}}$.

(f) **($\dim B < \infty$)** *If $\dim B < \infty$, then $\mathrm{L}^p(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ (with equivalent norms). If $\dim B_2 < \infty$, then $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ (with equivalent norms).*

(g1) Assume that $p = \infty$ or μ is σ-finite. Then $Hg \in \mathrm{L}$ for all $g \in \mathrm{L}^p(Q;B)$ iff $H \in \mathrm{L}_{\mathrm{strong}}$.

(g2) Assume that $p = \infty$ or μ is non-atomic. Then $H\cdot \in \mathcal{B}(\mathrm{L}^p)$ iff $H \in \mathrm{L}^\infty_{\mathrm{strong}}$.

"Usually" $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ are Banach spaces only for $p = \infty$; see Theorem F.1.9 and Example F.1.10 for details.

Proof: (a1)&(a2) These are obvious.

(b) Let $F \in \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$. W.l.o.g. we assume that $Q = \mathbf{R}^n$ (replace F by $F\chi_Q$). Set $M := \|F\|_{\mathcal{B}}$. For any $q \in \mathbf{R}^n$, the set $X_q := \{x \in B \mid q \in \mathrm{Leb}(LFx)\}$ is a subspace of B, and $\|LFx\| \le \|Fx\|_\infty \le M\|x\|_B$ on Q ($x \in B$), by Lemma B.5.3.

For each $q \in Q$, the map $x \mapsto LFx(q)$ is obviously linear on X_q, hence it has a norm-preserving extension $G(q) \in \mathcal{B}(B,B_2)$, by Lemma A.3.11, so that $\|G(q)\|_{\mathcal{B}(B,B_2)} \le M$.

Let $x \in B$. Then for a.e. $q \in Q$ we have $x \in X_q$ and hence $LF(q)x = G(q)x$; but $LFx = Fx$ a.e., hence $Fx = Gx$ a.e. Consequently, $G : Q \to \mathcal{B}(B,B_2)$ is strongly measurable and $\|F-G\|_{\mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))} = 0$.

Thus, we have constructed $G \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ s.t. $G = F$ as an element of $\mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$. Finally, G satisfies the additional condition $\|G(q)\|_{\mathcal{B}(B,B_2)} \le \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ for every $q \in Q$.

(c1)&(d) These are obvious (and the norms of the embeddings in (c1) are at most one).

(c2) The second claim follows from Theorem B.4.12(a); the first claim follows from the second (note that the norms may be infinite).

(e) This holds because $x(F^*\Lambda) = \Lambda F x$ for all $x \in B^{**} = B$.

(f) 1° *Case* $n := \dim B < \infty$: By Lemma F.1.3(d), we have $\mathrm{L}_{\mathrm{strong}} = \mathrm{L}$, so we only have to show that the two norms are equivalent. Let $b_1, ..., b_n$ be a vector base of B with $\|b_k\|_B = 1$ ($k = 1, \ldots, n$), and set $\varepsilon := \min_{|\alpha|_{\mathbf{K}^n}=1} \|\sum_k \alpha_k b_k\|_B > 0$. Then

$$\|T\|_{\mathcal{B}(B,B_2)} = \sup_{\|b\|=1} \|Tb\| \leq \sup_{|\alpha|_{\mathbf{K}^n}=1} \|T\varepsilon^{-1}\sum_k \alpha_k b_k\|, \leq \varepsilon^{-1}\sum_k \|Tb_k\|, \tag{F.3}$$

for all $T \in \mathcal{B}(B,B_2)$. Therefore, each $f \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ we have

$$\|f\|_p \leq \varepsilon^{-1}\sum_k \|fb_k\|_p \leq n\varepsilon^{-1}\|f\|_{\mathrm{L}^p_{\mathrm{strong}}}. \tag{F.4}$$

Trivially, $\|f\|_{\mathrm{L}^p_{\mathrm{strong}}} \leq \|f\|_p$, hence the two sets are equal and have equivalent norms.

2° *Case* $n := \dim B_2 < \infty$: As in 1° we see that there are $\Lambda_k \in B_2^*$, $\|\Lambda_k\| = 1$ ($k = 1, \ldots, n$) s.t.

$$\|fx\|_p \leq \varepsilon^{-1}\sum_{k=1}^n \|\Lambda_k fx\|_p \leq \varepsilon^{-1}\|f\|_{\mathrm{L}^p_{\mathrm{weak}}} \tag{F.5}$$

for all $f \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ and $x \in B$ s.t. $\|x\| \leq 1$.

(g1) "If" follows from Lemma F.1.3(a1). "Only if": If $p = \infty$, then we have $Hg \in \mathrm{L}$ when $g \equiv x$, for each $x \in B$, hence $H \in \mathrm{L}_{\mathrm{strong}}$. If $p < \infty$ and $Q = \cup_{n\in\mathbf{N}} Q_n$, $\mu(Q_n) < \infty$ for all n, then we can take $g = \chi_{Q_n} x$ for each $n \in \mathbf{N}$ and $x \in B$, hence $Hx \in \mathrm{L}$ for each $x \in B$, by Lemma B.2.5(d1).

(g2) By Theorem F.1.7(b), we have $\mathrm{L}^\infty_{\mathrm{strong}} \subset \mathcal{B}(\mathrm{L}^p, \mathrm{L}^p)$, isometrically. Conversely, if $(g \mapsto Hg) \in \mathcal{B}(\mathrm{L}^p)$, then $H \in \mathrm{L}_{\mathrm{strong}}$, by (g1), and $\|H\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|H\cdot\|_{\mathcal{B}(\mathrm{L}^p)}$, by Theorem F.1.7(b). □

The following lemma makes things simpler:

Lemma F.1.6 *Let $F : Q \to \mathcal{B}(B,B_2)$. We have $F \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ iff $Fx \in \mathrm{L}^p(Q;B_2)$ for all $x \in B$. We have $F \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ iff $\Lambda F x \in \mathrm{L}^p(Q)$ for all $x \in B$.*

Thus, if $Fx \in \mathrm{L}^p$ for each x, then "$Fx \in \mathrm{L}^p$ uniformly"; the proof is based on the Closed Graph Theorem.

Proof: 1° $\mathrm{L}^p_{\mathrm{strong}}$: Let $Fx \in \mathrm{L}^p(Q;B_2)$ for all $x \in B$. Then $x \mapsto Fx \in \mathrm{L}^p$ is linear. Let $x_n \to 0$ in B and $Fx_n \to f$ in L^p, as $n \to \infty$. Then $Fx_{n_k} \to f$ a.e. for some subsequence, by Theorem B.3.2, hence $f = \lim_k Fx_{n_k} = F0 = 0$ a.e. Consequently, $T : x \mapsto Fx$ is bounded, by Lemma A.3.4(E1). Therefore, $\|F\|_{\mathrm{L}^p_{\mathrm{strong}}} = \|T\|_{\mathcal{B}(B,\mathrm{L}^p)} < \infty$. The converse is obvious.

2° $\mathrm{L}^p_{\mathrm{weak}}$: For each $x \in B$, we haven $Fx \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B_2^*,\mathbf{K}))$, by 1°, hence $\|\Lambda F x\|_p \leq M_x \|\Lambda\|_{B_2^*}$ for some $M_x < \infty$. Thus, $Tx := Fx \in \mathcal{B}(B_2^*, \mathrm{L}^p(Q))$. Obviously, T is linear $B \to \mathcal{B}(B_2^*, \mathrm{L}^p)$.

Let $x_n \to 0$ in B and $Tx_n \to H$ in $\mathcal{B}(B_2^*, \mathrm{L}^p)$, as $n \to \infty$. Then, for each $\Lambda \in B_2^*$, we have $\Lambda F x_n = (Tx_n)\Lambda \to H\Lambda$ in $\mathrm{L}^p(Q)$ hence some subsequence converges pointwise a.e. But $\Lambda F x_n \to \Lambda F 0 = 0$ pointwise everywhere, hence $H\Lambda = 0$ a.e. Because $\Lambda \in B_2^*$ was arbitrary, we have $H = 0$. Consequently, $T : x \mapsto Tx$ is bounded, by Lemma A.3.4(E1). Therefore,

$$\|\Lambda F x\|_p = \|(Tx)\Lambda\|_p \le \|Tx\|_{\mathcal{B}(B_2^*,\mathrm{L}^p)} \|\Lambda\|_{B_2^*} \le \|T\|_{\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p))} \|x\|_B \|\Lambda\|_{B_2^*}, \tag{F.6}$$

hence $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|T\|_{\mathcal{B}(B,\mathrm{L}^p)} < \infty$. The converse is obvious. □

For "usual" L^p's, we have the following result with important applications:

Theorem F.1.7 ($\mathbf{L}^\infty_{\mathbf{strong}} \subset \mathcal{B}(\mathbf{L}^p)$) *Let* $F \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$, $1 \le p \le \infty$.

(a) We have $\|Ff\|_{\mathrm{L}^p(Q;B_2)} \le \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \|f\|_{\mathrm{L}^p(Q;B)}$ $\ (f \in \mathrm{L}^p(Q;B))$.

(b) Let μ be non-atomic or $p = \infty$, *and* $\mathrm{L}^p(Q;B) \ne \{0\}$. *Then*

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_{f \in \mathcal{E} \setminus \{0\}} \|Ff\|_p / \|f\|_p \le \infty, \tag{F.7}$$

where $\mathcal{E}$ is $\mathrm{L}^p(Q;B)$ *or* $\mathcal{E} = \mathcal{E}' \cdot X$, *where* $\mathcal{E}' \subset \mathrm{L}^p(Q)$ *and* $X \subset B$ *are dense.*

In particular, then $\mathrm{L}^\infty_{\mathrm{strong}}$ *is a subspace of* $\mathcal{B}(\mathrm{L}^p)$ *(with same norm).*

Proof: (a) 1° Since $F \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$, we have $Ff \in \mathrm{L}$ for all $f \in \mathrm{L}$, by Lemma F.1.3(a). W.l.o.g., we assume that $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} < \infty$ and $\|f\|_p > 0$.

2° *Case* $f \in \mathrm{CVM}^p$*:* Set $\mathrm{CVM}^p := \{f \in \mathrm{L}^p(Q;B) \mid f \text{ is countably-valued}\}$. Clearly $\|Ff\|_p \le \|F\|_\infty \|f\|_p$ for all $f \in \mathrm{CVM}$.

3° *Case* $f \in \mathrm{L}^p$*:* Let $\mathrm{CVM}^p \ni f_n \to f$ in L^p. Then $\{Ff_n\}$ is a Cauchy-sequence in $\mathrm{L}^p(Q;B_2)$, hence $Ff_n \to g$ in L^p for some $g \in \mathrm{L}^p$ with $\|g\|_p \le \|F\|_\infty \|f\|_p$, and a subsequence $\{Ff_{n_k}\}$ converges a.e. to g.

On the other hand, a subsubsequence converges to Ff, hence $Ff = g$ a.e., hence $\|Ff\|_p = \|g\|_p \le \|F\|_\infty \|f\|_p$ for all $f \in \mathrm{L}^p$.

(b) (Actually, even for $p < \infty$ it suffices that for any $E' \in \mathfrak{M}$ with $\mu(E') = \infty$, there is $E \subset E'$ s.t. $0 < \mu(E) < \mu(E')$. We show below that this condition is also necessary.) W.l.o.g., we assume that $0 < \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \le \infty$.

1° *A "counter-example":* Let $E' \in \mathfrak{M}$ be s.t. $\mu(E') = \infty$ and $\mu(E) \in \{0, \infty\}$ for all measurable $E \subset E'$, and let $p < \infty$ and $B \ne \{0\} \ne B_2$. Then we can take $F := \chi_{E'} T$, where $T \in \mathcal{B}(B, B_2) \setminus \{0\}$, so that $\|Ff\|_p = 0$ for all $f \in \mathrm{L}^p$.

Indeed, if $Ff \ne 0$, then $E'' := (Ff)^{-1}[B_2 \setminus \{0\}] \subset E'$ and $\mu(E'') > 0$. By Lemma B.2.8(b), there are $A \subset E''$ and $\Lambda \in B^*$ s.t. $\Lambda f > 1$ on A and $\mu(A) > 0$. Consequently, $\mu(A) = \infty$, hence $\|\Lambda f\|_p = \infty$, hence $f \notin \mathrm{L}^p$, a contradiction.

2° *The equality:* We assume that $0 \le M < \|F\|_{\mathrm{L}^i_{\mathrm{strong}} nfty}$ is arbitrary and find $\phi \in \mathcal{E} \setminus \{0\}$ s.t. $\|F\phi\|_p / \|\phi\|_p > M$. By (a), this establishes our claim.

Suppose that $0 < M < \|F\|_\infty \le \infty$. Take $x \in X$ (set $X := B$ if none is given) s.t. $\|Fx\|_\infty > M\|x\|$ and choose $E' \in \mathfrak{M}$ s.t. $\|F(q)x\| > M\|x\|$ for all $q \in E'$ & $0 < \mu(E')$ and choose $E \subset E'$ s.t. $0 < \mu(E) < \infty$.

If $p = \infty$, then $\|F\chi_E x\|_\infty > M\|x\| = M\|\chi_E x\|_\infty$; obviously the inequality is preserved if $\|\chi_E - g\|_\infty$ is small enough and χ_E is replaced by g (and x by $x' \in X$). Assume then that $p < \infty$.

For any $n \in \mathbf{N}$, there is $g \in \mathcal{E}'$ s.t. $\|g-\chi_E\|_{\mathrm{L}^p(E)} \le \|g-\chi_E\|_p < 1/n$. Set $R := \|\chi_E\|_p = \mu(E)^{1/p}$. Since $\|g\|_p/\|g\|_{\mathrm{L}^p(E)} < (R+1/n)/(R-1/n)$, we have

$$\|Fgx\|_p^p > M^p\|x\|^p\|g\|_{\mathrm{L}^p(E)}^p > M^p(R-1/n)(R+1/n)^{-1}\|gx\|_p. \tag{F.8}$$

Consequently, $\|Fgx\|_p > M\|gx\|_p$ for n big enough. Since $\|gx\|_p$ and $\|Fgx\|_p$ are continuous functions of $x \in B$, we can replace x by some $x' \in X$ close enough. □

When applying the Hölder inequality or similar results, one must make correct measurability requirements (cf. (d)):

Lemma F.1.8 *Let* $1 \le p \le \infty$ & $1/p + 1/q = 1$. *Let* $F : Q \to \mathcal{B}(B,B_2)$, $G : Q \to \mathcal{B}(B_2,B_3)$, $f : Q \to B$. *Then*

(a1) $F \in \mathrm{L}^p_{\text{strong}}$ & $G \in \mathrm{L}^\infty \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^\infty}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a2) $F \in \mathrm{L}^\infty_{\text{strong}}$ & $G \in \mathrm{L}^p \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^p}\|F\|_{\mathrm{L}^\infty_{\text{strong}}}$.

(a3) $F \in \mathrm{L}^p_{\text{strong}}$ & $G \in \mathrm{L}^q \Longrightarrow GF \in \mathrm{L}^1_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^1_{\text{strong}}} \le \|G\|_{\mathrm{L}^q}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(b) $F \in \mathrm{L}^\infty_{\text{strong}}$ & $f \in \mathrm{L}^p \Longrightarrow Ff \in \mathrm{L}^p$ & $\|Ff\|_{\mathrm{L}^p} \le \|F\|_{\mathrm{L}^\infty_{\text{strong}}}\|f\|_{\mathrm{L}^p}$.

(c) $G \in \mathrm{L}^\infty_{\text{strong}}$ & $F \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^\infty_{\text{strong}}}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(d) We may have $\|Ff\|_{\mathrm{L}^1} > M\|F\|_{\mathrm{L}^2_{\text{strong}}}\|f\|_{\mathrm{L}^2}$, $\|Ff\|_{\mathrm{L}^2} > M\|F\|_{\mathrm{L}^2_{\text{strong}}}\|f\|_{\mathrm{L}^\infty}$ *and* $\|GF\|_{\mathrm{L}^2_{\text{strong}}} > M\|G\|_{\mathrm{L}^2_{\text{strong}}}\|F\|_{\mathrm{L}^\infty_{\text{strong}}}$ *for any* $M > 0$, $Q = \mathbf{R}$ *and* $B = \ell^2(\mathbf{N})$ *(or* $B = \mathbf{K}^N$ *for* $N > M^2$*).*

(a') Claims (a1)–(a3) hold also with L_{weak} *in place of* $\mathrm{L}_{\text{strong}}$ *if G is scalar (i.e.,* $G : Q \to \mathbf{K}$*).*

(b') $F \in \mathrm{L}^\infty_{\text{weak}}$ & $f \in \mathrm{L}^p \Longrightarrow Ff \in \mathrm{L}^p_{\text{weak}}$ & $\|Ff\|_{\mathrm{L}^p_{\text{weak}}} \le \|F\|_{\mathrm{L}^\infty_{\text{weak}}}\|f\|_{\mathrm{L}^p}$.

(c') $G \in \mathrm{L}^\infty_{\text{weak}}$ & $F \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G\|_{\mathrm{L}^\infty_{\text{weak}}}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a1”) $F \in \mathrm{L}^p_{\text{strong}}$ & $G^* \in \mathrm{L}^\infty_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^\infty_{\text{strong}}}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a2”) $F \in \mathrm{L}^\infty_{\text{strong}}$ & $G^* \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^p_{\text{strong}}}\|F\|_{\mathrm{L}^\infty_{\text{strong}}}$.

(a3”) $F \in \mathrm{L}^p_{\text{strong}}$ & $G^* \in \mathrm{L}^q_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^1_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^1_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^q_{\text{strong}}}\|F\|_{\mathrm{L}^p_{\text{strong}}}$.

Note that we may take $G \in \mathrm{L}(Q) := \mathrm{L}(Q;\mathbf{K})$ (and $B_3 = B_2$). Note also that $F \in \mathrm{L}^p(Q;B_2)$ can be interpreted as $F \in \mathrm{L}^p_{\text{strong}}(Q;\mathcal{B}(\mathbf{K};B_2))$, so that (a1")–(a3") etc. apply.

Let $Q = \mathbf{R}^n$. Then, with the assumptions of (a3") (or (a3)), the weak convolution $G * F$ exists everywhere on $\mathbf{R}^n$ (see Theorem F.2.1(b)), hence the norm estimates of Lemma D.1.7 can be applied (to $(G^*\Lambda)$ and Fx, for each $x \in B$ and $\Lambda \in B_2^*$). However, without the assumptions of (a3") (even when, e.g., $G \in \mathrm{L}^2_{\text{strong}}$, $F \in \mathrm{L}^2$), we do not know whether $G * F$ exists as a function (with values in $\mathcal{B}(B,B_3)$).

Proof of Lemma F.1.8: By Lemma F.1.3(a)&(b), we have $GF, hF \in \mathrm{L}_{\text{strong}}$, $Ff \in \mathrm{L}$, hence only the claims on norms have to be shown.

(a1) Let $x \in B$. Then $\|Fx\|_{\mathrm{L}^p} \le \|F\|_{\mathrm{L}^p_{\text{strong}}}\|x\|_B$, hence $\|GFx\|_{\mathrm{L}^p} \le \|G\|_{\mathrm{L}^\infty}\|F\|_{\mathrm{L}^p_{\text{strong}}}\|x\|_B$.

(a2)&(a3) The proof is analogous to that of (a1) (use the Hölder Inequality for (a3)).

(b) 1° *Simple functions:* Let $f = \sum_{j=1}^k x_j\chi_{E_j}$ be simple with sets E_j disjoint. Then $\|Fx_j\| \le \|F\|\|x_j\|$ $(j \le k)$, hence $\|Ff\|_p^p \le \|F\|^p\|f\|_p^p$.

2° *General $f \in \mathrm{L}^p$:* By Theorems B.3.2 and B.3.11, there are simple functions $\{f_n\} \subset \mathrm{L}^p$ s.t. $f_n \to f$ a.e. and in L^p. Then $Ff_n \to Ff$ a.e. By 1°, $\{Ff_n\}$ is an L^p-Cauchy sequence, hence $Ff_n \to g$ in L^p for some $g \in \mathrm{L}^p$ with $\|g\|_p \le \|F\|\|f\|_p$. But a subsequence of $\{Ff_n\}$ converges a.e. to g, hence $g = Ff$ a.e., hence (b1) holds.

(c) Let $x \in B$. Then $\|Fx\|_{\mathrm{L}^p} \le \|F\|_{\mathrm{L}^p_{\text{strong}}}\|x\|_B$, hence $\|GFx\|_{\mathrm{L}^p} \le \|G\|_{\mathrm{L}^\infty_{\text{strong}}}\|F\|_{\mathrm{L}^p_{\text{strong}}}\|x\|_B$, by (b).

(d) Let $B = \ell^2(\mathbf{N})$. Define $F \in \mathrm{L}^2_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(B))$ by $F = \sum_{k\in\mathbf{N}}\chi_{[k,k+1)}P_k$, where $P_k(x_j)_{j\in\mathbf{N}} := x_k e_k$, and $f \in \mathrm{L}^2$ by $f = \sum_{k=1}^N \chi_{[k,k+1)}e_k$. Then

$$\|F\sum_k \alpha_k e_k\|_2^2 = \|\sum_k \alpha_k\chi_{[k,k+1)}e_k\|_2^2 = \sum_k |\alpha_k|^2 = \|\sum_k \alpha_k e_k\|_B^2, \tag{F.9}$$

hence $\|F\|_{\mathrm{L}^2_{\text{strong}}} = 1$. However, $Ff = f$, hence $\|Ff\|_2 = N^{1/2}$ and $\|Ff\|_1 = N$, although $\|f\|_\infty = 1$ and $\|f\|_2 = N^{1/2}$. (Note that we could take $f = \sum_{k=1}^\infty k^{-1}\chi_{[k,k+1)}e_k$ to obtain $\|f\|_2^2 = \sum_k k^{-2} < \infty$, $\|Ff\|_1 = \|f\|_1 = \sum_k k^{-1} = \infty$.)

Finally, set $G := \sum_{k=1}^N P_{1k}\chi_{[k,k+1)}$, where $P_{1k}(x_j)_{j\in\mathbf{N}} := x_1 e_k$. Then $\|G\|_{\mathrm{L}^\infty_{\text{strong}}} = 1$, but $Ge_1 = f$, hence $\|FGe_k\|_2 = N^{1/2}$, although $\|F\|_{\mathrm{L}^2_{\text{strong}}}\|G\|_{\mathrm{L}^\infty_{\text{strong}}}\|e_k\|_B = 1$.

(a')–(c') The proofs of (a1)–(c) apply mutatis mutandis.

(a1")–(a3") Let $\Lambda \in B_2^*$, $x \in B$. Then $\Lambda GFx = (G^*\Lambda)Fx$, $\|G^*\Lambda\| \le \|G^*\|\|\Lambda\|_{B_3^*}$ and $\|Fx\| \le \|F\|\|x\|_B$ for all $x \in B$, $\Lambda \in B_3^*$, hence (a1")–(a3") hold (by corresponding scalar results). □

Now we present a lifting result (claims (s3) and (w3)) and use it to show that $\mathrm{L}^\infty_{\text{strong}}$ and $\mathrm{L}^\infty_{\text{weak}}$ are complete:

Theorem F.1.9 ($\mathbf{L}^\infty_{\mathbf{strong}}$ and $\mathbf{L}^\infty_{\mathbf{weak}}$ are complete) *If (1.) $Q \subset \mathbf{R}^n$ is measurable, $\mu = m$, and B is a Hilbert space or $B_2 = \mathbf{K}$, or (2.) B is separable, then the following hold:*

(s1) $\mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2))$ *is a Banach space.*

(s2) Any Cauchy-sequence in $\mathrm{L}^\infty_{\mathrm{strong}}$ *converges uniformly outside some null set.*

(s3) For each $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2))$*, there is a representative* $G \in [F]_{\mathrm{L}_{\mathrm{strong}}}$ *s.t.* $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} = \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$.

(s4) $\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2)) = \mathcal{G}\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2))$ *if (1.) holds.*

If, instead, B and B_2^* are separable, then

(w1) $\mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$ *is a Banach space.*

(w2) Any Cauchy-sequence in $\mathrm{L}^\infty_{\mathrm{weak}}$ *converges uniformly outside some null set.*

(w3) For each $[F] \in \mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$*, there is a representative* $G \in [F]_{\mathrm{L}_{\mathrm{weak}}}$ *s.t.* $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$.

All above assumptions are unnecessary if μ is the counting measure on a set Q (then every function is measurable and $\mathrm{L}^\infty = \mathrm{L}^\infty_{\mathrm{strong}} = \mathrm{L}^\infty_{\mathrm{weak}} = \mathcal{C}_{\mathrm{b}}$, where we use the discrete topology on Q, hence all these spaces are complete and have $[0] = \{0\}$). However, we do not know whether the theorem holds for general (Q,μ), B and B_2; the main problem is the "lifting" claim (s3)/(w3); once it is established, the completeness claim requires just the latter assumption (cf. part II below).

Recall from Lemma F.1.5(b) that we have $\mathrm{L}^\infty_{\mathrm{strong}} = \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$ in case (1.).

Proof of Theorem F.1.9: *Part I: (s3)&(w3):* Obviously, $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} \ge \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ in (s3) and $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} \ge \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$ in (w3) so we only have to show the converses.

1° *Case (1.): $Q \subset \mathbf{R}^n$:* This was shown in the proof of Lemma F.1.5(b).

2° *Case (2.): separable B:* Remove all Λ's from 3°.

3° $\mathrm{L}^\infty_{\mathrm{weak}}$*:* Let $[F] \in \mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$, and let $S \subset B$ and $S_2 \subset B_2^*$ be dense and countable. Set $M := \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$. Then, for each $x \in B$ and $\Lambda \in B_2^*$, we have $|\Lambda F(q)x| \le M\|x\|\|\Lambda\|$ for a.e. $q \in Q$; choose a null set $N \subset Q$ s.t. this inequality holds for all $x \in S$, $\Lambda \in S_2$ and $q \in N^c$. By density and continuity, this inequality holds for all $x \in B$, $\Lambda \in B_2^*$ and $q \in N^c$, therefore, $G := \chi_{N^c}F$ is of the required form.

Part II: (s1)&(w1): Because $\mathrm{L}^\infty_{\mathrm{strong}}$ and $\mathrm{L}^\infty_{\mathrm{weak}}$ are normed spaces, we only have to prove their completeness. (Note that our proof does not require (1.) nor (2.) explicitly, but we rely on Part I.)

1° $\mathrm{L}^\infty_{\mathrm{strong}}$ *is a Banach space:* Let $\{[F_n]\}$ be a $\mathrm{L}^\infty_{\mathrm{strong}}$-Cauchy sequence. Assume that each F_n is chosen so that $\sup\|F_n\|_{\mathcal{B}(B,B_2)} \le \|F_n\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$, as in (s3). Set $\delta_n := \sup_{k\in\mathbf{N}}\|F_n - F_{n+k}\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ ($n \in \mathbf{N}$), so that $\delta_n \to 0$ as $n \in \infty$. Set

$$M := \lim_{n\to+\infty}\|F_n\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \in [0,\infty), \qquad g_x := \lim_{n\to+\infty} F_n x \in \mathrm{L}^\infty(Q;B_2) \quad (x \in B), \tag{F.10}$$

so that $\|g_x(t)\|_{B_2} \le M\|x\|_B$ for all $x \in B$ (set $g_x(t) = 0$ for t in the null set $N_x \subset N'_x$ where the limit does not exist).

Now $F_n x \to g_x$ uniformly on N_x^c, where $N_x := \cup_{n,k}\{q \in Q \,\big|\, \|F_n x - F_{n+k}x\| > \delta_n\}$.

Obviously, $X_q := \{x \in B \,\big|\, \lim_{n\to+\infty} F_n(q)x \text{ exists}\}$ is a subspace of B for all $q \in Q$. For each $q \in Q$, we let $F(q)$ be a norm-preserving (see Lemma A.3.11) extension of $(X_q \ni x \to g_x(q) \in B_2)$ so that $\|F(q)\|_{\mathcal{B}(B,B_2)} \le M$. Then, for any $x \in B$, we have for a.e. $q \in Q$ that $x \in X_q$, hence that $F(q)x = g_x(q) = \lim_n F_n(q)x$ (in particular, F is strongly measurable) and hence $\|F(q)x - F_n(q)x\| = \lim_k \|F_{n+k}(q)x - F_n(q)x\| \le \delta_n\|x\|$. Therefore, $\|F - F_n\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \le \delta_n \to 0$, so that $F_n \to F$ in $\mathrm{L}^\infty_{\mathrm{strong}}$, as $n \to +\infty$. Because $\{F_n\}$ was arbitrary, we have shown that $\mathrm{L}^\infty_{\mathrm{strong}}$ is complete, hence a Banach space.

2° $\mathrm{L}^\infty_{\mathrm{weak}}$ *is a Banach space:* Let $\{[F_n]\}$ be a Cauchy-sequence in $\mathrm{L}^\infty_{\mathrm{weak}}$ with each F_n chosen so that $\sup_{q\in Q}\|F_n(q)\| = \|F_n\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$. Set $\delta_n := \sup_{k\in\mathbf{N}}\|F_n - F_{n+k}\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$ ($n \in \mathbf{N}$), so that $\delta_n \to 0$ as $n \in \infty$.

Let $S \subset B$ and $S_2 \subset B_2$ be dense and set $N := \cup_{x\in S,\,\Lambda\in S_2} N_{x,\Lambda}$, where

$$N_{x,\Lambda} = \cup_{n,k\in\mathbf{N}}\{q \in Q \,\big|\, \|\Lambda F_n(q)x - \Lambda F_{n+k}(q)x\| > \delta_n\|\Lambda\|\|x\|\}, \tag{F.11}$$

so that $\mu(N) = 0$. Let $q \in N^c$. Then, for all $n,k \in \mathbf{N}$, we have $\|\Lambda F_n(q)x - \Lambda F_{n+k}(q)x\| \le \delta_n\|\Lambda\|\|x\|$ for all $\Lambda \in B_2^*$, $x \in B$, by density, hence $\|F_n(q) - F_{n+k}(q)\|_{\mathcal{B}(B,B_2)} \le \delta_n$. In particular, $\{F_n(q)\}$ is a Cauchy-sequence in $\mathcal{B}(B,B_2)$; let $F(q) \in \mathcal{B}(B,B_2)$ be its limit. Then $\|F(q) - F_n(q)\| = \lim_k \|F_{n+k}(q) - F_n(q)\| \le \delta_n$; but $q \in N^c$ was arbitrary, hence $F_n \to F$ uniformly on N^c.

Part III: (s2)&(w2): In Part II we showed that $F_n \to F$ uniformly on N^c for some null set $N \subset Q$, but we assumed $\{F_n\}$ chosen as in Part I. For general $\{F'_n\}$ we can choose the sequence $\{F_n\}$ as above, so that $F_n = F'_n$ outside some null set N_n for each $n \in \mathbf{N}$. Then $F'_n \to F$ uniformly outside the null set $N \cup \big(\cup_n N_n\big)$.

Part IV: (s4): By Lemma F.1.3(f2), $\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, isometrically, hence $\mathcal{G}\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \subset \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$. Conversely, if $F \in \mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, then $F(q) \in \mathcal{G}\mathcal{B}(B,B_2)$ for all $q \in Q$ and hence $[F]^{-1} = [F^{-1}]$, by two applications of Lemma F.1.3(f2) (since there is a bounded representative of $[F]^{-1}$, by (s3)), hence then $[F]^{-1} \in \mathcal{C}_{\mathrm{b}}$. $\square$

In contrast to the above theorem, the normed spaces $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ are usually not complete for $p < \infty$:

Example F.1.10 Let $Q := [0,1]$, $B := \ell^2(\mathbf{N})$. Then there is a sequence $\{F_n\} \subset \mathcal{C}(Q;\mathcal{B}(B)) \subset \mathrm{L}^2 \subset \mathrm{L}^2_{\mathrm{strong}} \subset \mathrm{L}^2_{\mathrm{weak}}$ s.t. $\{F_n\}$ is a Cauchy-sequence in $\mathrm{L}^2_{\mathrm{strong}}(Q;\mathcal{B}(B))$ and hence in $\mathrm{L}^2_{\mathrm{weak}}(Q;\mathcal{B}(B))$ too, but $\{F_n\}$ does not converge in either of these spaces (although it does converge in $\mathcal{B}(B;\mathrm{L}^2(Q;B))$, which is a Banach space). Moreover, $F_n(t) = F_n(t)^*$ for all $t \in [0,1]$ and $n \in \mathbf{N}$. $\triangleleft$

See also Example F.3.6.

Proof: 1° *The construction of* $\{F_n\}$*:* Set $g(t) := |t|^{-1/3}$, $g_n(t) := (|t| + 1/n)^{-1/3}$ ($t \in \mathbf{R}$) so that $0 \le g_n \le n$, $g_n \in \mathcal{C}_0(\mathbf{R})$, $g \in \mathrm{L}^2(Q)$ and $M_g :=$

$\sup_{q\in[0,1]}\|\tau^q g\|_2<\infty$. Furthermore, $g_n(t)\to g(t)$ monotonely for each $t\in\mathbf{R}$ and $\|g-g_n\|_{\mathrm{L}^2([-1,1])}\to 0$, by the Dominated Convergence Theorem.

Let $\{q_k\}\subset Q$ be dense. For $t\in Q$ and $n\in\mathbf{N}$, define $F_n(t):=\sum_{k\in\mathbf{N}} g_n(t-q_k)P_k$ (i.e., $F_n(t)x:=\sum_{k\in\mathbf{N}} g_n(t-q_k)x_k e_k$), where P_k is the projection $P_k:\sum_{j\in\mathbf{N}} x_j e_j\mapsto x_k e_k$ $(k\in\mathbf{N})$.

Note that for $f\in\mathrm{L}^2(Q;B)$, we have $\|f\|_2^2=\sum_k\|f_k\|_2^2$. Obviously, $F_n(t)^*=\sum_{k\in\mathbf{N}}\overline{g_n(t-q_k)}P_k^*=\sum_{k\in\mathbf{N}} g_n(t-q_k)P_k$ for all n and t.

2° $\|F_n\|_{\mathcal{B}(B)}\le n$: This is obvious.

3° $\{F_n\}\subset\mathcal{C}(Q;\mathcal{B}(B))$*:* Let $n\in\mathbf{N}$, $t\in Q$ and $\varepsilon>0$. The function g_n is uniformly continuous, because $g_n\in\mathcal{C}_0(\mathbf{R})$, hence there is $\delta>0$ s.t. $|g_n(t')-g_n(t'')|<\varepsilon$ for $|t'-t''|<\delta$. Let $|t'-t|<\delta$ and $x\in B$. Then

$$\|(F_n(t)-F_n(t'))x\|_B^2=\|\sum_k x_k(g_n(t-q_k)-g_n(t'-q_k))e_k\|_B^2\le\sum_k|x_k|^2\varepsilon^2=\varepsilon^2\|x\|_B. \tag{F.12}$$

Because x was arbitrary, we have $\|(F_n(t)-F_n(t'))\|\le\varepsilon$. Thus, F_n is continuous.

4° *$F_n\to F$ in $\mathcal{B}(B,\mathrm{L}^2(Q;B))$:* For $t\in Q$, $n\in\mathbf{N}$, and $x\in B$, we define $F(t):=\sum_{k\in\mathbf{N}} g(t-q_k)P_k$. Then $\|Fx\|_2^2\le\sum_k|x_k|^2M_g^2=M_g^2\|x\|_B^2$. Thus, $\|F\|\le M_g$. Moreover, given $\varepsilon>0$, there is $N\in\mathbf{N}$ be s.t. $\|g-g_m\|_{\mathrm{L}^2([-1,1))}<\varepsilon$ for all $n,m>N$, and, consequently,

$$\|F_nx-Fx\|_{\mathrm{L}^2(Q;B)}^2\le\sum_k|x_k|^2\|g_n(\cdot-q_k)-g(\cdot-q_k)\|_{\mathrm{L}^2(Q)}^2\le\varepsilon^2\|x\|_B^2\quad(x\in B). \tag{F.13}$$

Therefore, $F_n\to F$ in $\mathcal{B}$. (In particular, $\{F_n\}$ is $\mathrm{L}^2_{\text{strong}}$-Cauchy.)

5° *$\{F_n\}$ does not converge in $\mathrm{L}^2_{\text{weak}}$:* (I.e., F does not correspond to any function $Q\to\mathcal{B}(B)$.) To obtain a contradiction, assume that $F\in\mathrm{L}^2_{\text{weak}}(Q;\mathcal{B}(B))$ is such that $\langle F_nx,y\rangle\to\langle Fx,y\rangle$ in $\mathrm{L}^2(Q)$ for all $x,y\in B$.

We have $F_ne_k=\tau^{-q_k}g_ne_k\to\tau^{-q_k}ge_k$ in $\mathrm{L}^2(Q;B)$ and pointwise on Q, for each $k\in\mathbf{N}$. Thus, $\langle Fe_k,e_j\rangle=\langle\tau^{-q_k}ge_k,e_j\rangle=:f_{k,j}$ a.e. for all $k,j\in\mathbf{N}$, because $\langle Fe_k,e_j\rangle$ is a.e. the pointwise limit of a subsequence of $\{\langle F_ne_k,e_j\rangle\}$, and all subsequences of the latter converge to $f_{k,j}$.

Choose a null set N s.t. $\langle Fe_k,e_j\rangle=\langle\tau^{-q_k}ge_k,e_j\rangle$ on $Q\setminus N$ for all k,j. Then $Fe_k=\tau^{-q_k}ge_k$ on $Q\setminus N$ for all $k\in\mathbf{N}$. Let $t\in Q\setminus N$ and $M>0$. By the density of $\{q_k\}$ in Q, there is k s.t. $q_k<t$ and $(\tau^{-q_k}g)(t)=|t-q_k|^{-1/3}>M$, so that $\|F(t)e_k\|_B>M$. Consequently, $\|F(t)\|_{\mathcal{B}(B)}>M$. But this holds for all $t\in Q\setminus N$ and all $M>0$, hence F must be unbounded almost everywhere; in particular, F is not $\mathcal{B}(B)$-valued. □

(See the notes on p. 1023.)

F.2 Strong and weak integration ($\mathop{\mathrlap{\int}S}$, $\mathop{\mathrlap{\int}W}$)

Whenever anyone says, "theoretically," they really mean, "not really."
— Dave Parnas

Here we define the *strong and weak (operator) integrals*; in the rest of this section we treat corresponding convolutions.

Theorem F.2.1 (Strong and weak integrals $\mathop{\mathrlap{\int}S}$ and $\mathop{\mathrlap{\int}W}$) *Let $F : Q \to \mathcal{B}(B, B_2)$.*

(a) **(Strong integral)** *If $F \in \mathrm{L}^1_{\text{strong}}$, then there is a unique $L =: \mathop{\mathrlap{\int}S}_Q F\,d\mu \in \mathcal{B}(B, B_2)$ s.t. $Lx = \int_Q Fx\,d\mu$ for all $x \in B$. Moreover, $\|L\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathrm{L}^1_{\text{strong}}}$.*

(b) **(Weak integral)** *If $F \in \mathrm{L}^1_{\text{weak}}$, then there is a unique $L =: \mathop{\mathrlap{\int}W}_Q F\,d\mu \in \mathcal{B}(B, B_2^{**})$ s.t. $(Lx)\Lambda = \int_Q \Lambda F x\,d\mu$ for all $x \in B$, $\Lambda \in B_2^*$. Moreover, $\|L\|_{\mathcal{B}(B,B_2^{**})} \le \|F\|_{\mathrm{L}^1_{\text{weak}}}$.*

(c) A Bochner integral (a uniform integral) is a strong integral (i.e., $\int_Q F\,d\mu = \mathop{\mathrlap{\int}S}_Q F\,d\mu$ for $F \in \mathrm{L}^1$), and a strong integral is a weak integral (i.e., $\mathop{\mathrlap{\int}S}_Q F\,d\mu = \mathop{\mathrlap{\int}W}_Q F\,d\mu$ for $F \in \mathrm{L}^1_{\text{strong}}$).

(d) If $F \in \mathrm{L}^1_{\text{weak}}(Q; \mathcal{B}(B, B_2))$ and B is reflexive, then $F^* \in \mathrm{L}^1_{\text{weak}}(Q; \mathcal{B}(B_2^*, B))$ and $\mathop{\mathrlap{\int}W}_Q F^*\,d\mu = \left(\mathop{\mathrlap{\int}W}_Q F\,d\mu\right)^*$.

(e) Let $T \in \mathcal{B}(B_2, B_3)$ [and $x \in B$]. In (a) we have $\mathop{\mathrlap{\int}S}_Q TF\,d\mu = T\mathop{\mathrlap{\int}S}_Q F\,d\mu$ and $\int_Q TFx\,d\mu = T(\mathop{\mathrlap{\int}S}_Q F\,d\mu)x$; in (b) we have $\mathop{\mathrlap{\int}W}_Q TF\,d\mu = T\mathop{\mathrlap{\int}W}_Q F\,d\mu$ and $\mathop{\mathrlap{\int}W}_Q TFx\,d\mu = T(\mathop{\mathrlap{\int}W}_Q F\,d\mu)x$.

(f) Claims (a), (b), (c) and (e) also hold with replacements $\mathrm{L}^1_{\text{strong}} \mapsto \mathcal{B}(B, \mathrm{L}^1(Q; B_2))$ and $\mathrm{L}^1_{\text{weak}} \mapsto \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{L}^1(Q)))$.

With replacement $\mathrm{L}^1_{\text{weak}} \mapsto \mathcal{B}(B, \mathcal{B}(B_2, \mathrm{L}^1(Q; B_3)))$ we get that $\|\mathop{\mathrlap{\int}W}_Q F\,d\mu\|_{\mathcal{B}(B,\mathcal{B}(B_2,B_3))} \le \|F\|_{\mathcal{B}(B,\mathcal{B}(B_2,\mathrm{L}^1(Q;B_3)))}$.

Even if B_2 is a Hilbert space, the expression $\mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{L}^1(Q))$ refers to the linear dual of B_2.

(g) Let $p \in [1, \infty]$. Then the embedding $\mathrm{L}^p_{\text{strong}}(Q; \mathcal{B}(B, B_2)) \to \mathcal{B}(B, \mathrm{L}^p(Q; B_2))$ is a linear isometry to the subspace $\mathrm{L}_{\text{strong}}(Q; \mathcal{B}(B, B_2)) \cap \mathcal{B}(B, \mathrm{L}^p(Q; B_2))$. Analogously,

$$\mathrm{L}^p_{\text{weak}}(Q; \mathcal{B}(B, B_2)) = \mathrm{L}_{\text{weak}}(Q; \mathcal{B}(B, B_2)) \cap \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{L}^p(Q))) \tag{F.14}$$

isometrically.

Thus, if B_2 is reflexive (e.g., a Hilbert space) and $F \in \mathrm{L}^1_{\text{weak}}$, then $\mathop{\mathrlap{\int}W}_Q F\,d\mu \in \mathcal{B}(B, B_2)$ and $\|\mathop{\mathrlap{\int}W}_Q F\,d\mu\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathrm{L}^1_{\text{weak}}}$.

If B and B_2 are Hilbert spaces, then, of course, all asterisks can be removed.

If $B = \mathbf{K}$, then $\mathop{\mathrlap{\int}W}$ becomes the "weak*-integral" (or Gelfand Integral or Dunford Integral). The asterisk in (d) refers to Banach adjoints (since B and B_2 are only assumed to be Banach spaces); however, (d) is obviously true also for Hilbert adjoints (i.e., "F^{d}", not "F^{B}").

By Example F.1.10, the subspaces mentioned in (g) need not be closed in general.

Proof of Theorem F.2.1: The uniqueness is obvious in (a) and (b).

(a) It is obvious that $L : x \mapsto \int_Q Fx\,d\mu \in B_2$ satisfies $\|Lx\|_{B_2} \le \|Fx\|_{\mathrm{L}^1} \le \|F\|_{\mathrm{L}^1_{\mathrm{strong}}}\|x\|_B$.

(b) As in (a), we see that $L : x \mapsto (\Lambda \mapsto \int_Q \Lambda Fx\,d\mu \in \mathbf{K}) \in B_2^{**}$ satisfies $\|L\| \le \|F\|_{\mathrm{L}^1_{\mathrm{weak}}}$.

(c) This is obvious (see (B.18)).

(d) By Lemma F.1.5(e), $F^* \in \mathrm{L}^1_{\mathrm{weak}}$. Now $L^* := \left(⨍_Q F\,d\mu\right)^* \in \mathcal{B}(B_2^{***}, B^*) \subset \mathcal{B}(B_2^*, B^*)$. But

$$\int_Q (F^*\Lambda)x\,d\mu = \int_Q \Lambda Fx\,d\mu = (Lx)\Lambda = (L^*\Lambda)x \quad (x \in B,\ \Lambda \in B_2^*), \qquad \text{(F.15)}$$

hence $L^* = ⨍_Q F^*\,d\mu$.

(e) This follows from the definition (see (a) and (b)), because for $\Lambda \in B_3^*$ we have $\Lambda T \in B_2^*$.

(f) The above proofs will do mutatis mutandis (alternatively, use the fact that $\int$ is linear and bounded on L^1; in fact, (a) and (b) can then be deduced as special cases of (f); see (g)).

(One could also extend (d) by using the fact that $\mathcal{B}(B, \mathcal{B}(B_2, \mathrm{L}^1(Q; B_3))) = \mathcal{B}(B_2, \mathcal{B}(B, \mathrm{L}^1(Q; B_3)))$, isometrically.)

Recall from Remark A.3.22 that the linear (Banach) dual $B_2^* := B_2^{\mathrm{B}}$ is equipped with scalar multiplication $(\beta\Lambda)y := \beta(\Lambda y)$ $\alpha \in \mathbf{K}$, $\Lambda \in Y^*$, $y \in Y$), hence the isometry $B_2^{\mathrm{d}} \to B_2^*$, $y \mapsto \langle\cdot, y\rangle_{B_2}$ becomes conjugate-linear, cf. Remark A.3.22.

(g) Note first that the map $\mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2)) \mapsto (B \mapsto \mathrm{L}(Q; B_2))$ is well-defined (i.e., zero functions map to zero mapping), linear and one-to-one, hence an inclusion. Obviously, the $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathcal{B}$ norms are equal on $\mathrm{L}_{\mathrm{strong}}$, hence the strong claims of (g) hold. The weak claims can be proved analogously. $\square$

The rest of this section is dedicated for $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$.

Most of the above also holds for the Banach space $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$ (and the weak claims for $\mathcal{B}(B, \mathcal{B}(B_2, \mathrm{L}^p(Q))))$. Because $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$ contains $\mathrm{L}^p_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$ as a subspace, the following applies to $\mathrm{L}^p_{\mathrm{strong}}$ too:

Lemma F.2.2 (Strong convolutions) *Let $F \in \mathcal{B}(B, \mathrm{L}^p_\omega(\mathbf{R}^n; B_2))$ and $f \in \mathrm{L}^r_\omega(\mathbf{R}^n; B)$, $p, r \in [1, \infty]$, $n \in \mathbf{N}+1$, $\omega \in \mathbf{R}$ (we require that $\omega = 0$ if $n \ne 1$).*

(a) Assume that f is finite-dimensional and $p^{-1} + r^{-1} \ge 1$. Then

$$(F * f)(t) := \int_{\mathbf{R}^n} F(f(t-s))(s)\,ds = \int_{\mathbf{R}^n} F(f(s))(t-s)\,ds \in B_2 \qquad \text{(F.16)}$$

exists for a.e. $t \in \mathbf{R}^n$, and $\|F * f\|_{\mathrm{L}^p_\omega} \le \|F\|_{\mathcal{B}}\|f\|_{\mathrm{L}^1_\omega} \le \infty$.

(b) Thus, we can (and will) extend $*$ to $\mathcal{B}(B, \mathrm{L}^p_\omega(\mathbf{R}^n; B_2)) \times \mathrm{L}^1_\omega(\mathbf{R}^n; B) \to \mathrm{L}^p_\omega(\mathbf{R}^n; B_2)$.

(c) Moreover, $\tau^T(F*f)=(\tau^T F)*f=F*\tau^T f$ $(T\in\mathbf{R}^n)$ *(time-invariance), and if* $n=1$ *and* $\pi_-F,\pi_-f=0$*, then* $\pi_-(F*f)=0$ *(causality).*

(d1) **($\widehat{F}\in\mathrm{H}^\infty$)** *Assume that* $n=1=p$, $\mathbf{K}=\mathbf{C}$, $F=\pi_+F$ *and* $f=\pi_+f$*. Then*

$$\|\widehat{F}\|_{\mathcal{B}(\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}\leq\|F\|_{\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}_+;B_2))}. \tag{F.17}$$

Moreover, if $r=1$ *or* f *is finite-dimensional, then* $\widehat{F*f}=\widehat{F}\widehat{f}$ *on* $\mathbf{C}^+_\omega$*.*

(d2) Assume that $n=1=p$ *and* $\mathbf{K}=\mathbf{C}$*. Then* $\widehat{F}:\omega+i\mathbf{R}\to\mathcal{B}(B,B_2)$ *is strongly continuous and uniformly bounded:*

$$\|\widehat{F}(\omega+ir)\|_{\mathcal{B}(B,B_2)}\leq\|\widehat{F}\|_{\mathcal{B}(B,\mathcal{C}_b(\omega+i\mathbf{R};B_2))}\leq\|F\|_{\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))}\quad(r\in\mathbf{R}). \tag{F.18}$$

Moreover, if $r=1$ *[or* f *is finite-dimensional and* $r\leq 2$*], then* $\widehat{F*f}=\widehat{F}\widehat{f}$ *[a.e.] on* $\omega+i\mathbf{R}$*.*

(d3) Assume that $n=1=p$ *and that* B *and* B_2 *are complex Hilbert spaces. Then* $\|F\|_{\mathrm{TI}(B,B_2)}\leq\|F\|_{\mathcal{B}}$*. Moreover,* $F\in\mathrm{TIC}\Leftrightarrow F=\pi_+F$*, and (if* $\mathbf{K}=\mathbf{C}$*) the function* $\widehat{F}$ *coincides with the transform* $\widehat{F*}$ *of Theorems 6.2.1 and 3.1.3.*

(e) $\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}_+;B_2))\underset{c}{\subset}\mathcal{B}(B,\mathrm{L}^{p'}_{\omega'}(\mathbf{R}_+;B_2))$ *for* $p'\in[1,p]$, $\omega'>\omega$*.*

(f) The convolution also extends to $\mathcal{B}(B,\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+;B_2))\times\mathrm{L}^r_{\mathrm{loc}}(\mathbf{R}_+;B)\to\mathrm{L}^p_{\mathrm{loc}}(R_+;B_2)$*.*

(g) Parts (a)–(f) also hold with $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p_\omega(\mathbf{R}^n)))$ *in place of* $\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}^n;B_2))$ *(use* $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+)))$ *in (f),* B_2^{**} *in place of* B_2 *for the values of* $F*f$*, and* $\mathcal{B}(B,B_2^{**})$ *in place of* $\mathcal{B}(B,B_2)$ *for the values of* $\widehat{F}$*).*

In particular, $\mathcal{B}(B,\mathcal{B}(B_2^{\mathrm{B}},\mathrm{L}^1(\mathbf{R}_{[+]})))\subset\mathrm{TI[C]}(B,B_2)$ *when* B *and* B_2 *are Hilbert spaces.*

(h) Parts (a)–(f) also hold with $\mathcal{B}(B,\mathrm{e}^{\omega\cdot}\mathrm{MTI}_{B_2})$ *in place of* $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ *and* $\mathrm{e}^{\omega\cdot}\mathrm{MTI}_B$ *in place of* $\mathrm{L}^r_\omega(\mathbf{R}^n;B)$ *(for* $n=1$*).*

(Also the weak (cf. (g)) and multidimensional analogies holds.)

Note that (h) corresponds to class $\mathrm{SMTI}_\omega(B,B_2)$ of Definition 2.6.3, and that $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ is its closed subspace.

Here, of course, $\mathcal{L}F:=\widehat{F}\in\mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ is defined by

$$\widehat{F}(s)x:=\int_0^\infty\mathrm{e}^{-st}(Fx)(t)\,dt\in B_2\quad(x\in B,\ s\in\mathbf{C}^+); \tag{F.19}$$

analogously, $\widehat{F}\in\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2^{**}))$ for $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^1_\omega(\mathbf{R}_+)))$ (see (g)). In claim concerning $\mathcal{L}$, we assume that $\mathbf{K}=\mathbf{C}$.

We do not know whether the norm inequalities may have additional constants growing with the dimension of f (as in the proof of (a)) for $r>1$, thus preventing the generalization to infinite dimensions for such r; at least this is not the case in (d3) (where $p=1$ and $r=2$ but B and B_2 are required to be Hilbert spaces). An analogous phenomenom is illustrated in Example B.4.13.

Proof of Lemma F.2.2: W.l.o.g. we assume that $\omega = 0$.

(a) 1° $F * f$ *exists:* If $f = \phi x$, $\phi \in \mathrm{L}^r(\mathbf{R}^n)$, $x \in B$, then

$$\int_{\mathbf{R}^n} F(\phi(t-s)x)(s)\,ds = \int_{\mathbf{R}^n} \phi(t-s)(Fx)(s)\,ds = (Fx * \phi)(t), \tag{F.20}$$

which exists a.e. and $\|F * \phi x\|_p \le \|F\|_{\mathcal{B}} \|\phi x\|_1$, by Lemma D.1.7. By linearity, $(F * f)(t)$ exists for a.e. $t \in \mathbf{R}^n$ whenever f is finite-dimensional.

2° $\|F * \phi x\|_p \le \|F\|_{\mathcal{B}} \|\phi x\|_1$*:* Let $E_1, \ldots, E_k \subset \mathbf{R}^n$ be measurable and disjoint, and let $x_1, \ldots, x_k \in B$, $k \in \mathbf{N}+1$. Then

$$\|F * \sum_{j=1}^k \chi_{E_j} x_j\|_p \le \sum_{j=1}^k \|F\|_{\mathcal{B}} \|\chi_{E_j} x_j\|_1 = \|F\|_{\mathcal{B}} \|\sum_{j=1}^k \chi_{E_j} x_j\|_1. \tag{F.21}$$

By density (Theorem B.3.11 and Lemma A.3.10), $F*$ has an extension $T : \mathcal{B} \times \mathrm{L}^1 \to \mathrm{L}^p$ with $\|Tf\|_p \le \|F\|_{\mathcal{B}} \|f\|_1$. But as in (F.21), one verifies that if the values of f lie in a finite-dimensional subspace B' of B, then $\|F * f\|_p \le \|F\|_{\mathcal{B}} M_{B'} \|f\|_1$. By density, the two continuous extensions to k-dimensional $f \in \mathrm{L}^1$ must coincide, hence $\|F * f\|_p \le \|F\|_{\mathcal{B}} \|f\|_1$ for finite-dimensional f.

(b) This follows from (a).

(c) This extends from case of finite-dimensional f (see Lemma D.1.7), by continuity.

(d1) The norm bound on $\widehat{F}$ follows from Lemma D.1.11(a1'). By Lemma D.1.1(c), we have $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(B, B_2))$.

Let $\phi \in \mathrm{L}^r(\mathbf{R}^n)$, $x \in B$. Then

$$\mathcal{L}(F * \phi x) = \mathcal{L}(Fx * \phi) = \widehat{Fx}\widehat{\phi} = \widehat{F}\widehat{\phi}x \tag{F.22}$$

on $\mathbf{C}^+$, by Lemma D.1.11(c'). By linearity, the same applies to any finite-dimensional f in place of ϕx. If $r = 1$, then $\widehat{F * f} = \widehat{F}\widehat{f}$ holds for general $\widehat{f}$, by continuity.

(d2) The proof is analogous to that of (d1): $\|\widehat{F}(\omega + ir)\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathcal{B}}$ ($r \in \mathbf{R}$), by Theorem F.2.1(a). By Lemma D.1.11(a1)&(b), we have $\|\widehat{F}\|_{\mathcal{B}(B, \mathcal{C}_0(\omega + i\mathbf{R}; B_2))} \le \|F\|_{\mathcal{B}}$. As in (d1), we obtain $\widehat{F * f} = \widehat{F}\widehat{f}$ from Lemma D.1.11(c).

(d3) Theorem 3.1.3 (resp. 6.2.1) provides a unique continuous extension of $F*$ (restricted to finite-dimensional L^2 functions) to TI (resp. to TIC). Indeed, the operator defined by $\widehat{F}$ coincides with $F*$ for finite-dimensional functions, by (d2) (resp. by (d1)).

(e) This follows from Lemma D.1.4(b4).

(f) This follows from causality (as in the case of Lemma D.1.7 too).

(g) The above proofs apply mutatis mutandis. (Recall B_2^{B} from Remark A.3.22.)

(h) If $f = \sum_{j=1}^N x_j \delta_{T'_j}$, then

$$\|F * f\|_{\mathrm{MTI}} \le \sum_{j=1}^N \|Fx_k * \delta_{T'_j}\|_{\mathrm{MTI}} \le \sum_{j=1}^N \|x_k\| \|F\|_{\mathcal{B}} = \|F\|_{\mathcal{B}} \|f\|_{\mathrm{MTI}}. \tag{F.23}$$

By density, the map $F*$ (and the above inequality) has a unique continuous extension to measures of form $f=\sum_{j=1}^{\infty}x_j\delta_{T_j'}$; combine this with (a) to observe that any $f\in\mathrm{MTI}_B$ can be allowed in (a).

For (b)–(f), the proof is analogous to the original one (use Lemma D.1.12(c)&(c') in place of Lemma D.1.7), hence omitted. $\square$

The above "strong convolution" (of functions $Q\to\mathcal{B}(B,B_2)$ and $Q\to B$) can also be defined between two operator-valued functions ($Q\to\mathcal{B}(B_2,B_3)$ and $Q\to\mathcal{B}(B,B_2)$, or, slightly more generally, as follows:

Lemma F.2.3 ($G*F$) *Let $F\in\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}^n;B_2))$, $G\in\mathcal{B}(B_2,\mathrm{L}^p_\omega(\mathbf{R}^n;B_3))$.*

(a) $\|G*F\|_{\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}^n;B_3))}\le\|F\|_{\mathcal{B}}\|G\|_{\mathcal{B}}$, where $(G*F)(x)(t):=(G*Fx)(t)$. In particular, $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}^n;B))$ is a convolution Banach algebra.

(b) We have $\tau^T(G*F)=(\tau^T G)*F=G*\tau^T F$ ($T\in\mathbf{R}^n$), $(G*F)*f=G*(F*f)$, $(G*f)*g=G*(f*g)$ ($f,g\in\mathrm{L}^1_\omega(\mathbf{R}^n;*)$ with suitable $*$) and $n=1$ & $\pi_-F=0=\pi_-G\Rightarrow\pi_-(F*G)=0$.

(c) $(e^{s\cdot}G)*(e^{s\cdot}F)=e^{s\cdot}(F*G)$ for $s\in\mathbf{R}^n$.

(d1) $\widehat{G*F}=\widehat{G}\widehat{F}$ on $\mathbf{C}^+_\omega$ if $n=1$, $\mathbf{K}=\mathbf{C}$, $F=\pi_+F$ and $G=\pi_+G$.

(d2) $\widehat{G*F}=\widehat{G}\widehat{F}$ on $\omega+i\mathbf{R}$ if $n=1=p$ and $\mathbf{K}=\mathbf{C}$.

(e) If $F\in\mathrm{L}_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(B,B_2))$, $\|F(\cdot)\|_{\mathcal{B}(B,B_2)}\in\mathrm{L}^1_\omega$, $G\in\mathrm{L}_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(B_2,B_3))$ and $\|G(\cdot)\|_{\mathcal{B}(B_2,B_3)}\in\mathrm{L}^p_\omega$, then $F*G\in e^{-\omega\cdot}\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(B,B_3))$ and

$$\|\,\|G*F\|_{\mathcal{B}(B,B_3)}\|_{L^p_\omega}\le\|\,\|G(\cdot)\|_{\mathcal{B}(B_2,B_3)}\|_{L^p_\omega}\,\|\,\|F(\cdot)\|_{\mathcal{B}(B,B_2)}\|_{L^1_\omega}. \tag{F.24}$$

(h) Parts (a)–(e) also hold with $\mathcal{B}(B,e^{\omega\cdot}\mathrm{MTI}_{B_2})$ in place of $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ (for $n=1$).

(Also the multidimensional analogies holds.)

(i) Claims (a)–(h) also hold for $F\in\mathrm{L}^1(\mathbf{R}^n;B_2)$ (take $B:=\mathbf{K}$).

We are afraid that the weak analogies of the above claims do not hold.

Proof: (W.l.o.g. we assume that $\omega=0$.)

(a) Given $x\in B$, we have $Fx\in\mathrm{L}^1(\mathbf{R}^n;B_2)$, hence $\|G*Fx\|_p\le\|G\|_{\mathcal{B}}\|Fx\|_1\le\|G\|_{\mathcal{B}}\|F\|_{\mathcal{B}}\|x\|_B$.

(b)–(e) Analogously to (a), we obtain this from analogous claims on functions and from Lemma F.2.2 (one easily verifies that $(e^{s\cdot}G)*(e^{s\cdot}F)=e^{s\cdot}(F*G)$ holds when F and G are functions).

E.g., by Lemma D.1.7, we have $(G*f)*g=G*(f*g)$ for one-dimensional f and g, hence for finite-dimensional, hence for all $f,g\in\mathrm{L}^1$, by density and continuity. Consequently, for $f=\phi x$, $\phi\in\mathrm{L}^1(\mathbf{R}^n)$, we have

$$(G*F)*f=(G*F)x*\phi=(G*Fx)*\phi=G*(Fx*\phi)=G*(F*f). \tag{F.25}$$

By linearity and continuity, we may again allow for any $f\in\mathrm{L}^1$.

(h) The above proofs apply mutatis mutandis (use again Lemma D.1.12(c)).
(i) Note that $\mathcal{B}(\mathbf{K}, \mathrm{L}^1(\mathbf{R}^n; B_2)) = \mathrm{L}^1(\mathbf{R}^n; B_2)$ (with equal norms). □

The above "strong convolution of operator-valued functions" can be generalized to $\mathrm{L}^1_{\mathrm{loc}}$ too, if the supports of the functions are bounded to the left:

Lemma F.2.4 *Let* $F \in \mathcal{B}(B, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B_2))$, $G \in \mathcal{B}(B_2, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B_3))$, $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B)$.

The claims of Lemma F.2.3 can be generalized to this situation, and $\mathcal{B}(B, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B))$ *is a convolution algebra.*

Also the extension $X := \mathcal{B}(B, B\delta_0 + \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B))$ *is a convolution algebra with unit* $I\delta_0$*, where* $\delta_0 * f := f$ *for all* f*. Moreover, if* $L \in \mathcal{B}(B)$ *and* $B_2 = B$*, then* $L\delta_0 + F$ *is invertible in* X *iff* $L \in \mathcal{GB}(B)$.

Proof: 1° For each $T > 0$, we have $\pi_{[0,T)}(F * G) = \pi_{[0,T)}(\pi_{[0,T)}F * \pi_{[0,T)}G)$, by causality, hence we can apply Lemma F.2.3. Obviously, X is a convolution algebra (with $(L_1\delta_0 + F_1) * (L_2\delta_0 + F_2) = L_1L_2\delta_0 + L_1F_2 + F_1L_2 + F_1 * F_2$).

2° Obviously, $L \in \mathcal{GB}(B)$ is necessary for the invertibility. Sufficiency follows as in the finite-dimensional case (see, e.g., Theorem 2.3.1 on p 42 of [GLS]). □

We leave it to the reader to extend the above results for $\mathcal{B}(B, M(\mathbf{R}^n; B_2))$, where M refers to (uniform) measures; see [GLS], pp. 121–127 for the finite-dimensional case.

(See the notes on p. 1023.)

F.3 Weak Laplace transform ($\mathcal{L}_w$)

To generalize is to be an idiot.
— William Blake (1757–1827)

In this section, we define and study $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ spaces and the Laplace transform of strongly or weakly measurable functions.

Let U and Y be complex Hilbert spaces, and B, B_2 and B_3 complex Banach spaces.

Definition F.3.1 ($\mathrm{H}^p_{\mathrm{weak}}$ and the weak Laplace transform $\widehat{F}$) *Let* $1 \le p \le \infty$, $\omega \in \mathbf{R}$.

By $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *we denote the space of functions* $F : \mathbf{C}^+_\omega \to \mathcal{B}(B,B_2))$ *having a finite norm*

$$\|F\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))} := \sup_{\|\Lambda\|_{B_2^*}\le 1,\, \|x\|_B \le 1} \|\Lambda F x\|_{\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}. \tag{F.26}$$

The spaces $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{D}_r;\mathcal{B}(B,B_2))$, $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2))$ *and* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{D}_r;\mathcal{B}(B,B_2))$ *are defined analogously. We also set* $\mathrm{H}^p_{\mathrm{strong},\omega}(*) := \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;*)$, $\mathrm{H}^p_{\mathrm{strong},\infty} := \cup_{\omega\in\mathbf{R}}\mathrm{H}^p_{\mathrm{strong},\omega}$ *(*$p \in [1,\infty]$*).*

Let $F : \mathbf{R} \to \mathcal{B}(B,B_2)$ *and* $s \in \mathbf{C}$. *If* $\mathrm{e}^{-s\cdot}F \in \mathrm{L}^1_{\mathrm{weak}}$, *then we set* $\widehat{F}(s) := (\mathcal{L}_w F)(s) := \mathrm{w}\!\!\int_{\mathbf{R}} \mathrm{e}^{-st}F(t)\,dt$. *The function* $\mathcal{L}_w F$ *is the* (weak) Laplace transform *of* F. *The* strong Laplace transform $\mathcal{L}_s$ *is defined analogously.*

Obviously, $\mathcal{L}_w F$ is an extension of $\mathcal{L}F$. One easily verifies that $\|\cdot\|_{\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))} := \sup_{\|x\|_B\le 1}\|Fx\|_{\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$ and $\|\cdot\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$ are norms on corresponding spaces; in Lemma F.3.2(c) we shall show that $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ are Banach spaces.

Next we list the basic relations between different H^p_* spaces:

Lemma F.3.2 ($\mathrm{H}^p \subset \mathrm{H}^p_{\mathrm{strong}} \subset \mathrm{H}^p_{\mathrm{weak}} \subset \mathrm{H}^\infty$) *Let* $\omega \in \mathbf{R}$, $\varepsilon > 0$, $1 \le p_1 \le p \le p_2 \le \infty$.

- *(a1)* $\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \subset \mathrm{H}^\infty(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(B,B_2))$, *continuously. Moreover,* $\mathrm{H}^\infty_{\mathrm{weak}} = \mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$.
- *(a2)* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \underset{c}{\subset} \mathrm{H}^{p_2}_{\mathrm{strong}}(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(B,B_2))$,
 and $\mathrm{H}^{p_1}_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \cap \mathrm{H}^{p_2}_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$.
 These claims also hold with "weak" or void in place of "strong".
- *(b) Let* $G \in \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$. *Then* $\|G(s)\|_{\mathcal{B}(B,B_2)} \le (\pi(\mathrm{Re}\,s-\omega))^{-1/p}\,\|G\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$.
- *(c) The spaces* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *and* $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *are Banach spaces.*
- *(d)* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) = \mathcal{B}(B,\mathrm{H}^p(\mathbf{C}^+_\omega;B_2))$ *and* $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2^{**})) = \mathcal{B}(B,\mathcal{B}(B_2^{\mathrm{B}},\mathrm{H}^p(\mathbf{C}^+_\omega)))$.

(e) **($\dim B < \infty$)** *If* $\dim B < \infty$*, then* $\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) = \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *(with equivalent norms). If* $\dim B_2 < \infty$*, then* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) = \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *(with equivalent norms).*

Note from Theorem 6.2.1 that $\mathrm{TIC}_\omega(U,Y)$ operators correspond to $\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ functions through an isometric isomorphism onto. Thus, all $\mathrm{H}^p_{\mathrm{weak}}$ functions over Hilbert spaces are TIC_∞ operators, by Lemma F.3.2(a).

Recall from Remark A.3.22 that $B_2^{\mathrm{B}} = B_2^*$ means the Banach dual of B_2, not the "sesquilinear" ("Hilbert") dual B_2^{d} (which is usually denoted by B_2^* if B_2 is a Hilbert space).

Proof: (a1) By Lemma D.1.1(c), we have $\mathrm{H}^\infty_{\mathrm{weak}} = \mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$. The embedding $\mathrm{H}^p \underset{c}{\subset} \mathrm{H}^p_{\mathrm{strong}} \underset{c}{\subset} \mathrm{H}^p_{\mathrm{weak}}$ follows from Lemma F.1.5(c1) and Lemma D.1.2(b1); the embedding $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \underset{c}{\subset} \mathrm{H}^\infty_{\mathrm{weak}}(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(B,B_2))$ follows from Lemma D.1.4(d).

(a2) By (a1), we may replace $\mathrm{H}^{p_2}_{\mathrm{weak}}$ by H^∞ in the second claim. By Lemma D.1.4(d), then the claims hold for void in place of "strong". Apply this for each $x_0 \in B$ [and $\Lambda \in B_2^*$] to obtain the strong [weak] claim (note that we can use same embedding bounds as for the uniform case).

(b) Case $p = \infty$ follows from (a), so we assume that $p < \infty$. If G is scalar, then this holds, by (6.4.3) of [HP] (because then the nonstandard assumption (iii) of Definition 6.4.1 of [HP] is redundant, by Theorem 3.3.1(a3)). Thus, we can replace $G(s)$ by $\Lambda G(s)x$ for any $x \in B$ and $\Lambda \in B_2^*$ with $\|x\|,\|\Lambda\| \le 1$ (because then $\|\Lambda G x\|_{\mathrm{H}^p} \le \|G\|_{\mathrm{H}^p_{\mathrm{weak}}}$); the general inequality follows.

(c) Let $\{f_n\} \subset \mathrm{H}^p_{\mathrm{weak}}$ be a Cauchy sequence. By (a), for each $\alpha > \omega$ there is f_α s.t. $f_n \to f_\alpha$ in $\mathrm{H}^\infty(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$; let $f \in \mathrm{H}(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$ be the pointwise limit function. Given $\Lambda \in B_2^*$ and $x \in B$, the Cauchy-sequence $\Lambda f_n x$ converges in H^p; the limit is equal its pointwise limit $\Lambda f x$, hence $f \in \mathrm{H}^p_{\mathrm{weak}}$ and $f_n \to f$ in $\mathrm{H}^p_{\mathrm{weak}}$.

The proof for $\mathrm{H}^p_{\mathrm{strong}}$ is analogous, and it can be obtained from (d) too (the same applies to $\mathrm{H}^p_{\mathrm{weak}}$, though not as obviously).

(d) The left-hand-sides are obviously (isometrically) subspaces of right-hand-sides. If $p = \infty$, then the converses follow from Lemma D.1.1(d). Assume then that $p < \infty$.

By (a), we have $\mathcal{B}(B,\mathrm{H}^p(\mathbf{C}^+_\omega;B_2)) \underset{c}{\subset} \mathcal{B}(B,\mathrm{H}^\infty(\mathbf{C}^+_\alpha;B_2)) = \mathrm{H}^\infty(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$, for all $\alpha > \omega$. Thus, any $F \in \mathcal{B}(B,\mathrm{H}^p(\mathbf{C}^+_\omega;B_2))$ takes form of a function $F \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$. As noted at the beginning of the proof, the two norms on F are equal, hence $F \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$.

For $F \in \mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{H}^p(\mathbf{C}^+_\omega)))$, the proof is analogous.

(e) This follows from Lemma F.1.5(f). $\square$

The following is rather obvious:

Lemma F.3.3 *Let* $F : \mathbf{R}_+ \to \mathcal{B}(B,B_2)$*,* $f : \Omega \to \mathcal{B}(B,B_2)$*,* $\Omega = \mathbf{C}^+_\omega$ *or* $\Omega = \partial r\mathbf{D}$*, and* $1 \le p \le \infty$*. Then*

(a1) We have $f \in \mathrm{H}^p_{\mathrm{strong}}(\Omega;\mathcal{B}(B,B_2))$ *iff* $fx \in \mathrm{H}^p_{\mathrm{strong}}(\Omega;B_2)$ *for all* $x \in B$*.*

(a2) We have $f \in \mathrm{H}^p_{\text{weak}}(\Omega;\mathcal{B}(B,B_2))$ *iff* $\Lambda f x \in \mathrm{H}^p_{\text{strong}}(\Omega;B_2)$ *for all* $x \in B$ *and* $\Lambda \in B_2^*$.

(b) If $T \in \mathcal{B}(B_2,B_3)$ *and* $x \in B$, *then* $T\mathcal{L}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}}T$, $T\mathcal{L}_{\mathrm{w}} = \mathcal{L}_{\mathrm{w}}T$, *and* $T\widehat{F}x = \widehat{TFx}$ *wherever* $\widehat{F}$ *exists.*

(c) We have $\widehat{F^{\mathrm{B}}}(s) = F(s)^{\mathrm{B}}$ *and* $\widehat{F^{\mathrm{d}}}(s) = F(\overline{s})^{\mathrm{d}}$ *for any* s *for which either transform is defined if* B *and* B_2 *are reflexive (this applies to* $\mathcal{L}_{\mathrm{w}}$ *and* $\mathcal{L}$*).*

(Recall that "F^*" refers to "F^{B}" in Banach and to "F^{d}" in Hilbert space settings.)

Proof: We get (a1)&(a2) from Lemma F.1.6, (b) from Theorem F.2.1(e), and (c) from Lemma F.2.1(d). □

Functions in $\mathrm{L}^1_{\text{weak}}$ have bounded holomorphic weak Laplace transforms on the right half-plane:

Lemma F.3.4 ($\widehat{\mathbf{L}^1_{\text{weak}}} \subset \mathbf{H}^\infty$) *Let* $F : \mathbf{R}_+ \to \mathcal{B}(B,B_2)$, $s_0 \in \mathbf{C}$, $r := \operatorname{Re} s_0$, $\varepsilon > 0$, $1 \le p \le \infty$, $p^{-1} + q^{-1} = 1$.

(a1) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^1_{\text{weak}}$, *then* $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le \|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^1_{\text{weak}}}$.

(a2) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^1_{\text{strong}}$, *then* $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2))} \le \|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^1_{\text{strong}}}$.

(b1) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^p_{\text{weak}}$, *then* $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**})) \cap \mathrm{H}^\infty(\mathbf{C}^+_{r+\varepsilon}\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le M_{\varepsilon,p}\|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^p_{\text{weak}}}$.

(b2) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^p_{\text{strong}}$, *then* $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**})) \cap \mathrm{H}^\infty(\mathbf{C}^+_{r+\varepsilon}\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le M_{\varepsilon,p}\|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^p_{\text{strong}}}$.

(c1) If $p \le 2$ *and* $\mathrm{e}^{-r\cdot}F \in \mathrm{L}^p_{\text{weak}}(\mathbf{R}_+;\mathcal{B}(B,B_2))$, *then* $\widehat{F} \in \mathrm{H}^q_{\text{weak}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))$ *and* $\|\widehat{F}\|_{\mathrm{H}^q_{\text{weak}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le (2\pi)^{1/q}\|\mathrm{e}^{-r\cdot}F\|_{\mathrm{L}^p_{\text{weak}}}$, *with equality for* $p = 2$.

(c2) If $p \le 2$ *and* $\mathrm{e}^{-r\cdot}F \in \mathrm{L}^p_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(B,Y))$, *then* $\widehat{F} \in \mathrm{H}^q_{\text{strong}}(\mathbf{C}^+_r;\mathcal{B}(B,Y))$ *and* $\|\widehat{F}\|_{\mathrm{H}^q_{\text{strong}}(\mathbf{C}^+_r;\mathcal{B}(B,Y))} \le (2\pi)^{1/q}\|\mathrm{e}^{-r\cdot}F\|_{\mathrm{L}^q_{\text{strong}}}$, *with equality for* $p = 2$.

(d) The (strong/weak) Laplace transform is an isometric isomorphism modulo $\sqrt{2\pi}$ *of* $\mathrm{L}^2_\omega(\mathbf{R}^+;Y))$ *onto* $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;Y)$, *of* $\mathcal{B}(U,\mathrm{L}^2_\omega(\mathbf{R}^+;Y))$ *onto* $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) = \mathcal{B}(U,\mathrm{H}^2(\mathbf{C}^+_\omega;Y))$ *and of* $\mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}},\mathrm{L}^2_\omega(\mathbf{R}^+)))$ *onto* $\mathrm{H}^2_{\text{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) = \mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}},\mathrm{H}^2(\mathbf{C}^+_\omega)))$.

Recall that we may replace B_2^{**} by B_2 above if B_2 is reflexive (e.g., a Hilbert space).

From (d) we conclude that the inverse transform of $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$ covers in general more than $\mathrm{L}^2_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$, as shown in Example F.3.6.

Proof: (a2)&(b2)&(c2) Modify the proofs of (a1), (b1) and (c1) accordingly.

(a1) Let $s \in \mathbf{C}_r^+$. We have $e^{-st}F(t) = e^{-(s-s_0)t}e^{-s_0t}F(t)$, and $|e^{-(s-s_0)t}| \le 1$, hence $\|e^{-s\cdot}F\|_{L^1_{\text{weak}}} \le \|e^{-s_0\cdot}F\|_{L^1_{\text{weak}}}$. By Theorem F.2.1, it follows that $\|\widehat{F}(s)\| \le \|e^{-s_0\cdot}F\|_{L^1_{\text{strong}}}$. By Lemma D.1.10(a) and Lemma D.1.1(b), $\widehat{F} \in \mathrm{H}(\mathbf{C}_r^+;\mathcal{B}(B,B_2))$.

(b1) Set $M_{\varepsilon,p} := \|e^{-\varepsilon\cdot}\|_q$ and use (a1) (note that $\|e^{-s_0\cdot}e^{-\varepsilon\cdot}F\|_{L^1_{\text{weak}}}\| \le M_{\varepsilon,p}\|e^{-s_0\cdot}F\|_{L^p_{\text{weak}}}$).

(c1) Now $\widehat{F} \in \mathrm{H}(\mathbf{C}_r^+;\mathcal{B}(B,B_2^{**}))$, by (b1). For any $\Lambda \in B_2^*$, $x \in B$, we have $\Lambda\widehat{F}x = \mathcal{L}\Lambda F x$, by (d), and

$$\|\Lambda\widehat{F}x\|_{\mathrm{H}^q(\mathbf{C}_r^+)} \le (2\pi)^{1/q}\|\Lambda F x\|_{L^p_r}, \tag{F.27}$$

by Theorem E.1.7, with equality for $p = 2$.

Taking the supremum on both sides of (F.27) over $\|\Lambda\|, \|x\| \le 1$, we obtain that $\|\widehat{F}\|_{\mathrm{H}^q_{\text{weak}}(\mathbf{C}_r^+;\mathcal{B}(B,B_2^{**}))} \le (2\pi)^{1/q}\|e^{-r\cdot}\widehat{F}\|_{L^p_{\text{weak}}(\mathbf{R}_+;\mathcal{B}(B,B_2^{**}))}$, with equality for $p = 2$.

(Thus, $\mathcal{L}_w$ is an isometric isomorphism of $\pi_+ L^2_{\text{weak}}$ onto a closed subspace of $\mathrm{H}^2_{\text{weak}}$. We believe that this is a proper subspace of $\mathrm{H}^2_{\text{weak}}$; in particular, we believe that for any infinite-dimensional U and Y, some $\mathrm{H}^2_{\text{weak}}$ functions do not have boundary functions (although they have "boundary operators" $B \times B_2^* \to L^2(\omega + i\mathbf{R})$; the situation for L^2_{strong} seems to be analogous.)

(d) The identities were given in Lemma F.3.2(d). Because $\mathcal{L}$ is an isometric isomorphism times $\sqrt{2\pi}$ of L^2_ω onto H^2_ω, by Lemma D.1.15, it is also an isometric isomorphism times $\sqrt{2\pi}$ (onto) $\mathcal{B}(U, L^2_\omega(\mathbf{R}^+;Y)) \to \mathcal{B}(U, \mathrm{H}^2(\mathbf{C}_\omega^+;Y))$ and $\mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}}, L^2_\omega(\mathbf{R}^+))) \to \mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}},\mathrm{H}^2(\mathbf{C}_\omega^+)))$. □

The multiplication of elements of different H^p spaces works in the same way as that of L^p spaces:

Lemma F.3.5 ($\mathbf{H}^r_{\text{weak}} \cdot \mathbf{H}^p_{\text{strong}} \subset \mathbf{H}^p_{\text{strong},\varepsilon}$) *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Let $F : \mathbf{C}^+ \to \mathcal{B}(B,B_2)$, $G : \mathbf{C}^+ \to \mathcal{B}(B_2,B_3)$, $f : \mathbf{C}^+ \to B$. Then all claims (i.e., (a1)–(a3")) of Lemma F.1.8 hold with* H *in place of* L *and $G^*(\overline{\cdot})$ in place of G^*.*

Moreover, if $f \in \mathrm{H}^p_{\text{strong}}(\mathbf{C}_\omega^+;\mathcal{B}(B,B_2))$ and $g \in \mathrm{H}^r_{\text{weak}}(\mathbf{C}_\alpha^+;\mathcal{B}(B_2,B_3))$, where $r \in [1,\infty]$ and $\omega > \alpha$, then $gf \in \mathrm{H}^p_{\text{strong}}(\mathbf{C}_\omega^+;\mathcal{B}(B,B_3))$ and $\|gf\| \le M\|g\|\|f\|$; the same holds with H^p in place of $\mathrm{H}^p_{\text{strong}}$.

Naturally, by shifting one obtains an analogous $\mathbf{C}_\omega^+$ claim.

Proof: The product functions are holomorphic, by Lemma D.1.2(b3). The claims on norms follow from Lemma F.1.8.

The last claim follows from the fact that $g \in \mathrm{H}^\infty(\mathbf{C}_\omega^+;\mathcal{B}(B_2,B_3))$, by Lemma F.3.2(a). □

As noted above, the Laplace transform $L^2_{\text{strong}} \to \mathrm{H}^2_{\text{strong}}$ is not onto in general:

Example F.3.6 [$\widehat{\mathbf{L}^2_{\text{strong}}} \ne \mathbf{H}^2_{\text{strong}}$] Even for $U := \ell^2(\mathbf{N})$, there are $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U))$ functions and $L^2_{\text{strong}}(i\mathbf{R};\mathcal{B}(U))$ functions that are not Laplace transforms of any $L^2_{\text{strong}}(\mathbf{R};\mathcal{B}(U))$ functions, not even of any $L^2_{\text{weak}}(\mathbf{R};\mathcal{B}(U))$

functions (nor of any other $\mathcal{B}(U)$-valued functions, see 5° of the proof of Example F.1.10).

In fact, there is $\widehat{F} \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}_{\text{strong}}(i\mathbf{R};\mathcal{B}(U))$ s.t. $\widehat{F}(\overline{\cdot})^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}_{\text{strong}}(i\mathbf{R};\mathcal{B}(U))$ but $\widehat{\mathbb{D}}$ is not the Laplace (or Fourier) transform of any $F : \mathbf{R}_+ \to \mathcal{B}(U)$ (nor of any $F(\mathbf{R}_+ \setminus N) \to \mathcal{B}(U)$ where N is a null set). $\triangleleft$

Proof: Let $F_n \in \mathrm{L}^2_{\text{strong}}([0,1);U)$ ($n \in \mathbf{N}$), $F \in \mathcal{B}(U,\mathrm{L}^2([0,1);U))$ be as in Example F.1.10.

Because $\pi_{[0,1)}\mathrm{L}^2 \underset{c}{\subset} \mathrm{L}^2_\omega$ and $\mathcal{L}\mathrm{L}^2_\omega = \mathrm{H}^2_\omega$, we have $\widehat{F}_n,\widehat{F} \in \mathrm{H}(\mathbf{C};\mathcal{B}(U)) \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for all $\omega \in \mathbf{R}$, and $\widehat{F}_n \to \widehat{F}$ in each $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U))$. The claims on $\widehat{F}(\overline{\cdot})^*$ follow from duality and the fact that $\widehat{F}_n(\overline{\cdot})^* = \widehat{F^*_n}$ for each $n \in \mathbf{N}$.

Assume then that $\widehat{F} = \mathcal{L}G$ for some $G \in \mathrm{L}^2_{\text{strong}}(\mathbf{R};\mathcal{B}(U))$ (if $G : (\mathbf{R}_+ \setminus N) \to \mathcal{B}(U)$ where N is a null set, for some reasonable sense so that $\widehat{F}u_0 = \widehat{Gu_0}$ for all $u_0 \in U$, then Gu_0 equals the inverse transform of $\widehat{F}u_0$, hence then $Gu_0 \in \mathrm{L}^2(\mathbf{R}_+;U)$; consequently, then $G \in \mathrm{L}^2_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(U))$).

Given u_0, we have $\widehat{F}_n u_0 \to \widehat{F}u_0$ in H^2, hence $F_n u_0 \to Gu_0$ in L^2, as $n \to \infty$, hence $F_n \to G$ in $\mathrm{L}^2_{\text{strong}}(\mathbf{R};\mathcal{B}(U))$, which is a contradiction, by Example F.1.10. Thus, $\widehat{F}$ is not the transform of any $G : \mathbf{R}_+ \to \mathcal{B}(U)$. $\square$

We finish this appendix by a more technical lemma. A main observation of the lemma is that $\widehat{\mathbb{D}}(\overline{\cdot})^* \in \mathrm{H}^2_{\text{strong}}$, then $\mathbb{D}$ is smoothing (see (b2) below).

Lemma F.3.7 *Let $\omega \in \mathbf{R}$, $\varepsilon > 0$, $1 \le p \le \infty$.*

(a1) **(Inverse transform of $\mathbf{H^1}$)** *Let $g \in \mathrm{H}^1(\mathbf{C}^+_\omega;B) \cap \mathrm{L}^1(\omega + i\mathbf{R};B)$ and $\gamma \ge \omega$. Then $g = \widehat{f}$, where*

$$f(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\int_{\mathbf{R}} \mathrm{e}^{tir} g(\gamma + ir)\,dr \in B \quad (t \in \mathbf{R}). \tag{F.28}$$

Moreover, $\mathrm{e}^{-\omega\cdot} f \in \mathcal{C}_0(\mathbf{R};B)$, $\pi_- f = 0$, *and* $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} f\|_B \le \|g\|_{\mathrm{H}^1_\omega}/2\pi$.

(a1') Let $g \in \mathrm{H}^1(\mathbf{C}^+_\omega;B)$ and $\gamma > \omega$. Then $g = \widehat{f}$, where f is defined by (F.28). Moreover, $\mathrm{e}^{-\omega\cdot} f \in \mathcal{C}_\mathrm{b}(\mathbf{R};B)$, $\pi_- f = 0$, *and* $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} f\|_B \le \|g\|_{\mathrm{H}^1_\omega}/2\pi$.

(a2) Let $G \in \mathrm{H}^1_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ and $\gamma > \omega$. Then $G = \widehat{F}$, where

$$F(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\,\mathrm{s}\!\!\int_{\mathbf{R}} \mathrm{e}^{tir} G(\gamma + ir)\,dr \quad (t \in \mathbf{R}). \tag{F.29}$$

Moreover, $\pi_- F = 0$, $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} F\|_B \le \|G(\cdot - \omega)\|_{\mathrm{H}^1_{\text{strong}}}/2\pi$, *and* $\mathrm{e}^{-\omega\cdot} fx \in \mathcal{C}_\mathrm{b}(\mathbf{R},B_2)$ *for all $x \in B$.*

(a3) Let $G \in \mathrm{H}^1_{\text{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ and $\gamma > \omega$. Then $G = \widehat{F}$, where

$$F(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\,\mathrm{w}\!\!\int_{\mathbf{R}} \mathrm{e}^{tir} G(\gamma + ir)\,dr \quad (t \in \mathbf{R}). \tag{F.30}$$

Moreover, $\pi_- F = 0$, $\sup_{\mathbf{R}} \|e^{-\omega\cdot} F\|_B \le \|G(\cdot-\omega)\|_{H^1_{\mathrm{weak}}}/2\pi$, and $e^{-\omega\cdot}\Lambda f x \in C_0(\mathbf{R})$ for all $x \in B$ and $\Lambda \in B_2^*$.

(b1) **(Inverse transform of $\mathbf{H^2}$)** *Let $\widehat{\mathbb{D}} \in H^2(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. Then $\mathbb{D} : L^2_\omega(\mathbf{R}_+;U) \ni u \mapsto \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$ satisfies $e^{-\omega\cdot}\mathbb{D}u \in C_0(\mathbf{R};Y)$, and $\sup_{\mathbf{R}} \|e^{-\omega\cdot}\mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\cdot)\|_{H^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{L^2_\omega}$ for all $u \in L^2_\omega(\mathbf{R}_+;U)$ (for all $u \in L^2_\omega(\mathbf{R};U)$ if we extend $\mathbb{D}$ onto L^2_ω; this extension coincides with the operator $\mathbb{D} \in \mathrm{TIC}_\omega$ (see Theorem 6.2.1) if, in addition, $\widehat{\mathbb{D}} \in H^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$).*

(b2) Let $\widehat{\mathbb{D}}(\bar{\cdot})^* \in H^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$. Then the map $\mathbb{D} : u \mapsto \mathcal{L}_w^{-1}\widehat{\mathbb{D}}\widehat{u}$ satisfies $e^{-\omega\cdot}\mathbb{D}u \in C_b(\mathbf{R};Y)$ and $\sup_{\mathbf{R}} \|e^{-\omega\cdot}\mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{H^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{L^2_\omega}$ for all $u \in L^2_\omega(\mathbf{R}_+;U)$ (for all $u \in L^2_\omega(\mathbf{R};U)$ if we extend $\mathbb{D}$ onto L^2_ω; this extension coincides with the operator $\mathbb{D} \in \mathrm{TIC}_\omega$ (see Theorem 6.2.1) if, in addition, $\widehat{\mathbb{D}} \in H^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$).

As stated in Theorem 3.3.1(a1), expression $g \in H^1 \cap L^1$ means that $g \in H^1$ has a boundary function in L^1. By (a1), any element of $H^1_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(B,B_2))$ is the (strong) Laplace transform of a bounded and weakly continuous function $\mathbf{R}_+ \to \mathcal{B}(B,B_2)$.

By (b2), an I/O map with a transfer function in "dual" H^2_{strong} produces bounded and continuous output. This fact will be used in Theorem 6.9.1(b) to show that such maps are exactly the I/O maps with a WPLS realization with a bounded output operator.

We will see in Theorem 3.3.1(d2) that if $\widehat{\mathbb{D}}(\bar{\cdot})^* \in H^2_{\mathrm{strong}} \cap H^\infty$, then, actually, $\widehat{\mathbb{D}} \in L^2_{\mathrm{strong}} \cap L^\infty_{\mathrm{strong}}$, so that the map $u \mapsto \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$ defined in (b2) can be defined with the same formula (but with the (Fourier) transforms considered on $\omega + i\mathbf{R}$ only) for any $u \in L^2_\omega(\mathbf{R};U)$.

Proof of Lemma F.3.7: We take $\omega = 0$ w.l.o.g. (replace g by $g(\omega + \cdot)$ for the general case; see also Remark 2.1.6).

(a1) By the usual contour integration argument (use (6.4.4) of [HP] and note that $e^{t(\gamma+ir)}$ is bounded for bounded γ), expression (F.28) is independent of $\gamma > 0$ for any $t \in \mathbf{R}$. But (F.28) is continuous (from the right) at $\gamma = 0$, hence we may take $\gamma = 0$ too to obtain the same f. With $\gamma = 0$ we obtain that $\|f\|_\infty \le \|g\|_{H^1_\omega}/2\pi$. By Lemma D.1.11(a1)&(b), we have $f \in C_0(\mathbf{R};B)$.

By p. 230 of [HP], $g = \mathcal{L}\pi_+ f$. Let $T > 0$ and set $f_T := \tau^{-T}\pi_+ f \in L^\infty(\mathbf{R}_+;B)$. Then $\widehat{f_T} = e^{-T\cdot} g \in H^1$, so that $\widehat{f_T} = \mathcal{L}\pi_+ \widetilde{f}_T$, by the above, where

$$\widetilde{f}_T(t) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{tir}\widehat{f_T}(ir)\,dr = f(t-T) \quad (t \in \mathbf{R}). \tag{F.31}$$

By uniqueness (Lemma D.1.10(b)), $f_T = \pi_+ f(\cdot - T)$, hence $\pi_{[-T,0)} f = 0$. Because $T > 0$ was arbitrary, we have $\pi_- f = 0$.

(a1') Apply (a1) with $\omega + \varepsilon$ in place of ω for some $\varepsilon > 0$.

(a2) Apply (a1') for Gx ($x \in B$).

(a3) Apply (a1) for ΛGx ($x \in B$, $\Lambda \in B_2^*$) (note that $\Lambda Gx \in H^1 \cap L^1$, by Theorem 3.3.1(a1)).

(b1) 1° *Case* $u \in \pi_+\mathrm{L}^2$*:* Let $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, so that $\widehat{y} := \widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^1(\mathbf{C}^+;Y)$. Then

$$\|\widehat{y}\|_{\mathrm{L}^1(i\mathbf{R};Y)} = \|\widehat{y}\|_{\mathrm{L}^1(i\mathbf{R};Y)} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|\widehat{u}\|_{\mathrm{H}^2} \le 2\pi\|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|u\|_2, \tag{F.32}$$

by Theorem 3.3.1(a). Thus, $\sup_{\mathbf{R}}\|y\|_Y \le \|\widehat{y}\|_{\mathrm{H}^1}/2\pi = \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|u\|_2$ and $y \in \mathcal{C}_0(\mathbf{R};Y)$, by (a1).

2° *Case* $u \in \mathrm{L}^2$*:* By repeating the proof of Lemma 2.1.3, we can show that each such $\mathbb{D}$ extends to a time-invariant ($\tau^t\mathbb{D} = \mathbb{D}\tau^t$ for all $t \in \mathbf{R}$) operator $\mathbb{D}_0 \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathcal{C}_0(\mathbf{R};Y))$ with the same norm. By continuity, this extension satisfies the requirements of (c1).

3° *Case* $\mathbb{D} \in \mathrm{H}^\infty$*:* Let $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ too, so that $\mathbb{D} \in \mathrm{TIC}(U,Y)$, by Theorem 6.2.1. By time-invariance, $\mathbb{D}u = \mathbb{D}_0 u$ for any $u \in \mathrm{L}^2([T,+\infty);U)$, $T \in \mathbf{R}$. If $u_n \to u$ in L^2, then $\mathbb{D}u_n$ converges in both L^2 and $\mathcal{C}_0$, so that the limits must be equal (a.e.), by Theorem B.3.2, hence $\mathbb{D} = \mathbb{D}_0$ on L^2.

(b2) By Lemma F.3.5(a3”), we have $\|\widehat{\mathbb{D}}\widehat{u}\|_{\mathrm{H}^1_{\text{weak}}} \le \|\widehat{\mathbb{D}}(\overline{\cdot})^*\|_{\mathrm{H}^2_{\text{strong}}}\|\widehat{u}\|_{\mathrm{H}^2}$ for $u \in \mathrm{L}^2(\mathbf{R}_+;U)$. Consequently, $\Lambda\mathbb{D}u \in \mathcal{C}_0(\mathbf{R})$ for all $\Lambda \in Y^*$, and the norm estimate holds (cf. the proof of (b1)), by (a3).

If $u \in \mathcal{C}_c^\infty$, then $\mathbb{D}u \in \mathrm{W}^{1,2}_1 \subset \mathcal{C}$ (because $\mathbb{D} \in \mathrm{TIC}_1$), by Theorem 3.1.5 (alternatively $\mathbb{D}u \subset \mathcal{F}^{-1}[\mathrm{L}^1(1+i\mathbf{R};Y)] \subset e^{1\cdot}\mathcal{C}_0 \subset \mathcal{C}$, by Lemma D.1.11(e2)&(e1)). For general u, we can take $u_n \in \mathcal{C}^\infty$ ($n \in \mathbf{N}$) s.t. $u_n \to u$ in L^2; it follows from the norm estimate that $\mathbb{D}u_n$ converges in $\mathcal{C}_b$ and $\mathbb{D}u_n$ pointwise to $\mathbb{D}u$, hence $\mathbb{D}u \in \mathcal{C}_b$.

The rest of the proof is analogous to 2° and 3° of (b1). (Note that we can analogously show that $e^{-\omega\cdot}\Lambda\mathbb{D}u \in \mathcal{C}_0(\mathbf{R})$ for each $\Lambda \in Y^*$.) □

Notes for Appendix F

Dinculeanu [Dinculeanu], Chapter IV, has some results in this direction for functions $F : Q \to \mathcal{B}(B,B_2)$ s.t. $\|F\|$ is measurable (if $F \in \mathrm{L}^\infty_{\text{strong}}$ and B is separable, then F “weakly locally integrable” in Dinculeanus terms). In [CZ], $\mathrm{L}_{\text{weak}} = \mathrm{L}_{\text{strong}}$ is defined for separable Hilbert spaces, but the norm $\|\|F\|_{\mathcal{B}(B,B_2)}\|_{\mathrm{L}^p}$ is used instead of the $\mathrm{L}^p_{\text{strong}}$ norm; also the appendix [Sbook] will list similar results. However, for $p = \infty$ their results become special cases of those of ours, by Theorem F.1.9(s3). We do not know any studies in our generality.

For some purposes it would be more natural to define $\mathrm{L}^p_{\text{strong}}$ and $\mathrm{L}^p_{\text{weak}}$ as spaces of (equivalence classes of) functions, whose values are unbounded operators. Fortunately, the above setting suffices for our purposes.

See Chapter 3 for further results on $\mathrm{L}^\infty_{\text{strong}}$ in the Hilbert space setting and Appendices B and D and notes therein for uniform variants of the results of this appendix.