Appendix D

Laplace and Fourier Transforms ($\mathcal{L}u = \widehat{u}$)

The meeting of two personalities is like the contact of two chemical substances: if there is any reaction, both are transformed.
— Carl Jung (1875–1961)

In this appendix, we mainly study holomorphic vector-valued functions. This includes H^p spaces, Laplace and Fourier transforms and Poisson integral formulae. We also present some results on convolutions.

Throughout this chapter, B, B_2 and B_3 are complex Banach spaces, U, H and Y are complex Hilbert spaces, $\mathbf{K} = \mathbf{C}$, and $\Omega \subset \mathbf{C}$ is open.

A function $f : \Omega \to B$ is *holomorphic* ($f \in \mathrm{H}(\Omega; B)$) if the (complex) *derivative*

$$f'(s) := \lim_{h\to 0} \frac{f(s+h) - f(s)}{h} \tag{D.1}$$

of f exists at each $s \in \Omega$. By $f^{(k)}$ we denote the kth derivative of f. The Banach space $\mathrm{H}^\infty(\Omega; B)$ of bounded holomorphic funcitons is defined by

$$\mathrm{H}^\infty(\Omega; B) := \{ f \in \mathrm{H}(\Omega; B) \,\big|\, \|f\|_{\mathrm{H}^\infty} := \sup_{s\in\Omega} \|f(s)\|_B < \infty \}. \tag{D.2}$$

We start with the basic properties of holomorphic functions:

Lemma D.1.1 (Weakly holomorphic ⇒ holomorphic) *Let $\Omega \subset \mathbf{R}$ be open, $f : \Omega \to B$, and $F : \Omega \to \mathcal{B}(B, B_2)$.*

(a) If $\Lambda f \in \mathrm{H}(\Omega)$ for all $\Lambda \in B^*$, then $f \in \mathrm{H}(\Omega; B)$.

We may replace B^* by any $A \subset B^*$ satisfying $\|x\|_B = \sup\{|\Lambda x| \,\big|\, \Lambda \in \mathrm{span}(A),\ \|\Lambda\| \le 1\}$ for all $x \in B$.

(b) If $\Lambda F(\cdot)b \in \mathrm{H}(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $F \in \mathrm{H}(\Omega; \mathcal{B}(B, B_2))$.

(c) If $\Lambda F(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $F \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2))$.

(d) If $G \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}^\infty(\Omega)))$ (resp. $G \in \mathcal{B}(B, \mathrm{H}^\infty(\Omega; B_2)))$ and $\Lambda G(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2^{**}))$ (resp. $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2)))$.

Thus, if f is weakly differentiable on Ω, then it is differentiable on Ω.

Proof: (a)&(b) These follow from [HP, Theorem 3.10.1]; note that, by (a), we can replace B and B_2^* in (b) by $A \subset B$ and $A_2 \subset B_2^*$ as in (a). (Span means the set of linear combinations.)

(c) If $\Lambda F(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then F is uniformly bounded, by the uniform boundedness theorem (fix first Λ, then b).

(d) 1° Assume that $G \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}^\infty(\Omega)))$. Define $g(\cdot) \in \mathcal{B}(B, B_2^{**})$ by $(g(s)b)\Lambda := (Gb\Lambda)(s) \in \mathbf{C}$. Then $\sup_\Omega \|g\| = \|G\|_{\mathcal{B}}$ (obviously) and $g \in \mathrm{H}^\infty$, by (c). Naturally, we can and will identify G and g.

2° Assume that $G \in \mathcal{B}(B, \mathrm{H}^\infty(\Omega; B_2))$. Then $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2^{**}))$, by 1°. But $G(s)b\Lambda = \Lambda(Gb(s))$ for all b, Λ, s, hence $G(s)b = Gb(s) \in B_2$, for any $s \in \Omega$, $b \in B$. Thus, $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2))$.

(Note analogous altered versions of (a) and (b) also hold with, e.g., $f \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}(\Omega)))$ in (a), where $\mathrm{H}(\Omega)$ equipped with the topology of uniform convergence on compact subsets.) $\square$

Next we extend the standard properties of scalar holomorphic functions to the vector-valued case:

Lemma D.1.2 (Holomorphic functions) *Let $s_0 \in \Omega$. For $r \in [0, \infty]$ we set $D_r(s_0) := \{s \in \mathbf{C} \,\big|\, |s - s_0| < r\}$.*

Let $f, g \in \mathrm{H}(\Omega; B)$. Then $f' \in \mathrm{H}(\Omega; B)$, $f \in \mathcal{C}^\infty(\Omega, B)$, and the Cauchy integral formula applies; in particular, if Γ is a closed path in Ω, and the index of Γ around $s_0 \in \Omega$ is 1 (see [Rud73, p. 79]), then

$$\int_\Gamma f(s)\,ds = 0 \quad \text{and} \quad f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(z)\,dz}{(z - z_0)^{k+1}} \in H \ \text{ for all } k \in \mathbf{N}_0. \quad \text{(D.3)}$$

Moreover, we have the following:

- *(a) If $\{h_n\} \subset \mathrm{H}(\Omega; B)$ and $h_n \to h$ uniformly on compact subsets of Ω, then $h \in \mathrm{H}(\Omega; B)$ and, for each $k \in \mathbf{N}$, $h_n^{(k)} \to h^{(k)}$ uoc on Ω.*
- *(b1) If $T \in \mathcal{B}(B, B_2)$, then $Tf \in \mathrm{H}(\Omega; B_2)$ and $(Tf)'(s) = T(f'(s))$ for all $s \in \Omega$.*
- *(b2) If $F \in \mathrm{H}(\Omega; \mathcal{B}(B, B_2))$ and $F(s) \in \mathcal{GB}(B, B_2)$ for some $s \in \Omega$, then F^{-1} is analytic on a neighborhood of s, and $(F^{-1})'(s) = -F(s)^{-1}F'(s)F(s)^{-1}$.*
- *(b3) If $F \in \mathrm{H}(\Omega; \mathcal{B}(B_2, B_3))$, then $Ff \in \mathrm{H}(\Omega; B_3)$ and $(Ff)' = F'f + Ff'$.*
- *(b4) If $\phi \in \mathrm{H}(\Omega', \Omega)$, then $f \circ \phi \in \mathrm{H}(\Omega', B)$.*
- *(b5) Let $F, G \in \mathcal{C}(\Omega; B)$. We have $F \in \mathrm{H}(\Omega; B)$ and $F' = G$ iff $F(z) - F(a) = \int_{[a,z]} G\,dm$ whenever $[a, z] \subset \Omega$ (here $[a, z] := \{(1-t)a + tz \,\big|\, t \in [0, 1]\}$).*
- *(b6) If $F \in \mathrm{H}(\Omega; \mathcal{B}(U, Y))$, then $F(\overline{\cdot})^* \in \mathrm{H}(\Omega; \mathcal{B}(Y, U))$.*
- *(c)* **(Liouville)** *If $\Omega = \mathbf{C}$ and Λf is bounded for each $\Lambda \in B^*$, then f is a constant (in B).*
- *(d)* **(Morera)** *If(f) $F \in \mathcal{C}(\Omega; B)$ and $\int_\gamma F(s)\,ds = 0$ whenever $\gamma = \partial([x_1, x_2] \times [y_1, y_2])$ (a rectangle whose sides are parallel to the coordinate axes), then $F \in \mathrm{H}(\Omega; B)$.*

(e) **(Analytic continuation)** *If Ω is connected, and $f = g$ on a set $A \subset \Omega$ having a limit point in Ω, then $f = g$ on Ω.*

(f) **(Maximum modulus principle)** *If Ω is bounded and $f \in \mathcal{C}(\overline{\Omega}; B) \cap \mathrm{H}(\Omega; B)$, then $\|f\|_{\mathrm{H}^\infty} = \sup_{\partial\Omega} \|f(\cdot)\|_B$.*

Moreover, if this is the case, then the values of f on Ω can be obtained by the Poisson integral formula.

(g1) Let $\{b_n\} \subset B$, and set $\rho := \left(\limsup_{n\in\mathbf{N}} \|b_n\|^{1/n}\right)^{-1}$. Then $\sum_{n\in\mathbf{N}} b_n(s - s_0)^n =: F(s)$ converges absolutely and uoc to a function $F \in \mathrm{H}(D_\rho(s_0); B)$. Moreover, $F^{(k)}(s_0) = k! a_k$ for all k.

(g2) **(Taylor series)** *Let $F \in H(D_R(z_0), B)$ and $r < R$. Then $\|F^{(k)}(s_0)\| \le k! M r^{-k}$, where $M := \sup_{D_r(z_0)} \|F(\cdot)\|$, and*

$$F(s) = \sum_{k\in\mathbf{N}} \frac{F^{(k)}(s_0)}{k!}(s - s_0)^k \quad \text{when } |s - s_0| < R; \tag{D.4}$$

this presentation is unique.

(h) If $h \in \mathrm{H}(\Omega \setminus \{s_0\}; B)$ is bounded on a neighborhood of s_0, then $b := \lim_{s\to s_0} h(s)$ exists and $h \in \mathrm{H}(\Omega; B)$ if we set $h(s_0) := b$ (such points are often called removable singularities).

(i) Let $B \subset B_2$ continuously. Then $\mathrm{H}(\Omega; B) = \mathcal{C}(\Omega; B) \cap \mathrm{H}(\Omega; B_2)$.

(j) If $F \in \mathrm{H}(\Omega; B)$ and $F(s_0) = 0$, then $F/(s - s_0) \in \mathrm{H}(\Omega; B)$.

The fact (h) is equivalent to $\mathrm{H}^\infty(\Omega \setminus \{s_0\}; B) = \mathrm{H}^\infty(\Omega; B)$ (for arbitrary open sets Ω).

Proof: The first claims follow from [HP, Theorem 3.10.1], [Rud73, Theorem 3.31] and induction (alternatively, by applying the corresponding scalar claims to Λf, $\Lambda \in B^*$).

Claims (a), (e), (g1) and (g2) follow from [HP, pp. 96–100].

Claims (b1) and (b3) are obvious, claims (b4), (c), (d) and (f) follow from corresponding scalar claims (in (c) we also need uniform boundedness theorem, in (d) also Lemma D.1.1).

(b2) (N.B. This proof applies also if $\mathcal{B}(B, B_2)$ is replaced by a Banach algebra.) By Lemma A.3.3(A2), F^{-1} exists on a neighborhood of s. Therefore,

$$F(s+h)\Big(\frac{F(s+h)^{-1} - F(s)^{-1}}{h} + F(s)^{-1}F'(s)F(s)^{-1}\Big)F(s) \tag{D.5}$$

$$= \frac{F(s) - F(s+h)}{h} + F(s+h)F(s)^{-1}F'(s) \to 0, \tag{D.6}$$

as $h \to 0$, because $F(s+h)F(s)^{-1}F'(s) \to F'(s)$, by continuity, and $\frac{F(s)-F(s+h)}{h} \to -F'(s)$. To remove the outer terms from (D.5), multiply it by $F(s+h)^{-1}$ to the left and by $F(s)^{-1}$ to the right, and use Lemma A.3.1(j3) (see Lemma A.3.4(F1)).

(b5) 1° *"If":* Let $a \in \Omega$. Choose $\delta > 0$ s.t. $|z - a| < \delta \Rightarrow z \in \Omega$ & $\|G(z) - G(a)\|_B < \varepsilon$. Then $(z - a)^{-1}(F(z) - F(a)) - G(a) < \varepsilon$, for $|z - a| < \delta_\varepsilon$, hence $F'(a) = G(a)$. Because $a \in \Omega$ was arbitrary, we have $F \in \mathrm{H}$ and $F' = G$.

2° *"Only if":* This follows from the scalar case.

(b6) Now $\frac{F(\bar{s})^*-F(\bar{s}_0)^*}{s-s_0} = \big(\frac{F(\bar{s})-F(\bar{s}_0)}{\bar{s}-\bar{s}_0}\big)^* \to F'(\bar{s}_0)^*$, as $s \to s_0$.

(h) By the scalar case, Λh extends to $\mathrm{H}(\Omega;\mathbf{C})$ for all $\Lambda \in B^*$. The operator $h(s_0) : \Lambda \mapsto (\Lambda h)(s_0)$ is linear and bounded, hence $h \in \mathrm{H}(\Omega;B^{**})$, by Lemma D.1.1(b). By continuity, $h(s_0) \in B$.

(i) We have $\mathrm{H}(\Omega;B) \subset \mathcal{C}(\Omega;B) \cap \mathrm{H}(\Omega;B_2)$, by (b1). Conversely, if $F \in \mathcal{C}(\Omega;B) \cap \mathrm{H}(\Omega;B_2)$ then $\int_\gamma F(s)\,ds = 0$ in B_2, hence in B, by (B.18) applied to $I : B \to B_2$, when γ is as in (d). Therefore $F \in \mathrm{H}(\Omega;B)$.

(j) We have $\Lambda F/(s-s_0) \in \mathrm{H}(\Omega;B)$ for all $\Lambda \in B^*$, by Theorem 10.18 of [Rud86], hence $F/(s-s_0) \in \mathrm{H}(\Omega;B)$, by Lemma D.1.1. $\square$

We will use the following notation:

Definition D.1.3 ($\mathbf{H}^p$, $\mathbf{L}^p_r$, $\mathbf{H}^p_r$, $\mathbf{C}_J$, $\mathbf{C}^\pm_r$, $\mathbf{C}_{a,b}$) *Let $1 \le p \le \infty$, $-\infty \le a < b \le \infty$, and $r \in \mathbf{R}$, let $J \subset \mathbf{R}$ be an interval and let $\Omega \subset \mathbf{R}$ be open. Set $\mathbf{C}^\pm_r := \pm\{s \in \mathbf{C} \bigm| \operatorname{Re} s > r\}$, $\mathbf{C}_{a,b} := \{s \in \mathbf{C} \bigm| a < \operatorname{Re} s < b\}$,*

$$\mathrm{L}^p_r(J;B) := \mathrm{e}^{r\cdot}\mathrm{L}^p(J;B) = \{f : J \to B \bigm| \mathrm{e}^{-r\cdot} f(\cdot) \in \mathrm{L}^p(J;B)\}, \tag{D.7}$$

$$\mathrm{H}^p_r(B) := \mathrm{H}^p(\mathbf{C}^+_r;B) := \{g \in \mathrm{H}(\mathbf{C}^+_r;B) \bigm| \|g\|_{\mathrm{H}^p_r(B)} < \infty\},\ \textit{where} \tag{D.8}$$

$$\|g\|_{\mathrm{H}^p_r(B)} := \sup_{r'>r} \|g(r'+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)} = \lim_{r'\to r+} \|g(r'+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)}. \tag{D.9}$$

We also set $\mathrm{H}^p := \mathrm{H}^p_0$, $\mathrm{H}^p_\infty := \cup_\omega \mathrm{H}^p_\omega$, $\mathbf{C}^\pm := \mathbf{C}^\pm_0$, $\|f\|_{\mathrm{L}^p_r} := \|\mathrm{e}^{-r\cdot} f\|_p$. If $B = U$ is a Hilbert space, then, in H^2_r, we define the inner product

$$\langle f, g\rangle_{\mathrm{H}^2_r} := \lim_{r'\to r+} \langle f(r'+i\cdot), g(r'+i\cdot)\rangle_{\mathrm{L}^2(\mathbf{R};U)}. \tag{D.10}$$

The spaces $\mathrm{H}^p(r\mathbf{D};B)$ $(r > 0)$ are defined analogously, with $\|g\|_{\mathrm{H}^p(\mathbf{D}_r;B)} := \sup_{r'<r} \|g(r'\mathrm{e}^{i\cdot})\|_{\mathrm{L}^p([0,2\pi);B)}$.

Finally, $\|g\|_{\mathrm{H}^p(\mathbf{C}_J;B)} := \sup_{r\in J} \|g(r+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)}$ defines (and norms) a subspace of $\mathrm{H}(\mathbf{C}_J;B)$, where $\mathbf{C}_J := \{s \in \mathbf{C} \bigm| \operatorname{Re} s \in J\}$, if J is open.

Recall that $r\mathbf{D} = \mathbf{D}_r := \{z \in \mathbf{C} \bigm| |z| < r\}$. Note that $\mathbf{C}_{\mathbf{R}_\pm} = \mathbf{C}^\pm$, $\mathbf{C}_{(a,b)} = \mathbf{C}_{a,b}$, and that $F \in \mathcal{G}\mathrm{H}^p_\infty$ iff there is $\omega \in \mathbf{R}$ s.t. $F \in \mathcal{G}\mathrm{H}^p_\omega$.

With the aid of Section 6.2 of [HP], one easily verifies that Definition D.1.3 is correct, that the above inner product induces the original $\|\cdot\|_{\mathrm{H}^2}$ norm, and that the above definitions of L^p_0 and H^∞_r coincide with their previous definitions. Note also that we use the (one-dimensional) Lebesgue measure on $i\mathbf{R}$ and $[0,2\pi)$ (in particular, we have no 2π-normalization).

Note that each $\mathrm{H}^p(\mathbf{C}^+_\omega)$ result has a "mirror image result" for $\mathrm{H}^p(\mathbf{C}^-_{-\omega})$, because $f \mapsto f(-\cdot)$ is an isometric isomorphism between these two spaces ($p \in [1,\infty]$, $\omega \in \mathbf{R}$).

Lemma D.1.4 ($\mathbf{L}^p_r$, $\mathbf{H}^p_r$) *Let $1 \le p_1 \le p \le p_2 \le \infty$, and $-\infty < r < r' < \infty$, and let $\Omega \subset \mathbf{C}$ be open. Then the following holds:*

(a) The spaces L^p_r, $\mathrm{H}^\infty(\Omega;B)$ and H^p_r are Banach spaces; in particular, $\mathrm{H}^2(\mathbf{C}^+_r;U)$ is a Hilbert space.

By Theorem 3.3.1(b), we can use the boundary functions of f and g to write $\langle f,g\rangle_{\mathrm{H}^2_r} = \langle f,g\rangle_{\mathrm{L}^2(r+i\mathbf{R};U)}$.

(b1) The mapping $f \mapsto \mathrm{e}^{a\cdot} f$ is an isometric isomorphism of L^p_r onto L^p_{r+a}. The mapping $g \mapsto g(\cdot - a)$ is an isometric isomorphism of H^p_r onto H^p_{r+a}.

(b2) If J is bounded, then $\mathrm{L}^p_r = \mathrm{L}^p = \mathrm{L}^p_{r'}$ with equivalent norms. If $p < \infty$, then $\mathrm{L}^p_r(J;B) = \mathrm{L}^p(J,\mu_r;B)$ with equal norms, where $d\mu_r = \mathrm{e}^{-rp\cdot}\,dm$, for $p < \infty$.

(b3) If $f_n \to f$ in L^p_r and $f_n \to g$ in $\mathrm{L}^{p_2}_\omega$ (or $f_n \to g$ pointwise a.e.) for some functions f and g and some $\omega \in \mathbf{R}$, then $f = g$ a.e.

(b4) We have $\mathrm{L}^{p_2}_r(\mathbf{R}_+;B) \underset{c}{\subsetneq} \mathrm{L}^p_{r'}(\mathbf{R}_+;B)$, indeed, $\|f\|_{\mathrm{L}^p_{r'}} \le M_{p,p_2,r'-r}\|f\|_{\mathrm{L}^{p_2}_r}$ for all $f \in \mathrm{L}^{p_2}_r(\mathbf{R}_+;B)$.

(c) If $g \in \mathrm{H}^\infty(\mathbf{C}^+_r;B)$ is continuous to the boundary $r+i\mathbf{R}$, then $\|g\|_{\mathrm{H}^\infty_r} = \sup_{r+i\mathbf{R}}\|g(\cdot)\|_B$.

If $B = U$ or $B = \mathcal{B}(U,Y)$, then Theorem 3.3.1(a2)&(c1)&(c2)provide analogous results for an arbitrary $g \in \mathrm{H}^\infty_r$.

(d) We have $\mathrm{H}^p_r(B) \underset{c}{\subsetneq} \mathrm{H}^{p_2}_{r'}(B)$ and $\mathrm{H}^{p_1}_r(B) \cap \mathrm{H}^{p_2}_r(B) \underset{c}{\subsetneq} \mathrm{H}^p_r(B)$.

(e) **($\mathbf{H}^p(r\mathbf{D};B)$)** *Results analogous to (a), (c) and (d) hold for $\mathrm{H}^p(r\mathbf{D};B)$ too, and the mapping $g \mapsto g(t\cdot)$ is an isometric isomorphism of $\mathrm{H}^p(r\mathbf{D};B)$ onto $\mathrm{H}^p(tr\mathbf{D};B)$.*

(f) Let $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B)$, $1 \le p < \infty$, $\omega \in \mathbf{R}$, $\varepsilon > 0$. Then $\sup_{|\theta|\le\pi/2}\|f(\omega+\varepsilon+r\mathrm{e}^{i\theta})\|_B \to 0$ as $r \to \infty$.

See also Lemma F.3.2.

Proof: (a) For L^p_r this follows from (b1). A H^∞-Cauchy sequence is a pointwise uniformly a Cauchy-sequence, hence it converges uniformly, and the limit is holomorphic, by Lemma D.1.2(a). Thus $\mathrm{H}^\infty(\Omega;B)$ is complete.

A H^p_r-Cauchy sequence $\{f_n\}$ converges to some function f in each $\mathrm{L}^p(t+i\mathbf{R};B)$ $(t > r)$. By (6.4.3) of [HP] (applied to some $\sigma \in (r,t)$; cf. Theorem 3.3.1(a3)), this convergence is uniform on $\mathbf{C}^+_t$ for each $t > r$), hence f is holomorphic. Obviously, $f_n \to f$ in H^p_r.

(b1)&(b2) These are obvious.

(b3) By Theorem B.3.2, there are $n_1 < n_2 < n_3 < \cdots$ s.t. $f_{n_k} \to f$ and $f_{n_k} \to g$ pointwise a.e., hence $f = g$ a.e.

(b4) By (b1), we may assume that $r' = 0$. Assume that $r' = 0 > r$. Then

$$\|f\|_p \le \|\mathrm{e}^{-r\cdot} f\|_{p_2}\|\mathrm{e}^{r\cdot}\|_q \le M_{p,p_2,r}\|f\|_{\mathrm{L}^{p_2}_r} \tag{D.11}$$

by Lemma B.3.13, where $M_{p,p_2,r} := \|\mathrm{e}^{r\cdot}\|_q < \infty$, $q^{-1} := p^{-1} - p_2{}^{-1} \in [1,\infty]^{-1}$.

(c) This follows from the scalar case (alternatively, from the Maximum modulus principle by using a conformal map of $\mathbf{C}^+_r \to \mathbf{D}_{1+\varepsilon}$).

(d) The latter claim follows easily from the fact that $\mathrm{L}^{p_1} \cap \mathrm{L}^{p_2} \underset{c}{\subsetneq} \mathrm{L}^p$, by Lemma E.1.1.

We have a continuous embedding $\mathrm{H}^p_r(B) \subset \mathrm{H}^\infty_{r'}(B)$, by Theorem 6.4.2 of [HP] (shift it by $(r'-r)/2$); obviously also inclusion $\mathrm{H}^p_r(B) \subset \mathrm{H}^p_{r'}(B)$ is continuous, hence

$$\mathrm{H}^p_r(B) \underset{c}{\subset} \mathrm{H}^p_{r'}(B) \cap \mathrm{H}^\infty_{r'}(B) \underset{c}{\subset} \mathrm{H}^{p_2}_{r'}(B). \tag{D.12}$$

(e) The above proofs apply mutatis mutandis (see also Theorem 3.3.1).

(f) This follows from Theorem 6.4.2 of [HP] (replace ω by $\omega+\varepsilon/2$ and ε by $\varepsilon/2$ to overcome the nonstandard assumption (iii) of Definition 6.4.1 of [HP]; cf. Theorem 3.3.1(a3)). □

The norm of a holomorphic function is greatest on the boundary (cf. Lemma D.1.2(f)):

Lemma D.1.5 (Hadamard Three Line Theorem) *Let* $f \in \mathcal{C}_\mathrm{b}(\overline{\mathbf{C}_{a,b}};B) \cap \mathrm{H}(\mathbf{C}_{a,b};B)$. *Set* $M_r := \sup \|f(r+i\mathbf{R})\|_B$ *(*$r \in [a,b]$*). Then*

$$M_r \le M_a^{1-\theta_r} M_b^{\theta_r} \le \max\{M_a, M_b\} \quad (r \in [a,b]), \tag{D.13}$$

where $\theta_r := (r-a)/(b-a)$.

Use the mapping $s \mapsto \mathrm{e}^s$ to obtain the Hadamard Three Circle Theorem (cf. p. 264 of [Rud86]).

Proof: For $B = \mathbf{C}$, $a = 0$ and $b = 1$, this is Lemma 1.1.2 of [BL]. By setting $g(s) := f((b-a)s+a)$ we obtain the scalar version of the theorem. For a general f, we then have $\|\Lambda f(r+it)\| \le M_a^{1-\theta_r} M_b^{\theta_r}$ for all $r+it \in \overline{\mathbf{C}_{a,b}}$ and $\Lambda \in B^*$ s.t. $\|\Lambda\| \le 1$; the general claim follows from this. □

Definition D.1.6 (Convolution) *Let* $B \times B_2 \to B_3$ *be continuous and bilinear. The* convolution $f * g$ *of* $f : \mathbf{R}^n \to B$ *and* $g : \mathbf{R}^n \to B_2$ *is defined by*

$$(f*g)(t) := \int_{\mathbf{R}^n} f(t-r)g(r)\,dr = \int_{\mathbf{R}^n} f(r)g(t-r)\,dr \in B_3, \tag{D.14}$$

for those $t \in \mathbf{R}^n$ *for which* $f(t-\cdot)g(\cdot) \in \mathrm{L}^1(\mathbf{R}^n;B_3)$.

If, e.g., f (resp. g) is defined on $\Omega \subset \mathbf{R}^n$ only, we define the above convolution by declaring $f = 0$ (resp. $g = 0$) outside Ω.

Here $B = B_2 = B_3$ might be a Banach algebra, but even more common is the case where $B = \mathcal{B}(B_2, B_3)$ (e.g., $B = \mathbf{C}$ and $B_3 = B_2$, or $B = B_2^*$ and $B_3 = \mathbf{C}$); in fact the latter contains the former.

Standard convolution results hold also for vector-valued functions:

Lemma D.1.7 ($\|f*h\|_q \le M\|f\|_p\|h\|_r$) *Let* $p,q,r \in [1,\infty]$, $f \in \mathrm{L}^p(\mathbf{R}^n;B)$ *and* $g \in \mathrm{L}^q(\mathbf{R}^n;B_2)$, *where* $B \times B_2 \to B_3$ *is bilinear and* $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$.

If $1/p+1/q = 1$, *then* $f*g$ *exists on the whole* $\mathbf{R}^n$, $f*g \in \mathcal{C}_\mathrm{bu}(\mathbf{R}^n;B_3)$ *and* $\|f*g\|_\infty \le M\|f\|_p\|g\|_q$ *(and* $f*g \in \mathcal{C}_0(\mathbf{R}^n;B_3)$ *if* $1 < p < \infty$*).*

If $h \in \mathrm{L}^1(\mathbf{R}^n;B_2)$, *then* $f*h$ *exists a.e. and* $\|f*h\|_p \le M\|f\|_p\|h\|_1$. *If* $h \in \mathrm{L}^r(\mathbf{R}^n;B_2)$ *and* $p^{-1}+r^{-1} = 1+q^{-1}$, *then* $f*h$ *exists a.e. and* $\|f*h\|_q \le M\|f\|_p\|h\|_r$.

Moreover, in all combinations listed above (e.g., L^1*,* L^1 *and* L^p *or* L^1*,* L^p*,* L^q*, in any order), the laws* $\tau^T(G*F) = (\tau^T G)*F = G*\tau^T F$ $(T \in \mathbf{R}^n)$ *(time-invariance),* $(G*F)*H = G*(F*H)$ *(commutativity) are valid, and for* $n=1$ *we have* $\pi_- F = 0 = \pi_- G \Rightarrow \pi_-(F*G) = 0$ & $(F*G)(t) = (\pi_{[0,t)}F * \pi_{[0,t)}G)(t)$ *(causality).*

All above claims also hold with $\mathrm{L}^{\cdot}_{\omega}$ *in place of* $\mathrm{L}^{\cdot}$*. All above claims hold with* $\mathbf{Z}^n$ *in place of* $\mathbf{R}^n$*.*

If $B \times B_2 \to B_3$ is bilinear and continuous, then $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$ for some $M < \infty$, by Lemma A.3.4(J1). In most cases, one has $M = 1$ (e.g, when $B_2 = \mathcal{B}(B, B_3)$).

We also note that if $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+; B)$ and $g \in \mathrm{L}^q_{\mathrm{loc}}(R_+; B_2)$, then $f*g \in \mathcal{C}(\mathbf{R}_+; B_3)$ (apply the lemma to $\pi_{[0,T)}f$ and $\pi_{[0,T)}g$ for each $T \in \mathbf{R}_+$, and use causality to obtain continuity on $[0,T]$).

Proof of Lemma D.1.7: The proof of the third inequality (Young's Inequality) is analogous to that of Theorem E.1.7 and hence omitted; see Theorem 1.2.2 of [BL] for the proof (the $\mathbf{Z}^n$ case is analogous). The proofs of the other claims are analogous to that in the finite-dimensional case (for $\mathbf{R}^n$); see, e.g., pp. 39 & 63–64 of [GLS] and use Lemma B.3.9 for uniform continuity and Corollary B.3.8 for the $\mathcal{C}_0$ property. Note that because $\|(f*h)(t)\|_{B_3} \le M(\|f\|_B * \|h\|_{B_2})(t) < \infty$ for a.e. t, the convolution integrand is in L^1 for a.e. t.

We note that the commutativity claims follow easily from the Fubini Theorem.

Let $f \in \mathrm{L}^p_\omega$ and $h \in \mathrm{L}^r_\omega$, and set then $\widetilde{f} = e^{-\omega\cdot} f \in \mathrm{L}^p$, $\widetilde{h} = e^{-\omega\cdot} h$. Then, obviously, $f*h = e^{\omega\cdot}(\widetilde{f}*\widetilde{h})$, hence $\|f*h\|_{\mathrm{L}^q_\omega} \le M\|f\|_{\mathrm{L}^p_\omega}\|h\|_{\mathrm{L}^r_\omega}$ as above, etc. □

Let $\Omega = \mathbf{C}^+$ or $\Omega = \mathbf{D}$. Let $\{z_n\} \in \Omega$ and $z_n \to z_\infty$, where $z_\infty \in \partial\Omega$. We say that $z_n \to z_\infty$ *nontangentially* if there is an open cone $C \subset \Omega$ with vertex at z_∞ s.t. $\{z_n\} \subset C$ (see [Hoffman], [Rud86] or [Garnett] for equivalent definitions). In particular, $z_n \to 0$ nontangentially on $\mathbf{C}^+$, as $n \to +\infty$, iff $z_n \in \mathbf{C}$ for all n, $z_n \to 0$, as $n \to +\infty$, and there is $M < \infty$ s.t. $\operatorname{Im} z_n / \operatorname{Re} z_n < M$ for all n.

The Poisson integral $P_r * f$ of a function f converges to f in many ways:

Lemma D.1.8 (Poisson integral formula) *Let* $P_r(t) := \frac{r}{\pi(r^2+t^2)}$.

(a1) The operator $f \mapsto P_r * f$ *is a stable* C_0*-semigroup on* $\mathcal{C}_{\mathrm{bu}}(\mathbf{R}; B)$ *and on* $\mathrm{L}^p(\mathbf{R}; B)$ *when* $1 \le p < \infty$*.*

(a2) If $f \in \mathcal{C}(i\mathbf{R} \cup \{\infty\}; B)$*, then* $r + it \mapsto (P_r * f)(t)$ *is in* $\mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\}; B)$*. In particular,* $f \in \mathcal{C}_0(\mathbf{R}; B)$ *implies that* $(P_r * f)(t) \to 0$ *as* $|r + it| \to \infty$ *and* $r + it \in \overline{\mathbf{C}^+}$*.*

(a3) Let $f \in \mathrm{L}^p(\mathbf{R}; B)$, $1 \le p \le \infty$*. Then* $f \mapsto P_r * f$ *is a semigroup (*$P_r * P_{r'} = P_{r+r'}$ *for* $r, r' > 0$*), although not necessarily strongly continuous (in case* $p = \infty$*),* $\|P_r * f\|_p \le \|f\|_p$ *(*$r > 0$*),* $\|P_r * f\|_p \to \|f\|_p$*, and* $\int_{-R}^R g P_r * f\, dm \to \int_{-R}^R g f\, dm$*, as* $r \to 0+$*, for any* $R > 0$ *and* $g \in \mathrm{L}^\infty(\mathbf{R}; \mathcal{B}(B, B_2))$*. Moreover,* $(P_r * f)(t) \to f(t)$ *as* $r \to 0+$ *nontangentially, for every Lebesgue point* t *of* f*, hence a.e.*

(a4) If $f \in \mathrm{L}^\infty(\mathbf{R};B)$ and $g \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(B,B_2))$, then $\int_{\mathbf{R}} g \cdot (P_r * f)\,dm \to \int_{\mathbf{R}} g f\,dm$, as $r \to 0+$.

(b) Let $F \in \mathcal{C}(\overline{\mathbf{C}^+};B) \cap \mathrm{H}^\infty(\mathbf{C}^+;B)$. Then F is given on $\mathbf{C}^+$ by the Poisson integral formula

$$F(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{F(iy)\,dy}{r^2+(t-y)^2} = (P_r * F(i\cdot))(t) \quad (r>0,\ t\in\mathbf{R}). \qquad \text{(D.15)}$$

(c) If $F \in \mathrm{H}^\infty(\mathbf{C}^+;B)$, then $F \in \mathcal{C}_{\mathrm{bu}}(r+i\mathbf{R};B)$ for any $r>0$, and $r \mapsto F(r+i\cdot)$ is continuous $(0,\infty) \to \mathcal{C}_{\mathrm{bu}}(\mathbf{R};B)$.

(d) The above results hold mutatis mutandis for $\mathbf{D}$ in place of $\mathbf{C}^+$ and $\partial\mathbf{D}$ in place of $i\mathbf{R}$, with $P_r(t) := \frac{1-r^2}{1-2r\cos t+r^2}$ $(0 \le r < 1,\ t\in\mathbf{R})$.

(We have set here $P_0 := \delta$, i.e., $P_0 * f := f$.) Note that in the above results, the imaginary axis ($i\mathbf{R}$) can be shifted right or left, whichever is needed. If B is a Hilbert space, then an arbitrary H^p function F has an L^p boundary function (whose Poisson integral is F, by the scalar result; see also [HP, p. 227]), for any $p \in [1,\infty]$; see Theorem 3.3.1(a2)&(a1).

Proof: (a1) The semigroup claim can be proved as in the scalar case [Garnett, pp. 12–17]; note that $\int_{\mathbf{R}} P_r\,dm = 1$.

(a2) For $\mathbf{D}$ in place of $\mathbf{C}_+$, this follows from [Rud86, Theorem 11.8] if g is finite-dimensional; in the general case one can approximate g by a finite-dimensional continuous function (e.g., by a function that is "piecewise linear on $[0,2\pi]$") and deduce the continuity as in the proof of [Rud86, Theorem 11.8]). The original ("$\mathbf{C}^+$") claim follows through Cayley transform.

(a3) 1° *Basic properties:* By p. 13–14 of [Garnett], we have $P_r * P_{r'} = P_{r+r'}$ and $\int_{\mathbf{R}} P_r\,dm = 1$, hence $P_r *$ is a semigroup and $\|P_r * f\|_p \le \|f\|_p$.

2° *Nontangential limits:* For $f \in \mathrm{L}^p(\mathbf{D};B)$ (cf. (d)), the existence of a nontangential limit at each Lebesgue point follows exactly as in the proof of (scalar) Theorem 11.23 of [Rud86]. Therefore, the same holds for $\mathrm{L}^p(i\mathbf{R};B)$, because the Cayley transform maps $\mathrm{L}^p(i\mathbf{R};B)$ one-to-one into $\mathrm{L}^p(\partial\mathbf{D};B)$, and preserves Lebesgue points and nontangential limits, by Lemma 13.2.1(e). (Recall that $r \to 0+$ nontangentially means that $r \to 0$ on a subset of $\mathbf{C}^+$ where $\mathrm{Im}\,r/\mathrm{Re}\,r$ is bounded (e.g., on $(0,+\infty)$).)

3° *$\|P_r * f\|_p \to \|f\|_p$ as $r \to 0+$:* For $p<\infty$ this follows from Theorem 6.4.3(ii) of [HP]. For $p=\infty$ this follows 2° and the inequality $\|P_r * f\|_p \le \|f\|_p$.

4° *$\int_{-R}^R gP_r * f\,dm \to \int_{-R}^R g f\,dm$:* For $p=\infty$ this follows from (a4); for $p<\infty$ this follows from the fact that $\chi_{[-R,R]}g \in \mathrm{L}^q$ and $P_r * f \to f$ in L^p, where $p^{-1}+q^{-1}=1$.

(a4) $\int_{\mathbf{R}} g(t)\int_{\mathbf{R}} P_r(t-s)f(s)\,ds - g(t)f(t)\,dt = \int_{\mathbf{R}}\int_{\mathbf{R}} g(t)P_r(t-s)\,dt\,f(s)\,ds \to \int_{\mathbf{R}} g(s)f(s)\,ds \to 0$, by The Fubini Theorem (and The Hölder Inequality), as $r \to 0+$, since $\int_{\mathbf{R}} g(t)P_r(t-s)\,dt \to g$ in L^1, by (a1) (note that $P_r(t-s) = P_r(s-t)$).

(b) $\Lambda F = \Lambda(P_r * F(i\cdot))$ on $\mathbf{C}^+$ for all $\Lambda \in B^*$, by [Garnett, Lemma 3.4], hence (D.15) holds.

(c) By shifting F to the left by $r/2$, we see that we can assume that $\mathcal{C}(\overline{\mathbf{C}^+};B)\cap \mathrm{H}^\infty(\mathbf{C}^+;B)$. Because $P_r\in \mathrm{L}^1$ and $F(i\cdot)\in \mathcal{C}_{\mathrm{b}}\subset \mathrm{L}^\infty$, we have $P_r * F(i\cdot)\in \mathcal{C}_{\mathrm{bu}}(\mathbf{R};B)$, by Lemma D.1.7.

From (a1) it follows that $t\mapsto P_t * F(i\cdot) = P_{t-r} * P_r * F(i\cdot)\in \mathcal{C}_{\mathrm{bu}}$ is continuous on (r,∞); because $r>0$ was arbitrary, the claim follows.

(d) The proof are analogous (see the same references). □

We use the standard definition of the Laplace transfrom; it follows that the Fourier transform becomes its restriction to $i\mathbf{R}$. (Thus, this is the standard Fourier transform rotated clockwise by $\pi/2$.)

Definition D.1.9 (Laplace and Fourier transforms) *Let $u:\mathbf{R}\to B$ be measurable. The* Laplace transform *of u is the function*

$$\mathcal{L}u := \widehat{u}(s) := \int_{\mathbf{R}} \mathrm{e}^{-st}u(t)\,dt, \tag{D.16}$$

defined on those $s\in\mathbf{C}$ for which the integral converges (i.e., for which $\mathrm{e}^{-s\cdot}u\in \mathrm{L}^1(\mathbf{R};B)$). If $u\in\mathrm{L}^1(\mathbf{R};B)$, we call the restriction $\widehat{u}|_{i\mathbf{R}}: i\mathbf{R}\to B$ the Fourier transform *of u.*

Naturally, the Laplace and Fourier transforms inherit the properties of the Bochner integral; in particular, they are linear and commute with bounded linear operators.

It is shown in Lemma D.1.11 that $\widehat{u}\in \mathrm{L}^1(\mathbf{R};B)\implies \widehat{u}(i\cdot)\in\mathcal{C}_0(i\mathbf{R};B)$, and that $\widehat{u}\in\mathrm{L}^1(\mathbf{R}_+;B)\implies \widehat{u}(i\cdot)\in\mathrm{H}^\infty(\mathbf{C}^+;B)$.

Next we list some basic properties of Laplace transforms:

Lemma D.1.10 ($\widehat{f}$) *With the notation of Definition D.1.3, we have the following:*

(a1) If $f\in\mathrm{L}^1_r(\mathbf{R}_+;B)$, then $\|\widehat{f}(s)\|_B\le\|f\|_{\mathrm{L}^1_r}$ for $s\in\overline{\mathbf{C}}^+_r$,

(a2) If $f\in\mathrm{L}^p_r(\mathbf{R}_+;B)$ and $t>r$, then $\widehat{f}\in\mathrm{H}^\infty(\mathbf{C}^+_t;B)$, and $\|\widehat{f}\|_B\le \|\mathrm{e}^{-(t-r)\cdot}\|_q\|f\|_{\mathrm{L}^p_r}<\infty$ on $\mathbf{C}^+_t$.

(a3) Let $a<b$. If $\|f\|_{\mathrm{L}^p_r(\mathbf{R};B)}\le M$ for $r\in(a,b)$, then $f\in\mathrm{L}^p_a\cap\mathrm{L}^p_b$ and $\|f\|_{\mathrm{L}^p_r}\in\mathcal{C}([a,b])$.

(a4) If $f\in\mathrm{L}^p_a\cap\mathrm{L}^p_b$, then $\|f\|_{\mathrm{L}^p_r(\mathbf{R};B)}\le M:=\max\{\|f\|_{\mathrm{L}^p_a},\|f\|_{\mathrm{L}^p_b}\}$ for all $r\in[a,b]$ (hence (a3) applies).

(b) **(Uniqueness)** *Let $u,v:\mathbf{R}_+\to B$ be measurable and let $\mathrm{e}^{r\cdot}u,\mathrm{e}^{r\cdot}v\in\mathrm{L}^1$ for some $r\in\mathbf{R}$. Then $u=v$ a.e. iff $\widehat{u}=\widehat{v}$ on $\mathbf{C}^+_r$ (iff $\widehat{u}=\widehat{v}$ on $r+i\mathbf{R}$).*

See also Theorem 3.3.1.

Proof: (a1) This is obvious.

(a2) Choose q s.t. $1/p+1/q=1$. Because $\|\mathrm{e}^{-s\cdot}\mathrm{e}^{r\cdot}\|_q\le\|\mathrm{e}^{-(t-r)\cdot}\|_q<\infty$ for $s\in\mathbf{C}^+_t$, the function $\mathrm{e}^{-s\cdot}f$ is in L^1 and we get the norm bound (by Hölder's inequality). By the dominated convergence theorem, we may differentiate under the integral sign, thus we see f is holomorphic. See, e.g., [Sbook] for details.

(a3) 1° *Case* $p<\infty$*:* Let $g\in \mathrm{L}^p(\mathbf{R}_+;B)$ and $\|g\|_{\mathrm{L}^p_r}\le M$ for $r\in(a,b)$. Then

$$\|g\|_{\mathrm{L}^p_r}^p := \int_{\mathbf{R}} \mathrm{e}^{-rpt}\|g(t)\|_B^p\,dt \to \int_{\mathbf{R}} \mathrm{e}^{-bpt}\|g(t)\|_B^p\,dt =: \|g\|_{\mathrm{L}^p_b}^p, \tag{D.17}$$

as $t\to b-$, by the Monotone Convergence Theorem, hence $g\in \mathrm{L}^p_b$ and $\|g\|_{\mathrm{L}^p_b} = \lim_{r\to b-}\|g\|_{\mathrm{L}^p_r}$.

The same holds for any $g\in \mathrm{L}^p(\mathbf{R}_-;B)$, by the Dominated Convergence Theorem; consequently, an arbitrary $g\in\mathrm{L}^1(\mathbf{R};B)$ will do. Apply this to $\text{Я} g$ (and $(-b,-a)$) to obtain the same at a.

Set $F(t):=\|f\|_{\mathrm{L}^p_t}$. By the above, $F(t)=\lim_{s\to t}F(s)$ for $t\in\{a,b\}$; apply this for $[a,t]$ and $[t,b]$ to see that F is continuous at every $t\in[a,b]$.

2° *Case* $p=\infty$*:* Let $g\in\mathrm{L}^\infty_c(\mathbf{R}_+)$ for some $c<0$. Set $M_r:=\|g\|_{\mathrm{L}^\infty_r} := \|\mathrm{e}^{-r\cdot}g\|_\infty$ for $r\in\mathbf{R}$. Then there is $T>0$ s.t. $\mathrm{e}^{c/2}|g|<M_0$ for a.e. $t>T$. It follows that $M_r\le M\mathrm{e}^{-rT}$ for $r>c/2$, hence $M_0\ge\limsup_{r\to 0-}M_r$. Therefore, $M_0=\lim_{r\to0-}M_r$ (because M_r is obviously decreasing).

If $g\in\mathrm{L}^\infty(\mathbf{R}_-)$, then M_r is increasing and, so M is, obviously, again continuous at 0 from the left. The same holds for any $g\in\mathrm{L}^\infty(\mathbf{R})$, because $\max:\mathbf{R}^2\to\mathbf{R}$ is continuous.

Apply the above for $g:=\|\mathrm{e}^{-b\cdot}f\|_B$ to obtain continuity at b. The rest follows as in 1°.

(a4) W.l.o.g. we assume that $a=0$, $B=\mathbf{C}$, $f\ge0$, $f\ne0$ and $p<\infty$ (case $p=\infty$ is obvious). Set

$$g(r):=\|f\|_{\mathrm{L}^p_r}^p=\int_{\mathbf{R}}\mathrm{e}^{-prt}f(t)\,dt \quad (r\in[0,b]). \tag{D.18}$$

Obviously, $g'(r) = -\int_{\mathbf{R}} pt\mathrm{e}^{-prt}f(t)\,dt$, $g''(r) = -pg'(r) + p^2\int_{\mathbf{R}} t^2\mathrm{e}^{-prt}f(t)\,dt > -pg'(r)$ for all $r\in(0,b)$. Thus, if $g'(r_0)=0$ for some $r_0\in(0,b)$, then $g''(r_0)>0$, hence g has no maximum on $(0,b)$, hence (a4) holds.

(b) For separable U, this result is contained in [CZ, Theorem A.6.19], and we may always replace U by the closed span of $f[\mathbf{C}^+]$, which is a separable Hilbert space (because $f[\mathbf{C}^+]$ is a continuous image of a separable set, hence separable).

(c) This follows from the scalar case (or from Lemma D.1.11(a3')) (and of Lemma D.1.11(a3). □

As in the scalar case, the Fourier transform maps $\mathrm{L}^1(\mathbf{R};B)$ to continuous functions on $i\mathbf{R}$, vanishing at infinity:

Lemma D.1.11 ($\mathbf{L^1}$ Fourier transform) *Let B be a Banach space, $f\in\mathrm{L}^1(\mathbf{R};B)$ and $\omega\in\mathbf{R}$. Then we have the following:*

(a1) $\widehat{f}\in\mathcal{C}_0(i\mathbf{R};B)$*, and* $\|\widehat{f}(ir)\|_B\le\|f\|_{\mathrm{L}^1}$ *for* $r\in\mathbf{R}$.

(a2) The Fourier transform $f\mapsto\widehat{f}$ *is linear and continuous, i.e., in* $\mathcal{B}(\mathrm{L}^1,\mathcal{C}_0)$.

(a3) **(Uniqueness)** *If* $\widehat{f}=0$ *on* $i\mathbf{R}$*, then* $f=0$ *(in* L^1*, that is, a.e.).*

(b) **(Riemann–Lebesgue)** *We have* $\|\widehat{f}(ir)\|_B\to0$ *as* $|r|\to\infty$ *and* $r\in\mathbf{R}$.

(c) $\widehat{f*g} = \widehat{f}\widehat{g}$ *on* $i\mathbf{R}$ *when* $g \in \mathrm{L}^p(\mathbf{R};B_2)$ *(*$p \in \{1,2\}$*) and* $B \times B_2 \to B_3$ *is bilinear and continuous. See also Lemma D.1.12(c).*

(d) If $f, f' \in \mathrm{L}^1$*, then* $\widehat{f'}(ir) = ir\widehat{f}(ir) - f(0)$.

(e1) **(Inverse Fourier transform)** *If* $\widehat{f} \in \mathrm{L}^1(i\mathbf{R};B)$*, then* $f(\cdot) = \frac{1}{2\pi}\int_{\mathbf{R}} \mathrm{e}^{ix\cdot}\widehat{f}(ix)\,dx \in \mathcal{C}_0(\mathbf{R};B)$ *a.e. (and in all points t of continuity of f).*

(e2) If $f, f', f'' \in \mathrm{L}^1(\mathbf{R};B)$ *(e.g., if* $f \in \mathcal{C}_c^2(\mathbf{R};B)$*), then* $\widehat{f} \in \mathrm{L}^p(i\mathbf{R};B)$ *for all* $p \in [1,\infty]$.

(f1) $\widehat{\tau(t)f}(s) = \mathrm{e}^{st}\widehat{f}(s)$ *(*$s \in i\mathbf{R}$*) (for* $s \in \overline{\mathbf{C}^+}$ *if* $f = \pi_+ f$*).*

(f2) If $f, \mathrm{e}^{\omega\cdot} f \in \mathrm{L}^1$*, then* $\widehat{\mathrm{e}^{\omega\cdot}f}(s) = \widehat{f}(s-\omega)$ *(*$s \in i\mathbf{R}$*) (for* $s \in \overline{\mathbf{C}^+}$ *if* $f = \pi_+ f$*).*

Assume, in addition, that $f = \pi_+ f$*. Then we have the following:*

(a1') $f \in \mathrm{H}^\infty(\mathbf{C}^+;B) \cap \mathcal{C}_0(\overline{\mathbf{C}^+};B)$*, and* $\sup_{\mathbf{C}^+}\|\widehat{f}\| = \sup_{i\mathbf{R}}\|\widehat{f}\| \le \|f\|_{\mathrm{L}^1}$.

(a2') The Laplace transform $f \mapsto \widehat{f}$ *is linear and continuous, i.e., in* $\mathcal{B}(\pi_+\mathrm{L}^1, \mathrm{H}^\infty)$.

(a3') **(Uniqueness)** *If* $\widehat{f} = 0$ *on* $\mathbf{C}^+$*, then* $f = 0$ *(in* L^1*, that is, a.e.).*

(b') We have $\|\widehat{f}(s)\|_B \to 0$ *as* $|s| \to \infty$ *and* $s \in \overline{\mathbf{C}^+}$.

(c') $\widehat{f*g} = \widehat{f}\widehat{g}$ *on* $\mathbf{C}^+$ *if the assumptions of (c) are satisfied and* $\pi_- g = 0$.

(d') One can obtain $\widehat{f}(r+it)$ *from the* Poisson integral formula

$$\widehat{f}(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{\widehat{f}(iy)\,dy}{r^2 + (t-y)^2} \quad (r > 0,\ t \in \mathbf{R}). \tag{D.19}$$

(d'') If $f, f' \in \mathrm{L}^1$*, then* $\widehat{f'}(s) = s\widehat{f}(s) - f(0)$ *for* $s \in \mathbf{C}^+$.

As above, we do not distinguish between the Fourier and Laplace transforms unless it is necessary.

If one would define the Fourier transform as $\mathcal{F}f := \widehat{f}(i\cdot)$, then one would obtain from (e1) that $\mathcal{F}^{-1} = \frac{1}{2\pi}\text{Я}\mathcal{F} = \frac{1}{2\pi}\mathcal{F}\text{Я}$. However, we consider the Fourier transform of f in the Laplace sense, hence its domain is $i\mathbf{R}$, not $\mathbf{R}$. Despite this rotation by $\pi/2$, the properties of the Fourier transform are shared by its inverse, e.g., $\widehat{f} \in \mathrm{L}^1$ implies that $f \in \mathcal{C}_0$.

Proof: (a1) The uniform continuity is most easily obtained from (b); the rest is obvious.

(a1') The holomorphicity is noted in [HP, pp. 215–216]. The continuity is obtained from the Dominated Convergence Theorem. The norm claim follows from the scalar case (alternatively, from the Poisson formula). The uniform continuity is most easily obtained from (b');

(a2)&(a2') These are obvious.

(a3)&(a3')&(d)&(d')&(d") These follow from the scalar case (cf. Lemma B.2.6) (the Poisson kernel is an L^1 function; pages 229 and 227 of [HP] give an alternative proof of (g')). (In (d) and (d"), the derivative f' need not exist everywhere, it is enough that $f \in \mathrm{W}^{1,1}$; analogously, in (e2) it suffices that $f \in \mathrm{W}^{2,1}$.)

(b') Let $\varepsilon > 0$. Choose $\phi \in \mathcal{C}_c^\infty((0,+\infty);B)$ s.t. $\|F-\phi\|_1 < \varepsilon/2$. Because $\|\widehat{\phi}(s)\|_B = \|\widehat{\phi'}/s\|_B \le \|\phi'\|_1/|s|$ on $\mathbf{C}^+$, we have $\|\widehat{\phi}(s)\|_B < \varepsilon/2$, hence $\|F(s)\|_B < \varepsilon$, for $|s|$ large enough (by continuity (see (a1')), $\|F(s)\| \le \varepsilon$ for $s \in i\mathbf{R}$ too).

(b) If $f \in \mathrm{L}^1(\mathbf{R};B)$, then $\widehat{f} = \widehat{\pi_+ f} + \boldsymbol{ЯЯ}\widehat{\pi_- f}$ vanishes at infinity, by (b).

(c) This can be proved as in the scalar case [Rud86, Theorem 9.2(c)].

(c') This is contained in [HP, Theorem 6.2.4].

(e1) Use the scalar case (and (a1)&(b) for $\mathcal{C}_0$).

(e2) By (d) and (a1), $ir\widehat{f}(ir) - f(0)$ and $(ir)^2\widehat{f}(ir) - (ir+1)f(0)$ are bounded, hence so is $r^2\widehat{f}(ir)$. Because $\widehat{f} \in \mathcal{C}_0$, it follows that $\widehat{f} \in \mathrm{L}^1 \cap \mathrm{L}^\infty$, hence in each L^p.

(f) These are obvious. □

Many of the properties treated in the previous lemma hold for general vector-valued measures. To make things simple, we only treat "MTI", which corresponds to certain (all if $\dim B < \infty$) measures having only a discrete part plus an absolutely continuous part:

Lemma D.1.12 (MTI Fourier transform) *Let B be a Banach space and set*

$$\mathrm{MTI}_B := \{\mu = \sum_{j\in\mathbf{Z}} a_j\delta_{t_j} + f \,\big|\, \|\mu\|_{\mathrm{MTI}} := \sum_j \|a_j\|_B + \|f\|_{\mathrm{L}^1(\mathbf{R};B)} < \infty\}, \quad \text{(D.20)}$$

$$\mathrm{MTIC}_B := \{\mu = \sum_j a_j\delta_{t_j} + f \in \mathrm{MTI}_B \,\big|\, f \in \mathrm{L}^1(\mathbf{R}_+;B) \ \&\ t_j \ge 0 \textit{ for all } j\}. \quad \text{(D.21)}$$

Let $\mu = \sum_{j\in\mathbf{Z}} a_j\delta_{t_j} + f \in \mathrm{MTI}_B$.

We use the standard definitions of the Fourier transform of $\mu \in \mathrm{MTI}_B$ and the Laplace transform of $\mu \in \mathrm{MTIC}_B$:

$$\widehat{\mu}(s) := \sum_{j\in\mathbf{Z}} a_j e^{-st_j} + \widehat{f}(s), \quad \text{(D.22)}$$

for $s \in i\mathbf{R}$ (for $s \in \overline{\mathbf{C}^+}$, if $\mu \in \mathrm{MTIC}_B$).

We have the following:

(a1) $\widehat{\mu} \in \mathcal{C}_{\mathrm{bu}}(i\mathbf{R};B)$, and $\|\widehat{\mu}(ir)\|_B \le \|\mu\|_{\mathrm{MTI}}$ for $r \in \mathbf{R}$.

(a2) The Fourier transform $\mu \mapsto \widehat{\mu}$ is linear and continuous, i.e., in $\mathcal{B}(\mathrm{MTI}_B;\mathcal{C}_{\mathrm{bu}})$.

(a3) **(Uniqueness)** *If $\widehat{\mu} = 0$ on $i\mathbf{R}$, then $\mu = 0$.*

(b1) We have $a_j = \lim_{T\to+\infty} \int_{-T}^{T} \widehat{\mu}(ir)e^{-it_j r}\,dr$ for all $j \in \mathbf{Z}$.

(b2) If $f = 0$, then $\widehat{\mu} \in \mathrm{AP}(i\mathbf{R};B)$ (i.e., $\widehat{\mu}(i\cdot) \in \mathrm{AP}(\mathbf{R};B)$).

(c) Let $\nu \in \mathrm{MTI}_{B_2}$, and let $B \times B_2 \to B_3$ be bilinear and $\|bb_2\|_{B_3} \le \|b\|_B\|b_2\|_{B_2}$. Then (c1)–(c3) hold.

(c1) $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ on $i\mathbf{R}$ and $\|\mu * \nu\|_{\mathrm{MTI}} \le \|\mu\|_{\mathrm{MTI}}\|\nu\|_{\mathrm{MTI}}$, where $\delta_t * \delta_r := \delta_{r+t}$ and $\delta_t * f := \tau(-t)f$.

(c2) For all $1 \le p \le \infty$ *and* $g \in \mathrm{L}^p(\mathbf{R};B_2)$*, the convolution* $\mu * g$ *exists a.e. and* $\|\mu * g\|_{\mathrm{L}^p(\mathbf{R};B_3)} \le \|\mu\|_{\mathrm{MTI}}\|g\|_p$.

(c3) We have $(\mu*\nu)*\lambda = \mu*(\nu*\lambda)$ *and* $(\mu*\nu)*g = \mu*(\nu*g)$ *when* $\lambda \in \mathrm{MTI}_{\mathcal{B}(B_3,B_4)}$ *and* $g \in \mathrm{L}^p(\mathbf{R};\mathcal{B}(B_3,B_4))$.

(d) Assume that $B = \mathcal{B}(U,Y)$*, where* U *and* Y *are Hilbert spaces. Then* $\|\mu*\|_{\mathrm{TI}(U,Y)} \le \|\mu\|_{\mathrm{MTI}}$, $(\mu*)^* = (\text{Я}\mu^*)* \in \mathrm{TI}(Y,U)$, $(\mu*)^{\mathrm{d}} = \mu^* * \in \mathrm{TI}(Y,U)$, $\text{Я}(\mu*)\text{Я} = (\text{Я}\mu)* \in \mathrm{TI}(U,Y)$*, and* $\mathrm{e}^{\omega\cdot}(\mu*)\mathrm{e}^{-\omega\cdot} = (\mathrm{e}^{\omega\cdot}\mu)* \in \mathrm{TI}_\omega(U,Y)$*. In particular, then* $\mu* \in \mathrm{MTIC}_B \Leftrightarrow (\mu*)^{\mathrm{d}} \in \mathrm{MTIC}_B$.

Moreover, then $\widehat{\mu * g} = \widehat{\mu}\widehat{g}$ *a.e. on* $i\mathbf{R}$ *for all* $g \in \mathrm{L}^1(\mathbf{R}_+;U) \cup \mathrm{L}^2(\mathbf{R};U)$ *(and on* $\mathbf{C}^+$ *if* $\pi_- g = 0 = \pi_-\mu$*). Therefore,* $\widehat{\mu}$ *coincides with* $\widehat{\mu*}$ *of of Theorem 3.1.3 (and with that of Theorem 6.2.1 if* $\pi_-\mu = 0$*).*

Assume, in addition, that $\mu,\nu \in \mathrm{MTIC}_B$*. Then we have the following:*

(a1') $\widehat{\mu} \in \mathrm{H}^\infty(\mathbf{C}^+;B) \cap \mathcal{C}_{\mathrm{bu}}(\overline{\mathbf{C}^+};B)$*, and* $\sup_{\mathbf{C}^+}\|\widehat{\mu}\| = \sup_{i\mathbf{R}}\|\widehat{\mu}\| \le \|\mu\|_{\mathrm{MTI}}$.

(a2') The Laplace transform $\mu \mapsto \widehat{\mu}$ *is linear and continuous, i.e., in* $\mathcal{B}(\mathrm{MTIC}_B;\mathrm{H}^\infty)$.

(a3') **(Uniqueness)** *If* $\widehat{\mu} = 0$ *on* $\mathbf{C}^+$*, then* $\mu = 0$.

(b') $\|\widehat{\mu}(s) - M\|_B \to 0$ *as* $\operatorname{Re} s \to +\infty$ *and* $s \in \mathbf{C}^+$*, where* $M \in B$ *is the coefficient of* δ_0.

(c') We have $\widehat{\mu*\nu} = \widehat{\mu}\widehat{\nu}$ *on* $\overline{\mathbf{C}^+}$ *if* $\nu \in \mathrm{MTIC}_{B_2}$ *and the assumptions of (c) hold.*

(d') For $r + it \in \mathbf{C}^+$ *one can obtain* $\widehat{\mu}(r+it)$ *from the* Poisson integral formulaPoisson integral formula!for measures

$$\widehat{\mu}(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{\widehat{\mu}(iy)\,dy}{r^2 + (t-y)^2}. \tag{D.23}$$

Because B is not necessarily an algebra, we have defined MTI_B to consist of “measures”, not of the corresponding convolution operators (we identify L^1 functions and corresponding absolutely continuous measures $E \mapsto \int_E f\,dm$).

If we had $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$, $M > 0$, in (c), then the conlusions of (c) would still be valid, provided that we added M also to the other inequalities in (c).

To clarify (d), we note that $\mathrm{e}^{\omega\cdot}\delta_{t_k} = \mathrm{e}^{\omega t_k}\delta_{t_k}$, hence

$$\mathrm{e}^{\omega\cdot}(F + \sum_{k\in\mathbf{N}} T_k\delta_{t_k}) = \mathrm{e}^{\omega\cdot}F(\cdot) + \sum_{k\in\mathbf{N}} \mathrm{e}^{\omega t_k}T_k\delta_{t_k}. \tag{D.24}$$

Thus, the “stability” of μ can be shifted (cf. Remark 6.1.9). Most of (d) is valid for general B, B_2 and B_3 too. Here “$(\text{Я}\mu)$” refers to Я operating μ (on MTI), not on L^2.

The functions in the space generated by the sums of AP functions and continuous functions having limits at infinity (which contains the Fourier transforms of MTI_B functions) are called *semi-almost periodic functions* (see, e.g., [Sarason] for more on such functions).

Proof: The parts (a1), (a2), (a1') and (a2') follows easily from those of Lemma D.1.11.

(a3) This follows from (b1) and Lemma D.1.11(a3).

(a3') This follows from (a1') and (a3).

(b1)&(b2) These follow from Lemma C.1.2(h2), because f has no effect on the limit in (b1), by Lemma D.1.11(b).

(b') W.l.o.g. we assume that $M = 0$ and $f = 0$ (by Lemma D.1.11(b')).

Given $\varepsilon > 0$, take first $N \in \mathbf{N}$ s.t. $\sum_{|k|>N} \|a_k\| < \varepsilon/3$. Because $e^{-st_k} \to 0$, as $\operatorname{Re} s \to +\infty$, for each k, there is $R > 0$ s.t. $\sum_{|k|\leq N} \|a_k\| < \varepsilon/3$ for $\operatorname{Re} s > R$.

(c1)–(c2) These are not hard to verify (the hardest part is contained in Lemma D.1.11(c); use formulae analogous to (2.55) and Theorem 2.6.4(d) for $\|\mu * \nu\| \leq \|\mu\|\|\nu\|$). Note that we use the standard (associative and distributive) definition of $*$, which coincides with

$$(\mu * f)(t) := \int_{\mathbf{R}} d\mu(\cdot) \cdot f(t - \cdot) \tag{D.25}$$

(if the integral is defined in some reasonable sense; cf. [Dobrakov]; we do not need this).

(c3) Decompose μ, ν, λ and verify the claims for distributed parts (Obviously, the convolution is distributive).

(Note that one can take the values of λ or g on X and $B_3 = \mathcal{B}(X, B_4)$ for some Banach space X, so that X is isometrically isomorphic to a closed subspace of $\mathcal{B}(B_3, B_4)$. More generally, we could as well assume that $B \times B_2 \times X \to B_4$ is "trilinear" and continuous.)

(d) Clearly $\mu * \in \mathrm{TI}$. By (c2) we have $\|\mu *\|_{\mathrm{TI}(U,Y)} \leq \|\mu\|_{\mathrm{MTI}}$. The second claim follows from equations (here $f, g \in \mathrm{L}^2$, $F \in \mathrm{L}^1(\mathbf{R}; B)$, $T_k \in B$, $t_k \in \mathbf{R}$)

$$\langle T_k \tau(-t_k) f, g \rangle = \langle f, T_k^* \tau(t_k) g \rangle \text{ and} \tag{D.26}$$

$$\int_{\mathbf{R}} \langle \int_{\mathbf{R}} F(t-r) f(r)\, dr, g(t) \rangle dt = \int_{\mathbf{R}} \langle f(r), \int_{\mathbf{R}} (\boldsymbol{Я} F^*)(r-t) g(t)\, dt \rangle dr, \tag{D.27}$$

(use the Fubini Theorem). Note that $(\boldsymbol{Я}\mu)^* = \boldsymbol{Я}(\mu^*)$, so the order of these operations does not matter. Equations

$$(F * \boldsymbol{Я} f)(-s) = \int_{\mathbf{R}} F(-t-s) f(-s)\, ds = \int_{\mathbf{R}} F(-t+s) f(s)\, ds = ((\boldsymbol{Я} F) * f)(s) \tag{D.28}$$

and $\boldsymbol{Я}(\tau(t_k) \boldsymbol{Я} f) = \tau(-t_k) f$ imply that $\boldsymbol{Я}(\mu * \boldsymbol{Я} f) = (\boldsymbol{Я}\mu) * f$. By combining the two identities proved above we obtain that $(\mu *)^{\mathrm{d}} = \mu^* *$, which implies the final claim.

The $e^{-\omega \cdot}$ claim is most easily obtained from the equation $\mathcal{L}(e^{\omega \cdot} \mu) = \widehat{\mu}(s - \omega)$ (cf. Remark 2.1.6).

If $g \in \mathrm{L}^1$, then the claim on $\widehat{\mu * g}$ is contained in (c1) and (c'). If $g \in \mathrm{L}^2 \setminus \mathrm{L}^1$, then this follows from the L^1 case by density: choose $\{g_n\} \subset \mathrm{L}^1 \cap \mathrm{L}^2$ s.t. $g_n \to g$ in L^2 and use (c2) and Lemma D.1.15. The claims on TI and TIC follow from this.

(c')&(d') These can be proved as parts (c') and (d') of Lemma D.1.11. □

See, e.g., [DU, pp. 1–6] and [Dinculeanu] for more general vector-valued measures.

If $f \in L^1(\mathbf{R}; L^p)$, then $\widehat{f} \in \mathcal{C}_0(i\mathbf{R}; L^p)$, by Lemma D.1.11(a1). Therefore, $\widehat{f}(ir) \in L^p$, so that $\widehat{f}(ir)$ is "well-defined a.e.", i.e., $(\widehat{f}(ir))(\cdot)$ makes sense as an equivalent class. But can we fix some point t and then transform $(f(\cdot))(t)$? The answer is positive for a.e. t:

Proposition D.1.13 ($(\mathcal{L}f)(t) = \mathcal{L}(f(\cdot)(t))$) *Let $(Q, \mathfrak{M}, \mu)$ be σ-finite. Let $\omega \in \mathbf{R}$, $p, q \in [1, \infty]$.*

If $f \in L^1_\omega(\mathbf{R}; L^p(Q; B)) \cap L(\mathbf{R} \times Q; B)$ and $s \in \omega + i\mathbf{R}$, then $\int_{\mathbf{R}} e^{-sr} f(r)(t)\, dr$ exists for a.e. $t \in Q$, and

$$\widehat{f}(s)(t) = \int_{\mathbf{R}} e^{-sr} f(r)(t)\, dr \in B \quad \text{for a.e. } t \in Q. \tag{D.29}$$

In particular, if $f \in L^q_\omega(\mathbf{R}_+; L^p(Q; B)) \cap \mathcal{C}(\mathbf{R}_+; L^p(Q; B))$, then, for each $s \in \mathbf{C}^+_\omega$, $\widehat{f}(s)(t) = \int_{\mathbf{R}_+} e^{-sr} f(r)(t)\, dr \in B$ for a.e. $t \in Q$; (the integrand in in $L^1(Q; B)$ for a.e. $t \in Q$).

Note that this is result is nontrivial, although it is unfortunaly widely used without reference (we do not know any).

Proof: Let $s \in \omega + i\mathbf{R}$, so that $e^{-s\cdot} f \in L^1(\mathbf{R}; L^p(Q; B)) \cap L(\mathbf{R} \times Q; B)$. By Lemma B.4.17, $g_s(t) := \int_{\mathbf{R}} e^{-sr} f(r)(t)\, dr$ exists a.e., and $g_s = \int_{\mathbf{R}} e^{-sr} f(r)\, dr \in L^p(Q; B)$. In the second claim we have $e^{-s\cdot} f \in\in L^1(\mathbf{R}; L^p(Q; B))$ for $\operatorname{Re} s > \omega$ (cf. the proof of Lemma D.1.10(a2)). □

The set of bounded complex Borel measures on a locally compact Hausdorff space Q equals $\mathcal{C}_0(Q)^*$, by the (a) Riesz representation theorem [Rud86, Theorem 6.19]. Therefore, one often defines measures as the set $\mathcal{C}_0^*$ (see, e.g., [Dinculeanu]). We shall need some basic results on even more general "measures":

Lemma D.1.14 (M) *Let H be a Hilbert space, and let $\Omega \subset \mathbf{R}^n$ be open. Set $M := \mathcal{B}(\mathcal{C}_0(\Omega; B); H)$.*

(a) For each $\mu \in M$ and $\varepsilon > 0$, there is a compact $K_\varepsilon \subset \Omega$ s.t.

$$\|\mu\phi\|_\infty \le \varepsilon \|\phi\|_\infty + \|\mu\| \sup_{K_\varepsilon} \|\phi\|_B \quad \text{for all } \phi \in \mathcal{C}_0(\Omega; B). \tag{D.30}$$

(b) Each $\mu \in M$ has a unique norm-preserving extension $\mu \in \mathcal{B}(\mathcal{C}_b(\Omega; B); H)$.

(c) $\widetilde{\mu} : t \mapsto \mu(e^{it\cdot}) \in H$ satisfies $\widetilde{\mu} \in \mathcal{C}_{bu}(\mathbf{R}^n; \mathcal{B}(B, H))$ for all $\mu \in \mathcal{B}(\mathcal{C}_0(\mathbf{R}^n; B); H)$.

(d) $\mu\tau^t\phi \to 0$ in H as $|t| \to \infty$, $t \in \mathbf{R}^n$, for all $\mu \in M$ and $\phi \in \mathcal{C}_0(\mathbf{R}^n; B)$.

(e) This lemma also holds with $\mathbf{Z}^n$ in place of $\mathbf{R}^n$.

By (b), $\mathcal{C}_b(\Omega; B)$ is a closed subspace of $\mathcal{B}(M, H)$.

Proof: (a) 1° *For each $T \in M$ and $\varepsilon > 0$, there is a compact $K'_\varepsilon \subset \Omega$ s.t. $\|T\phi\| < \varepsilon\|\phi\|$ for $\phi \in \mathcal{C}_0$ s.t. $\phi = 0$ on K'_ε:* (This does not hold if, e.g., $H = \mathcal{C}_0$ (a Banach space) and $T = I$.)

Let $\|T\| = 1$, w.l.o.g. Find $\psi \in \mathcal{C}_0$ s.t. $\|\psi\| \le 1$ and $\|T\psi\|^2 > 1 - \varepsilon^2$. Set $K'_\varepsilon := \operatorname{supp}(\psi)$. If $\phi = 0$ on K'_ε and $\|\phi\| \le 1$, then $\|\alpha\phi + \psi\| \le 1$ for $|\alpha| \le 1$, hence

$$1 \ge \sup_{|\alpha|\le 1} \|\alpha T\phi + T\psi\|^2 \ge \|T\phi\|^2 + \|T\psi\|^2 > \|T\phi\|^2 + 1 - \varepsilon^2. \tag{D.31}$$

Thus, $\|T\phi\| < \varepsilon$.

2° *The rest of (a):* Choose a compact $K_\varepsilon \subset \Omega$ s.t. $K'_\varepsilon \subset K^o_\varepsilon$ (e.g., use Lemma A.2.3). Let $\phi \in \mathcal{C}_0(\Omega; B)$ be arbitrary.

By Lemma B.3.10, there is $\psi_k \in \mathcal{C}^\infty_c(\Omega)$ s.t. $\chi_{K'_\varepsilon} \le \psi \le \chi_{K^o_\varepsilon}$. By 1°, we have $\|T(1-\psi)\phi\|_H \le \varepsilon \|(1-\psi)\phi\|_\infty \le \varepsilon\|\phi\|_\infty$, hence

$$\|T\phi\| \le \|T(1-\psi)\phi\| + \|T\psi\phi\| \le \varepsilon\|\phi\|_\infty + \|T\| \sup_{K_\varepsilon} \|\phi\|_B. \tag{D.32}$$

(b) 1° *Isometric extension:* Let $\{K_k\} \subset \Omega$ be as in Lemma A.2.3. For each $k \in \mathbf{N}+1$, there is $\psi_k \in \mathcal{C}^\infty_c(\Omega)$ s.t. $\chi_{K_k} \le \psi_k \le \chi_{K^o_{k+1}}$, by Lemma B.3.10.

Let $\phi \in \mathcal{C}_b(\Omega; B)$. Set $\phi_k := \phi\psi_k \in \mathcal{C}_c(\Omega; B)$. For each $T \in M$, the sequence $\{T\phi_k\}$ is a Cauchy-sequence in H, by (a)1° (because $\phi_k - \phi_{k+j} = 0$ on K_k and $\|\phi_k\| \le \|\phi\|$ for $k, j \in \mathbf{N}+1$). Let $\phi T \in H$ be the limit of this sequence.

Then $\phi : M \to H$ becomes linear and $\|\phi T\| \le \|T\|\|\phi\|$. On the other hand, $\|\phi\|_{\mathcal{B}(M,H)} \ge \|\phi\|_{\mathcal{C}_b}$, because given $\varepsilon > 0$, we can choose $q \in \Omega$ and $S \in \mathcal{B}(B,H)$ s.t. $\|S\| \le 1$ and $\|S\phi(q)\|_H \ge \|\phi\| - \varepsilon$ (note that $S\delta_q : \widetilde{\phi} \mapsto S\widetilde{\phi}(q)$ is in M and $\phi S\delta_q = S\phi(q)$). Thus, $\|\phi\|_{\mathcal{B}(M,H)} = \|\phi\|_{\mathcal{C}_b}$.

2° *Uniqueness:* Assume w.l.o.g. that $\mu \in M$ is extended as above, and $\|\mu\| = 1 = \|\mu'\|$, where also μ' is a continuous extension of μ.

Assume $\mu' \ne \mu$ to obtain a contradiction. Then $\varepsilon := \|\mu'(\phi) - \mu(\phi)\|_H > 0$ for some $\phi \in \mathcal{C}_b$ with $\|\phi\| = 1$.

Choose $r \in (0,2)$ s.t. $(1-\varepsilon/2r)^2 + (\varepsilon/2)^2 > 1$. Choose $k \in \mathbf{N}+2$ s.t. $K_{\varepsilon/r} \subset K^o_{k-1}$. Then $\phi_k - \phi = 0$ on K_k, hence $\mu(\phi_k - \phi) < \varepsilon/r < \varepsilon/2$, hence $\mu'(\phi_k - \phi) > \varepsilon/2$.

There is $\widetilde{\psi} \in \mathcal{C}_0$ s.t. $\|\widetilde{\psi}\| = 1$ and $\mu(\widetilde{\psi}) > 1 - \varepsilon/r$. Set $\psi := \psi_{k-1}\widetilde{\psi}$, so that $|\mu(\psi)| > 1 - \varepsilon/2r$. Now $\psi = 0$ on K^c_k and $\phi_k - \phi = 0$ on K_k, hence $\widetilde{\phi}_\alpha := \alpha\psi + \phi_k - \phi \in \mathcal{C}_b$ satisfies $\|\widetilde{\phi}\| \le 1$ when $|\alpha| = 1$. But

$$\sup_{|\alpha|=1} |\mu'(\widetilde{\phi}_\alpha)| \ge (1-\varepsilon/2r)^2 + (\varepsilon/2)^2 > 1, \tag{D.33}$$

hence $\|\mu'\| > 1$, a contradiction.

(c) The vector $\mu(e^{it\cdot})$ in the lemma refers to the map $\mu_0 \in \mathcal{B}(\mathcal{C}_0(\mathbf{R}^n); \mathcal{B}(B,H))$ induced by $\mu \in M$ through $\mu_0(\phi)x := \mu(\phi x)$ for $\phi \in \mathcal{C}_0(\mathbf{R}^n)$, $x \in B$. Obviously, $\|\mu_0\|_{\mathcal{B}} \le \|\mu\|_M$.

Given $\varepsilon > 0$, choose $k \in \mathbf{N}+1$ s.t. $K_{\varepsilon/4} \subset K_k$. Choose $\delta > 0$ s.t. $|e^{ih\cdot} - 1| \le \varepsilon/2\|\mu\|$ on K_{k+1} when $|h| < \delta$. We have $|e^{i(t+h)q} - e^{itq}| = |e^{ihq} - 1|$ ($q \in \mathbf{R}^n$), hence

$$\|\widetilde{\mu}(t+h)-\widetilde{\mu}(t)\|_H := \|\mu_0(\mathrm{e}^{i(t+h)\cdot})-\mu_0(\mathrm{e}^{it\cdot})\| \le \|\mu_0(\phi_k \mathrm{e}^{it\cdot}(\mathrm{e}^{ih\cdot}-1))\| \tag{D.34}$$

$$+\|\mu_0((1-\phi_k)\mathrm{e}^{it\cdot}(\mathrm{e}^{ih\cdot}-1))\| < \|\mu\|\varepsilon/2\|\mu\|+2\varepsilon/4=\varepsilon. \ \ (t\in\mathbf{R}^n,\ |h|_{\mathbf{R}^n}<\delta). \tag{D.35}$$

Thus, $\widetilde{\mu}\in\mathcal{C}_{\mathrm{bu}}$.

(d) This follows from (a).

(e) This analogous but easier. □

Next we present the Fourier–Plancherel and Paley–Wiener Theorems. Functions in $\mathrm{L}^1\cap\mathrm{L}^2$ can be Fourier transformed with ease, the transforms being continuous, by Lemma D.1.11(a1). Fortunately, in a Hilbert space setting, the L^2 norm of the function is preserved modulo the factor $\sqrt{2\pi}$ (see (D.36)), hence the Fourier transform $\mathrm{L}^1\to\mathcal{C}_0$ can be extended to a transform (isomorphism) $\mathrm{L}^2\to\mathrm{L}^2$, by Lemma A.3.10. Thus, for $f\in\mathrm{L}^2(\mathbf{R};H)$, the transform is defined by $\lim\widehat{f_n}$ (the limit being taken in L^2) for any sequence $\{f_n\}\subset\mathrm{L}^1\cap\mathrm{L}^2$ converging to f in L^2 (note that the integral (D.16) need not converge).

Lemma D.1.15 (Fourier–Plancherel transform) *Let H be a Hilbert space. If $f:\mathbf{R}\to H$ is in $\mathrm{L}^1\cap\mathrm{L}^2$, then $\|\widehat{f}\|_2=\sqrt{2\pi}\|f\|_2$. Therefore, the Fourier(–Plancherel) transform can be extended to an (isometric times $\sqrt{2\pi}$) isomorphism of $\mathrm{L}^2(\mathbf{R};H)$ onto $\mathrm{L}^2(i\mathbf{R};H)$.*

An analogous result holds on L^2_ω ($\omega\in\mathbf{R}$), hence, for all $F,G\in\mathrm{L}^2_\omega(\mathbf{R}_+;H)$ and $f,g\in\mathrm{L}^2(\mathbf{R};H)$, we have that

$$\langle\hat{f},\hat{g}\rangle_{\mathrm{L}^2}=2\pi\langle f,g\rangle_{\mathrm{L}^2},\quad \|\hat{f}\|_{\mathrm{L}^2}=\sqrt{2\pi}\|f\|_{\mathrm{L}^2},\quad \langle\hat{F},\hat{G}\rangle_{\mathrm{H}^2_\omega}=2\pi\langle F,G\rangle_{\mathrm{L}^2_\omega}. \tag{D.36}$$

Similarly, the mapping of $a=\sum_{k\in\mathbf{Z}}a_k\in\ell^2(\mathbf{Z};H)$ to $\widehat{a}(z):=\sum_{k\in\mathbf{Z}}a_kz^k\in\mathrm{L}^2(\partial\mathbf{D};H)$ is an (isometric times $\sqrt{2\pi}$, as above) isomorphism of $\ell^2(\mathbf{Z};H)$ onto $\mathrm{L}^2(\partial\mathbf{D};H)$, and it maps $\ell^2(\mathbf{N};H)$ onto $\mathrm{H}^2(\mathbf{D};H)$.

Here the norm on $\mathrm{L}^2(r\partial\mathbf{D};H)$ is given by $\|\widehat{f}\|_2^2:=\int_0^{2\pi}\|\widehat{f}(r\mathrm{e}^{it})\|_H^2\,dt$ (cf. [Rud86], p. 89 & 337), hence $\|1\|_2=\sqrt{2\pi}$, and

$$\|\widehat{f}\|_{\mathrm{H}^2(r\mathbf{D};H)}:=\sup_{0<t<r}\|\widehat{f}\|_{\mathrm{L}^2(t\partial\mathbf{D};H)}=\lim_{t\to r-}\|\widehat{f}\|_{\mathrm{L}^2(t\partial\mathbf{D};H)}=\|\widehat{f}\|_{\mathrm{L}^2(r\partial\mathbf{D};H)}. \tag{D.37}$$

□

Thus, $\int_0^{2\pi}\|\widehat{f}(\mathrm{e}^{i\theta})\|_H^2\,d\theta=2\pi\sum_{n\in\mathbf{Z}}\|f_n\|_H^2<\infty$ for all $f\in\ell^2(\mathbf{Z};H)$. (Recall that we have defined the Lebesgue measure of $\partial r\mathbf{D}$ to be 2π.)

The above facts can be verified as in the scalar case (see e.g., [Rud86]), and they are presented (in varying generalities; note that H can be assumed to be separable w.l.o.g.) in [RR], [Nikolsky], [HP] and [CZ]. We omit the details. The last equation in (D.37) refers to the boundary function of $\widehat{f}$, cf. Theorem 3.3.1.

By "isometric times $\sqrt{2\pi}$" we mean "$\|\hat{f}\|_{\mathrm{L}^2}=\sqrt{2\pi}\|f\|_{\mathrm{L}^2}$" (recall that isometric means norm-preserving).

If H is an arbitrary Banach space (contrary to standing assumptions of this section), even $\mathrm{L}^1\cap\mathrm{L}^2(\mathbf{R};H)$ functions are not in general mapped into L^2, as illustrated in Example 3.3.4 (see its second remark); however, if $f \in \mathrm{L}^2$ is finite-dimensional, then its range is isomorphic to some Hilbert space $\mathbf{C}^n$, hence then $\widehat{f} \in \mathrm{L}^2$ is well-defined (and coincides with $\widehat{f} \in \mathcal{C}_0$ a.e. if $f \in \mathrm{L}^1 \cap \mathrm{L}^2$).

We call $\mathcal{S}(\mathbf{R};B) := \{ f \in \mathcal{C}^\infty(\mathbf{R};B) \,|\, \|f\|_{k,n} := \|x^k f^{(n)}\|_\infty < \infty \text{ for all } k,n \in \mathbf{N}\}$, equipped with the (Fréchet space) topology induced by the $\|\cdot\|_{k,n}$ seminorms (cf. [Rud73, Theorem 1.37]), the space of *rapidly decreasing functions*.

Lemma D.1.16 ($\mathcal{S}(\mathbf{R};B)$) *The space $\mathcal{S} := \mathcal{S}(\mathbf{R};B)$ is a complete topological vector space (Fréchet space), and $\mathcal{C}_c^\infty$ is a dense subset of $\mathcal{S}$. Moreover, $\mathcal{S} \subset \mathrm{L}^p$ for all $p \in [1,\infty]$.*

Set $\mathcal{F} f := (\mathcal{L} f)(-i\cdot)$ for $f \in \mathrm{L}^1(\mathbf{R};B)$. Then $\mathcal{F}[\mathcal{S}] = \mathcal{S}$, $\mathcal{F}$ is an isomorphism (linear continuous bijection) on $\mathcal{S}$, and $\mathcal{F}^2 f = \text{Я} f$ for all $f \in \mathcal{S}$. Moreover, $\text{Я} \in \mathcal{B}(\mathcal{S})$ and $\tau\phi \in \mathcal{C}(\mathbf{R};\mathcal{S})$ for all $\phi \in \mathcal{S}$. □

(The proof for the scalar case is given in, e.g., Sections 7.3–7.10 of [Rud73], and in Section 2 of [Rauch]. Those proofs cover also the vector-valued case, mutatis mutandis, so we omit the proof.)

Here we extend the typical tool for Cauchy integrals:

Lemma D.1.17 *Assume that γ is a σ-finite, complete, positive measure space, $\Omega \subset \mathbf{C}$ is open, $g \in \mathrm{L}^1(\gamma;B)$, $f : \gamma \times \Omega \to \mathcal{B}(B,B_2)$, $f(t,\cdot) \in \mathrm{H}(\Omega;*)$ for each $t \in \gamma$, $f(\cdot,z) \in \mathrm{L}^\infty(\gamma;*)$ and $\sup_{z\in\Omega} \|f(\cdot,z)\|_\infty < \infty$. Then $F \in \mathrm{H}^\infty(\Omega;B_2)$, where*

$$F(z) := \int_\gamma f(t,z)g(t)\,dt. \tag{D.38}$$

By taking $f(t,z) := (2\pi i(t-z))^{-1}$ and letting γ be curve in $\mathbf{C}$, we get a Cauchy integral.

Proof: Set $M := \sup_{z\in\Omega} \|f(\cdot,z)\|_\infty$. Since $\|f(\cdot,z)g\|_1 \le M\|g_1\|_1 =: M'$ ($z \in \Omega$), we have $\|F\|_{B_2} \le M'$ on Ω. By The Dominated Converge Theorem (with majorant $M\|g\|_B \in \mathrm{L}^1(\gamma)$), F is continuous on Ω. For every rectangle R in Ω, we have, by The Fubini Theorem, that

$$\int_R F(z)\,dz = \int_\gamma \int_R f(t,z)\,dz g(t)\,dt = \int_\gamma g(t)0\,dt = 0, \tag{D.39}$$

(note that $\int_R \int_\gamma \|f(t,z)g(t)\|\,dt\,dz \le \int_R M\|g\|_1\,dz < \infty$ and that $f \in \mathrm{L}(\gamma \times \Omega; \mathcal{B}(B,B_2)$, by Lemma B.4.8). Therefore, $F \in \mathrm{H}(\Omega;B_2)$, by The Morera Theorem. Since $\|F\|_\infty \le M\|g\|_1$, we have $F \in \mathrm{H}^\infty(\Omega;B_2)$. □

If f is holomorphic on both sides of $i\mathbf{R}$ and weakly $\mathrm{L}^1_{\mathrm{loc}}$-continuous to $i\mathbf{R}$, then f is holomorphic on $i\mathbf{R}$ too:

Proposition D.1.18 ($\mathrm{H}(\mathbf{C}_{a,b}) \cap \mathrm{H}(\mathbf{C}_{b,c}) = \mathrm{H}(\mathbf{C}_{a,c})$) *Let $\Omega \subset \mathbf{C}$ be open. Assume that $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus (b+i\mathbf{R});B)$, and that $\Lambda f(t+i\cdot) \to \Lambda f(b+i\cdot)$ in $\mathrm{L}^1([u,v])$, as $t \to b$, for all $\Lambda \in B^*$ and $u,v \in \mathbf{R}$ s.t. $b+[u,v]i \subset \Omega$.*

Then f can be redefined on a null subset of $b+i\mathbf{R}$ so that $f \in \mathrm{H}(\Omega;B)$.

Naturally, we can replace $b+i\mathbf{R}$ by any other straight line (rotate the plane). An analogous result holds for $\partial r\mathbf{D}$ in place of $b+i\mathbf{R}$.

(Recall that $L^p([u,v]) \underset{c}{\subset} L^1([u,v])$ for $p \in [1,\infty]$ and that $\mathbf{C}_{a,b} := \{z \in \mathbf{C} \mid a < \operatorname{Re} z < b\}$.)

In particular, we actually have $f(t+i\cdot) \to f(b+i\cdot)$ uniformly on each such $[-u,v]$ (only outside a null set if we do not redefine f), hence in $L^p([u,v])$ for any $p \in [1,\infty]$.

Proof: We take $b=0$ w.l.o.g.

$1°$ *Case $B=\mathbf{C}$:*

$1.1°$ *Choosing u,v,ε,a,b:* Let $u_0,v_0 \in \mathbf{R}$ be s.t. $u_0 < v_0$ and $[u_0,v_0]i \subset \Omega$. By Lemma A.2.1(c), we have $\varepsilon := d([u_0,v_0]i,\Omega^c) > 0$. By Lemma B.4.19, there are $u \in [u_0,u_0-\varepsilon)$ and $v \in [v_0,v_0+\varepsilon)$ s.t. $f(\cdot+iu), f(\cdot+iv) \in L^1((-\varepsilon,\varepsilon))$. Set $a := -\varepsilon/2$, $c := \varepsilon/2$, so that $[a,c]\times[u,v] \subset \Omega$. Set $A := (a,c)\times(u,v)$. Then $f \in L^1(\gamma)$, where $\gamma := \partial A$ (with the $\mathbf{R}^1$– Lebesgue measure).

$1.2°$ *Defining $F \in H^\infty(A)$:* By Lemma D.1.17 (applied to each $(a+\delta,c'-\delta)\times(u+\delta,v-\delta) \subset A$, $\delta > 0$, to keep $(s-z)^{-1}$ bounded), we have $F \in H^\infty(A)$, where

$$F(z) := \int_\gamma h(s,z)\,f(s)\,ds, \quad h := (2\pi i(s-z))^{-1} \tag{D.40}$$

$1.3°$ *Showing that $f = F$ on $A \setminus i\mathbf{R}$:* For any $z \in A_c^\delta$, we have $\int_{\gamma_c^\delta} h(s,z)\,f(s)\,ds = f(z)$ and $\int_{\gamma_a^\delta} h(s,z)\,f(s)\,ds = 0$, where $\gamma_c^\delta = \partial A_c^\delta$, $A_c^\delta := (\delta,c)\times(u,v) \subset A$, $\gamma_a^\delta = \partial A_a^\delta$, $A_a^\delta := (a,-\delta)\times(u,v) \subset A$, $\delta \in (0,c)$, by the Cauchy Formula ([Rud86, 10.15]). An analogous claim holds for $z \in A_a^\delta$.

Let $z \in A_c^0$. Since $h(\cdot,z)$ is continuous and bounded near $i\mathbf{R}$ and $f(\delta+\cdot) \to f(\cdot)$ in $L^1([u,v])$, as $\delta \to 0$, we have $h(\delta+\cdot,z)\,f(\delta+\cdot) \to h(\cdot,z)\,f(\cdot)$ in $L^1([u,v])$ too.

Consequently, $\int_{\gamma_c^\delta} h(s,z)\,ds = f(z)$ and $\int_{\gamma_a^\delta} h(s,z)\,ds = 0$ hold also for $\delta = 0$, by continuity. An analogous claim holds for $z \in A_a^0$. But $\int_{\gamma_a^0+\gamma_c^0} = \int_\gamma$, hence $F(z) = \int_\gamma h = f(z)$ for every $z \in A_a^0 \cup A_c^0 = A \setminus i\mathbf{R}$.

$1.4°$ *$f = F$ a.e. on $A \cap i\mathbf{R}$:* This follows from 1.3°, since $F(\delta+\cdot) - f(\delta+\cdot) \to 0$ in $L^1([u,v])$, as $\delta \to 0$ (due to the continuity of F), so that $F - f = 0$ as an element of $L^1([u,v])$.

$1.5°$ *Case where $\Gamma := i\mathbf{R}\cap\Omega$ is connected:* Set $\widetilde{u} := \inf\Gamma/i$, $\widetilde{v} := \sup\Gamma_i$, and choose sequences $\{u_n\},\{v_n\} \subset \Gamma/i$ s.t. $u_n \to \widetilde{u}$ and $v_n \to \widetilde{v}$. Then by applying 1.1°–1.4° to each $[u_n,v_n]i$, we see that $f(it) := \lim_{\Omega\setminus i\mathbf{R}\ni z\to it} f(z)$ coincides with the original f a.e. (on each $[u_n,v_n]i$, hence a.e. on $(\widetilde{u},\widetilde{v})i$) and makes f holomorphic on Ω.

$1.6°$ *The original claim for $B = \mathbf{C}$:* Set $\Gamma := i\mathbf{R}\cap\Omega$. By Lemma A.2.2, $\Gamma = \cup_{n\in\mathbf{N}}\Gamma_n$, where the sets $\Gamma_n \subset i\mathbf{R}$ are disjoint open intervals. Redefine f by continuity as in 1.5°. For each n and each $z \in \Gamma_n$, we have $d(z,\cup_{k\neq n}\Gamma_k) > 0$, hence (the new) f is holomorphic at z; consequently (the new) f is holomorphic on the whole Ω. Since $\mathbf{N}$ is countable, f becomes redefined on a null set only.

$2°$ *General B:*

$2.1°$ *$G \in H(\Omega;B)$ s.t. $G = f$ on $\Omega\setminus i\mathbf{R}$:* Let $ir \in \Omega$. Then $G(ir)\Lambda := \lim_{t\to 0}\Lambda f(ir+t) \in \mathbf{C}$ exists for all $\Lambda \in B^*$. By Theorem 2.8 of [Rud73], it follows that $G(ir) \in \mathcal{B}(B^*,\mathbf{C}) =: B^{**}$.

Set $G := f$ on $\Omega \setminus i\mathbf{R}$ and define G on $i\mathbf{R} \cap \Omega$ as above. By 1.6°, $G\Lambda \in \mathrm{H}(\Omega)$ for all $\Lambda \in B^*$. By Lemma D.1.1(b), we have $G \in \mathrm{H}(\Omega; B^{**})$. By continuity, $G(z) \in B$ for $z \in i\mathbf{R} \cap \Omega$ too, hence $G \in \mathrm{H}(\Omega; B)$.

2.2° *$G = f$ a.e. on $\Omega \cap i\mathbf{R}$:* Because $f[\Omega] \subset B$ is separable (since Ω is separable), we can w.l.o.g. assume that B is separable. By Lemma A.3.9, there is a countable norming $\{\Lambda_k\}_{k \in \mathbf{N}} \subset B^*$. By the methods of 1.5°–1.6°, it suffices to consider an arbirary $[u,v]i \subset \Omega$ only. We shall use the diagonal argument.

Since $\Lambda_k f(t + i\cdot) \to \Lambda_k f(i\cdot)$ in $\mathrm{L}^1([u,v])$ for all $k \in \mathbf{N}$, there is a null set $N_0 \subset [u,v]$ and a sequence $t_j^0 \to 0+$ s.t. $\Lambda_0 f(t_j^0 + ir) \to \Lambda_0 f(ir)$, as $j \to \infty$, for all $r \in [u,v] \setminus N_0$, by Theorem B.3.2.

Analogously, there is a subsequence $\{t_j^1\}$ of $\{t_j^0\}$ and a null set $N_1 \subset [u,v]$ s.t. $\Lambda_1 f(t_j^1 + ir) \to \Lambda_1 f(ir)$ for all $r \in [u,v] \setminus N_1$. Given $N_0, \ldots, N_k$ and $\{t_j^0\}, \cdots, \{t_j^k\}$ as above, choose a subsequence $\{t_j^{k+1}\}$ of $\{t_j^k\}$ and a null set $N_{k+1} \subset [u,v]$ s.t. $\Lambda_{k+1} f(t_j^{k+1} + ir) \to \Lambda_{k+1} f(ir)$ for all $r \in [u,v] \setminus N_{k+1}$.

Consequently, $\Lambda_k f(t_j^j + ir) \to \Lambda_k f(ir)$, as $j \to \infty$, for all $r \in [u,v] \setminus N$, where $N := \cup_k N_k$. Thus, for such r (hence a.e.) we have $\Lambda_k[f(ir) - G(ir)] = 0$ for all k, hence $f(ir) = G(ir)$.

3° *The final claims:* If we rotate $i\mathbf{R}$, then we just have to use the Lebesgue measure on this oblique line.

If $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus \partial b\mathbf{D}; B)$, and $\Lambda f(r\mathrm{e}^{i\cdot}) \to \Lambda f(b\mathrm{e}^{i\cdot})$ in $\mathrm{L}^1([u,v])$, as $t \to b$, for all $\Lambda \in B^*$ and $u, v \in \mathbf{R}$ s.t. $b\mathrm{e}^{[u,v]i} \subset \Omega$, then f can be redefined on a null subset of $\Omega \cap \partial b\mathbf{D}$ so that $f \in (\Omega; B)$.

(The above proof applies, mutatis mutandis; se polar coordinates for the application of Lemma B.4.19 in 1.1° (locally, near an arc of $\partial(r\mathbf{D})$).)

Remark: Analogously, if $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus \Gamma; B)$, $\Gamma \subset \Omega$ is a continuous image $g[J]$ of an interval J, and any subinterval $[u,v] \subset g^{-1}[J \cap \Omega]$ has a neighborhood homeomorphic to $(-\varepsilon, \varepsilon) \times (u,v)$, with the homeomorphism satisfying $h(0,\cdot) = g$ on (u,v), and $\Lambda f(h(t,\cdot)) \to \Lambda f(h(0,\cdot))$ in $\mathrm{L}^1([u,v])$, as $t \to 0$, for all $\Lambda \in B^*$, then $f \in \mathrm{H}(\Omega; B)$ after a redefinition on a null subset of Γ. □

Analogously, if f is H^p on both sides of $i\mathbf{R}$ and has same boundary function from both sides, then f is holomorphic and H^p on the whole region:

Lemma D.1.19 ($\mathrm{H}^p(\mathbf{C}_{a,b}) \cap \mathrm{H}^p(\mathbf{C}_{b,c}) = \mathrm{H}^p(\mathbf{C}_{a,c})$) *Let $a < b < c$. Assume that $f : \mathbf{C}_{a,c} \to B$ is in $\mathrm{H}^p(\mathbf{C}_{a,b}; B) \cap \mathrm{H}^p(\mathbf{C}_{b,c}; B)$ and that f is the boundary function of itself on $b + i\mathbf{R}$ from both sides, in the sense of (6.) of Theorem 3.3.1. Then f can be redefined on a null subset of $b + i\mathbf{R}$ so that we get $f \in \mathrm{H}^p(\mathbf{C}_{a,c}; B)$.*

Proof: By Proposition D.1.18, we get $f \in \mathrm{H}(\mathbf{C}_{a,c}; B)$ (since $\|\cdot\|_{\mathrm{L}^1([u,v])} \le M\|\cdot\|_{\mathrm{L}^p(\mathbf{R})}$). By assumption (6.), $f \in \mathrm{H}^p(\mathbf{C}_{a,c}; B)$ (and $\|f\|_{\mathrm{H}^p(\mathbf{C}_{a,c};B)} = \max\{\|f\|_{\mathrm{H}^p(\mathbf{C}_{a,b};B)}, \|f\|_{\mathrm{H}^p(\mathbf{C}_{b,c};B)}\}$). □

If f is in H^2 on both $\mathbf{C}^+$ and on $\mathbf{C}^-$, then f is a constant:

Proposition D.1.20 ($\mathrm{H}^p_+ \cap \mathrm{H}^p_- = \mathbf{C}$) *Let $f_\pm \in \mathrm{H}^p(\mathbf{C}^\pm_\omega; B)$, where $\omega \in \mathbf{R}$, $p \in [1, \infty]$.*

If $\Lambda f_+ = \Lambda f_-$ on $\omega + i\mathbf{R}$ for each $\Lambda \in B^*$, then $f_+ \equiv x \equiv f_-$ for some $x \in B$ ($x = 0$ if $p < \infty$).

Analogously, if $f_\pm \in \mathrm{H}^p(\mathbf{D};B)$ and $\Lambda f_+(z) = \Lambda f_-(z^{-1})$ for a.e. $z \in \partial\mathbf{D}$, for each $\Lambda \in B^*$, then $f_+ \equiv x \equiv f_-$ for some $x \in B$.

Proof: 1° *Case* $\mathrm{H}^p(\mathbf{C}^\pm_\omega;B)$*:* Naturally, by "$\Lambda f_+ = \Lambda f_-$" we mean that the $\mathrm{L}^p(\omega+i\mathbf{R})$ boundary functions are equal a.e.; such functions exist, by Theorem 3.3.1(a2).

Assume w.l.o.g. that $\omega = 0$. Let $\Lambda \in B^*$. By Lemma D.1.19, we have $\Lambda f_+ = f_\Lambda = \Lambda f_-$ for some $f_\Lambda \in \mathrm{H}^p(\mathbf{C}^+)$.

But then $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C}^+_0)$, by (6.4.3) of [HP]; analogously, $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C}^-_1)$, hence $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C})$.

By the Liouville Theorem, $f_\Lambda \equiv x_\Lambda$ for some $x_\Lambda \in \mathbf{C}$. This holds for each $\Lambda \in B^*$, hence f_+ and f_- are equal to a single constant $x \in B$. If $p < \infty$, then, obviously, $x = 0$.

2° *Case* $\mathrm{H}^p(\mathbf{D};B)$*:* Set $f := f_+$ on $\mathbf{D}$ and $f(z) := f_-(z^{-1})$ for $z \in \overline{\mathbf{D}}^c$. As above, we see that we Λf can be extended $f_\Lambda \in \mathrm{H}(\mathbf{D})$. Since $f_\Lambda(\infty) = \Lambda f_-(0)$, f_Λ is bounded, hence f_Λ is a constant. Thus, f is a constant, as in 1°. □

If f belongs to L^p_r for two different r's, then $\widehat{f}$ is holomorphic on the corresponding strip:

Proposition D.1.21 ($\mathcal{L}[\mathrm{L}^p_a \cap \mathrm{L}^p_b] \subset \mathrm{H}(\mathbf{C}_{a,b};*)$) *Let $f \in \mathrm{L}^p_a(\mathbf{R};B) \cap \mathrm{L}^p_b(\mathbf{R};B)$, $p \in [1,\infty]$, $a < b$. Then*

- *(a1) We have $\widehat{f} \in \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$ whenever $a < a' < b' < b$; in particular, $\widehat{f} \in \mathrm{H}(\mathbf{C}_{a,b};B)$. Moreover, $f \in \mathrm{L}^1_r(\mathbf{R};B)$ for all $r \in (a,b)$, hence $\widehat{f}(s)$ converges absolutely on $\mathbf{C}_{a,b}$.*
- *(a2) The mapping $\mathrm{L}^p_a(\mathbf{R};B) \cap \mathrm{L}^p_b(\mathbf{R};B) \to \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$ is continuous.*
- *(b) If $p = 1$, then $\widehat{f} \in \mathcal{C}_{\mathrm{bu}}(\overline{\mathbf{C}_{a,b}};B) \cap \mathrm{H}^\infty(\mathbf{C}_{a,b};B)$ and $\|\widehat{f}(s)\|_B \le \|\pi_+ f\|_{\mathrm{L}^p_a} + \|\pi_- f\|_{\mathrm{L}^p_b}$ for all $s \in \overline{\mathbf{C}_{a,b}}$.*
- *(c)* **($\mathcal{L}[\mathrm{L}^2_a \cap \mathrm{L}^2_b] = \mathrm{H}^2(\mathbf{C}_{a,b};H)$)** *Assume that $p = 2$ and that $B = H$ is a Hilbert space. Then $\widehat{f} \in \mathrm{H}^2(\mathbf{C}_{a,b};H)$, and $\widehat{f}$ has the nontangential boundary function $\widehat{f}$ (the Plancherel transform of f) on $a + i\mathbf{R}$ and on $b + i\mathbf{R}$ (in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a)) and $(2\pi)^{-1/2}\|\widehat{f}(r + i\cdot)\|_2 \le \|\pi_+ f\|_{\mathrm{L}^2_a} + \|\pi_- f\|_{\mathrm{L}^2_b}$ ($r \in [a,b]$).*

  Conversely, if $\widehat{g} \in \mathrm{H}^2(\mathbf{C}_{a,b};H)$, then there is $g \in \mathrm{L}^2_a(\mathbf{R};H) \cap \mathrm{L}^2_b(\mathbf{R};H)$ s.t. $\widehat{g} = \mathcal{L}g$.

Recall that $\mathrm{H}^p(\mathbf{C}_{a,b};B) := \{f \in \mathrm{H}(\mathbf{C}_{a,b};B) \mid \sup_{r\in(a,b)} \|f(r+i\cdot)\|_p < \infty\}$.

Proof: Note that $f \in \pi_+\mathrm{L}^p_a(\mathbf{R};B) \cap \pi_-\mathrm{L}^p_b(\mathbf{R};B)$.

(a1) By Lemma D.1.10(a1)&(a2), we have $\widehat{\pi_+ f} \in \mathrm{H}^\infty(\mathbf{C}^+_{a'};B)$ and $\widehat{\pi_- f} \in \mathrm{H}^\infty(\mathbf{C}^-_{b'};B)$, hence $\widehat{f} \in \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$. The second claim follows from the proof of Lemma D.1.10(a2).

(a2) See the proof of (a1) and that of Lemma D.1.10(a2) for a bound of form $\|\widehat{f}\|_{\mathrm{H}^\infty} \le M(\|f\|_{\mathrm{L}^p_a} + \|f\|_{\mathrm{L}^p_b}$.

(b) Cf. the proof of (a) (use Lemma D.1.11(a1')).

(c) 1° *Properties of $\widehat{f}$:* Note that $\mathcal{L}\pi_+ f \in \mathrm{H}^2(\mathbf{C}^+_a; B)$ and $\mathcal{L}\pi_- f \in \mathrm{H}^2(\mathbf{C}^-_b; B)$, by Lemma D.1.15. By Theorem 3.3.1(a2), $\mathcal{L}\pi_+ f$ has a boundary function on $a + i\mathbf{R}$; by continuity, so does $\mathcal{L}\pi_- f$ too (in the sense of (1.) and (2.) of Theorem 3.3.1(a); claim (6.) (which implies (5.) and (4.)) follows from the strong continuity of Lemma D.1.8(a1)).

The "mirror image" boundary function on $b + i\mathbf{R}$ (from the left) is obtained analogously. The norm estimate is obvious.

2° *The converse claim:* For each $r \in (a,b)$, there is $g_r \in \mathrm{L}^2_r(\mathbf{R}; H)$ s.t. $\widehat{g} = \mathcal{L}g_r$ a.e. on $r + i\mathbf{R}$, by the Fourier–Plancherel Theorem.

Let $d \in (a,b)$. Set $F := \pi_+ g_d$, $G := \pi_- g_d$. Then $\mathcal{L}G \in \mathrm{H}^2(\mathbf{C}^-_d; H)$ is the boundary function of itself, by Theorem 3.3.1(b) (whose proof is obviously independent on this lemma, relying on [RR] and the part of Appendices preceding this lemma), and so is $\widehat{g} \in \mathrm{H}^2(\mathbf{C}_{a,d}; H)$ on $d + i\mathbf{R}$, hence $\widehat{g} - \mathcal{L}G = \mathcal{L}F$ is the boundary function of $\widehat{h}$ (a.e.) on $d + i\mathbf{R}$. But $\mathcal{L}F \in \mathrm{H}^2(\mathbf{C}^+_d; H)$, hence $\widehat{h}$ extends to a function $\widehat{h} \in \mathrm{H}^2(\mathbf{C}^+_a; H)$, by Lemma D.1.19.

It follows that $\widehat{h} = \mathcal{L}h$ on $\overline{\mathbf{C}}^+_a$ for some $h \in \mathrm{L}^2_a(\mathbf{R}_+; H)$. But then $F = h$. Analogously, we can show that $G \in \mathrm{L}^2_b(\mathbf{R}_-; H)$. It follows that $g_d = F + G \in \mathrm{L}^2_a \cap \mathrm{L}^2_b$. Because $\widehat{g} = \mathcal{L}g_d$ on $d + i\mathbf{R}$, we have $\widehat{g} = \mathcal{L}g_d$ on the whole $\mathbf{C}_{a,b}$, by Lemma D.1.2(e). $\square$

If $f \in \mathrm{L}^\infty(\mathbf{R}; B)$, then $\|f\chi_A\|_1 \le \|f\|_\infty m(A) < \infty$ whenever $m(A) < \infty$. Thus, we can then apply the following test:

Lemma D.1.22 *Let B_0 be a closed subspace of the Banach space B, and $f \in \mathrm{L}^\infty(\mathbf{R}; B)$. If $\mathcal{L}f\chi_A \in \mathcal{C}_0(\mathbf{R}; B_0)$ whenever $m(A) < \infty$, then $f \in \mathrm{L}^\infty(\mathbf{R}; B_0)$ i.e., $f(t) \in B_0$ for a.e. $t \in \mathbf{R}$.*

Note that always $\mathcal{L}f\chi_A \in \mathcal{C}_0(\mathbf{R}; B)$, because $f\chi_A \in \mathrm{L}^1$.

Proof: (We could have as well assume that $f \in \mathrm{L}^1_{\mathrm{loc}}$ and that A is bounded and measurable.)

Set $E := f^{-1}[B \setminus B_0]$. We assume that $m(E) > 0$, derive a contradiction, and deduce that $f(t) \in B_0$ for a.e. $t \in \mathbf{R}$.

Because $f_{|E}$ is nowhere zero, Lemma B.2.8(b) provides us $A \subset E$ and $\mathrm{L} \in B^*$ s.t. $0 < m(A)$, $\Lambda = 0$ on B_0, and $\mathrm{Re}\,\Lambda > 1$ on A. Choose $A' \subset A$ s.t. $0 < m(A') < \infty$. Then $0 \ne \Lambda f\chi_{A'} \in \mathrm{L}^1(\mathbf{R}; B)$, hence $0 \ne \mathcal{L}\Lambda f\chi_{A'} = \Lambda\mathcal{L}f\chi_{A'} \in \mathcal{C}_0(\mathbf{R})$, by Lemma D.1.11(a3). But $\mathcal{L}f\chi_{A'}$ is B_0-valued, hence $\Lambda\mathcal{L}f\chi_{A'} = 0$, a contradiction. $\square$

If a function is holomorphic around $\partial\mathbf{D}$, then it is the inverse Cayley transform of some $\mathrm{MTI}^{\mathrm{L}^1}$ operator.

Lemma D.1.23 *Assume that $\Omega \subset \mathbf{C}$ is open and s.t. $\partial\mathbf{D} \subset \Omega$, and that $g \in \mathrm{H}(\Omega; B)$. Then there are $b \in B$ and $f \in \mathrm{L}^1(\mathbf{R}; B)$ s.t. $g = b + \widehat{f} \circ \phi^{-1}_{\mathrm{Cayley}}$.*

Proof: 1° $b_0 = b_1 = b_2 = b_3 = 0$ *w.l.o.g.:* Write g as $\sum_{k=0}^\infty b_k(z+1)^k$ on a neighborhood of $K := \{z \in \mathbf{C} \,\big|\, |z+1| \le \varepsilon\} \subset \Omega$ (for some $\varepsilon > 0$). Because

$$(\cdot + 1) = \widehat{2\pi_+ e^{-\cdot}} \circ \phi_{\text{Cayley}}^{-1} = \widehat{F} \circ \phi_{\text{Cayley}}^{-1}, \tag{D.41}$$

where $\widehat{F} = \widehat{2\pi_+ e^{-\cdot}} \in \mathcal{L}[\mathrm{L}^1(\mathbf{R}_+; B)]$, we can take $\widehat{F_3} := \sum_{k=0}^3 b_k \widehat{F}^k$, to obtain that $g - \widehat{F_3} \circ \phi_{\text{Cayley}}^{-1} = \sum_{k=4}^\infty b_k(z+1)^k$. By Lemma D.1.12(c'), $\widehat{F_3} = b' + \widehat{f_3}$ for some $b' \in B$ and $f_3 \in \mathrm{L}^1(\mathbf{R}_+; B)$ (since $\pi_+ e^{-\cdot} \in \mathrm{L}^1(\mathbf{R}_+; B)$). Thus, we have reduced the problem to the case where $b_0 = b_1 = b_2 = b_3 = 0$.

2° Set $h := g \circ \phi_{\text{Cayley}}$. Assuming $b_0 = b_1 = b_2 = b_3 = 0$, it follows that $z \mapsto g(z)/z^4$ is holomorphic around K; let $M := \max_{z \in K} \|g(z)/z^4\|_B$. Then $\|g(z)\|_B \le |z+1|^4 M$ for $z \in K$, i.e., $\|h(s)\|_B \le |2/(1+s)|^4 M$ when $|2/(1+s)| < \varepsilon$; in particular $h|_{i\mathbf{R}} \in \mathrm{L}^1(i\mathbf{R}; B)$, because h is continuous on $i\mathbf{R}$.

3° Because ϕ_{Cayley}' and ϕ_{Cayley}'' are bounded on $i\mathbf{R}$, it follows (as in 2°) from the chain rule that also h' and h'' are in $\mathrm{L}^1(i\mathbf{R}; B)$, hence

$$f := \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix\cdot} h(ix)\, dx = \frac{1}{2\pi} \widehat{h(i\cdot)}(-i\cdot) \in \mathrm{L}^1(\mathbf{R}; B) \tag{D.42}$$

and $h = \widehat{f}$, by Lemma D.1.11(e2)&(e1). $\square$

Sometimes the properties of a transfer function $\widehat{\mathbb{D}} : i\mathbf{R} \to B$ can be recovered by multiplying it with the following function:

Lemma D.1.24 *Let $\varepsilon > 0$, $p \in [1, \infty]$. For each $t > 0$ and $r \in \mathbf{R}$, we set*

$$f_{t,r}(x) := \pi_+ 2t^{3/2} x e^{-(t-ir)x}, \quad \text{i.e.,} \quad \widehat{f_{t,r}}(s) = \widehat{f_{t,0}}(s - ir) = 2t^{3/2}(s + t - ir)^{-2}. \tag{D.43}$$

We have $f_{t,r} \in \mathrm{L}^2_{-t/2}(\mathbf{R}_+)$, $\|f_{t,r}\|_2 = 1$, and $\widehat{f} \in \mathrm{L}^p(\omega + i\mathbf{R}) \cap \mathcal{C}_0(\omega + i\mathbf{R})$ ($\omega \ge 0$). Moreover, the following hold:

(a) There is $\delta > 0$ s.t.

$$\sup_{s \in \mathbf{C},\, |s - ir| \ge \varepsilon} |\widehat{f_{t,r}}(s)| < \varepsilon \;\;\&\;\; \|\widehat{f_{t,r}}\|_{\mathrm{L}^p(\{i\rho \,\big|\, |\rho - r| \ge \varepsilon\})} < \varepsilon \;\;\&\;\; \|\widehat{f_{t,r}}\|_{\mathrm{L}^p(\omega + i\mathbf{R})} < \varepsilon \tag{D.44}$$

whenever $0 < t \le \delta$, $r \in \mathbf{R}$, $\omega \ge \varepsilon$.

(b) If $E \subset i\mathbf{R}$ is measurable and $p \in [1, 2]$, then, for a.e. $r \in E$, there is $\delta = \delta_{f,E,r,\varepsilon} > 0$ s.t.

$$\|\widehat{f_{t,r}}\|_{\mathrm{L}^p(i\mathbf{R} \setminus E_{\varepsilon,r})} < \varepsilon \quad (0 < t \le \delta), \tag{D.45}$$

where $E_{\varepsilon,r} := \{i\rho \in E \,\big|\, |\rho - r| < \varepsilon\}$.

(c) For any $g \in \mathrm{L}^p(i\mathbf{R}; B)$ and $p \in [1, \infty)$ (or $g \in \mathcal{C}_0(i\mathbf{R}; B)$ and $p = \infty$) and $t > 0$, there is $R \in \mathbf{R}$ s.t. $\int_{i\mathbf{R}} \|\widehat{f_{t,r}} g\|_B\, dm < \varepsilon$ and $\int_{i\mathbf{R}} |\widehat{f_{t,r}}|^2 \|g\|_B\, dm < \varepsilon$ whenever $r \in \mathbf{R}$, $|r| \ge R$.

Proof: The claims at the beginning of the lemma follow from straightforward computations (which we omit).

(a) We shall assume that $r=0$ (use translation), w.l.o.g.

1° $\sup_{s\in\mathbf{C},\,|s|\geq\varepsilon}|\widehat{f}_{t,0}(s)|<\varepsilon$: Set $\delta_1 := \min\{\varepsilon/2,(\varepsilon^2/9)^{2/3}\}$. Then, for all $s\in\mathbf{C}$ and $t\in(0,\delta_1]$, we have

$$|s|\geq\varepsilon \implies |\widehat{f}(s)|\leq\frac{2t^{3/2}}{(|s|-t)^2}\leq\frac{2\varepsilon^2/9}{(\varepsilon-\varepsilon/2)^2}=\frac{2\varepsilon^2/9}{\varepsilon^2/4}<\varepsilon, \tag{D.46}$$

which provides the first inequality in (D.45).

2° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(\omega+i\mathbf{R})}\to 0$: For $p=\infty$ this follows from 1°. For $p<\infty$, we have

$$\|\widehat{f}_{t,0}(\omega+i\cdot)\|_p^p\leq\|2^p|\varepsilon+i\cdot|^{-2p}t^{3p/2}\|_1\to 0, \tag{D.47}$$

as $t\to 0+$, for any $\omega\geq\varepsilon$, because $|\varepsilon+i\cdot|^{-2p}=|\varepsilon^2+\cdot^2|^{-p}\in\mathrm{L}^1$. Therefore, there is $\delta_3>0$ s.t. third inequality in (D.45) is achieved for all $t\in(0,\delta_3]$.

3° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(\{i\rho\,|\,|\rho|\geq\varepsilon\})}<\varepsilon$: We have $|\widehat{f}_{t,0}(i\rho)|=2t^{3/2}/(\rho^2+t^2)\leq 2t^{3/2}/\varepsilon^2\to 0$, as $t\to 0+$, whenever $|\rho|\geq\varepsilon$. Case $p=\infty$ follows directly from this; case $p<\infty$ follows from this and the Dominated Convergence Theorem, because $\widehat{f}_{t,0}\in\mathrm{L}^p$, as noted in 2°.

(b) 1° *Assumptions:* By Section 7.11 of [Rud86], we have

$$\lim_{\varepsilon\to 0+}\frac{m(E^c\cap(r-\varepsilon,r+\varepsilon))}{m(r-\varepsilon,r+\varepsilon)}=0 \tag{D.48}$$

for a.e. $r\in E$. We assume that r is such and $\varepsilon>0$, and find $\delta>0$ s.t. (D.45) holds. W.l.o.g. we assume that $r=0$ (use translation of E and f).

2° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(i\mathbf{R}\setminus E_{\varepsilon,r})}<\varepsilon$: Choose $R>0$ s.t. $2^{p+1}\int_R^\infty|y^2+1|^{-p}\,dy<\varepsilon^p/4^p$. Choose $\eta\in(0,\varepsilon)$ s.t. $m(E^c\cap(-\gamma,\gamma))/2\gamma<\varepsilon_1:=\varepsilon^p/3^p2^{p+1}R$ for all $\gamma\in(0,\eta]$. Set $\delta:=\min\{1,\eta/R,\varepsilon^4/8,\varepsilon/2,\varepsilon^3\}$. Then, for any $t\in(0,\delta]$, we have

$$\int_{|y|\geq Rt}|\widehat{f}_{t,0}(iy)|^p\,dy=2^{p+1}t^{3p/2}\int_{y\geq Rt}|y^2+t^2|^{-p}\,dy=2^{p+1}t^{3p/2}\int_{u\geq R}|u^2t^2+t^2|^{-p}t\,du \tag{D.49}$$

$$=2^pt^{3p/2-2p+1}\int_{u\geq R}|u^2+1|^{-p}\,du<2^pt^{1-p/2}\varepsilon^p/4^p\leq\varepsilon^p/2^p, \tag{D.50}$$

because $1-p/2\geq 0$. Moreover, $Rt<\eta$, hence

$$\int_{\{|y|<Rt,y\in E^c\}}|\widehat{f}_{t,0}(iy)|^p\,dy\leq\|\widehat{f}_{t,0}\|_\infty^p\cdot 2Rt\cdot\varepsilon_1\leq\left(\frac{2t^{3/2}}{t^2}\right)^p\cdot 2tR\varepsilon_1 \tag{D.51}$$

$$\leq 2^{p+1}t^{-p/2+1}R\varepsilon_1=\varepsilon^p/3^p \tag{D.52}$$

Because $\varepsilon^p/2^p+\varepsilon^p/3^p<\varepsilon^p$, we have established (b).

(c) Because $\widehat{f},|\widehat{f}|^2\in\mathrm{L}^{p/(p-1)}(i\mathbf{R})\cap\mathcal{C}_0(i\mathbf{R})$, this follows from Lemma B.3.13. □

The following function is handy when dealing with Fourier transforms:

Lemma D.1.25 *Define* $\phi(t) := e^{-t^2/2}$. *Then* $\widehat{\phi}$ *and* $\widehat{\phi}^{-1}$ *are entire functions and* $\widehat{\phi}(s) = \sqrt{2\pi}e^{s^2/2} \neq 0$ *for* $s \in \mathbf{C}$*; in particular,* $\widehat{\phi}(ir) = \sqrt{2\pi}e^{-r^2/2} > 0$ *for* $r \in \mathbf{R}$.

Moreover, $\phi \in \mathcal{S}(\mathbf{R}) \cap L^p_\omega$ *for all* $p \in [1,\infty]$ *and* $\omega \in \mathbf{R}$*, and the functions of the form* $\sum_{k=1}^n \tau(t_k)\phi b_k$*, where* $n \in \mathbf{N}+1$*, and* $t_k \in \mathbf{R}$ *and* $b_k \in B$ *for all* $k \in \mathbf{N}$*, are dense in* $L^1(\mathbf{R};B)$ *as well as in* $L^2(\mathbf{R};B)$.

Proof: By the dominated convergence theorem, we can exchange the order of integration and differentiation to obtain that $\widehat{\phi}$ is holomorphic everywhere ($\widehat{\phi} \in H(\mathbf{C})$). By [Rauch, pp. 64–65], $\widehat{\phi}(s) = \sqrt{2\pi}e^{s^2/2}$ holds for $s \in i\mathbf{R}$, hence it holds everywhere, by Lemma D.1.2(e). One easily verifies that $\phi \in \mathcal{S}(\mathbf{R}) \cap L^p_\omega(\mathbf{R})$ for all p, ω.

Because the Fourier transform of ϕ is nowhere zero, the density claim holds for $B = \mathbf{C}$, by, e.g., Theorem 9.5 of [Rud73] (the L^1 case) and p. 145 of [Katzn] (the L^2 case); the general case follows from the density of finite-dimensional functions in L^p (Theorem B.3.11). $\square$

We finish this section by presenting one more vector-valued extension of a standard result:

Lemma D.1.26 *The span of* $\{\pi_- e^{s\cdot} u_0 \mid s \in (\omega, \omega+1),\ u_0 \in U\}$ *is dense in* $L^2_\omega(\mathbf{R}_-;U)$.

Proof: We take $\omega = 0$ w.l.o.g. Simple functions are dense in $L^2(\mathbf{R}_-;U)$, by Theorem B.3.11, hence so is the span of $\{\phi u_0 \mid \phi \in L^2(\mathbf{R}_-),\ u_0 \in U\}$. Therefore, we may and will assume that $U = \mathbf{C}$ w.l.o.g.

Let $u \in L^2(\mathbf{R}_-)$. If $u \in \{\pi_- e^{s\cdot} \mid s \in (0,1)\}^\perp$, i.e.,

$$0 = \langle e^{s\cdot}, u\rangle_{L^2(\mathbf{R}_-)} = \int_{-\infty}^0 e^{st}u(t)\,dt = \int_0^\infty e^{-st}u(-t)\,dt = \widehat{\mathbf{Я}u}(s), \tag{D.53}$$

for all $s \in (0,1)$, then $\widehat{\mathbf{Я}u} = 0$ on $\mathbf{C}^+$, hence then $u = 0$, by Lemma D.1.2(e). Therefore, $\{e^{s\cdot} \mid s \in (0,1)\}$ is dense in $L^2(\mathbf{R}_-)$. $\square$

Notes

As obvious from the proofs, many of the results of this appendix are well known at least in the scalar case. Lemma D.1.23 is due to O. Staffans.

Further results on holomorphic vector-valued functions are given in, e.g., [HP]. General vector-valued measures (with MTIC as a special case; cf. Lemma D.1.12 and Section 2.6) are treated in [DU], [Dinculeanu], [Dobrakov], [Park] and in references therein.