Appendix C

Almost Periodic Functions (AP)

An aphorism is never exactly true; it is either a half-truth or one-and-a-half truths.
— Karl Kraus

In this appendix we briefly present a few facts about vector-valued almost periodic functions.

Because the Fourier transform of a discrete measure (e.g., of an I/O map consisting purely of delays) is an almost periodic function (by Lemma C.1.2(h2)), this has important applications in control theory. We shall use this theory to combine the $\mathrm{MTI_d}$ and $\mathrm{MTIC}^{\mathrm{L}^1}$ spectral factorization results for ones for MTI in Section 5.2 by using the results of this section.

Definition C.1.1 (AP) *Let B be a Banach space, and let $f : \mathbf{R} \to B$ be continuous. If $\varepsilon > 0$, then a number $T \in \mathbf{R}$ is called an* ε-almost period *of f if*

$$\sup_{t\in\mathbf{R}} \|f(t+T) - f(t)\| \le \varepsilon. \tag{C.1}$$

The function f is called almost periodic *($f \in \mathrm{AP}(\mathbf{R};B)$) if for each $\varepsilon > 0$ there is $R > 0$ s.t. each interval $(r, r+R) \subset \mathbf{R}$ ($r \in \mathbf{R}$) contains at least one ε-almost period of f.*

Lemma C.1.2 (AP) *Let B and $B_1,\dots,B_n$ be Banach spaces. Then we have the following:*

(a) The set $\mathrm{AP}(\mathbf{R};B)$ is a closed subspace of $\mathcal{C}_{\mathrm{bu}}(\mathbf{R};B)$ (with the norm $\|f\| := \sup_{\mathbf{R}} \|f\|_B$). If B is a Banach algebra, then so is $\mathrm{AP}(\mathbf{R};B)$.

(b) **Bochner's criterion** *A function $f \in \mathcal{C}_{\mathrm{b}}(\mathbf{R};B)$ is* AP *iff $\{f(\cdot - h)\}_{h\in\mathbf{R}}$ is compact in $\mathcal{C}_{\mathrm{b}}(\mathbf{R};B)$.*

(c) If $f_k \in \mathrm{AP}(\mathbf{R};B_k)$ for $k = 1,\dots,n$, then $(f_k)_{k=1}^n \in \mathrm{AP}(\mathbf{R};B_1 \times \cdots \times B_n)$; if, in addition, $\phi \in \mathcal{C}(B_1 \times \cdots \times B_n;B)$, then $\phi(f_1, f_2,\dots, f_n) \in \mathrm{AP}(\mathbf{R};B)$.

(d1) If $f \in \mathrm{AP}(\mathbf{R};B)$, then $K := \overline{f[\mathbf{R}]}$ is compact.

(d2) If f and K are as in (d1), and $\phi \in \mathcal{C}(K;B_1)$, then $\phi \circ f \in \mathrm{AP}(\mathbf{R};B_1)$.

(e) **(AP is inverse-closed in $\mathcal{C}_b$:)** *If $f \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B_1,B_2))$ and f is pointwise invertible on $\mathbf{R}$ with f^{-1} bounded (i.e., $f \in \mathcal{G}\mathcal{C}_b$), then $f^{-1} \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B_2,B_1))$ (i.e., $f \in \mathcal{G}\mathrm{AP}$).*

(f1) **(Bohr transformation)** *Let $f \in \mathrm{AP}(\mathbf{R};B)$. For each $\lambda \in \mathbf{R}$, the limit*

$$f_\lambda := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)\mathrm{e}^{-i\lambda t}\,dt \in B \tag{C.2}$$

exists, the values f_λ are bounded by $\sup_{\mathbf{R}}\|f\|_B$ and determine $f \in \mathrm{AP}$ uniquely, at most countably many of them are nonzero, and 0 is the only limit point of the set $\{f_\lambda\}_{\lambda\in\mathbf{R}}$ (called the AP-spectrum *of f).*

We formally denote $f \sim \sum_{\lambda\in\mathbf{R}} f_\lambda \mathrm{e}^{i\lambda t}$.

(f2) The function $f \mapsto f_\lambda$ is linear and of norm 1.

(f3) If $f \in \mathrm{AP}(\mathbf{R};B_1)$, $g \in \mathrm{AP}(\mathbf{R};B_2)$, $\phi \in \mathcal{B}(B_1\times B_2,B)$, and either $\sum_{\lambda\in\mathbf{R}}\|f_\lambda\|_{B_1} < \infty$ or $\sum_{\lambda\in\mathbf{R}}\|g_\lambda\|_{B_2} < \infty$ (in particular, when either AP-spectrum is finite), then for $\lambda \in \mathbf{R}$ we have

$$\phi(f,g)_\lambda = \sum_{\mu\in\mathbf{R}} \phi(f_\mu, g_{\lambda-\mu}), \tag{C.3}$$

In particular, if $B = B_1 = B_2$ is a Banach algebra, then $(fg)_\lambda = \sum_{\mu\in\mathbf{R}} f_\mu g_{\lambda-\mu}$ for $\lambda \in \mathbf{R}$.

(f4) If H is a Hilbert space, then $\|f\|_{\overline{\mathrm{AP}}} := \sum_\lambda \|f_\lambda\|_H^2 = (\|f\|_H^2)_0 \le \sup_{\mathbf{R}}\|f\|_H^2$, and $(f,g) := (\langle f,g\rangle_H)_0$ makes $\mathrm{AP}(\mathbf{R};H)$ an inner product space (i.e., a pre-Hilbert space), whose completion "$\overline{\mathrm{AP}}$" is isomorphic to $\ell^2(\mathbf{R};H)$. (Note that $\|f\| := \|f\|_\infty$, and that we write $\|f\|_{\overline{\mathrm{AP}}}$ when we refer to this Besicovitch space norm.*)*

(f5) Let $F \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B,B_2))$. Then, in $\overline{\mathrm{AP}}$, $f \mapsto Ff$ has the norm $\|F\|_{\mathcal{B}(\overline{\mathrm{AP}}(\mathbf{R};B),\overline{\mathrm{AP}}(\mathbf{R};B_2))} \le \|F\|_\infty$.

(g) If $f \in \mathrm{AP}(\mathbf{R};B)$ and $\varepsilon > 0$, then there are $n \in \mathbf{N}$, $b_k \in B$, $\lambda_k \in \mathbf{R}$ $(k = 1,\dots,n)$ s.t. $\|f - \sum_{k=1}^n b_k \mathrm{e}^{i\lambda_k\cdot}\|_\infty < \varepsilon$ and $f_{\lambda_k} \neq 0$ for $k = 1,\dots,n$.

(h1) Periodic functions are almost periodic.

(h2) Let $(a_\lambda)_{\lambda\in\mathbf{R}} \in \ell^1(\mathbf{R};B)$. Then $f := \sum_{\lambda\in\mathbf{R}} a_\lambda \mathrm{e}^{i\lambda t}$ is in $\mathrm{AP}(\mathbf{R};B)$ and $f_\lambda = a_\lambda$ for all $\lambda \in \mathbf{R}$ (this characterizes f uniquely, by (f1)).

(i) The class of treated in (h2) is inverse-closed AP *and in $\mathcal{C}_b$.*

Proof: (a)–(d2) We have $\mathrm{AP} \subset \mathcal{C}_{bu}$, by Property 2 of [LZ, Section 1.1]. By Property 6, AP is a vector space, and by Property 3 it is closed. By Properties 7, 4 and 6, claims (c), (d1) and (d2) hold, respectively; by Theorem 1.2.1, (b) holds.

By Property 7, the first claim in (c) holds; combine this with (d2) to observe that $\phi\circ(f_1, f_2,\dots, f_n) \in \mathrm{AP}(\mathbf{R};B)$, i.e., that the rest of (c) holds too.

Now (c) implies that the rest of (a) holds.

(e) This follows from (d2), because $K \subset \mathcal{G}\mathcal{B}$ by Lemma A.3.3(A3).

Part (f1) follows from [LZ, pp. 22–24]; part (f2) is obvious.

(f3) Clearly $[\phi(ae^{i\mu t}, be^{i\nu t})]_\lambda = \phi(a,b)(e^{i(\mu+\nu)t})_\lambda = \phi(a,b)\chi_{\{\lambda\}}(\mu+\nu)$, when $\lambda, \mu, \nu \in \mathbf{R}$, $a \in B_1$, $b \in B_2$.

Therefore, (C.3) holds for f, g with a finite AP-spectrum. For such a fixed g, both sides of (C.3) are continuous functions of f from AP to B (because the sum on the right is finite) for each $\lambda \in \mathbf{R}$, hence (C.3) holds whenever the AP-spectrum of g is finite, by (g).

If $\sum_{\lambda\in\mathbf{R}} \|f_\lambda\| < \infty$, then (C.3) holds for each g with a finite AP-spectrum and both sides of (C.3) are a continuous function of g, hence (C.3) holds for any g.

The case for $\sum_{\lambda\in\mathbf{R}} \|g_\lambda\| < \infty$ follows by exchanging the roles of f and g.

(We have no need to study whether these ℓ^1 conditions are necessary; at least they are not when B is a Hilbert space: Choose $P_n \to f$ as in (g). By (f4), $(g_\lambda)_{\lambda\in\mathbf{R}} \in \ell^2$ and $((P_n)_\lambda - f_\lambda)_{\lambda\in\mathbf{R}} = ((P_n - f)_\lambda))_{\lambda\in\mathbf{R}} \to 0$ in ℓ^2, hence also the right side of equation (C.3) with P_n in place of f converges as $n \to \infty$ (because $\|\phi(f_\mu, g_{\lambda-\mu})\|_B \le M\|f_\mu\|\|g_{\lambda-\mu}\|$ for some $M < \infty$).)

(f4) The first claim is given on p. 31 of [LZ] and it implies the isomorphism to $\ell^2(\mathbf{R}; B)$ (the trigonometric polynomials map naturally to a dense subspace of $\ell^2(\mathbf{R}; B)$); the rest is obvious (note that $\langle f, g\rangle \in \text{AP}$ by (c) and (d2)).

(f5) Let $f \in \text{AP}$. By (c) and (d2), we have $Ff \in \overline{\text{AP}}$. Obviously, $(\|Ff\|_H^2)_0 \le \|F\|_\infty^2 (\|f\|_H)_0^2$. Since $f \in \text{AP}$ was arbitrary, we conclude from (f4) that $\|F\|_{\mathcal{B}(\overline{\text{AP}}(\mathbf{R};B),\overline{\text{AP}}(\mathbf{R};B_2))} \le \|F\|_\infty$.

(g) See pages 17–18 and 24 of [LZ].

(h1) This is obvious (take $R = 2T$ in Definition C.1.1).

(h2) By (c), we have $a_k e^{it_k\cdot} \in \text{AP}(\mathbf{R}; B)$ for all k, hence $\sum_{k\in\mathbf{N}} a_k e^{it_k\cdot} \in \text{AP}(\mathbf{R}; B)$, by (a).

(i) Combine Wiener's Lemma (11.6 of [Rud73] or Theorem 4.1.1(a) for MTI_d) with (c). □

Notes

All results in this section are well known. A canonical reference on almost periodic functions is [LZ]. Further questions have been treated in [Basit] and [Zhang] and in the articles that they refer to.