# Appendix B

# Integration and Differentiation in Banach Spaces

*Bring me my bow of burning gold!*
*Bring me my arrows of desire!*
*Bring me my spear! O clouds unfold!*
*Bring me my chariot of fire!*
— William Blake (1757–1827)

In this appendix, we arm us with magic weapons to fight evil integral equations in final frontiers of unexplored Banach spaces. We treat (Bochner) integration, differentiation, function spaces $\mathcal{C}$ and $\mathrm{L}^p$ $(p \in [1,\infty])$ and similar concepts for functions with values in Banach spaces. We extend standard and extended results on scalar-valued functions for vector-valued ones.

Lebesgue measurability, integration and $\mathrm{L}^p$ spaces in the scalar-valued case are treated in Section B.1. In the rest of this appendix we treat the extensions of these concepts to the vector-valued case.

In Section B.2, we treat Bochner measurable functions $f : Q \to B$, where $B$ is a Banach space and $Q$ is a positive measure space. In Section B.3, we define and study the $\mathrm{L}^p$ and $\mathcal{C}$ spaces of such functions. In Section B.4, a generalization of the Lebesgue integral (*the Bochner integral*) is defined and studied for such functions. The reader might wish to have just a look at the beginnings of the sections mentioned above and skip the rest of this appendix until suggested to look up a specific fact by a proof in the main part of this monograph.

Differentiation of integrals and Lebesgue points are treated in Section B.5, vector-valued distributions ($\mathbb{D}'(\Omega;B) := \mathcal{B}(\mathcal{D}(\Omega);B)$) are treated in Section B.6, and Sobolev spaces ($\mathrm{W}^{k,p}(\Omega;B)$ and $\mathrm{W}_0^{k,p}(\Omega;B)$) in Section B.7.

Throughout this appendix, $B$, $B_2$ and $B_3$ denote Banach spaces with scalar field $\mathbf{K}$ ($\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$), $U$, $H$, and $Y$ denote Hilbert spaces, $\mu$ is a complete positive measure on a set $Q$, and $\mathfrak{M}$ is the corresponding $\sigma$-algebra.

We use the (standard) terminology of [Rud86]: a *positive measure* on a set $Q$ is a function $\mu : \mathfrak{M} \to [0,+\infty]$ (or the pair $(\mathfrak{M},\mu)$ or the triple $(Q,\mathfrak{M},\mu)$) s.t. $\mathfrak{M}$ is a $\sigma$-*algebra* on $Q$ (i.e., $\mathfrak{M}$ is a collection of subsets of $Q$ s.t. $Q \in \mathfrak{M}$ and $\mathfrak{M}$ is closed under complements and countable unions), and $\mu$ is *countably additive*, i.e., $\mu(\cup_{k=0}^{\infty} E_k) = \sum_{k=0}^{\infty} \mu(E_k)$ whenever $(E_k)_{k=0}^{\infty} \subset \mathfrak{M}$ is disjoint, and $\mu(\emptyset) = 0$. (In this chapter, we only treat scalar-valued measures. See Lemma D.1.12 and Section 2.6 for vector-valued measures.)

We call $\mu$ (or $Q$) $\sigma$-*finite* if $Q = \cup_{k\in\mathbf{N}} Q_k$, where $Q_k \in \mathfrak{M}$ and $\mu(Q_k) < \infty$ for all $k \in \mathbf{N}$. We call $\mu$ *complete* if all subsets of null sets are measurable.

When we assume $Q \subset \mathbf{R}^n$ (or $Q \subset \partial\mathbf{D}$, where we identify $\partial\mathbf{D}$ with $[0, 2\pi)$) and omit $\mu$, we tacitly assume that $\mu = m$, the *Lebesgue measure* (see Theorem 2.20 of [Rud86]).

A *null set* is a measurable set $N$ with $\mu(N) = 0$. A property (e.g. $f = g$ for functions $f, g : Q \to B$) is said to hold *almost everywhere (a.e.) on* $Q$ if it holds on $N^c := Q \setminus N$ for some null set $N$. Analogously, we can say that $f(q) = g(q)$ holds for *almost every (a.e.)* $q \in Q$.

Even though we have allowed a general $(Q, \mu)$ for completeness, we shall need the results only for 1. the *counting measure* $\mu$, for which every $A \subset Q$ is measurable and $\mu(A)$ is the cardinality of elements in $A$, and for 2. $(J, \mu)$, where $J \subset \mathbf{R}$ is an interval (i.e., a connected subset of $\mathbf{R}$) and $\mu = m$ or $d\mu = \mathrm{e}^{-2\omega t}\, dm$ for some $\omega \in \mathbf{R}$, and even so most results given below are used only for some technical details. Note that these both are complete positive measures, and the latter measure is $\sigma$-finite (so is the former too for countable $Q$).

For readers interested in the control of the systems with finite-dimensional input and output spaces only (this is very restrictive in cases where the output equals the state), it suffices to consider the (componentwise) Lebesgue measurability and integral; the Bochner measurability and integral are just the infinite-dimensional counterparts with $\|\cdot\|_B$ in place of $|\cdot|$.

We remark that almost all results given for Banach spaces in this appendix are valid for Fréchet spaces, mutatis mutandis (letting $B$ to be an arbitrary Fréchet space would make $\mathrm{L}^p(Q; B)$ spaces Fréchet spaces).

## B.1 The Lebesgue integral and $\mathrm{L}^p(\mathbf{R};[0,+\infty])$ spaces

*The subspace $W$ inherits the other 8 properties of $V$. And there aren't even any property taxes.*

— J. MacKay, Mathematics 134b

Here we shortly recall the Lebesgue integral and $\mathrm{L}^p$ spaces from [Rud86].

Let $R = \mathbf{C}$, or let $R$ be a connected subset of $[-\infty,+\infty]$ (open subsets of $[-\infty,+\infty]$ are arbitrary unions of sets of form $(a,b)$, $[-\infty,b)$, $(a,+\infty]$ with $a,b \in [-\infty,+\infty]$; note that $\mathbf{R}$ inherits is usual (metric) topology as a subset of $[-\infty,+\infty]$). A function $f : Q \to R$ is called *(Lebesgue) measurable* iff $f^{-1}[G]$ is measurable for each open $G \subset R$. (By Lemma B.2.5(b3), Lebesgue measurability is a special case of Bochner measurability for $R = \mathbf{K}$.)

If $f,g : Q \to R$ are Lebesgue measurable, then so is $\max(f,g)$, by Theorem 1.14(b) of [Rud86]; in particular, so are $f^{\pm}$, where $f^+ := \max(f,0)$, $f^- := \max(0,-f)$.

If $(\mu,\mathfrak{M})$ is any positive measure on $Q$, and $\mathfrak{M}'$ is the collection of sets $E \subset Q$ s.t. $A \subset E \subset A'$ and $\mu(A' \setminus A) = 0$ for some $A, A' \in \mathfrak{M}$ and we set $\mu'(E) := \mu(A)$ for such $E$, then the *completion* $(\mu',\mathfrak{M}')$ of $\mu$ is a (well-defined) complete positive measure on $Q$, by Theorem 1.36 of [Rud86]. Note also that $(r\mu,\mathfrak{M})$ is also a positive measure on $Q$ for any $r \in \mathbf{R}_+$.

The *Borel (measurable) sets* of a topological space $Q$ are the members of the minimal $\sigma$-algebra containing the open sets of $Q$. A function $f : Q \to \mathbf{K}$ is a *Borel (measurable) function* if $f^{-1}[V]$ is Borel measurable for all open $V \subset \mathbf{K}$. A *Borel measure* on $Q$ is a measure $(\mu,\mathfrak{M})$ s.t. $\mathfrak{M}$ contains the Borel sets (equivalently, s.t. $\mathfrak{M}$ contains the open sets). The Lebesgue measure $m$ is the completion of a measure whose domain is the collection of Borel sets, hence a Borel measure.

For $f : Q \to [-\infty,+\infty]$ we set $\operatorname{ess\,sup} f := \inf\{r \in [-\infty,+\infty] \mid f \le r \text{ a.e.}\}$, $\operatorname{ess\,inf} f := -\operatorname{ess\,sup} -f$. We also set $r/+\infty := 0$ $(0 \le r < +\infty)$, $r \cdot +\infty := +\infty$, $r/0 := +\infty$ $(0 < r < +\infty)$ and $0 \cdot +\infty := 0$. The function

$$\chi_E(q) := \begin{cases} 1, & q \in E; \\ 0, & q \notin E \end{cases} \tag{B.1}$$

is called the *characteristic function* of the set $E$.

If $n \in \mathbf{N}$, $E_0,\ldots,E_n$ are disjoint and measurable, and $\alpha_0,\ldots,\alpha_n \in [0,+\infty]$ or $\in B$, then $s := \sum_{k=0}^n \alpha_k \chi_{E_k}$ is a *simple measurable function* and the *Bochner integral* of such $s$ is given by

$$\int_Q s\,d\mu := \sum_{k=0}^n \alpha_k \mu(E_k). \tag{B.2}$$

For general measurable $f : X \to [0,+\infty]$ we set $\int_Q f\,d\mu := \sup \int_Q s\,d\mu$, the supremum being taken over all simple measurable functions $s$ s.t. $0 \le s \le f$. This integral is usually called the *Lebesgue integral*, and the term Bochner integral is reserved for functions whose values are not scalar (see Definition B.4.1).

If $Q' \subset Q$ is measurable, then we set $\int_{Q'} f\,d\mu := \int_Q \chi_{Q'} f\,d\mu$.

Let $1 \le p < \infty$. A measurable function $f : Q \to [0, +\infty]$ belongs to $\mathrm{L}^p(Q; [0, +\infty])$ if $\|f\|_p := \left(\int_Q f^p \, d\mu\right)^{1/p} < \infty$, and to $\mathrm{L}^\infty(Q; [0, +\infty])$ if $\|f\|_\infty := \operatorname{ess\,sup}_Q |f| < \infty$ (we sometimes write $\|f\|_{\mathrm{L}^p}$ or $\|f\|_{\mathrm{L}^p(Q;[0,+\infty])}$ instead of $\|f\|_p$).

To be exact, the $\mathrm{L}^p$ spaces ($1 \le p \le \infty$) are quotient spaces over with the set of functions that are 0 a.e. (i.e., $\mathrm{L}^p$ is the space of equivalence classes $[f]$, where $g \in [f]$ iff $g = f$ a.e.). Thus, $\mathrm{L}^p(Q; [0, +\infty])$ becomes "a normed space with scalar field $[0, +\infty]$". (One easily verifies that $\mathrm{L}^p(Q; [0, +\infty])$ is a vector space with scalar field $[0, +\infty]$ and that the axioms of a normed space are satisfied w.r.t. this scalar field; the latter requires the *Minkovski Inequality*, Theorem 3.19 of [Rud86], which says that $\|f + g\|_p \le \|f\|_p + \|g\|_p$ for all measurable $f, g : Q \to [0, +\infty]$.)

We also note the *Hölder Inequality*: $\|fg\|_1 \le \|f\|_p \|g\|_q$ for $p, q \in [1, +\infty]$ s.t. $1/p + 1/q = 1$. The reader will later note that these two inequalities extend to vector-valued functions, because for measurable (see Definition B.2.1) $f : Q \to B$ we shall define $\|f\|_p := \| \|f\|_B \|_p$.

*Lebesgue's Monotone Convergence Theorem* says that if $\{f_n\}$ are measurable, $0 \le f_1 \le f_2 \le \cdots \le \infty$ a.e. on $Q$ and $f_n(q) \to f(q)$ for a.e. $q \in Q$, then $f$ is measurable and $\int_Q f_n \, d\mu \to \int_Q f_n \, d\mu$.

**Lemma B.1.1** *Let $Q \subset \mathbf{R}^n$ be measurable. For any measurable sets $\{E_n\}$ s.t. $m(E_n) > 0$ for all $n \in \mathbf{N}$, there are disjoint measurable $\{E'_n\}$ s.t. $E'_n \subset E_n$ and $\infty > m(E'_n) > 0$ for each $n \in \mathbf{N}$.*

**Proof:** Choose $F_0 \subset E_0$ s.t. $\infty > m(F_0) > 0$. For each $n \in \mathbf{N} + 1$, choose $F_n \subset E_n$ s.t. $0 < m(F_n) < 2^{-n} \max_{k=0,\ldots,n-1} m(F_k)$ and set $E'_n := F_n \setminus \cup_{k=n+1}^\infty F_k$. Then $m(\cup_{k=n+1}^\infty F_k) < \sum_{k=n+1} 2^{-k} m(F_n) \le m(F_n)$, hence $m(E'_n) > 0$, for any $n \in \mathbf{N}$. $\square$

(See the notes on p. 947.)

## B.2 Bochner measurability ($f \in \mathrm{L}(Q;B)$)

*You cannot have a science without measurement.*
— R. W. Hamming

In this section we treat Bochner measurability in order to be able to define $\mathrm{L}^p$ spaces and the Bochner integral in the next two sections. For most readers, it suffices to just have a look at Definition B.2.1 and Lemma B.2.5 so as to be convinced that Bochner measurability is analogous to Lebesgue measurability.

A function $s : Q \to B$ is called a *countably-valued measurable function* iff it can be written as $s = \sum_{k\in\mathbf{N}} x_k \chi_{E_k}$, where the sets $E_k \subset Q$ are measurable and $x_k \in B$ ($k \in \mathbf{N}$). In general, Bochner measurability is defined as follows:

**Definition B.2.1 (Bochner measurability)** *A function $f : Q \to B$ is* (Bochner) measurable[1], *denoted by $f \in \mathrm{L}(Q;B)$, if some sequence of countably-valued measurable functions converges to $f$ a.e.*

*A function $f : Q \to B$ is called* almost separably-valued *if it can be redefined on a* null set *(a set of measure zero) so that $f[Q]$ becomes separable.*

In Lemma B.2.5(g) we shall show that a function $f : Q \to \mathbf{K}$ (or $f : Q \to [-\infty, +\infty]$) is measurable iff it is Lebesgue measurable.

We follow the standard convention to identify functions equal a.e. as members of L. Thus, the elements of L are actually equivalence classes. We sometimes write "$[f]$" instead of "$f$", when it would otherwise not be obvious that we refer to the class, not to the function.

Let $f \in \mathrm{L}(Q;B)$ and $F \in \mathcal{C}(B;B_3)$. Then $F \circ f$ is measurable (because $F \circ s_n \to F \circ f$ a.e.). If $f = g$ a.e., then $F \circ f = F \circ g$ a.e. Thus, we can follow the standard convention to define $F \circ [f] := [F \circ f]$. Most common examples of this are the definitions $\alpha[f] := [\alpha f]$ and $[f] + [g] := [f + g]$ for $f, g \in \mathrm{L}(Q;B)$, $\alpha \in \mathbf{K}$. We also extend to classes any other operations that can be well defined through representatives.

For separability results we shall need the following:

**Lemma B.2.2** *Let the sets $A_k$ be at most countable ($k \in \mathbf{N}$), and let $n \in \mathbf{N}$. Then the sets $A_0 \times A_1 \times \cdots \times A_n$, $\cup_{k\in\mathbf{N}} A_k$, $\{B \subset A_0 \mid B$ has $n$ elements$\}$ and $\{B \subset A_0 \mid B$ is finite$\}$ are at most countable.*

*For any set $A$, $\operatorname{card} A < \operatorname{card} 2^A$ (recall that $2^A$ is the set of all subsets of $A$). If $A$ and $B$ are nonempty sets, (at least) one of which is infinite, then* $\operatorname{card}(A \times B) = \max\{\operatorname{card} A, \operatorname{card} B\} = \operatorname{card}(A \cup B)$. □

(The countability claims are easily deduced from Theorems 2.12 and 2.13 of [Rud76]; the general cardinality claims are also well known and follow from, e.g., Theorems 176 and 180, pp. 276 and 280 of [Kelley].)

Next we list a few separability results in order to be able to prove Lemma B.2.5.

**Lemma B.2.3 (Separability)** *Let $E_n \subset B$ be separable for all $n \in \mathbf{N}$. Then*

[1]The term "strongly measurable" is sometimes used, but we reserve it for Definition F.1.1.

*(a) The closure and span of $E_0$ are separable.*

*(b) If $E \subset E_0$, then $E$ is separable.*

*(c) The union $\cup_n E_n$ is separable.*

*(d) If $E \subset B$ is weakly separable, then it is separable.*

*(e) Subsets of $\mathbf{R}^n$ are separable.*

*(e) If $Q$ is separable and $f \in \mathcal{C}(Q,Q')$, then $f[Q']$ is separable.*

**Proof:** Let $S_n \subset E_n$ be dense in $E_n$ and countable, for each $n$.

(a) The set $S_0$ is dense in $\overline{E_0}$ and the set of finite linear combinations of elements of $S_0$ with coefficients of form $r+iq$ ($q,r \in \mathbf{Q}$) is dense in the span of $E_0$.

(b) Let $S_0 = \{x_k\}_{k\in\mathbf{N}}$. For each $n,k \in \mathbf{N}$, choose an element $x_k^n \in E_k^n := \{y \in E \,\big|\, \|x_k - y\| < 1/(n+1)$, if such an element exists. Then the union $S$ of these elements is dense in $E$, because if $x \in E$ and $N > 0$, we can choose $x_k$ s.t. $|x - x_k| < 1/2N$, so that $E_k^n$ is nonempty for $n < 2N$, hence $\|x - x_k^{2N}\| \le \|x - x_k\| + \|x_k - x_k^{2N}\| < 1/N$. Thus, $S$ is dense in $E$.

(c) The union $\cup_n S_n$ is dense in $\cup_n E_n$.

(d) Let $S \subset E$ be countable and dense in the weak topology of $B$. Let $M$ be the closure of the span of $E$. Then $M$ is weakly separable, by (a) (whose proof is valid for any topological vector space), hence $M$ is separable, by Theorem 3.12 of [Rud73], hence $E$ is separable, by (b).

(e) The countable set $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$, hence (e) follows from (b).

(f) If $S \subset Q$ is dense, then $f[S] \subset Q'$ is obviously dense. □

**Lemma B.2.4** *Let $B$ be separable and $\Omega \subset \mathbf{C}$. If $f \in \mathcal{C}(\Omega;\mathcal{B}(B,B_2))$, then $f \in \mathcal{C}(\Omega;\mathcal{B}(B,B_2'))$ for some separable closed subspace $B_2' \subset B_2$.*

**Proof:** Let $\Omega' \subset \Omega$ and $Q \subset B$ be dense and countable. Then $B_2'' := f[\Omega'][Q] \subset f[\Omega][B]$ is dense and countable, by continuity ($x_k \to x$ & $s_k \to s \Rightarrow f(s_k)x_k \to f(s)x$). Consequently, the closed span $B_2'$ of $B_2''$ is a separable Banach space, and it contains $f[\Omega][B]$, i.e., $f(s)[B] \subset B_2'$ (i.e., $f(s) \in \mathcal{B}(B,B_2')$) for any $s \in \Omega$. □

Now we are ready to list the standard properties of Bochner measurability:

**Lemma B.2.5 (Bochner Measurability)** *Let $f_n, g, h : Q \to B$ be measurable ($n \in \mathbf{N}$) and $\alpha,\beta \in \mathbf{K}$.*

*(a1) The function $\alpha g + \beta h$ is measurable.*

*(a2) If $T \in \mathcal{B}(B,B_2)$ (or $T \in \mathcal{C}(B,B_2)$), then $Tg$ is measurable.*

*(a3) If $f : Q \to \mathcal{B}(B,B_2)$ is measurable, then so is $f^*$.*

*(a4) If also $f : Q \to B_2$ is measurable, then so is $(f,g) : Q \to B \times B_2$.*

*Thus, then $B(f,g) : Q \to B_3$ is measurable for any continuous $B : B \times B_2 \to B_3$.*

*(b1) A function $f : Q \to B$ is measurable iff $\Lambda f$ is measurable for all $\Lambda \in B^*$ (or for all $\Lambda$ in a norming subset of $B^*$) and $f$ is almost separably-valued.*

*(b2) A function $f : Q \to B$ is measurable iff some sequence of countably-valued measurable functions converges to $f$ uniformly outside some null set.*

*(b3) Let $f : Q \to B$. Then (i)⇒(ii)⇒(iii); if $B$ is separable, then (i)–(iii) are equivalent:*

*(i) $f$ is measurable;*

*(ii) the set $f^{-1}[V] := \{q \in Q \,\big|\, f(q) \in V\}$ is measurable for each open $V \subset B$;*

*(iii) $\Lambda f$ is measurable for all $\Lambda \in B^*$.*

*This holds also with $[-\infty, +\infty]$ in place of $B$.*

*(c) If $f_n : Q \to B$ ($n \in \mathbf{N}$) are measurable and $f_n(t) \to f(t)$ for a.e. $t \in Q$, then $f$ is measurable.*

*(d1) Let $Q_n \subset Q$ be measurable for all $n \in \mathbf{N}$ and $Q = \cup_{n\in\mathbf{N}} Q_n$. Then $f : Q \to B$ is measurable iff $f_{|Q_n}$ is measurable $Q_n \to Q$ for all $n \in \mathbf{N}$.*

*(d2) If $B_2$ is a closed subspace of $B$ and $fQ \subset B_2$, then $f$ is measurable $Q \to B_2$ iff $f$ is measurable $Q \to B$.*

*(d3) A function $f : Q \to B_1 \times B_2 \times \cdots \times B_k$ is measurable iff $f_j$ is measurable for $j = 1, \ldots, k$.*

*(e) Any continuous function $Q \to B$ is measurable if $Q$ is separable and $\mu$ is a Borel measure (e.g., $\mu = m$ and $Q \subset \mathbf{R}^n$ is measurable).*

*(f) If $B = \mathbf{K}$, then $(\mathrm{Re}\, g)^{\pm}$, $(\mathrm{Im}\, g)^{\pm}$ are measurable.*

*(g) Let $n \in \mathbf{N} + 1$. A function $f : Q \to \mathbf{K}^n$ is Bochner measurable iff each of its components is Lebesgue measurable.*

In (a4) the map $B$ can be multiplication, inner product, a continuous bilinear map, or similar. By (a2), $\|g\|_B : Q \to [0, \infty)$ is measurable.

Note for (d1) that the restriction of $\mu$ to some measurable $Q' \subset Q$ is also a complete positive measure. Note also that condition (ii) in (b3) depends on $\mathfrak{M}$ (and $B$) only, whereas conditions (i) and (iii) depend on the measure too.

See Lemma B.4.10 for the measurability of $f \circ \phi$ with $\phi \in \mathcal{C}^1$ increasing.

By (d1), piecewise continuous functions are Borel-measurable (hence Lebesgue-measurable if $Q \subset \mathbf{R}^n$ with $\mathbf{R}^n$'s topology).

**Proof of Lemma B.2.5:** (a)&(d) Choose $g_n \to g$ and $h_n \to h$ (a.e.) as in the definition of measurability. Then $\alpha g_n + \beta h_n$ is countably-valued and measurable for all $n \in \mathbf{N}$ and converges to $\alpha g + \beta h$ a.e. The proofs of (a2), (a3), (d1), (d2) and (d3) are more or less analogous.

Note for (d1) that $\mathfrak{M}_{Q'} := \{E \in \mathfrak{M} \,\big|\, E \subset Q'\}$ is a $\sigma$-algebra on $Q'$, and $\mu_{|\mathfrak{M}_{Q'}}$ is a complete, positive measure on $\mu_{Q'}$.

(Claim (a3) is a special case of (a2), it holds to both Hilbert space adjoint $f^* : Q \to \mathcal{B}(B_2, B)$ (in case that $B$ and $B_2$ are Hilbert spaces) and to the Banach space adjoint $f^* : Q \to \mathcal{B}(B_2^*, B^*)$ (and to any other continuous involution operator). Claim (a2) holds even when $T$ is only continuous from $B$ to the weak topology of $B_2$ (use (b1) and Lemma B.2.3(d)).)

(b1) Without our reference to norming sets, the proof on pp. 72–73 of [HP] applies. Assume then that $f$ is almost separably-valued and $\Lambda f$ is measurable for all $\Lambda \in C$, where $C \subset B^*$ is a norming set.

Redefine $f$ on a null set so that $f[Q]$ is separable, and replace $B$ by the closed span of $f[Q]$, which is separable. Let $S \subset B$ be dense, and choose countable $C' \subset C$ s.t. $\|x\|_B = \sup_{\Lambda \in C'} |\Lambda x|$ for all $x \in S$ Then $\|x\|_B = \sup_{\Lambda \in C'} |\Lambda x|$ for all $x \in B$. It follows that $\|f(t)\|_B = \sup_{\Lambda \in C'} |\Lambda f(t)|$ for all $t \in Q$, therefore, $\|f\|_B$ is measurable; analogously, so is $\|f - x\|_B$ for any $x \in B$. Thus, the rest of the proof on pp. 72–73 of [HP] is valid.

(b2) This is Corollary 1 on p. 73 of [HP].

(b3) The second claim follows from the first and (b1), so we only need to prove the first claim.

1° *(i)⇔(ii)⇔(iii) for $B = \mathbf{K}$ and for $B = [-\infty, +\infty]$:* This can be deduced from Theorems 1.14 and 1.17 of [Rud86] (note that (i)⇔(iii) is trivial, because $\mathbf{K}^* = \mathbf{K}$).

2° *(ii)⇒(iii):* If $f$ satisfies (ii), $\Lambda \in B^*$, and $V \subset \mathbf{K}$ is open, then $(\Lambda \circ f)[V] = f^{-1}[\Lambda^{-1}[V]]$ is measurable, because $\Lambda^{-1}[V]$ is open. Thus, $\Lambda f$ is measurable, by 1°.

3° *(i)⇒(ii):* Assume (i), so that some sequence $\{s_n\}$ of countably-valued, measurable functions converge to $f$ everywhere (redefine $s_n$ and $f$ to be zero on a null set, if necessary). Let $V \subset B$ be open, and define $F \in \mathcal{C}(B;\mathbf{K})$ by $F(x) := d(x, V^c) := \inf_{y \in V_c} \|x - y\|_B$. Then $F \circ s_n \to F \circ f$ everywhere, hence $F \circ f \in \mathrm{L}(Q)$, because $F \circ s_n$ is countably-valued and measurable for all $n \in \mathbf{N}$.

Therefore, the set $f^{-1}[V] = f^{-1}[F^{-1}[\mathbf{K} \setminus \{0\}]] = (F \circ f)^{-1}[\mathbf{K} \setminus \{0\}]$ is open, because $F \circ f$ satisfies (ii), by the implication "(i)⇒(ii)" of 1°.

4° *(iii)⇒(i) for separable $B$:* This follows from (b1).

(Note that $I : B \to B$ satisfies (ii) but not (i) if $B$ is an unseparable Banach space with the counting measure.)

(c) By Theorem 1.14 of [Rud86], $\Lambda h$ is measurable, so we only have to show that $h$ is almost separably-valued.

Choose a null set $N \subset Q$ s.t. $h_n(t) \to h(t)$ for $t \in N^c$. For each $n$, choose a null set $N_n$ s.t. $h_n[N_n^c]$ is separable. Let $N' := N \cup \cup_n N_n$, $Q' := Q \setminus N'$. Then $E := \cup_n h_n[Q']$ is separable, hence so is the closed subspace $M$ of $B$ spanned by $E$. But $h(t) = \lim_{n \to +\infty} h_n(t) \in M$ for each $t \in Q'$, hence $h$ is almost separably-valued.

(e) By Lemma B.2.3(e), $f[Q]$ is separable. Because $f^{-1}[V]$ is open, hence measurable for each open $V \subset B$, $f$ is measurable, by (b3).

(f) By (a2), $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ are measurable; by Section B.1, so are $(\mathrm{Re}\, g)^{\pm}$, $(\mathrm{Im}\, g)^{\pm}$.

(g) This follows from (b3). □

The rest of this section consists of less important results, hence the reader might wish to skip them.

We will use the following lemma to generalize several scalar results to the vector-valued case.

**Lemma B.2.6** *Let $f$ be Bochner measurable. Then $f = 0$ a.e. iff $\Lambda f = 0$ a.e. for all $\Lambda \in B^*$ (or for all $\Lambda$ in a norming subset of $B^*$).*

If $f$ is only "weakly vector measurable", i.e., $\Lambda f$ is measurable for all $\Lambda \in B^*$, then we may have $\Lambda f = 0$ a.e. for all $\Lambda \in B^*$ even though $f \neq 0$ everywhere (set $f(t) := e_t$, where $\{e_t\}_{t\in\mathbf{R}}$ is the natural base of $B := \ell^2(\mathbf{R})$ (i.e., $e_t := \chi_{\{t\}}$), so that for any $y := \sum_k \alpha_k e_{t_k} \in \ell^2(\mathbf{R})$ we have $\langle f(t), y\rangle_B = 0$ for $t \notin \cup_k\{t_k\}$).

**Proof of Lemma B.2.6:** The necessity is clear, so we assume that $\|f\| \geq \varepsilon > 0$ on $E \subset \mathbf{R}$ and $m(E) > 0$, and find $\Lambda \in B^*$ s.t. $\Lambda f \neq 0$ on a set of positive measure.

W.l.o.g. we assume that $f[E]$ has a dense countable subset $\{b_k\}_{k\in\mathbf{N}}$. For some $k \in \mathbf{N}$, we have $m(A_k) > 0$, where $A_k := \{r \in E \,\big|\, \|f(r) - b_k\| < \varepsilon/3\}$. If $|\Lambda b_k| > \|b_k\|/2$ and $\|\Lambda\| \leq 1$, then $|\Lambda f(r)| > |\Lambda b_k| - \varepsilon/3 =: M > \|b_k\|/2 - \varepsilon/2 > 0$ for $r \in A_k$. Thus, $\|\Lambda f\|_\infty \geq M \gg 0$. □

For $x \in B$ and $r > 0$ we set $D_r(x) := \{x' \in B \,\big|\, \|x - x'\| < r\}$.

**Lemma B.2.7 (Ess range($f$))** *Let $f : Q \to B$ be measurable. Then*

$$\operatorname{ess\,range}(f) := \{x \in B \,\big|\, r > 0 \Rightarrow \mu(f^{-1}[D_r(x)]) > 0\} \qquad \text{(B.3)}$$

*is closed and $\mu(f^{-1}[\operatorname{ess\,range}(f)^c]) = 0$. Moreover, $\operatorname{ess\,range}(f)$ is the smallest set with these properties.*

**Proof:** One easily verifies that $E_f := \operatorname{ess\,range}(f)$ is closed. We assume w.l.o.g. (see Lemma B.2.5(b1)) that $B$ is separable. Let $\{x_k\}$ be dense in $E_f^c$. For each $k$, set $r_k := \sup\{r > 0 \,\big|\, \mu(f^{-1}[D_r(x_k)]) = 0\}$. Obviously, $E_f^c \subset V := \cup_k D_{r_k/2}(x_k)$; but $\mu(f^{-1}[V]) = 0$, because $\mu(f^{-1}[D_{r_k/2}(x_k)]) = 0$ for all $k$; hence $\mu(f^{-1}[E_f^c]) = 0$.

By the definition of $E_f$, the set $E_f^c$ contains any open set $G \subset B$ s.t. $\mu(f^{-1}[G]) = 0$, i.e., $E_f^c$ is the biggest of such sets. □

**Lemma B.2.8** *Let $f : Q \to B$ be measurable.*

*(a) If $\mu(Q) > 0$, then there is $a_0 \in Q$ s.t. for each $\varepsilon > 0$ we have $\mu(A_\varepsilon) > 0$, where $A_\varepsilon := \{a \in Q \,\big|\, \|f(a) - f(a_0)\|_B < \varepsilon\}$.*

*(b) If(f) $f$ is not zero a.e., then there are $A \subset Q$ and $\Lambda \subset B^*$ s.t. $\mu(A) > 0$ and $\operatorname{Re}\Lambda f > 1$ on $A$.*

*If, in addition, $B_0$ is a closed subspace of $B$ and $\mu(E) > 0$, where $E := \{t \in Q \,\big|\, f(t) \notin B_0\}$, then we can choose $\Lambda$ and $A$ so that $\Lambda = 0$ on $B_0$.*

**Proof:** (a) Choose any $a_0 \in f^{-1}[\operatorname{ess\,range}(f)]$.

(b) If $B_0$ has not been given, set $B_0 = \{0\}$. Choose $n \in \{1, 2, 3, \ldots\}$ s.t. $A_n := \{q \in Q \,\big|\, d(f(q), B_0) > 1/n\}$ has a positive measure. Choose then $a_0 \in A_n$

for $f|_{A_n}$ and $\varepsilon := 1/2n$ as in (a). Then $V := \{b \in B \,\big|\, \|b - f(a_0)\|_B < 1/2n\}$ is convex, open and nonempty, $V \cap B_0 = \emptyset$, $A_0' := f^{-1}[V]$ has a positive measure (by (a), because $A_0 = A_0' \cap A_n$), and also $B_0$ is convex and nonempty, hence there are $\Lambda \in B^*$ and $\gamma \in \mathbf{R}$ s.t.

$$\mathrm{Re}\,\Lambda x < \gamma \leq \mathrm{Re}\,\Lambda y \quad \text{for all} \;\; x \in V,\; y \in B_0, \tag{B.4}$$

by Theorem 3.4 of [Rud73]. Thus, $\Lambda[B_0]$ is a proper subspace of $\mathbf{K}$, hence $\Lambda[B_0] = \{0\}$. Find $k \in \{1,2,3,\ldots\}$ s.t. $A := \{a \in A_0' \,\big|\, \mathrm{Re}\,\Lambda f(a) < -1/k\}$ has a positive measure, and then replace $\Lambda$ by $-k\Lambda$. □

The following result can be used for convolutions:

**Lemma B.2.9** *Let $f : \mathbf{R}^n \to B$ be measurable. Then $(r,s) \mapsto f(r-s)$ is measurable $\mathbf{R}^n \times \mathbf{R}^n \to B$.*

**Proof:** Let $s_n(r) \to f(r)$, as $n \to +\infty$, for all $r \in \mathbf{R}^n \setminus N$, where $m(N) = 0$ and $s_n$ is countably-valued and measurable ($n \in \mathbf{N}$) (see Definition B.2.1). We may and do require that $s_n = \sum_{k=0}^{\infty} x_k^n \chi_{E_k^n}$ where $E_k^n$ is Borel measurable for each $k,n$ (by redefining each $s_n$ on a null set).

Set $N_2 := \{(r,s) \,\big|\, r - s \in N\}$. Then (see Theorems 8.11 and 8.12 of [Rud86])

$$m(N_2) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{N_2}(r,s)\,dr\,ds = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \chi_{s+N}(r)\,dr\,ds = 0. \tag{B.5}$$

But $s_n(r-s) \to f(r-s)$ for all $(r,s) \in N_2^c$, and $(r,s) \mapsto s_n(r-s)$ is countably-valued and Borel measurable, because $(r,s) \mapsto r-s$ is a Borel function. □

(See the notes on p. 947.)

## B.3 Lebesgue spaces ($\mathrm{L}^p(Q,\mu;B)$)

*If God is perfect, why did He create discontinuous functions?*

In this section, we first define the $\mathrm{L}^p$ spaces ("Lebesgue spaces") and then go on to prove several technical results, some of which are considered to be well-known although not easily found in the literature.

The continuity of $\|\cdot\|_B : B \to [0,+\infty]$ implies that if $f$ is measurable, then so is $\|f\|_B$. Thus we can make the following definition:

**Definition B.3.1 ($\mathbf{L}^p(Q;B)$)** *Let* $1 \le p \le \infty$. $\mathrm{L}^p(Q;B) := \mathrm{L}^p(Q,\mu;B)$ *is the space of (equivalence classes of) measurable functions* $f : Q \to B$ *whose norm* $\|f\|_p := \|\|f(\cdot)\|_B\|_{\mathrm{L}^p(Q)}$ *is finite. We set* $\mathrm{L}^p(Q,\mu) := \mathrm{L}^p(Q,\mu;\mathbf{K})$.

*If* $B$ *is a Hilbert space, then we set* $\langle f,g\rangle_{\mathrm{L}^2} := \int_Q \langle f,g\rangle_B \, d\mu$. *By* $\ell^p(Q;B)$ *we mean* $\mathrm{L}^p(Q;B)$ *with the counting measure.*

*If compact subsets of* $Q$ *are measurable, then we set*

$$\mathrm{L}^p_{\mathrm{loc}}(Q,\mu;B) := \{ f \in \mathrm{L}(Q;B) \,\big|\, \|f\|_{\mathrm{L}^p(K,\mu;B)} < \infty \text{ for all compact } K \subset Q\}. \quad \text{(B.6)}$$

*If, in addition,* $Q$ *is* $\sigma$*-compact, then we equip* $\mathrm{L}^p_{\mathrm{loc}}$ *with the topology induced by the seminorms* $\|f\|_{\mathrm{L}^p(K,\mu;B)}$ *(see Theorem 1.37 of [Rud73]); in particular, then* $f_n \to f$ *in* $\mathrm{L}^p_{\mathrm{loc}}$ *if* $\|f_n - f\|_{\mathrm{L}^p(K,\mu;B)} \to 0$ *for all compact* $K \subset Q$ *(and* $f_n, f \in \mathrm{L}^p_{\mathrm{loc}}$ *for all* $n$*).*

Note that $\|f\|_p = 0$ iff $f = 0$ a.e., i.e., if $[f] = [0]$. Thus, $\|\cdot\|_p$ becomes a norm on $\mathrm{L}^p$ (with equivalence classes as elements). We define the "quasinorms" $\|f\|_p := \left(\int_Q \|f\|_B^p \, d\mu\right)^{1/p}$ for $p \in (0,1)$ too (but the vector spaces $\mathrm{L}^p$, $p \in (0,1)$ are not normed spaces, cf. [Rud73]). The topology of $\mathrm{L}^p_{\mathrm{loc}}$ is only rarely needed, hence the reader may well skip it; the resulting convergence condition given at the end of the above definition is only slightly more useful.

Obviously, $\mathrm{L}^p(Q;B) \subset \mathrm{L}^p_{\mathrm{loc}}(Q;B)$ $(p \in [1,\infty])$. If $\mu(K) < \infty$ for all compact subsets of $Q$, then $\mathrm{L}^p_{\mathrm{loc}}(Q;B) \subset \mathrm{L}^r_{\mathrm{loc}}(Q;B)$ $(\infty \ge p \ge r \ge 1)$, by the Hölder inequality; if, in addition, $Q$ or $B$ is separable and $\mu$ is a Borel-measure, then $\mathcal{C}(Q;B) \subset \mathrm{L}^p_{\mathrm{loc}}(Q;B)$. One usually equips $\mathcal{C}(Q;B)$ with $\mathrm{L}^\infty_{\mathrm{loc}}(Q;B)$ topology, but we do not need this.

**Theorem B.3.2** *The space* $\mathrm{L}^p(Q;B)$ *is a Banach space (a Hilbert space if* $p = 2$ *and* $B$ *is a Hilbert space). If* $f_n \to f$ *in* $\mathrm{L}^p$*, then some subsequence of* $\{f_n\}$ *converges to* $f$ *a.e.*

*If* $\Omega \subset \mathbf{R}^n$ *is open and* $\mu$ *is a Borel measure on* $\Omega$*, then* $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mu;B)$ *is a Fréchet space (hence a complete metric TVS).*

Note that if $f_n \to f$ in $\mathrm{L}^p$, $f_n \to g$ in $\mathrm{L}^q$ and $f_n \to h$ a.e. pointwise, as $n \to \infty$, then $f = g = h$ a.e. (take a subsequence of $\{f_n\}$ converging pointwise a.e. to $f$ and $g$).

**Proof:** 1° The proof of the first paragraph is identical to the scalar case (e.g., [Rud86, Theorems 3.11 & 3.12]), and hence omitted.

2° *Claims on* $\mathrm{L}^p_{\mathrm{loc}}$: Let $\{K_k\}_{k\in\mathbf{N}}$ be as in Lemma A.2.3. One easily verifies that the norms $\|\cdot\|_{\mathrm{L}^p(K_k;B)}$ generate the topology of $\mathrm{L}^p_{\mathrm{loc}}(\Omega;B)$; in particular, $\mathrm{L}^p_{\mathrm{loc}}$ is metrizable, by Remark 1.38(c) of [Rud73].

Let $\{f_n\}$ be a Cauchy-sequence in $\mathrm{L}^p_{\mathrm{loc}}$. Then $\{f_n\}$ has a limit $f^k$ in each $\mathrm{L}^p(K_k,\mu;B)$. Set $f := \sum_k \chi_{K_k\setminus K_{k-1}} f^k$. Obviously, $f = f^k$ a.e. on each $K_k$ and $f \in \mathrm{L}(\Omega;B)$, hence $f_n \to f$ in each $\mathrm{L}^p(K_k;B)$, so that $f_n \to f$ in $\mathrm{L}^p_{\mathrm{loc}}$. Because $\{f_n\}$ was arbitrary, $\mathrm{L}^p_{\mathrm{loc}}$ is complete. By Theorem 1.37 of [Rud73], $\mathrm{L}^p_{\mathrm{loc}}$ is a Fréchet space (i.e., a complete locally convex metrizable TVS). □

Also most other standard analysis results extend to the vector-valued case; in particular, standard properties of $\mathrm{L}^p(\mathbf{K})$ spaces (e.g., reflexivity (when $1 < p < \infty$) and separability (when $1 \le p < \infty$)) are inherited by $\mathrm{L}^p(\mathbf{R}^n;B)$ if(f) also $B$ possesses the same property; see [DU]. The most important exceptions (fortunately not much needed in this monograph) are that the dual of $\mathrm{L}^p(Q;B)$ need not be $\mathrm{L}^q(Q;B^*)$ (see [DU, Theorem IV.1.1, p. 98]; cf. Lemma B.4.15), and that an absolutely continuous measures (and functions) with values in $B$ need not be differentiable, unless $B$ is a *Radon–Nikodym space* (and $\mu$ is $\sigma$-finite and $1 \le p < \infty$). For most purposes, it suffices to know that Hilbert spaces and other reflexive spaces are Radon–Nikodym spaces [DU, p. 61, p. 98 & p. 82].

Derivatives are defined as in the scalar case:

**Definition B.3.3 ($\frac{df}{dt}$)** *Let $J$ be an interval of* $\mathbf{R}$. *The* derivative *of a function $f : J \to B$ at $t \in J$ is*

$$f'(t) := \lim_{h\to 0,\ t+h\in J} \frac{f(t+h) - f(t)}{h}. \tag{B.7}$$

*Let $\Omega \subset \mathbf{R}^n$ be open, $n \in \mathbf{N}+1$, $q \in \Omega$, $j \in \{1,2,\dots,n\}$. Then the $j$*th partial derivative $(D_j g)(q) := g_j(q)$ *of $g : \Omega \to B$ at $q$ is the derivative of $h \mapsto g(q + he_j)$ at* 0.

*If $g$ has all $n$ partial derivatives at $q$ [and they are continuous], then $g$ is* [continuously] differentiable at $q$. *If $g$ is [continuously] differentiable at each point of $\Omega$, then $g$ is* [continuously] differentiable *(on $\Omega$).*

*The partial derivatives of $g$ of* order $k \in \mathbf{N}$ *are the functions $D_1^{\alpha_1}\cdots D_n^{\alpha_n} g$ for which $\alpha \in \mathbf{N}^n$, $\sum_{j=1}^n \alpha_j = k$. If all partial derivatives of $g$ of order $k$ exist [at $q$], then $g$ is $k$* times differentiable [at $q$].

(Here $e_j$ is the $j$th unit vector $(e_1 := (1,0,0,\dots,0)$, $e_2 := (0,1,0,0,\dots,0)$, ...).)

Obviously, the definition of $f'(t)$ implies that $(Tf)'(t) = T(f'(t))$ for $T \in \mathcal{B}(B,B_2)$ whenever $f'(t)$ exists. Note that we allow one-sided derivatives at the endpoints of $J$ (if they belong to $J$).

Let $Q$ be a topological space. The space $\mathcal{C}(Q;B)$ is the vector space of continuous functions $f : \Omega \to B$. We equip the following subspaces of $\mathcal{C}(Q;B)$ (see p. 1045 for details) with supremum norm $f \mapsto \sup_{q\in Q}\|f(q)\|_B$: $\mathcal{C}_{\mathrm{b}}$ ($f$ bounded), $\mathcal{C}_{\mathrm{bu}}$ ($f$ bounded and uniformly continuous; this requires that $Q$ is metric), $\mathcal{C}_0 := \{f \in \mathcal{C}_{\mathrm{b}} \mid \text{for any } \varepsilon > 0, \text{ there is compact } K \subset Q \text{ s.t. } \|f\| < \varepsilon \text{ on } Q \setminus K\}$ (often called as “the functions vanishing at infinity”), and $\mathcal{C}_{\mathrm{c}} := \{f \in \mathcal{C} \mid \operatorname{supp} f \text{ is compact}\}$. Obviously, $\mathcal{C}_{\mathrm{c}} \subset \mathcal{C}_0 \subset \mathcal{C}_{\mathrm{b}} \subset \mathcal{C}$.

Let $\Omega \in \mathbf{R}^n$ be open (or an interval). Let $\mathcal{V}$ be one of the symbols $\mathcal{C}$, $\mathcal{C}_{\rm b}$, $\mathcal{C}_{\rm bu}$, $\mathcal{C}_0$, $\mathcal{C}_{\rm c}$. Then we set $\mathcal{V}^0 := \mathcal{V}$, $\mathcal{V}^{k+1}(\Omega;B) := \{f \in \mathcal{V}(\Omega;B) \,|\, D_j f \in \mathcal{V}^k(\Omega;B)$ for all $j \in \{1,2,\ldots,n\}\}$ (when $k \in \mathbf{N}$), $\mathcal{V}^\infty(\Omega;B) := \cap_{k\in\mathbf{N}} \mathcal{V}^k(\Omega;B)$.

**Lemma B.3.4 ($\mathcal{C}_{\rm c} \subset \mathcal{C}_0 \subset \mathcal{C}_{\rm bu} \subset \mathcal{C}_{\rm b}$)** *Let $\Omega$ be a metric space. Then $\mathcal{C}_{\rm c}(\Omega;B) \subset \mathcal{C}_0(\Omega;B) \subset \mathcal{C}_{\rm bu}(\Omega;B) \subset \mathcal{C}_{\rm b}(\Omega;B)$; the spaces $\mathcal{C}_0$, $\mathcal{C}_{\rm bu}$ and $\mathcal{C}_{\rm b}$ are Banach spaces (under the supremum norm), and $\mathcal{C}_{\rm c}$ is dense in $\mathcal{C}_0$.*

**Proof:** $1^\circ$ *Claims $\mathcal{C}_{\rm c}(\Omega;B) \subset \mathcal{C}_0(\Omega;B) \subset \mathcal{C}_{\rm bu}(\Omega;B) \subset \mathcal{C}_{\rm b}(\Omega;B)$:* These are obvious except the uniform continuity of function $f \in \mathcal{C}_0(\Omega;B)$. Given $f \in \mathcal{C}_0(\Omega;B)$ and any $\varepsilon > 0$, set $K := \{q \in \Omega \,|\, \|f(q)\| \geq \varepsilon/3\}$, so that $K$ is a closed subset of a compact set, hence compact. Analogously, $K' := \{q \in \Omega \,|\, \|f(q)\| \geq \varepsilon/2\}$ is compact. Since $K' \subset V := \{q \in \Omega \,|\, \|f(q)\| > \varepsilon/3\} \subset K$ and $V$ is open, we have $\delta' := d(K', K^c) \geq d(K', V^c) > 0$, by Lemma A.2.1(c).

Choose $\delta > 0$ s.t. $\|f(q) - f(q')\|_B < \varepsilon$ for all $q, q' \in K$ s.t. $d(q,q') < \delta$. Then, given $q, q' \in \Omega$ s.t. $d(q,q') < \min(\delta, \delta')$, we have $\|f(q) - f(q')\|_B < \varepsilon$.

Indeed, if $q \notin K$, then $q' \notin K'$, hence then $\|f(q) - f(q')\|_B < \varepsilon/3 + \varepsilon/2 < \varepsilon$; the case with $q' \notin K$ is analogous, and the case $q, q' \in K$ follows from the definition of $\delta$.

$2^\circ$ *The completeness of $\mathcal{C}_0$, $\mathcal{C}_{\rm bu}$ and $\mathcal{C}_{\rm b}$:* Let $\{f_n\}$ be a $\mathcal{C}_{\rm b}$-Cauchy-sequence. Then so is $\{f_n(q)\}$, hence $f(q) := \lim_{n\to\infty} f_n(q) \in B$ exists, for each $q \in Q$. Given $\varepsilon > 0$, there is $N \in \mathbf{N}$ s.t. $\|f_n - f_m\|_{\mathcal{C}_{\rm b}} := \sup_{q\in Q} \|f_n(q) - f_m(q)\|_B < \varepsilon/2$ for all $n, m \geq N$, so that $\sup_{q\in Q} \|\lim_{n\to\infty} f_n(q) - f_m(q)\|_B \leq \varepsilon/2 < \varepsilon$ for all $m \geq N$. Consequently, $f_m \to f$ uniformly as $m \to \infty$. As one easily verifies, it follows that $f$ is continuous and bounded. hence $f_n \to f$ in $\mathcal{C}_{\rm b}$.

If $f_n \in \mathcal{C}_0$ for all $n$, then, for each $\varepsilon > 0$ we can choose $N$ s.t. $\|f_N - f\| < \varepsilon/2$ and $K$ s.t. $\|f_N\| < \varepsilon/2$ on $K^c$, so that $\|f\| < \varepsilon/2 + \varepsilon/2$ on $K^c$; thus, then $f \in \mathcal{C}_0$.

Finally, assume that $f_n \in \mathcal{C}_{\rm bu}$ for each $n$. Given $\varepsilon > 0$, choose $N$ s.t. $\|f_N - f\| < \varepsilon/3$, and choose then $\delta > 0$ s.t. $\|f_N(q) - f_N(q')\| < \varepsilon/3$ whenever $q, q' \in \Omega$, $d(q,q') < \delta$. Then $\|f(q) - f(q')\| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ whenever $q, q' \in \Omega$, $d(q,q') < \delta$. Thus, then $f \in \mathcal{C}_{\rm bu}$.

$3^\circ$ *$\mathcal{C}_{\rm c}$ is dense in $\mathcal{C}_0$:* (This holds also when $\Omega$ is a locally compact or normal (possibly non-metrizable) Hausdorff space.)

Let $f \in \mathcal{C}_0(\Omega;B)$ and $\varepsilon > 0$. Choose $K, K', \delta'$ as in $1^\circ$. Set $g(q) := 1$ in $K'$ and $g(q) := 0$ in $K^c$, so that $g \in \mathcal{C}(K' \cup K^c, [0,1])$. By Tietze's Extension Theorem [Kelley], $g$ has an extension $h \in \mathcal{C}(\Omega;[0,1])$. But $\operatorname{supp} h \subset K$, hence $h \in \mathcal{C}_{\rm c}(\Omega)$. It follows that $hf \in \mathcal{C}_{\rm c}(\Omega;B)$ and $\|hf - f\| \leq \varepsilon/2$ (since $hf = f$ on $K'$ and $\|hf - f\| \leq \|f\| < \varepsilon/2$ on $(K')^c$). Since $f \in \mathcal{C}_0(\Omega;B)$ and $\varepsilon > 0$ were arbitrary, $\mathcal{C}_{\rm c}$ is dense in $\mathcal{C}_0$. □

The sequence spaces $\mathrm{c_c}$ (of finite sequences) and $\mathrm{c_o}$ (of vanishing sequences) are sometimes needed; we briefly introduce them below as special cases of $\mathcal{C}_{\rm c}$ and $\mathcal{C}_0$.

For any set $Q$, the function $d(q,q') := \begin{cases} 1, & \text{when } q \neq q'; \\ 0, & \text{when } q = q' \end{cases}$ is a metric, called the *discrete metric* of $Q$. In the corresponding *discrete topology* every subset of $Q$ is open, hence every function $Q \to B$ is continuous. A subset of $Q$ is compact iff it is finite. We set $c_o(Q;B) := \mathcal{C}_0(Q;B)$ and $c_c(Q;B) := \mathcal{C}_c(Q;B) = \{f : Q \to B \,|\, f(q) = 0$ except for finitely many $q \in Q\}$, where $Q$ is equipped by its discrete topology. (Recall that empty set is finite, i.e., $0 \in c_c(Q;B)$ (for $Q \neq \emptyset$).) Thus, $c_o(Q;B)$ is the closure of $c_c(Q;B)$ under the supremum norm (i.e., in $\ell^\infty(Q;B)$, the set of bounded functions $Q \to B$), by Lemma B.3.4.

The following two lemmas contain important facts:

**Lemma B.3.5 ($f'$ is measurable)** *Let $J \subset \mathbf{R}$ be an interval and let $f : J \to B$ be Lebesgue measurable. If $f'(t)$ exists for a.e. $t \in J$, then $f'$ is Lebesgue measurable.* □

(This follows from Lemma B.2.5(c).)

**Lemma B.3.6** *Let $p \in [1,\infty]$. If $f \in L^p(Q;B)$ and $T \in \mathcal{B}(B,B_2)$, then $Tf \in L^p(Q;B_2)$. If $F \in L^p(Q;\mathcal{B}(B,B_2))$, then $F^* \in L^p(Q;\mathcal{B}(B_2^*,B^*))$.* □

(This follows easily from Lemma B.2.5(a2)&(a3). This implies that the Bochner integral is a special case of so called Pettis integral.)

A casual reader might wish skip the rest this section except Lemma B.3.9 and Theorem B.3.11(a1)&(b1). Most other results below are rather technical and less often needed.

Most "$L^p$ mass" of a $L^p$ function lies on a compact set (unless $p = \infty$):

**Lemma B.3.7** *Let $f \in L^p(\mathbf{R}^n;B)$ and $p \in (0,\infty)$, or $f \in \mathcal{C}_0(\mathbf{R}^n;B)$ and $p = \infty$. Then $\|\chi_{\{q\in\mathbf{R}^n \,|\, |q|>R\}} f\|_p \to 0$ as $R \to \infty$. In particular, for any $\varepsilon, r > 0$, there is $R > 0$ s.t. $\|\chi_{\{q\in\mathbf{R}^n \,|\, |q|\le r\}} \tau^s f\|_p < \varepsilon$ for $s \in \mathbf{R}^n$ s.t. $|s| > R$.* □

(This follows from the (scalar) Monotone Convergence Theorem (note that $|q+s| > R-r$ when $|q| \le r$), except in the (trivial) $\mathcal{C}_0$ case.)

**Corollary B.3.8 ($\pi_+\tau^t f \to 0$)** *Let $f \in L^p(\mathbf{R};B)$ and $1 \le p < \infty$.*

*Then $\tau^t f \to \tau^T f$ in $L^p$, as $t \to T \in \mathbf{R}$, and $\pi_{[-T,t)} f \to f$, $\pi_{[0,t)} f \to \pi_+ f$, $\pi_{[0,t^{-1})} f \to 0$ and $\pi_+\tau(t) f \to 0$ in $L^p$, as $t, T \to +\infty$.*

*If $g \in L^p_{\text{loc}}(\mathbf{R};B)$, then $\|\pi_+\tau^{-t} g\|_p \to \|g\|_p$ as $t \to +\infty$.* □

(One obtains this easily from Lemma B.3.7.)

**Lemma B.3.9** *Let $p \in [1,\infty)$ and $f \in L^p(\mathbf{R}^n;B)$. Then*

$$\|f - \tau(h) f\|_p \to 0 \quad \text{as } h \to 0 \text{ in } \mathbf{R}^n. \tag{B.8}$$

**Proof:** By uniform continuity, (B.8) holds for $f \in \mathcal{C}_c(\mathbf{R}^n;B)$. For general $f \in L^p$ and $\varepsilon > 0$, choose $\phi \in \mathcal{C}_c$ s.t. $\|f - \phi\|_p < \varepsilon/3$, and then choose $\delta > 0$ s.t. $\|\phi - \tau(h)\phi\|_p < \varepsilon/3$ for $|h| < \delta$. Then $\|f - \tau(h) f\|_p < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ for $|h| < \delta$. □

Characteristic functions can be approximated by smooth functions:

**Lemma B.3.10** *If $K \subset \Omega \subset \mathbf{R}^n$, $K$ is compact, and $\Omega$ is open, there is $\phi \in \mathcal{C}_c^\infty(\Omega)$ s.t. $\chi_K \le \phi \le \chi_\Omega$.*

*If $\Omega' \subset \mathbf{R}^n$ is open, $E \subset \Omega' \subset \mathbf{R}^n$, $m(E) < \infty$, $p \in [1,\infty)$, [and $w \ge 0$ is $\mathrm{L}^1$ on a neighborhood of $E$], and $\varepsilon > 0$, then we can choose $K$ and $\Omega$ so that $K \subset E \subset \Omega \subset \Omega'$ and $m(\Omega \setminus K) < \varepsilon^p$ [$\int_{\Omega\setminus K} w\,dm < \varepsilon^p$] to obtain $\|\chi_E - \phi\|_p < \varepsilon$ [$\|\chi_E - \phi\|_{\mathrm{L}^p(\Omega, w\,dm)} < \varepsilon$].*

Note that $\phi$ is infinitely differentiable and with a compact support $\subset \Omega$, $\phi = 1$ on $K$, and $0 \le \phi \le 1$. Here $w$ is a nonnegative weight function, i.e., we refer to the measure $E \mapsto \int_E w\,dm$.

**Proof:** (We only sketch the proof; see [Adams, Section 2.17] for details on mollifiers.)

1° *$w \equiv 1$:* Because $r := d(K, \Omega^c) > 0$, we can set $K' := \{x \in \Omega \,|\, d(x,K) \le r/2\}$, and then take $\phi := \phi_k * \chi_{K'}$ for some large $k$, where $\{\phi_k\} \subset \mathcal{C}_c^\infty$ converges to the delta distribution $\delta_0$.

The last claim follows from the regularity of $m$ [Rud86, Theorem 2.20].

2° *The general case:* Let $w \in \mathrm{L}^1(\Omega'')$, where $E \subset \Omega'' \subset \Omega$ and $\Omega''$ is open. By [Rud86, Exercise 1.12], $\int_{\Omega\setminus K} w\,dm < \varepsilon^p$, when $m(\Omega \setminus K)$ is small enough. Consequently,

$$\int_\Omega |\chi_E - \phi|^p w\,dm \le \int_\Omega w\,dm < \varepsilon^p. \tag{B.9}$$

□

The space of simple $\mathrm{L}^p$ functions (as well as $\mathcal{C}_c^\infty$) is dense in $\mathrm{L}^p$, even in $\mathrm{L}^{p_1} \cap \mathrm{L}^{p_2} \cap \cdots \cap \mathrm{L}^{p_n}$ (and even with different weight functions), when $p < \infty$:

**Theorem B.3.11 ($\mathcal{C}_c^\infty$ is dense in $\mathrm{L}^p$)** *Let $\Omega \subset \mathbf{R}^n$ be open and $1 \le p < \infty$.*

*(a1) Simple $\mathrm{L}^p$ functions are dense in $\mathrm{L}^p(Q;B)$, and countably-valued $\mathrm{L}^\infty$ functions are dense in $\mathrm{L}^\infty(Q;B)$.*

*(a2) If we are given $n \in \mathbf{N}+1$, exponents $p_1,\dots,p_n \in [1,\infty)$ and $\varepsilon > 0$, then, for any $f \in \cap_{k=1}^n \mathrm{L}^{p_k}(Q;B)$, there is a simple function $s \in \mathrm{L}^p$ s.t. $\|f - s\|_{\mathrm{L}^{p_k}(Q;B)} < \varepsilon$.*

*(a3) At least in (a1)–(b2) and (d), given a dense subspace $B_0$ of $B$, we may choose the (dense set of) functions so that they have their values in $B_0$.*

*(b1) Finite-dimensional $\mathcal{C}_c^\infty(\Omega;B)$ functions are dense in $\mathrm{L}^p(\Omega;B)$.*

*(b2) If we are given $n \in \{1,2,\dots\}$, exponents $p_1,\dots,p_n \in [1,\infty)$, nonnegative (weight) functions $w_1,\dots,w_n \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;B)$, and $\varepsilon > 0$, then, for any $f \in \cap_{k=1}^n \mathrm{L}^{p_k}(\Omega, w_k\,dm;B)$, there are a simple function $s \in \mathrm{L}^p$ and a finite-dimensional $\mathcal{C}_c^\infty(\Omega;B)$ function $\phi$ s.t.*

$$\|f-s\|_{\mathrm{L}^{p_k}(\Omega,w_k\,dm;B)} < \varepsilon \ \text{ and } \ \|f-\phi\|_{\mathrm{L}^{p_k}(\Omega,w_k\,dm;B)} < \varepsilon \ \text{ for all } k \in \{1,\dots,n\}. \tag{B.10}$$

*(b3) $\{\phi \in \mathcal{S}(\mathbf{R};B) \,|\, \widehat{\phi} \in \mathcal{C}_c^\infty(i\mathbf{R};B)\}$ is dense in $\mathrm{L}^p(\mathbf{R};B)$.*

*(c) The closure of $\mathcal{C}_c(Q;B)$ in $\mathrm{L}^\infty$ is $\mathcal{C}_0(Q;B)$ when $Q$ is metrizable.*

*(d) The closure of simple measurable functions in* $\mathrm{L}^\infty(Q;B)$ *equals*

$$\mathrm{L}^\infty_{\mathrm{K}}(Q;B) := \{ f \in \mathrm{L}^\infty(Q;B) \,\big|\, \text{there is a compact } K \subset B \text{ s.t. } f(q) \in K \text{ for a.e. } q \in Q \}. \tag{B.11}$$

*We have* $\mathcal{C}_0(Q;B) \subset \mathrm{L}^\infty_{\mathrm{K}}(Q;B)$ *when* $Q$ *is metrizable. Moreover,* $\mathrm{L}^\infty_{\mathrm{K}}(\Omega;B) = \mathrm{L}^\infty(\Omega;B)$ *iff* $\dim B < \infty$*, and* $\mathrm{L}^\infty_{\mathrm{K}}(Q;B) = \mathrm{L}^\infty(Q;B)$ *if* $\dim B < \infty$.

(By using mollifiers one could obtain further density results; cf. pp. 29–52 of [Adams].)

Recall that simple means finite-valued, hence $s$ is a simple $\mathrm{L}^p(Q;B)$ function iff $s = \sum_{k=0}^n \chi_{E_k} x_k$ for some $n \in \mathbf{N}$, $\{x_k\} \subset B$, $E_k \subset \mathfrak{M}$, and $\|s\|_p < \infty$.

**Proof:** (a1) 1° *Case* $p = \infty$*:* This follows from Lemma B.2.5(b2).

2° *Case* $p < \infty$*:* Let $f \in \mathrm{L}^p(Q;B)$ and $\varepsilon > 0$. By Theorem 3.13 of [Rud86], there is a simple function $s : Q \to \mathbf{R}_+$ s.t. $\|\|f\|_B - s\|_p < \varepsilon/3$. Set $K := \{q \in Q \,|\, s(q) \neq 0\}$, so that $m(K) < \infty$ and $\|f - \widetilde{f}\| < \varepsilon/3$, where $\widetilde{f} := f\chi_K \in \mathrm{L}^p$. By the (scalar) Monotone Convergence Theorem, we have $\widetilde{f}\chi_{\{\|f\| \le n\}} \to \widetilde{f}$ in $\mathrm{L}^p$, as $n \to +\infty$, hence $g := f\chi_K\chi_{\{\|f\| \le M\}}$ satisfies $\|f - g\| < \varepsilon/4$ for $M$ big enough. Note that $\|g\|_\infty \le M$.

By 1°, there is a (countably-valued measurable) function $h = \sum_{k=1}^\infty b_k \chi_{E_k} : K \to B$ ($b_k \in B$, $E_k \subset K$ measurable and disjoint, $\chi_{E_k}$ its characteristic function for each $k$) s.t. $\|g - h\|_\infty$ is arbitrarily small. Because $m(K) < \infty$, it follows that we can take $\|g - h\|_p$ arbitrarily small, say $< \min(\varepsilon/4, 1)$ (and, simultaneously $\|h\|_\infty \le M + 1 < \infty$).

By applying the (scalar) dominated convergence theorem to $\|h\|_B^p$ (with the constant function $M + 1 \in \mathrm{L}^1(K)$ as the majorant), we see that $\int_K \|h - \sum_{k=1}^n b_k \chi_{E_k}\|_B^p \, dm < (\varepsilon/4)^p$ for some $n$. Thus, $\|f - \sum_{k=1}^n b_k \chi_{E_k}\|_p < 3\varepsilon/4 < \varepsilon$.

(b1) By Lemma B.3.10, we may approximate each $\chi_{E_k}$ above by some $\phi_k \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ to get

$$\|\sum_{k=1}^n b_k(\chi_{E_k} - \phi_k)\|_p \le \sum_{k=1}^n \|b_k\|_B \|\chi_{E_k} - \phi_k\|_p < \varepsilon/4 \tag{B.12}$$

(by the Minkovski inequality), hence $\|f - \sum_{k=1}^n b_k \phi_k\|_p < 4\varepsilon/4 = \varepsilon$.

(a2) Work as in (a1) for each $p_k$. Let $K$ be the union of "$K$'s" and let $M$ be the maximum of "$M$'s". Require $\|g - h\|_\infty$ to be small enough for each $p_k$. Let $n$ be the maximum of "$n$'s".

(b2) Write $\Omega = \cup_l K_l$, where $K_1 \subset K_2 \subset \cdots$ are compact sets (e.g., $K_l := \{x \in \Omega \,\big|\, |x| \le l \wedge d(x, \Omega^c) \ge 1/l\}$), then, for some $M \in \{1, 2, \ldots\}$, we have

$$\int_\Omega \|f - f\chi_{K_M}\chi_{\{\|f\|_B \le M\}}\|_B^{p_k} w_k \, dm < \varepsilon^{p_k} \quad (k = 1, \ldots, n), \tag{B.13}$$

by the dominated convergence theorem, because $\|f\|^{p_k} w_k \in \mathrm{L}^1$ ($k = 1, \ldots, n$), by the assumption on $f$; this gives the function $g := f\chi_K\chi_{\{\|f\|_B \le M\}}$, where $K := K_M$.

By taking $\|g - h\|_\infty < \min(\delta, 1)$, where $\delta^{p_k} \int_K w_k \, dm < \varepsilon/4$ for all $k$, we get a suitable countably-valued function $h$, and obtain then $s$ as some partial sum of $h$, as in the proof of (a1). The rest goes approximately as in (b1) (when applying Lemma B.3.10, use unions (over $k \in \{1,2,\ldots,n\}$) of compact sets and intersections of open sets).

(a3) This is obvious in the sense of simple (or countably-valued) functions. The procedure in (b1) obviously keeps the values in $B_0$, so does that in (b2) too.

(b3) This is given in Lemma 2.3 of [Zimmermann] and its proof (which shows that we may additionally require that $0 \notin \operatorname{supp}\widehat{\phi}$ if $p > 1$).

(See Appendix D for $\mathcal{S}$ and the Fourier transform $\widehat{\phi}$. Note that the claim follows easily from (b1) if $p = 2$ and $B$ is a Hilbert space.)

(c) This follows from Lemma B.3.4.

(d) (Note that $\mathrm{L}^\infty_\mathrm{K}$ equals the space $\mathrm{L}_\infty$ used in the interpolation theory of [BL].)

1° Obviously, $\mathcal{C}_\mathrm{c} \subset \mathrm{L}^\infty_\mathrm{K}$, hence $\mathcal{C}_0 \subset \mathrm{L}^\infty_\mathrm{K}$ when $Q$ is metrizable, by (c).

2° SMF *is a dense subset of* $\mathrm{L}^\infty_\mathrm{K}$*:* Let SMF denote the set of simple measurable functions (obviously, $\mathrm{SMF} \subset \mathrm{L}^\infty_\mathrm{K}$).

Let $\varepsilon > 0$ and $f \in \mathrm{L}^\infty_\mathrm{K}$ be arbitrary. Choose $K$ for $f$ as in the definition of $\mathrm{L}^\infty_\mathrm{K}$. Choose $n \in \mathbf{N}$, $x_0,\ldots,x_n \in K$ s.t. $K \subset \cup_{k=0}^n D(x_k,\varepsilon)$.

Set $E'_k := f^{-1}[D(x_k,\varepsilon)]$, $E_0 := E'_0$, $E_{k+1} := E'_{k+1} \setminus \cup_{j=0}^k E'_j$ $(k = 0,\ldots,n-1)$, $s := \sum_{k=0}^n x_k \chi_{E_k}$ to obtain that $\|s - f\|_B < \varepsilon$ a.e., hence $\|f - s\|_\infty \le \varepsilon$. Since $\varepsilon > 0$ and $f \in \mathrm{L}^\infty_\mathrm{K}$ were arbitrary, we observe that SMF is dense in $\mathrm{L}^\infty_\mathrm{K}$.

3° $\mathrm{L}^\infty_\mathrm{K}$ *is the closure of* SMF*:* By 2°, we only need to show that $f \in \mathrm{L}^\infty_\mathrm{K}$ assuming that $\{s_n\} \subset \mathrm{SMF}$ and $\|s_n - f\|_\infty < 1/n$ for all $n \in \mathbf{N}+1$ (so that $\mathrm{L}^\infty_\mathrm{K}$ is closed).

For each $n \ge 1$, choose a null set $N_n$ s.t. $\|s_n - f\| < 1/n$ on $N_n^c$. Set $N := \cup_{n\ge1} N_n$, $A := f[N^c] \subset B$. Given $\varepsilon > 0$, choose $n > 1/\varepsilon$, so that $\|s_n - f\| < \varepsilon$ on $N^c$. Write $s_n$ as $s_n = \sum_{k=1}^n x_k \chi_{E_k}$ with $E_k \cap E_j = \emptyset$ for $k \ne j$.

Then $A \subset D(0,\varepsilon) \cup \big(\cup_{k=1}^n D(x_k,\varepsilon)\big)$, because $\|f - x_k\| < \varepsilon$ on $E_k$ and $\|f\| < \varepsilon$ on $(\cup_k E_k)^c \setminus N$. Because $\varepsilon > 0$ was arbitrary, the set $A$ is totally bounded (i.e., precompact), hence so is $K = \overline{A}$ (use $2\varepsilon$ in place of $\varepsilon$), hence $K$ is compact (use, e.g., Theorem 9.4 of [Bredon] and the completeness of $B$). Moreover, $f(q) = \lim_n s_n(q) \in K$ for $q \in N^c$, hence a.e. Therefore, $f \in \mathrm{L}^\infty_\mathrm{K}$.

4° $\mathrm{L}^\infty_\mathrm{K}(Q;B) = \mathrm{L}^\infty(Q;B)$ *if* $\dim B < \infty$*:* If $\dim B < \infty$, then we can take $K := D(0,\|f\|_\infty)$ for any $f \in \mathrm{L}^\infty$ to observe that $f \in \mathrm{L}^\infty_\mathrm{K}$.

5° $\mathrm{L}^\infty_\mathrm{K}(\Omega;B) = \mathrm{L}^\infty(\Omega;B)$ *iff* $\dim B < \infty$*:* Assume that $\dim B = \infty$. Let $\{E_k\} \subset \Omega$ are disjoint sets of positive measure (in fact, we need not have $\Omega \subset \mathbf{R}^n$ as long as this property is satisfied).

The unit ball $D_1$ of $B$ is not compact, by Theorem 1.23 of [Rud73], hence there is a sequence $\{x_k\} \subset D_1$ without limit points (by Exercise 2.26 and Theorem 2.37 of [Rud76], this is equivalent to noncompactness in any metric space). Set $f := \sum_{n\in\mathbf{N}} x_k \chi_{E_k} \in \mathrm{L}^\infty(\Omega;B)$ to obtain that $f[N^c] = \{x_k\}$ whenever $N$ is a null set; in particular, $f[N^c]$ is not contained in any compact subset of $B$, hence $f \notin \mathrm{L}^\infty_\mathrm{K}$.

(N.B. the above example also shows that $\mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega;B) \not\subset \mathrm{L}^\infty_\mathrm{K}(\Omega;B)$; e.g., choose $\{E_k\}$ s.t. $\sum_k m(E_k) < \infty$ to have $f \in \mathrm{L}^1(\Omega;B)$ too.) □

If $\mu$ is the completion of another measure $\mu'$, then the simple functions constructed above can be chosen to be "$\mu'$-measurable":

**Lemma B.3.12** *Let $X = B$ or $X = [-\infty, \infty]$. Let $s = \sum_{k\in\mathbf{N}} x_k \chi_{E_k}$, where $x_k \in X$ and $E_k \in \mathfrak{M}$ for all $k \in \mathbf{N}$. If $\mathfrak{M}$ is the completion of another $\sigma$-algebra $\mathfrak{M}'$, then there are sets $\{E'_k\} \subset \mathfrak{M}'$ s.t. $\sum_{k\in\mathbf{N}} x_k \chi_{E'_k} = s$ a.e.* □

(We omit the trivial proof.)

We generalize the Hölder inequality to the case $r > 1$:

**Lemma B.3.13 ($\|fg\|_r \le \|f\|_p \|g\|_q$)** *Let $f \in \mathrm{L}^p(Q;B)$, $g \in \mathrm{L}^q(Q;B_2)$, $p,q \in (0,\infty]$ and $\|bb_2\|_{B_3} \le \|b\|_B \|b_2\|_{B_2}$ ($b \in B$, $b_2 \in B_2$). Then $\|fg\|_r \le \|f\|_p \|g\|_q$, where $r^{-1} := p^{-1} + q^{-1}$.*

*Moreover, if $Q = \mathbf{R}^n$, $\mu = m$, $p,q < \infty$ and $\varepsilon > 0$, then there is $R > 0$ s.t. $\|f\tau^t g\|_r < \varepsilon$ for all $t \in \mathbf{R}^n$ s.t. $|t| > R$ (if $f \in \mathcal{C}_0$, then we can allow $p = \infty$; if $g \in \mathcal{C}_0$, then we can allow $q = \infty$).*

Thus, we may have, e.g., $B = \mathbf{K}$ ($B_3 = B_2$), $B = B_2^*$ ($B_3 = \mathbf{K}$) or $B = \mathcal{B}(B_2, B_3)$. Note that we may have $r < 1$, but $\min\{p,q\}/2 \le r \le \min\{p,q\}$.

**Proof:** 1° *Inequality $\|fg\|_r \le \|f\|_p \|g\|_q$:* If $p = \infty$ or $q = \infty$, then this is obvious (and $r = \min\{p,q\}$), so we assume that $p,q \in [1,\infty)$. Set $F := \|f\|_B$, $G := \|g\|_{B_2}$, $t := (p+q)/q$, $t' := (p+q)/p$, so that $rt = p$, $rt' = q$ and $t^{-1} + t'^{-1} = 1$. Then, by the Hölder inequality, we have

$$\|fg\|_r^r = \int_Q F^r G^r \, d\mu \le \|F^r\|_t \|G^r\|_{t'} \le \|F\|_{rt}^r \|G\|_{t'r}^r \le \|F\|_p^r \|G\|_q^r, \tag{B.14}$$

hence $\|fg\|_r \le \|f\|_p \|g\|_q$.

2° *Finding $R$:* Assume that $\|f\|_p \le 1$ and $\|g\|_q \le 1$. By Lemma B.3.7, there is $R > 0$ s.t. $\|\chi_{D_R^c} f\|_p < \varepsilon/2$ and $\|\chi_{D_R} \tau^t g\|_q < \varepsilon/2$ for $t \in \mathbf{R}^n$ s.t. $|t| > q$. Consequently, $\|f\tau^t g\| < \varepsilon$ for such $t$. □

If $p \in [p_0, p_1]$, then $\mathrm{L}^p \subset \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$:

**Lemma B.3.14 ($\|f\|_p \le \max\{\|f\|_{p_0}, \|f\|_{p_1}\}$)** *Let $f \in \mathrm{L}^{p_0}(Q;B) \cap \mathrm{L}^{p_1}(Q;B)$, $1 \le p_0 \le p \le p_1 \le \infty$. Then*

$$\|f\|_p \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta} \le \max\{\|f\|_{p_0}, \|f\|_{p_1}\}, \tag{B.15}$$

*where $\theta := \frac{p^{-1} - p_0^{-1}}{p_1^{-1} - p_0^{-1}}$.*

**Proof:** The scalar case is Theorem 5.1.1 of [BL]; see also Theorem 4.1.2 and p. 27 of [BL]. (The definition of $\mathrm{L}^\infty$ in [BL] coincides with the standard one in the scalar case, by Theorem 1.17 of [Rud86].) Apply the scalar case to $\|f\|_B \in \mathrm{L}^{p_0}(Q)$ to obtain the general case. □

Next we note that, roughly speaking, $\mathrm{L}^p$ is separable iff $p < \infty$ and $B$ is separable:

**Lemma B.3.15** *Let $B$ be separable and $1 \le p < \infty$. Then $\ell^p(Q;B)$ is separable iff $Q$ is at most countable. If $Q \subset \mathbf{R}^n$ is measurable and $d\mu = |f|\,dm$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$,then $\mathrm{L}^p(Q;B)$ is separable.*

*If $\mu$ is as above, $\mu(Q) \neq 0$ and $B_2 \neq \{0\}$, then $\mathrm{L}^\infty(Q,\mu;B_2)$ is unseparable.*

*If $0 < \mu(E) < \infty$ for some measurable $E \subset Q$, then $\mathrm{L}^p(Q;B_2)$ and $\mathrm{L}^\infty(Q;B_2)$ are unseparable for unseparable $B_2$.*

**Proof:** 1° *Preparations:* For each $k \in \mathbf{N}$, we let $P_k$ be the set of points in $\mathbf{R}^n$ whose coordinates are integral multiples of $2^{-k}$, and $V_k$ the collection of $2^{-k} \times \cdots 2^{-k}$ boxes with corners at points of $P_k$. Set $V := \cup_k V_k$, so that $V$ is countable and any open $G \subset \mathbf{R}^n$ is the union of disjoint elements of $V$ (see [Rud86, 2.19] for details). Let $S \subset B$ be dense and countable.

2° *The closed span of $\mathcal{F} := \{x\chi_G \,|\, x \in S,\ G \in V\}$ is dense in $\mathrm{L}^p(\mathbf{R}^n,\mu;B)$:* Let $A \subset \mathbf{R}^n$ have a finite measure and $\varepsilon > 0$. By the Monotone Convergence Theorem, there is $R > 0$ s.t. $A_R := \{q \in A \,|\, |q| < R\}$ satisfies $\|\chi_A - \chi_{A_R}\|_1 < \varepsilon/4$. Choose open $G \subset \mathbf{R}^n$ s.t. $\|\chi_{A_R} - \chi_G\|_1 < \varepsilon/4$ (see (B.28) and use the fact that $\mu$ is absolutely continuous w.r.t. $m$ on $\{|q| < R\}$).

Write $G$ as a disjoint union of elements $G_0, G_1, G_2, \ldots \in V$, so that $khi_G = \sum_{k\in\mathbf{N}} \chi_{G_k}$, and use the Monotone Convergence Theorem to choose $k \in \mathbf{N}$ s.t. $\|\chi_G - \sum_{j=0}^k \chi_{G_j}\|_1 < \varepsilon/4$. Then, $\|\chi_A - \sum_{j=0}^k \chi_{G_j}\|_1 < 3\varepsilon/4$.

Given $x \in B$, we have $\|x\chi_A - x' \sum_{j=0}^k \chi_{G_j}\|_1 < \varepsilon$ for some $x' \in S$. By linearity, it follows that the closed span of $\mathcal{F}$ contains that of simple measurable functions, i.e., it is equal to $\mathrm{L}^p(\mathbf{R}^n;B)$.

3° *The closed span of $\mathcal{F} := \{x\chi_G\chi_Q \,|\, x \in S,\ G \in V\}$ is dense in $\mathrm{L}^p(Q,\mu;B)$:* Apply 2° to the zero extensions of $f$ and $\mu$ onto $\mathbf{R}^n$.

4° *$\ell^p(Q;B)$:* The closed span of $\{\chi_{\{q\}} x \,|\, x \in S,\ q \in Q\}$ is obviously dense in $\ell^p(Q,X)$ (e.g., use the proof of 2° with $V := \{\{q\} \,|\, q \in Q\}$).

5° *$\ell^\infty(Q;B_2)$ is unseparable for uncountable $Q$ and $B_2 \neq \{0\}$:* Let $\varepsilon < 1/2$. Choose $x \in B_2$ s.t. $\|x\| = 1$. Then the $\varepsilon$-neighborhoods of no countable subset of $\ell^\infty$ can contain every $x\chi_q$, $q \in Q$, because $\|x\chi_q - x\chi_{q'}\|_\infty = 1$ for $q' \neq q$.

6° *$\mathrm{L}^\infty(Q,\mu;B_2)$ is unseparable:* Let $Q_0 := Q$, $r_0 := \infty$. Given $(Q_k, r_k)$, choose $r_{k+1} \in (0, r_k)$ s.t. $0 < \mu(Q_{k+1}) < \mu(Q_k)$, where $Q_{k+1} := \{q \in Q_k \,|\, |q| < r_{k+1}\}$, and set $Q'_k := Q_k \setminus Q_{k+1}$. Then $Q = \cup_{k\in\mathbf{N}} Q'_k$, and the sets $Q'_k$ are disjoint and of positive (or infinite) measure.

Choose $x \in B_2$ s.t. $\|x\|_{B_2} = 1$. For each $E \subset \mathbf{N}$, set $f_E := x\chi_{\cup_{n\in E} Q'_n}$. Then $\|f_E - f_{E'}\|_\infty = 1$ whenever $E \neq E'$, so that the $\varepsilon$-neighborhoods of no countable set can contain every $f_E$ for $\varepsilon < 1/2$. Thus, $\mathrm{L}^\infty(Q,\mu;B)$ is unseparable.

7° *$\mathrm{L}^p(Q;B_2)$ is unseparable for unseparable $B_2$ if $0 < \mu(E) < \infty$ for some measurable $E \subset Q$:* The subspace $\{x\chi_E \,|\, x \in B_2\}$ of $\mathrm{L}^p$ is isometrically isomorphic to $B_2$, hence unseparable. Therefore, so is $\mathrm{L}^p$. □

Recall from Lemma A.3.1(a1)&(a2) that $\dim H$ means the cardinality of an arbitrary orthonormal basis of $H$. We have $\dim \mathrm{L}^2(Q;H) = \dim H$ for infinite-dimensional $H$'s and most $Q$'s:

**Lemma B.3.16 ($\dim \mathrm{L}^2(Q,\mu;H) = \dim H$)** *Assume that $H$ is a Hilbert space. Then* $\dim \mathrm{L}^2(Q;H) = \dim \mathrm{L}^2(Q) \times \dim H$*; in particular*, $\dim \ell^2(Q;H) = \operatorname{card} Q \times \dim H$ *(*$= \dim H$ *whenever $Q \neq \emptyset$, $Q$ is at most countable and $H$ is infinite-dimensional).*

*If $Q \subset \mathbf{R}^n$ or $Q \subset \partial\mathbf{D}$ is measurable, $\mu(Q) \neq 0$, and $d\mu = |f|\,dm$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(Q,m)$,then $\dim \mathrm{L}^2(Q,\mu;H) = \operatorname{card}\mathbf{N} \times \dim H$ ($= \dim H$ whenever $H$ is infinite-dimensional).*

**Proof:** 1° Let $\{x_a\}_{a\in A}$ be an orthonormal base of $H$ (so that $\dim H = A$) Let $F$ be an orthonormal base of $\mathrm{L}^2(Q,\mu)$. Then $\{fx_a\}_{f\in F, a\in A} \subset \mathrm{L}^2(Q,\mu;H)$ is an orthonormal base of $\mathrm{L}^2(Q,\mu;H)$ (its closed span is $\mathrm{L}^2$, by the density of simple $\mathrm{L}^2$ functions) of cardinality of $F \times A$. Thus, $\dim \mathrm{L}^2(Q;H) = \operatorname{card}(F \times A) = \dim \mathrm{L}^2(Q,\mu) \times \dim H$.

2° Since the set of simple $\ell^2(Q)$ functions is exactly $\mathrm{c}_{\mathrm{c}}(Q)$, the set $\{\chi_{\{q\}}\}_{q\in Q}$ is an orthonormal base of $\ell^2(Q)$. Consequently, $\dim \ell^2(Q) = \operatorname{card} Q$, so that $\dim \ell^2(Q;H) = \operatorname{card} Q \times \dim H$, by 1°.

3° Let $Q$ be as in the latter paragraph. By Lemma B.3.15, $\mathrm{L}^2(Q,\mu)$ is separable. It is obviously infinite-dimensional, hence $\dim \mathrm{L}^2(Q,\mu) = \operatorname{card}\mathbf{N}$. Consequently, $\dim \mathrm{L}^2(Q,\mu;H) = \operatorname{card}\mathbf{N} \times \dim H$, by 1°.

4° By Lemma B.2.2, $A \times \dim H = \dim H$ when $A \neq \emptyset$ and $A \leq \dim H \geq \operatorname{card}\mathbf{N}$. □

(See the notes on p. 947.)

## B.4 The Bochner integral ($\int_Q : \mathrm{L}^1(Q;B) \to B$)

*Adde parvum parvo manus acervus erit.*
— Ovid (43 B.C. – 17 A.D.)

In this section we define the Bochner integral $\mathrm{L}^1(Q;B) \to B$ and present the Bochner integral extensions of several more and less known Lebesgue integral results. A casual reader might wish just to have a look at Subsections B.4.1–B.4.3 and Theorem B.4.6 and then skip the rest of this section, just remembering that the Bochner integral is "the Lebesgue integral with '$\|\cdot\|$' in place of '$|\cdot|$'".

By Theorem B.3.11(a1) and Lemma A.3.10, we may use the natural definition and density to define the Bochner integral:

**Definition B.4.1 (Bochner integral)** *We recall that for simple functions* $s := \sum_{k=0}^n x_k \chi_{E_k}$ *(*$x_k \in B$*,* $E_k$ *measurable for all* $k$*) we have set*

$$\int_Q s\, d\mu := \sum_{k=0}^n x_k \mu(E_k) \in B. \tag{B.16}$$

*The unique continuous extension of* $\int_Q \cdot\, d\mu$ *onto* $\mathrm{L}^1(Q;B)$ *is called the* Bochner integral.

*Let* $f \in \mathrm{L}^1$*. Then* $f$ *is called* Bochner integrable *and the* integrand *of* $\int_Q f\, d\mu$*, which, in turn, is said to* converge absolutely.

Obviously, $\|\int_Q s\, d\mu\|_B \le \|s\|_1$, hence the same holds for any integrable function $s : Q \to B$, by Lemma A.3.10. One easily verifies that if $f \in \mathrm{L}^1(Q;\mathbf{K})$ and $f \ge 0$, then this coincides with the (Lebesgue) integral defined in Section B.1.

Sometimes we write $\int_Q f(t)\, d\mu(t) := \int_Q f\, d\mu$ (e.g., $\int_Q t^2\, d\mu(t)$). If $-\infty \le a \le b \le +\infty$ and $\mu = m$ (or $m(E) = 0 \Rightarrow \mu(E) = 0$, i.e., $\mu$ is *absolutely continuous w.r.t.* $m$), then we set $\int_a^b f(t)\, dt := \int_a^b f\, dm := \int_{(a,b)} f\, dm$. For $b < a$ we set $\int_a^b := -\int_b^a$.

If $B = \mathbf{K}$ (or "$B = [0, +\infty]$"), then the Bochner integral is called the Lebesgue integral, etc. By Theorem 11.33 of [Rud76], a function $f : Q \to \mathbf{K}$ is Riemann integrable iff it is bounded and continuous a.e. It follows that Riemann integrable functions belong to $\mathrm{L}^1$. Moreover, the Riemann integral coincides with the Lebesgue integral. We shall not be using the Riemann integral.

An equivalent way to define the integral is to define it for simple functions in the natural way and then use Lemma A.3.10 and Theorem B.3.11 to extend it to all $\mathrm{L}^1$ functions. This definition is used in [HP] and illustrated below:

**Lemma B.4.2** *The Bochner integral is in* $\mathcal{B}(\mathrm{L}^1, B)$*, its norm is* $1$ *(unless* $\mathrm{L}^1 = \{0\}$*); in particular,*

$$\|\int_Q f\, d\mu\| \le \int_Q \|f\|_B\, d\mu =: \|f\|_1, \tag{B.17}$$

*and the integral commutes with bounded linear transformations:*

$$T \int_Q f\, d\mu = \int_Q T f\, d\mu \quad (T \in \mathcal{B}(B, B_2)). \tag{B.18}$$

*Moreover,* $\int_Q f\,d\mu$ *is the unique element of* $B$ *satisfying* $\Lambda\int_Q f\,d\mu = \int_Q \Lambda f\,d\mu$ *for all* $\Lambda \in B^*$.

*Finally, if* $s = \sum_{k\in\mathbf{N}} x_k\chi_{E_k} \in \mathrm{L}^1$*, with the sets* $E_k$ *being disjoint and measurable, then* $\int_Q s\,d\mu = \sum_{k\in\mathbf{N}} x_k\mu(E_k)$.

It follows that our definitions of integrable functions and the Bochner integral are equivalent to those of [HP], Section 3.7.

**Proof:** The $BL(\mathrm{L}^1,B)$ claim and (B.17) follow from Definition B.4.1. If $\mathrm{L}^1 \neq \{0\}$, then there are $x \in B\setminus\{0\}$ and $E \subset Q$ s.t. $\mu(E) \in (0,+\infty)$, and we have $\|\int_Q x\chi_E\,d\mu\| = \|x\|_B\mu(E) = \|x\chi_E\|_1$.

We have $T\int = \int T$ for simple functions, hence for all $\mathrm{L}^1$ functions, by continuity; (B.18). Elements $\{\Lambda x \,|\, \Lambda \in B^*\}$ determines $x \in B$ uniquely.

If $s$ is as in the final claim, then $s_n := \sum_{k=0}^n x_k\chi_{E_k} \to s$ in $\mathrm{L}^1$, by the Monotone Convergence Theorem, hence $\int_Q s_n\,d\mu \to \int_Q s\,d\mu$. □

The standard results can be extended with ease:

**Theorem B.4.3 (Lebesgue's Dominated Convergence Theorem)** *Assume that* $1 \le p < \infty$*, that the functions* $f_n : Q \to B$ *be measurable (*$n \in \mathbf{N}$*), that the limit* $f(q) := \lim_{n\to+\infty} f_n(q)$ *exists a.e., and that there is* $g \in \mathrm{L}^p(Q;[0,+\infty])$ *s.t.* $\|f_n(q)\|_B \le g(q)$ *a.e. for each* $n \in \mathbf{N}$.

*Then* $f \in \mathrm{L}^p(Q;B)$ *and* $f_n \to f$ *in* $\mathrm{L}^p$*. In particular, if* $p = 1$*, then* $\lim_{n\to+\infty}\int_Q f_n\,d\mu = \int_Q f\,d\mu$. □

(This follows by applying the scalar LCD Theorem with $F_n := \|f - f_n\|_B^p$, $F := 0$, $\mathrm{L}^p \ni G := (2g)^p \ge F_n$. Obviously, this does not holds for $p = \infty$.)

From the above and the Monotone Convergence Theorem applied to $\sum_{n=1}^N \chi_{E_k} f$ we obtain that if the sets $E_n \subset Q$ $(n \in \mathbf{N})$ are measurable and disjoint, and $f : Q \to B$ is measurable, then $\int_{\cup_n E_n} f\,d\mu = \sum_{n\in\mathbf{N}}\int E_n f\,d\mu$ whenever either sides converges absolutely (i.e., with $\|f\|_B$ in place of $f$).

Next we extend the standard definition of a product measure:

**Definition B.4.4 ($\overline{\mu\times\nu}$)** *Assume that* $\mu : \mathfrak{M} \to [0,\infty]$ *and* $\nu : \mathfrak{M}' \to [0,\infty]$ *are* $\sigma$*-finite, positive measures on* $Q$ *and* $R$*, respectively. By* $\mathfrak{M}_{\mu\times\nu}$ *we denote the smallest* $\sigma$*-algebra containing* $\{E \times E' \,|\, E \in \mathfrak{M},\ E' \in \mathfrak{M}'\}$*. The* product measure *of* $\mu$ *and* $\nu$ *is given by*

$$(\mu\times\nu)(E) := \int_Q \nu(\{r \,|\, (q,r) \in E\})\,d\mu \quad (E \in \mathfrak{M}_{\mu\times\nu}). \tag{B.19}$$

*By* $\overline{\mu\times\nu} : \mathfrak{M}_{\overline{\mu\times\nu}} \to [0,\infty]$ *we denote the completion of* $\mu\times\nu$*. By* $\mathrm{L}(Q\times R;B)$ *we refer to* $\overline{\mu\times\nu}$*-measurable functions* $Q\times R \to B$.

Since in the definitions and results of this chapter we have assumed $\mu$ to be complete (as in [HP]), we shall use $\overline{\mu\times\nu}$ (not $\mu\times\nu$) on $Q\times R$. The basic properties of this measure are listed below:

**Lemma B.4.5** *Assume that* $\mu : \mathfrak{M} \to [0,\infty]$ *and* $\nu : \mathfrak{M}' \to [0,\infty]$ *are* $\sigma$*-finite, positive measures on* $Q$ *and* $R$*, respectively.*

*(a) The measure $\overline{\mu\times\nu}$ is σ-finite.*

*(b) We have $(\overline{\mu\times\nu})(E) := \int_R \mu(\{q \,|\, (q,r)\in E\})\,d\nu$ for all $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$.*

*(c) Moreover, if $m_k$ is the Lebesgue measure on $\mathbf{R}^k$ ($k\geq 1$), then $m_{n+k} = \overline{m_n\times m_k}$ ($n,k\geq 1$).*

*(d1) If $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$, then $E_q := \{r\in R \,|\, (q,r)\in E\}$ is measurable for a.e. $q\in Q$, and $E^r := \{q\in Q \,|\, (q,r)\in E\}$ is measurable for a.e. $r\in R$.*

*(d2) If $N$ is a null set, then $\nu(N_q)=0$ for a.e. $q\in Q$ and $\mu(N^r)=0$ for a.e. $r\in R$.*

*(e)* **(“$\mathrm{cl}(\mathfrak{M}\times\mathfrak{M}') = \mathfrak{M}_{\overline{\mu\times\nu}}$”)** *Let $\mathcal{E}$ be the collection of finite unions of elements of $\mathfrak{M}\times\mathfrak{M}'$. Let $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$ and $\overline{\mu\times\nu}(E)<\infty$. Then, for any $\varepsilon>0$, there is $F\in\mathcal{E}$ s.t. $\|\chi_E-\chi_F\|<\varepsilon$.*

*(f) If $f\in \mathrm{L}^p(Q\times R;B)$, $p\in(0,\infty)$, then simple measurable functions $\{s_n\}$ of form $s_n=\sum_{k=0}^{N_n}\chi_{E_{n,k}\times F_{n,k}}b_{n,k}$ and $E_{n,k}\times F_{n,k}\in\mathfrak{M}\times\mathfrak{M}'$, $b_{n,k}\in B$ ($n,k\in\mathbf{N}$) s.t. $\|s_n-f\|_p\to 0$ as $n\to\infty$.*

*Note that each $s_n$ can be written as $\sum_{k=0}^{N_n'}\chi_{E_{n,k}'}s_{n,k}'$, where $E_{n,k}'\in\mathfrak{M}$ and $s_{n,k}': R\to B$ is simple ($n,k\in\mathbf{N}$).*

**Proof:** Claims (a)–(c) are proved in Chapter 8 of [Rud86]. Claims (d1) and (d2) follow from the Fubini Theorem (with $f=\chi_E$).

(e) 1° *Case $\mu(Q),\nu(R)<\infty$:* Set $\mathfrak{M}_0 := \{E\in\mathfrak{M}_{\overline{\mu\times\nu}} \,|\, E$ the claim in (e) holds for $E\}$, so that we only need to show that $\mathfrak{M}_0=\mathfrak{M}_{\overline{\mu\times\nu}}$, i.e., that $\mathfrak{M}_0$ is a σ-algebra (because, trivially, $\mathfrak{M}\times\mathfrak{M}'\subset\mathcal{E}\subset\mathfrak{M}_0$). This follows from 1.1° and 1.2°.

1.1° One easily verifies that $\|\chi_E-\chi_F\|_1 = \|\chi_{E^c}-\chi_{F^c}\|_1$ and $\|\chi_{E\cup E'}-\chi_{F\cup F'}\|_1\leq\|\chi_E-\chi_F\|_1+\|\chi_{E'}-\chi_{F'}\|_1$, hence $\mathfrak{M}_0$ is closed under complements and finite unions.

1.2° Let $E_j\in\mathfrak{M}_0$ ($j\in\mathbf{N}$) and $E=\cup_{j\in\mathbf{N}}E_j$. Set $E_{j+1}' := E_{j+1}\setminus\cup_{k=0}^{j}E_k$, so that $E=\cup_j E_j'$ and the sets $E_j'$ are disjoint. Because $\overline{\mu\times\nu}(Q\times R)<\infty$, we have $\|\chi_E-\chi_{\cup_{j\leq N}E_j'}\|<\varepsilon/2$ for some $N\in\mathbf{N}$. But $\|\chi_{\cup_{j\leq N}E_j'}-\chi_F\|_1<\varepsilon/2$ for some $F\in\mathcal{E}$, hence $E\in\mathfrak{M}_0$.

2° *General Case:* Let $Q=\cup_{n\in\mathbf{N}}Q_n$, $R=\cup_{n\in\mathbf{N}}R_n$, where $Q_0\subset Q_1\subset\cdot$ and $R_0\subset R_1\subset\cdot$. Given $\varepsilon>0$ and $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$ s.t. $\overline{\mu\times\nu}(E)<\infty$, there is $N\in\mathbf{N}$ s.t. $\|\chi_E-\chi_{E\cap(Q_n\times R_n)}\|_1<\varepsilon/2$. By 1°, there is $F\in\mathcal{E}$ s.t. $\|\chi_{E\cap(Q_n\times R_n)}-\chi_F\|_1<\varepsilon/2$, hence (e) holds.

(f) The first claim follows from (e) and Theorem B.3.11. For the second claim, given $n\in\mathbf{N}$, let $\{E_{n,k}' \,|\, k=0,\ldots,N_n'\}$ consist of the sets $\{(\cup_{k\in s}E_{n,k})\setminus\cup_{k\notin s}E_{n,k} \,|\, s\subset\{0,1,\ldots,N_n\}\}$, and note that $s_n(q,r)=s_n(q',r)=: s_{n,k}(r)$ for all $q,q'\in E_{n,k}'$ ($k\in\mathbf{N}$). □

As in the scalar case, the norm of a $\overline{\mu\times\nu}$-measurable function may be integrated in any order, and for $\mathrm{L}^1$ functions the same applies to the function itself:

**Theorem B.4.6 (Fubini)** *Assume that μ and ν are σ-finite, complete, positive measures on $Q$ and $R$, respectively. Let $f\in\mathrm{L}(Q\times R;B)$.*

*(a1) We have* $\int_R \|f\|_B \, d\nu \in \mathrm{L}(Q;[0,+\infty])$ *and* $\int_Q \|f\|_B \, d\mu \in \mathrm{L}(R;[0,+\infty])$.

*(a2) If* $\int_Q \int_R \|f\|_B \, d\nu \, d\mu < \infty$ *or* $\int_R \int_Q \|f\|_B \, d\mu \, d\nu < \infty$, *then* $f \in \mathrm{L}^1(Q \times R; B)$.

*(b) If* $f \in \mathrm{L}^1(Q \times R; B)$, *then* $g(q) := \int_R f(q,r) \, d\nu(r)$ *and* $h(r) := \int_Q f(q,r) \, d\mu(q)$ *are defined a.e. and satisfy* $g \in \mathrm{L}^1(Q;B)$, $h \in \mathrm{L}^1(R;B)$ *and*

$$\int_{Q\times R} f \, d\overline{\mu\times\nu} = \int_Q g \, d\mu = \int_R h \, d\nu. \tag{B.20}$$

**Proof:** (a1)&(a2) This follows from the classical Fubini Theorem (Theorem 8.8 of [Rud86]).

(b) This is Theorem 3.7.13 of [HP]. □

The rest of this section is rather technical.

In order to satisfy the $\overline{\mu\times\nu}$-measurability assumption of the Fubini Theorem and its several important applications, we need to show that our function is (product) measurable and study the relation between $f : Q \times R \to B$ and $f : Q \to \mathrm{L}(R;B)$. We start from basic facts:

**Lemma B.4.7** *Assume that* $\mu : \mathfrak{M} \to [0,\infty]$ *and* $\nu : \mathfrak{M}' \to [0,\infty]$ *are* $\sigma$*-finite, complete, positive measures on* $Q$ *and* $R$*, respectively. Let* $p \in [1,\infty]$.

*(a) If* $f \in \mathrm{L}(\overline{\mu\times\nu}; B)$, *then* $f(q,\cdot) \in \mathrm{L}(R;B)$ *for a.e.* $q \in Q$.

*(b) If* $f \in \mathrm{L}(Q;B)$, $g \in \mathrm{L}(R;B_2)$, *and* $B \times B_2 \to B_3$ *is continuous, then* $fg \in \mathrm{L}(Q \times R; B_3)$.

*(c) Let* $f, g \in \mathrm{L}(Q \times R; B) \cap \mathrm{L}(Q; \mathrm{L}^p(R;B))$. *If* $f = g$ *a.e. on* $Q \times R$, *then* $[f] = [g] \in \mathrm{L}(Q;\mathrm{L}^p(R;B))$. *Conversely, if* $[f] = [g]$ *as elements of* $\mathrm{L}(Q;\mathrm{L}^p)$, *i.e.,* $f(q,\cdot) = g(q,\cdot)$ *a.e. on* $R$ *for a.e.* $q \in Q$, *then* $f = g$ *a.e. on* $Q \times R$.

*(d) Let* $f \in \mathrm{L}(Q \times R; B)$ *and* $p \in [1,\infty)$. *Then* $f \in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$ *iff* $f \in \mathrm{L}^p(Q \times R; B)$. *If* $f \in \mathrm{L}^p(Q \times R;B)$, *then* $\|f\|_{\mathrm{L}^p(Q\times R;B)} = \|f\|_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))}$.

**Proof:** (a) Let $s_n \to f$ pointwise on $N^c$, where $N \subset Q \times R$ is a null set and $s_n = \sum_{k\in\mathbf{N}} \chi_{E_{n,k}} b_{n,k}$, where $E_{n,k} \in \mathfrak{M}_{\overline{\mu\times\nu}}$, $b_{n,k} \in B$ and $k \neq j \Rightarrow E_{n,k} \neq E_{n,j}$ for all $n,k,j \in \mathbf{N}$.

Let $n \in \mathbf{N}$. For each $q \in Q$, the function $s_n(q,\cdot) : R \to B$ is countably-valued. But there is a null set $N' \subset Q$ s.t. $(E_{n,k})_q$ is measurable for all $q \in Q \setminus N'$ and all $k \in \mathbf{N}$, by Lemma B.4.5(d1). Moreover, $\nu(N_q) = 0$ for a.e. $q \in Q$, say, for $q \in Q \setminus N''$, where $N''$ is a null set. Set $N''' := N' \cup N''$. Then $s_n(q,\cdot) \to f(q,\cdot)$ and $s_n(q,\cdot)$ is countably-valued and measurable for all $q \in Q \setminus N'''$. Thus, $f(q,\cdot) \in \mathrm{L}(R;B)$ for all $q \in Q \setminus N'''$, hence a.e.

(b) Let $f_n : Q \to B$ and $g_n : R \to B_2$ be countably-valued and measurable ($n \in \mathbf{N}$), and let $f_n \to f$ on $Q \setminus N_Q$, $R \setminus N_R$, where $\mu(N_Q) = 0 = \nu(N_R)$. Then $f_n g_n \to fg$ on $N_Q^c \times N_R^c$, and $\overline{\mu\times\nu}((Q\times R) \setminus (N_Q^c \times N_R^c)) = 0$.

(c) (The assumptions on $f$ mean that $f : Q \times R \to B$ is measurable, $f(q,\cdot) \in \mathrm{L}^p(R;B)$ for a.e. $q \in Q$, and $(q \mapsto f(q,\cdot)) \in \mathrm{L}(Q;\mathrm{L}^p)$. The first claim shows that $f \in \mathrm{L}$ is independent of the representative of $[f] \in \mathrm{L}(Q \times R;B)$. However, $[f] \in \mathrm{L}(Q;\mathrm{L}^p(R;B))$ may have a representative $h : Q \to \mathrm{L}^p(R;B)$ that is not measurable $Q\times R\to B$, see Example B.4.18; but if $h$ is measurable, then $h=f$ a.e. on $Q\times R$, by the converse claim.)

If $(f-g)=0$ a.e., say on $Q\times R\setminus N$, where $\overline{\mu\times\nu}(N)=0$, then $(f-g)(q)=0$ a.e. on $R$ for a.e. $q$, by Lemma B.4.5(d2), hence $[f]=[g]\in \mathrm{L}^a(Q;\mathrm{L}^p)$.

Conversely, assume that $[f]=[g]\in \mathrm{L}^a$ and set $N:=\{(q,r)\in R\times Q \mid f(q,r)\neq g(q,r)\}$. Then $f(q)=g(q)$ a.e. on $R$ for all $q\in Q\setminus N'$, where $\mu(N')=0$, i.e., $\nu(N_q)=0$ for all $q\in Q\setminus N'$. But

$$\overline{\mu\times\nu}(N):=\int_Q \nu(N_q)\,d\mu=0, \tag{B.21}$$

i.e., $f=g$ a.e. on $Q\times R$.

(d) By (b), we can consider $f$ also as a function $Q\to \mathrm{L}(R;B)$ (by redefining $f$ on a null subset of $Q$; this affects $f$ only on a null subset of $Q\times R$).

1° Let $f\in \mathrm{L}^p(Q\times R;B)$. Choose simple measurable functions $\{s_n\}$ as in Lemma B.4.5(f). It follows that $s_n:Q\to \mathrm{L}^p(R;B)$ is measurable ($n\in\mathbf{N}$). But

$$\|s_n-s_m\|^p_{\mathrm{L}^p(Q;\mathrm{L}^p)}=\int_Q\int_R \|s_n-s_m\|_B^p\,d\nu\,d\mu=\|s_n-s_m\|^p_{\mathrm{L}^p(Q\times R;B)}\to 0 \tag{B.22}$$

as $n,m\to\infty$, hence $s_n$ converges in $\mathrm{L}^p(Q;\mathrm{L}^p)$ to some $g:Q\to \mathrm{L}^p(R;B)$.

Replace $\{s_n\}$ by a subsequence s.t. $s_n(q)\to g(q)$ for $q\in Q\setminus N'$ and $s_n(q,r)\to f(q,r)$ for $(q,r)\in Q\times R\setminus N$, where $\mu(N')=0=\overline{\mu\times\nu}(N)$. By Lemma B.4.5(d2), there is $N''\subset Q$ s.t. $\nu(N_q)=0$ for all $q\in Q\setminus N''$. Set $N''':=N'\cup N''$.

Let $q\in Q\setminus N'''$. Then $s_n(q,r)\to f(q,r)$ for a.e. $r\in R$ and $s_n(q)\to g(q)$ in $\mathrm{L}^p(R;B)$, hence $f(q)=g(q)$ a.e., i.e., as elements of $\mathrm{L}^p(R;B)$, Because $\mu(N''')=0$, we have $f=g$ as elements of $\mathrm{L}(Q;\mathrm{L}^p(R;B))$; thus, $f=g\in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$.

2° Let $f\in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$. Then, by the Fubini Theorem,

$$\|f\|_p^p=\int_Q\int_R \|f\|_B^p\,d\nu\,d\mu=\int_Q \|f\|^p_{\mathrm{L}^p(R;B)}\,d\mu=:\|f\|^p_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))}, \tag{B.23}$$

hence then $\|f\|_p=\|f\|_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))}<\infty$. □

If $f(\cdot,\cdot)$ is continuous w.r.t. one argument and measurable w.r.t. the other, then $f$ is product measurable:

**Lemma B.4.8** *Assume that $\mu:\mathfrak{M}\to[0,\infty]$ and $\nu:\mathfrak{M}'\to[0,\infty]$ are σ-finite, complete, positive measures on $Q$ and $R$, respectively. Assume, in addition, that $Q$ is a separable metric space and the open subsets of $Q$ are measurable.*

*If $f:Q\times R\to B$ is s.t. $f(q,\cdot)\in \mathrm{L}(R;B)$ for a.e. $q$ and $f(\cdot,r)\in \mathcal{C}(Q;B)$ for a.e. $r$, then $f\in \mathrm{L}(Q\times R;B)$.*

**Proof:** Let $N_{-1}\subset R$ be a null set s.t. $f(\cdot,r)\in\mathcal{C}(Q;B)$ for $r\in R':=N_{-1}^c$. Let $Q'\subset Q$ be s.t. $\mu(Q\setminus Q')=0$ and $f(q,\cdot)\in \mathrm{L}(R;B)$ for all $q\in Q'$.

Let $\{q_k\}_{k\in\mathbf{N}}$ be dense in $Q'$. For each $k\in\mathbf{N}$, find a sequence of countably-valued measurable functions $s_{k,n}:R\to B$ s.t. $s_{k,n}(\cdot)\to f(q_k,\cdot)$ uniformly on $N_k^c$ for some null set $N_k\subset R$ (use Lemma B.2.5(b2)). Set $R':=R\setminus\cup_{k=-1}^{\infty}N_k$. We require that

$$\|f(q_k,r)-s_{k,n}(r)\|_B < 1/(n+1) \quad (k,n\in\mathbf{N},\ r\in N_k^c) \tag{B.24}$$

(replace $\{s_{k,n}\}_{n\in\mathbf{N}}$ by its subsequence, for each $k\in\mathbf{N}$, if necessary).

Set $B_{k,n} := \{q\in Q' \,\big|\, d(q,q_k) < 1/n)\}$, $A_{0,n} := B_{0,n}$, $A_{k+1,n} := B_{k+1,n} \setminus \cup_{j\le k} B_{j,n}$ for all $k\in\mathbf{N}$ (the sets $\{A_{k,n}\}_{k\in\mathbf{N}}$ are measurable and disjoint, their union is $Q'$, and $d(q,q_k) < 1/n$ for all $q\in A_{k,n}$).

Then the functions

$$s_n(q,r) := \sum_{k\in\mathbf{N}} s_{k,n}(r)\chi_{A_{k,n}}(q) \tag{B.25}$$

form a sequence of countably-valued $\overline{\mu\times\nu}$-measurable functions that converge to $f$ on $Q'\times R'$, as $n\to\infty$. Indeed, given $(q,r)\in Q'\times R'$ and $\varepsilon>0$, choose $N>2/\varepsilon$ s.t. $\|f(q,r)-f(q',r)\|_B<\varepsilon/2$ for $d(q,q')<1/(N+1)$. Then, for $n>N$, we have

$$\|f(q,r)-s_n(q,r)\|_B \le \|f(q,r)-f(q_k,r)\|_B + \|f(q_k,r)-s_{k,n}(r)\|_B \tag{B.26}$$

$$< \varepsilon/2 + 1/(n+1) < \varepsilon \tag{B.27}$$

(here $k$ is chosen s.t. $q\in A_{k,n}$, so that $d(q,q_k)<1/n\le 1/(N+1)$ and hence $\|f(q,r)-f(q_k,r)\|_B<\varepsilon/2$; note that $s_{k,n}(r)=s_n(q,r)$).

But $(Q'\times R')^c \subset ((Q\setminus Q')\times R)\cup(Q\times(R\setminus R'))$ is a null set, hence $f$ is measurable. □

If $Q$ is also a topological space, then $\mu$ is called *regular* if

$$\mu(E) = \inf\{\mu(V) \,\big|\, V\supset E,\ V \text{ open}\} = \sup\{\mu(K) \,\big|\, K\subset E,\ K \text{ compact}\} \tag{B.28}$$

($\mu$ is outer regular if the former and inner regular if the latter condition is satisfied). If $Q\subset\mathbf{R}^n$ is open, then, by Theorem 2.18 of [Rud86], any locally finite Borel-measure on $Q$ is regular; in particular, $m$ is regular on $Q$.

One often defines a measure from another one by using a weight function; the "formula $d\nu = f\,d\mu$" works also in the vector-valued case:

**Lemma B.4.9 ($\nu = f\,d\mu$)** *Let $f: Q\to[0,+\infty]$ be measurable. Set*

$$\tilde{\nu}(E) := \int_E f\,d\mu \quad (E\in\mathfrak{M}), \tag{B.29}$$

*Then $\tilde{\nu}$ is a positive measure on $Q$; let $\nu$ be the completion of $\tilde{\nu}$. Then*

$$\int_Q g\,d\nu = \int_Q g f\,d\mu \tag{B.30}$$

*when $g$ is measurable $Q\to[0,\infty]$ or $g\in \mathrm{L}^1(Q,\nu;B)$ (equivalently, $gf\in \mathrm{L}^1(Q,\mu;B)$).*

*If $Q\subset\mathbf{R}^n$ is open, $\mu=m$ and $\int_K f\,dm<\infty$ for compact $K\subset Q$, then $\nu$ is regular.*

**Proof:** $1^\circ$ *Case $g: Q\to[0,\infty]$:* The claim on $\tilde{\nu}$ is Theorem 1.29 of [Rud86], which also contains the claim on $g$ provided that $g$ is $\tilde{\nu}$-measurable. By Lemma B.3.12, general $g: Q\to[0,\infty]$ will do.

2° *Case* $g : Q \to B$: By 1°, we have $g \in \mathrm{L}^1(Q,\nu;B) \Leftrightarrow gf \in \mathrm{L}^1(Q,\mu;B)$. Apply 1° to $(\mathrm{Re}\, g)^\pm$ and $(\mathrm{Im}\, g)^\pm$ to cover the case $B = \mathbf{K}$. Replace $g$ by $\Lambda g$ to obtain the general case.

3° *Regularity:* By Theorem 2.18 of [Rud86], measures $\nu$ and $\widetilde{\nu}$ are regular,

□

Standard changes of variable can be applied without problems for measurable functions that nonnegative or integrable:

**Lemma B.4.10 (Change of variable)** *Assume that* $\phi \in \mathcal{C}^1(J_1;\mathbf{R})$ *is s.t.* $\phi' > 0$ *on* $J_1$ *(or that* $\phi' \geq 0$ *and* $\phi'$ *has at most a countable number of zeros), where* $J_1$ *is an interval. Set* $J_2 = \phi[J_1]$. *Let* $X = B$ *or* $X = [0,+\infty]$.

*Then a function* $f : J_2 \to X$ *is measurable iff* $f \circ \phi : J_1 \to X$ *is measurable. Moreover,* $\|f\|_\infty = \|f \circ \phi\|_\infty \leq \infty$. *Let* $p \in [1,\infty]$. *Then* $f \in \mathrm{L}^p_{\mathrm{loc}} \Leftrightarrow f \circ \phi \in \mathrm{L}^p_{\mathrm{loc}}$. *If* $\varepsilon < \phi' < \varepsilon^{-1}$ *on* $J_1$ *for some* $\varepsilon > 0$, *then* $f \in \mathrm{L}^p \Leftrightarrow f \circ \phi \in \mathrm{L}^p$.

*If* $f \in \mathrm{L}^1(J_2;B)$ *or* $f$ *is measurable* $J_2 \to [0,+\infty]$, *then*

$$\int_{J_2} f(t)\,dt = \int_{\phi^{-1}[J_2]} f(\phi(s))\phi'(s)\,ds \tag{B.31}$$

Here, as elsewhere, intervals are equipped with the Lebesgue measure.

By Lemma B.5.5, the functions $f$ and $f \circ \phi$ have "same" Lebesgue points.

**Proof:** We shall replace $J_1$ by $[a,b] \subset J_1$ s.t. $\phi' \geq r > 0$ on $(a,b)$ (note $J_1' := J_1 \setminus \{t \in J_1 \,|\, \phi'(t) = 0\}$ is a countable union of such intervals, and it is enough to treat $J_1'$). Let $[\alpha,\beta] = \phi[[a,b]]$.

1° Note that $\phi^{-1} \in \mathcal{C}^1([\alpha,\beta];[a,b])$ and $\phi^{-1'} \geq 1/r$ (recall that we use one-sided derivatives at endpoints), and $\phi^{-1}[E]$ is a Borel set for each Borel set $E$.

2° *If* $N \subset [a,b]$ *a null set, then so is* $\phi[N]$*:* Set $R := \max \phi'$. Let $\varepsilon > 0$ and assume w.l.o.g. that $N \subset (a,b)$. Now $N \subset V$ for some open $V$ with $m(V) < \varepsilon/R$ (see Theorem 2.20 of [Rud86]). Write $V$ as the union of a countable number of disjoint open intervals: $V = \cup_{n\in\mathbf{N}}(a_n,b_n)$. Then $m(\phi[(a_n,b_n)]) = m((\phi(a_n),\phi(b_n))) \leq |b_n - a_n| R = m((a_n,b_n))R$, so that $m(\phi[N]) \leq m(\phi[V]) < \varepsilon$. Because $\varepsilon > 0$ was arbitrary, $m(\phi[N]) = 0$.

3° *Case* $f : J_2 \to X$ *is measurable:* Let $\{s_n\}$ be countably-valued and measurable and s.t. $s_n \to f$ a.e. Redefine each $s_n$ on a null set so that they become Borel-measurable; then still $s_n \to f$ outside some null set $N \subset J_2$. Consequently, $s_n \circ \phi \to f \circ \phi$ outside $\phi^{-1}[N]$, which is a null set, by 2°. But $s_n \circ \phi$ is a countably-valued Borel-measurable function, by 1°, for each $n$, hence $f \circ \phi$ is measurable.

4° *Measurability: the general case:* Exchange the roles of $\phi$ and $\phi^{-1}$ to obtain a converse for 3°. (Note also that $(f \circ \phi)\phi'$ is measurable iff $f \circ \phi$ is measurable.)

5° $\|f\|_\infty = \|f \circ \phi\|_\infty \leq \infty$*:* This follows from 2°.

6° *(B.31) for* $X = [0,+\infty]$*:* This is contained in (15) on p. 156 of [Rud86].

7° $\mathrm{L}^1$ *and* $\mathrm{L}^1_{\mathrm{loc}}$ *claims:* Apply 6° to $\|f\|_B$ (for $\mathrm{L}^1_{\mathrm{loc}}$ we note that $\phi[K] \subset J_2$ is compact for compact $K \subset J_1$ and $\phi^{-1}[K] \subset J_1$ is compact for compact $K \subset J_2$; moreover, $\phi'$, $\phi^{-1'}$ are bounded on compact sets).

8° *(B.31) for $X = B$:* If $X = \mathbf{K}$, then this follows from 6° (for $(\operatorname{Re} f)^{\pm}$, $(\operatorname{Im} f)^{\pm}$); in the general case we replace $f$ by $\Lambda f$ ($\Lambda \in B^*$) and use Lemma B.4.2.

9° $\mathrm{L}^p$ *and* $\mathrm{L}^p_{\mathrm{loc}}$ *claims:* If $p < \infty$, this follows from 7° applied to $\|f\|_B^p$. If $p = \infty$, then this follows 4° and 5°.

10° *Case* $\phi' \not> 0$*:* Let $f$ be measurable. Now $G := \{t \in J_1 \,|\, \phi'(t) > 0\}$ consists of a countable number of disjoint open intervals, by Lemma A.2.2. The image of the null set $J_1 \setminus G$ is a null set, so it follows from Lemma B.2.5(d1) that $f \circ \phi$ is measurable. The proof of the converse implication is analogous (set $G := \{t \in J_1 \,|\, \phi^{-1\prime}(t) < +\infty\}$ etc.).

Now for $X = [0, +\infty]$, we can write the integral as a countable sum of integrals, by the Monotone Convergence theorem; for $X = B$ the results follow as in 6°–9°. □

By Theorem B.3.2, $\mathrm{L}^2$ is a Hilbert space, hence $(\mathrm{L}^2)^* = \mathrm{L}^2$. The multidimensional version of this goes as follows:

**Lemma B.4.11** *Let $H$ be a Hilbert space and $n \in \{1, 2, \ldots\}$. Then, for each $\Lambda \in \mathcal{B}(\mathrm{L}^2(Q; H), \mathbf{K}^n)$, there is a unique $F \in \mathrm{L}^2(Q; \mathcal{B}(H, \mathbf{K}^n))$ s.t.*

$$\Lambda f = \int_Q F f \, d\mu \quad (f \in \mathrm{L}^2(Q; H)). \tag{B.32}$$

*Moreover,* $\|\Lambda\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^n)} \le \|F\|_2 \le \sqrt{n} \|\Lambda\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^n)}$.

Conversely, by the Hölder Inequality, each $F \in \mathrm{L}^2(Q; \mathcal{B}(H, \mathbf{K}^n))$ defines $\Lambda_F \in \mathcal{B}(\mathrm{L}^2(Q; H), \mathbf{K}^n)$, by (B.32). In Example B.4.13(c), we have $\|F\|_2 = \sqrt{n}$, $\|F\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^2)} = 1$, hence the constant $\sqrt{n}$ is optimal. By Example B.4.13(d), we may have $\|F\|_2 = \infty$ if $\mathbf{K}^n$ is replaced by an infinite-dimensional (Hilbert) space.

**Proof:** Let $P_k \in \mathcal{B}(\mathbf{K}^n, \mathbf{K})$ be the projection $P_k : x \mapsto x_k$. By Theorem B.3.2, there are $F_1, \ldots, F_n \in \mathrm{L}^2(Q; H)$ s.t. $P_k \Lambda_k = \langle F_k, f \rangle_{\mathrm{L}^2}$ ($k = 1, \ldots, n$) and $\|F_k\| = \|P_k \Lambda\|$. Set $F := \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ to obtain (B.32).

Obviously, $F \mapsto \Lambda$ is linear and $\Lambda = 0 \Rightarrow F = 0$, hence $F$ is unique. By the Hölder Inequality, we have $\|\Lambda\| \le \|F\|$. On the other hand, $\|F_k\| \le \|\Lambda\|$, hence $\|F\|^2 \le n \|\Lambda\|^2$. □

The $\mathrm{L}^p$ norm of a measurable function $f : Q \to B$ is the supremum of $|\int_Q f \phi \, d\mu|$, where $\phi \in \mathrm{L}^q(Q; B^*)$, $1/p + 1/q = 1$. We can even take $\phi$ to be simple, and, with some extra assumptions, smooth:

**Theorem B.4.12** *Let $B$ be a Banach space, let $\mu$ be a complete $\sigma$-finite positive measure on $Q$, let $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let $X$ be a norming subspace of $B^*$ for $B$. Let $f : Q \to B$ be measurable. Then the following hold:*

*(a) We have* $\|f\|_\infty = \sup_{\Lambda \in X_0,\ \|\Lambda\| \le 1} \|\Lambda f\|_\infty \le \infty$ *if $X_0$ is a norming subset of $B^*$ for $B$.*

*(b1) We have*

$$\|f\|_p = \sup_{\phi \in \mathcal{F}} \left| \int_Q \phi f \, d\mu \right| \leq \infty, \tag{B.33}$$

*where either* $\mathcal{F} = \mathcal{F}_1$ *or* $\mathcal{F} = \mathcal{F}_2$*, and* $\mathcal{F}_1 := \{\phi \in \mathrm{L}^q(Q;X) \,\big|\, \|\phi\|_q \leq 1 \;\&\; \phi f \in \mathrm{L}^1\}$*,* $\mathcal{F}_2 := \{\sum_{k=1}^n x_k \chi_{E_k} \in \mathcal{F}_1 \,\big|\, n \in \mathbf{N},\ x_k \in X,\ m(E_k) < \infty$ *for all* $k\}$*.*

*(b2) If* $Q$ *is an open subset of* $\mathbf{R}^n$*,* $\mu$ *is regular, and* $\int_K \|f\|_B \, d\mu < \infty$ *for compact* $K \subset \mathbf{R}^n$ *(i.e.,* $f \in \mathrm{L}^1_{\mathrm{loc}}(Q;B)$*), then we can take* $\mathcal{F} = \mathcal{F}_3 := \mathcal{C}_c^\infty(Q;X) \cap \mathcal{F}_1$ *in (b1).*

*(b3) If* $\mu$ *is the counting measure (i.e.,* $\mathrm{L}^p(Q;B) = \ell^p(Q;B)$*), then* $\mathcal{F}_2 = \mathrm{c_c} \cap \mathcal{F}_1 = \mathrm{c_c}$ *in (b1), where* $\mathrm{c_c} := \{x : Q \to B \,\big|\, x_q = 0$ *except for finitely many* $q \in Q\}$*. Moreover, then we can allow* $Q$ *to be an arbitrary set (i.e.,* $\mu$ *need not be* $\sigma$*-finite).*

*(b4) In (b1)–(b3), we may require, in addition, that* $\phi = \sum_{k=1}^m \phi_k x_k$*, where* $m \in \mathbf{N}$*,* $\phi_k$ *is scalar and* $x_k \in X$*, for each* $k$*.*

*(c) If* $g : Q \to [0,\infty]$ *is measurable, we can take* $X = [0,\infty]$ *and have*

$$\|g\|_p = \sup_{\phi \in \mathcal{F} \;\&\; \phi \geq 0} \int_Q \phi g \, d\mu \leq \infty, \tag{B.34}$$

*where* $\mathcal{F}$ *is as in any of (b1)–(b3); we can even drop the requirement* $\phi g \in \mathrm{L}^1$*, as well as the requirement* $g \in \mathrm{L}^1_{\mathrm{loc}}$ *(in (b2)).*

*(d) If* $p = \infty$*, then, in (b1)–(b3) above, we may let* $X$ *be any normed space s.t.* $X \times B \to B_2$ *is bilinear and* $\sup_{\|x\|_X \leq 1} \|xb\|_{B_2} = \|b\|_B$ *for some normed space* $B_2$*.*

*For* $p < \infty$*, we must replace "$=$" by "$\geq$" in (B.33) for such a general* $X$ *(see Example B.4.13), but the supremum is still nonzero, i.e.,* $f \neq 0 \Longrightarrow \int_Q \phi f \, d\mu \neq 0$ *for some* $\phi \in \mathcal{F}$*.*

*(e) We have* $f = 0$ *a.e. if* $f \in \mathrm{L}^1(Q;B)$ *and* $\int_E f \, d\mu = 0$ *for all measurable* $E \subset \mathbf{R}$*, or* $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ *and* $\int_E f \, dm = 0$ *for all bounded measurable* $E \subset \mathbf{R}$*.*

(The sets $\mathrm{L}^q$ and $\mathcal{C}_c^\infty$ are defined for incomplete normed spaces exactly as for the complete ones.)

The assumption $\phi f \in \mathrm{L}^1$ is required to make the integral $\int_Q \phi f \, d\mu$ well defined; for $f \in \mathrm{L}^1_{\mathrm{loc}}$ and $\mathcal{F} = \mathcal{F}_3$ it is redundant. Clearly it is not needed for (B.34).

In (d), we may have, e.g., $X = \mathcal{B}(B,B_2)$, $B = \mathcal{B}(X,B_2)$ or $X = \mathbf{K}$.

See Examples B.4.13 and B.4.14 for "counter-examples".

**Proof:** (a) Clearly "$\geq$" holds, so it is enough to assume that $\|f\| > M$ on a set $E \subset Q$ of positive measure, and find $\Lambda \in X_0$ s.t. $\|\Lambda\| \leq 1$ and $|\Lambda f| > M$ on a set of positive measure.

Pick $\varepsilon > 0$ s.t. $E' := \{t \in Q \,\big|\, \|f(t)\| > M + \varepsilon\}$ has a positive measure. Choose $a_0 \in E'$, $A_\varepsilon \subset E'$ for $f|_{E'}$ and $\varepsilon$ as in Lemma B.2.8(a). Choose $\Lambda \in X_0$ s.t. $|\Lambda f(a_0)| > \|f(a_0)\| - \varepsilon/2$. Then (recall that $\|\Lambda\| \leq 1$)

$$|\Lambda f(a)| > |\Lambda f(a_0)| - \varepsilon/2 > \|f(a_0)\| - \varepsilon > M \tag{B.35}$$

for $a \in A_\varepsilon$, hence $\|\Lambda f\|_\infty > M$.

(b1) If $f \in \mathrm{L}^p$ and $\phi \in \mathcal{F}$, then $\| \int_Q \phi f \, d\mu \| \leq \| f \|_p$ by the Hölder inequality; trivially, we have $\| \int_Q \phi f \, d\mu \| \leq \| f \|_p$ for $f \notin \mathrm{L}^p$ (i.e., $\| f \|_p = \infty$). Thus, we only need to assume that $0 < M < \| f \|_p \leq \infty$ and find $\phi \in \mathcal{F}$ s.t. $\| \int_Q \phi f \, d\mu \| > M$. We divide the proof of this fact into three parts.

**Case I — $p = \infty$, $\mathcal{F} = \mathcal{F}_2$:** Take $M' > M$ s.t. $\mu(E) > 0$, where $E := \{ t \in Q \,\big|\, \| f(t) \| > M' \}$. Because $\mu$ is $\sigma$-finite, there is $A \subset E$ s.t. $0 < \mu(A) < \infty$. Choose $a_0 \in A$ and $A' := A_\varepsilon \subset A$ for $\varepsilon := (M' - M)/2$ as in Lemma B.2.8. Choose $x \in X$ s.t. $\|x\| = 1$ and $x f(a_0) > \| f(a_0) \| - \varepsilon > M + \varepsilon$. Then for $t \in A'$, we have $\| f(t) \|_B \in (f(a_0) - \varepsilon, f(a_0) + \varepsilon)$, and $\| x f(t) - x f(a_0) \| < \varepsilon$. Therefore, $\phi := x \chi_{A'} / \mu(A') \in \mathcal{F}_2$, and

$$\| \int_Q \phi f \, d\mu \| = \mu(A')^{-1} \| \int_{A'} x f \, d\mu \| \geq \| x f(a_0) \| - \varepsilon > M. \tag{B.36}$$

**Case II — $p < \infty$, $\mathcal{F} = \mathcal{F}_2$:** $1^\circ$ Assume that $\mu(Q) < \infty$, $\| f \|_p = 1$, and $f = \sum_{j=1}^k b_j \chi_{Q_j}$, where the sets $Q_j$, $j \in \mathbf{N}$ are disjoint.

For each $j$ choose $x_j \in X$ s.t. $\|x_j\| = 1$ and $x_j b_j > \|b_j\| - \delta$, where $\delta := (1 - M) / \sum_{j=1}^k \|b_j\|^{p-1} \mu(Q_j) < 1 - M$,

For $\phi := \sum_{j=1}^k \|b_j\|^{p-1} x_j \chi_{Q_j}$, we have (because $q(p-1) = p$) that

$$\int_Q \|\phi(t)\|^q \, d\mu = \int_Q \sum_{j=1}^k \|b_j\|^p \chi_{Q_j}(t) \, d\mu = \| f(t) \|_p^p = 1 \quad \text{for all } t \in Q, \tag{B.37}$$

for $q < \infty$, and $\|\phi\|_\infty = 1$ as well for $q = \infty$. Moreover,

$$\| \int_Q \phi f \, d\mu \| = \sum_{j=1}^k \|b_j\|^{p-1} (x_j b_j) \mu(Q_j) \geq \int \| f \|_p^p - h(\delta) = 1 - h(\delta), \tag{B.38}$$

where $h(\delta) := \sum_{j=1}^k \|b_j\|^{p-1} (\|b_j\| - x_j b_j) \mu(Q_j) < \delta \sum_{j=1}^k \|b_j\|^{p-1} \mu(Q_j) = 1 - M$, hence $\| \int_Q \phi f \, d\mu \| > M$.

$2^\circ$ By scaling and density (Theorem B.3.11), any $f \in \mathrm{L}^p$ will do in $1^\circ$, if $\mu(Q) < \infty$.

$3^\circ$ For the general case, let $Q = \cup_{j \in \mathbf{N}} Q_j$, where the sets $Q_j$, $j \in \mathbf{N}$ are disjoint and $\mu(Q_j) < \infty$ for all $j$. Set $E_j := \{ t \in Q_1 \cup \cdots \cup Q_j \,\big|\, \| f(t) \|_B < j \}$.

By the monotone convergence theorem, $\int_Q \chi_{E_j} \| f \|_B^p \, d\mu > M$ for some $j \in \mathbf{N}$. Use now $2^\circ$ to find $\phi = \phi \chi_{E_j} \in \mathcal{F}$ for $E_j$ and $\chi_{E_j} f$; the same $\phi$ will clearly do for $Q$ and $f$.

**Case III — $\mathcal{F} = \mathcal{F}_1$:** This follows easily from case $\mathcal{F} = \mathcal{F}_2$.

(b2) There is a nested sequence of open sets $\Omega_j \subset Q$ with compact closure s.t. $\cup_{j \in \mathbf{N}} \Omega_j = Q$. By the monotone convergence theorem, it follows that for some $j$, we have $\| f \|_{\mathrm{L}^p(\Omega_j, \mu; B)} > M$. Find $g = g \chi_{\Omega_j} = \sum_{i=1}^k x_i \chi_{E_i} \in \mathcal{F}_2$ s.t. $M_f := \| \int_{\Omega_j} g f \, d\mu \| > M$. Set $M_g := \max_i \|x_i\| > 0$.

Because $f \in \mathrm{L}^1(\Omega_j, \mu; B)$, we can find, for each $i$, a compact $K_i \subset \Omega_j$ and an open $V_i \subset \Omega_j$ s.t. $K_i \subset E_i \subset V_i$ and $\int_{V_i \setminus K_i} \| f \|_B \, d\mu < \delta := (M_f - M)/2kM_g$. By Lemma B.3.10, there is some $\phi_i \in C_c^\infty(\Omega_j)$ s.t. $\chi_{K_i} \leq \phi_i \leq \chi_{V_i}$. Set $\phi :=$

$\sum_{i=1}^k x_i \phi_i \in \mathcal{F}_2$ It follows that

$$\|\int_{\Omega_j} \phi f\, d\mu\| \geq \|\int_{\Omega_j} g f\, d\mu\| - \sum_{i=1}^k \int_{V_i \setminus K_i} \|g-\phi\|_X \|f\|_B\, d\mu \geq \|\int_{\Omega_j} g f\, d\mu\| - \sum_{i=1}^k 2M_g \delta > M. \tag{B.39}$$

(b3) 1° *Case* $p = \infty$*:* If $M < \|f\|_\infty = \sup_{q\in Q} \|f\|$, then there is $q \in Q$ s.t. $\|f(q)\|_B > M$, so that $\|f(q)x\| > M$ for some $x \in X$ with $\|x\| = 1$. Thus, then we can take $\phi := \chi_{\{q\}} x$.

2° *Case* $1 \leq p < \infty$*:* Set $A := \text{supp}(f) := \{q \in Q \,|\, f(q) \neq 0\}$. If $A$ is countable, then (b1) applies; obviously, $\mathcal{F}_2 = \text{c}_\text{c} \cap \mathcal{F}_1 = \text{c}_\text{c}$. Assume then that $A$ is uncountable. Choose $n \in \mathbf{N}+1$ s.t. $A_n := \{a \in A \,|\, \|f(a)\| > 1/n\}$ is uncountable (hence infinite). Choose distinct elements $\{a_k\}_{k\in\mathbf{N}} \subset A_n$. For each $k$, choose $x_k \in X$ s.t. $\|x_k\| = 1$ and $f(a_k)x_k > 1/n$. Then $\int_Q \phi_m f\, d\mu = \sum_{k=1}^m f(a_k)x_k > m/n$, where $\phi_m := \sum_{k=1}^m x_k \chi_{\{a_k\}} \in \text{c}_\text{c}$ and $m \in \mathbf{N}$ is arbitrary, hence then (B.33)$= \infty = \|f\|_p$.

(b4) The functions $\phi$ constructed in the proofs of (b1)–(b3) are of this form.

(c) Finally, for $g : Q \to X$, $X = [0,\infty]$, we can choose $\phi \in \mathcal{F}$ to prove (B.34) as above. By Hölder inequality, $\phi g \in \text{L}^1$ may be dropped; by the trick used in "3°" above (with $E_j := \{t \in Q \,|\, f(t) < j\}$) we may drop the assumption $f \in \text{L}^1_\text{loc}$.

(d) Case I above applies here too (mutatis mutandis). For $p < \infty$, we still have the Hölder inequality ("$\geq$"); naturally for $B_2 = \mathbf{K}$ this reduces to (b1)–(b3).

Finally, if $f \neq 0$, then some $\phi = \sum_{k=1}^m \phi_k b_k^* \in \mathcal{F}$ (here $b_k^* \in B^*$ for all $k$) satisfies $\int_Q \phi f\, d\mu \neq 0$, by (b4), hence then $b := \int_Q \phi_k f\, d\mu \neq 0$ for some $k$. Choose $x \in X$ s.t. $xb \neq 0$, and set $\phi = \phi_k x$. □

We now show that part (d) does not hold for $p < \infty$:

**Example B.4.13** Let $Q = [0,2]$, $\mu = m$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

(a) $B = \mathcal{B}(\mathbf{K},\mathbf{K}^2)$, $X = \mathbf{K}$, $f = \chi_{[0,1]} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \chi_{[1,2]} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \text{L}^p(Q; \mathcal{B}(\mathbf{K},\mathbf{K}^2)))$, Then

$$\|f\|_p = 2^{1/p} > \sup_{\phi \in \text{L}^q(Q;X),\ \|\phi\|_q = 1} \|\int_Q f\phi\, d\mu\|_{\mathbf{K}^2}. \tag{B.40}$$

Thus, $\|f\|_p > \|f\|_{\mathcal{B}(\text{L}^q(Q;\mathbf{K}),\mathbf{K}^2)}$.

However, $\|f\|_p = \|f^*\|_{\text{L}^p(Q;\mathcal{B}(\mathbf{K}^2,\mathbf{K}))} = \|f^*\|_{\mathcal{B}(\text{L}^q(Q;\mathbf{K}^2),\mathbf{K})}$, hence $\|f^*\|_{\mathcal{B}(\text{L}^q(Q;\mathbf{K}^2),\mathbf{K})} > \|f\|_{\mathcal{B}(\text{L}^q(Q;\mathbf{K}),\mathbf{K}^2)}$.

(b) Analogously, $\|f\|_p > \|f\|_{\mathcal{B}(\text{L}^q(Q;\mathbf{K}^2),\mathbf{K}^2)}$ if we set $f = \chi_{[0,1]} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \chi_{[1,2]} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in \text{L}^p(Q; \mathcal{B}(\mathbf{K}^2))$.

(c) Moreover, by setting $f = \sum_{k=1}^n \chi_{[n-1,n]} P_n^*$, where $P_n^* \in \mathcal{B}(\mathbf{K},\mathbf{K}^n)$ is defined by $P_n^* \alpha = \alpha e_n$, we get $\|f\|_{\text{L}^2([0,n];B)} = \sqrt{n}$, $\|\int f u\, dm\| \leq 1$ when $\|u\|_2 \leq 1$, where we can take $B = \mathcal{B}(X,\mathbf{K}^n)$ for any nontrivial Banach space $X$ (replace $f$ by $f\Lambda$ for some $\Lambda \in X^* \setminus \{0\}$).

(d) Finally, if $f = \sum_{k=1}^\infty \chi_{[n-1,n]} P_n^*$, $B_2 := \ell^2(\mathbf{N})$ where $P_n^* \in \mathcal{B}(\mathbf{K}, B_2)$ is defined by $P_n^* \alpha = \alpha e_n$, then $\|\int f u\, dm\| \leq \|u\|_{\text{L}^2(\mathbf{R}_+;\mathbf{K})}$, i.e., $\|f\|_{\mathcal{B}(\text{L}^2(Q;\mathbf{K}),B_2)} = 1$, although $\|f\|_{L^2(\mathbf{R}_+;\mathcal{B}(\mathbf{K};B_2))} = \infty$. Note that $\|f^*\|_{\mathcal{B}(L^2(Q;B_2),\mathbf{K})} = \|f^*\|_2 = \|f\|_2 = \infty$.

$\triangleleft$

**Proof:** (a) Let $q < \infty$. Obviously, $\|f\|_p^p = 2$, hence $\|f\|_p = 2^{1/p}$. Let $\phi \in L^q(Q;X)$, $\|\phi\|_q = 1$. Set $P_1 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\|P_1 \int_Q f\phi\, dm\|_X = \|\int_0^1 P_1\phi\, dm\|_X \le \|\phi\|_{L^q([0,1];X)} =: a, \tag{B.41}$$

by the Hölder inequality. Analogously, $\|P_2 \int_Q f\phi\, dm\|_X \le \|\phi\|_{L^q([1,2];X)} =: b$, hence $\|\int_Q f\phi\, dm\|_X^2 \le a^2 + b^2$. For $q \le 2$ we have $a^2 + b^2 \le a^q + b^q \le \|\phi\|_q^q = 1$ and hence $\|\int_Q f\phi\, dm\|_X^2 \le 1$. For $q > 2$, the maximum of $a^2 + b^2$ given $a^q + b^q \le 1$ is $2^{-2/q} + 2^{-2/q} = 2^{1/p-1/q} < 2^{1/p} < (2^{1/p})^2 = \|f\|_p^2$. Finally, for $q = \infty$ we have $\|P_k \int_Q f\phi\, dm\|_X \le 1$ $(k = 1,2)$, hence $\|\int_Q f\phi\, dm\|_X \le \sqrt{2} < 2 = \|f\|_p$.

(b) This follows from (a).

(c)&(d) These can be proved as for (a) (for $p = 2 = q$ this is obvious: $\|\int f\phi\|^2 \le \sum_n \|\phi\|^2_{L^2([n-1,n];X)} = \|\phi\|_2^2$, as in (B.41)).

(e) 1° For $f \in L^1(Q;B)$ this follows from Lemma B.2.8(b) (since $\Lambda \int_A f\, d\mu = \int_A \Lambda f\, d\mu > 0$ for such $A$).

2° If $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ and $\int_E f\, dm = 0$ for all bounded measurable $E \subset \mathbf{R}$, then $\chi_{[-R,R]^n} f = 0$ a.e. for all $R > 0$, by 1°, hence $f = 0$ a.e. $\square$

Even if $f$ is strongly measurable, "weak $L^p$" differs from $L^p$:

**Example B.4.14 ($\Lambda f \in L^p$ for all $\Lambda \not\Rightarrow f \in L^p$)** Set $H := \ell^2(\mathbf{N})$. Define $f \in \ell^2(\mathbf{N};H)$ by $f(k) := e_k$. Then, for any $x \in H$, we have $\|\langle f, x\rangle\|_2 = \|x.\|_2 = \|x\|_H$. Thus, $\|\Lambda f\|_2 = \|\Lambda\|$ for all $\Lambda \in H^*$, although $\|f\|_2 = \infty$. (The same holds for $\widetilde{f} := \sum_{k\in\mathbf{N}} \chi_{[k,k+1)} e_k \in L^2(\mathbf{R}_+;H)$.) $\triangleleft$

By The Hölder Inequality, the formula $\ell^p(Q;B) \ni f \mapsto \sum_{t\in Q} F_t f_t \in \mathbf{K}$ determines a contractive map $\ell^q(Q;B^*) \ni F \mapsto \sum F\cdot \in \ell^p(Q;B)^*$. Usually, all elements of $\ell^p(Q;B)^*$ are of this form:

**Lemma B.4.15 ($\boldsymbol{\ell}^{\mathbf{p}}(\mathbf{Q};\mathbf{B})^* = \boldsymbol{\ell}^{\mathbf{q}}(\mathbf{Q};\mathbf{B}^*)$)** *Let $Q$ be a set, $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. Then $\ell^p(Q;B)^* = \ell^q(Q;B^*)$ and $c_o(Q;B)^* = \ell^1(Q;B)$ (with equal norms).*

Recall that $\ell^p(Q;B)$ refers to $L^p(Q,\sigma;B)$, where $\sigma$ is the counting measure. Note that it follows that $\ell^p_r(S;B)^* = \ell^q_{1/r}(S;B^*)$ for $S \subset \mathbf{Z}$, where $\ell^p_r := r^{\cdot}\ell^p$ (see (13.2)).

**Proof:** 0° *Remarks:* We allow $B$ to be an arbitrary Banach space; this cannot be done in the case of, e.g., $L^p([0,1];B)$; cf. [DU]. For infinite $Q$ and $B \ne \{0\}$, the closed subspace $c_o(Q;B)$ of $\ell^\infty(Q;B)$ is not dense, hence there is $F \in \ell^\infty(Q;B)^*$ s.t. $Ff = 0$ for all $f \in c_o$, in particular, $F \notin \ell^1(Q;B^*)$ (a constructive example is given in Exercise 3.4 of [Rud73]).

1° *Sufficiency:* Let $1 \le p \le \infty$. Let $F \in \ell^q(Q;B^*)$. By Lemma B.4.12(b3), $F \in \ell^p(Q;B)^*$ and $\|F\|_{\ell^p(Q;B)^*} = \|F\|_{\ell^q(Q;B^*)}$ (and the same holds with $c_o$ in place of $\ell^p$ if $p = \infty$).

2° *Necessity, case $p < \infty$:* For the converse, assume that $F \in \ell^p(Q;B)^*$ and $p < \infty$. Since $\pi^*_{\{t\}} \in \mathcal{B}(B, \ell^p(Q;B))$ (here $\pi^*_{\{t\}}x$ is $x$ at $t$ and zero elsewhere), we have $G_t := F\pi_{\{}t\}^* \in \mathcal{B}(B,\mathbf{K}) = B^*$.

This function $G : Q \to B^*$ satisfies $Ff = \sum_{t\in Q} G_t f_t$ for each $f \in c_c(Q;B)$, by linearity. Since $|\sum_{t\in Q} G_t f_t| = |Ff| \le \|F\|\|f\|$ for all $f \in c_c(Q;B)$, we have $\|G\|_{\ell^q(Q;B^*)} \le \|F\|$, by Lemma B.4.12(b3) (with $\mathcal{F} = \mathcal{F}_2 = c_c$). Consequently, $G \in \ell^p(Q;B)^*$, by 1°.

It follows that $G = F$ on the closure of $c_c(Q;B)$, i.e., on $\ell^p(Q;B)$. Thus, $F$ is of the required form. By density (see Theorem B.3.11(a)), $G = F$.

3° *Necessity for* $c_o(Q;B)$*:* If $F \in c_o(Q;B)^*$ and $p = \infty$, then the above proof applies, mutatis mutandis, and $G = F$ on $c_o(Q;B)$, by density (see Theorem B.3.11(c)), hence $F$ is again of the required form. □

The Minkovski Integral Inequality (not to be mixed to the Minkovski inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$) says that $\|\int_R f\,d\nu\|_p \le \int_R \|f\|_p\,d\nu$; also this can be extended to vector-valued functions

**Theorem B.4.16 (Minkovski Integral Inequality)** *Let $(Q,\mu)$ and $(R,\nu)$ be complete, positive, $\sigma$-finite measure spaces. Equip $Q\times R$ with $\overline{\mu\times\nu}$, the completion of $\mu\times\nu$. Let $1 \le p \le \infty$.*

*(a) If $f : Q\times R \to [0,\infty]$ is measurable, then*

$$\|\int_R f\,d\nu\|_{L^p(Q)} \le \int_R \|f\|_{L^p(Q)}\,d\nu \le \infty. \tag{B.42}$$

*(b) If $B$ is a Banach space, $f : Q\times R \to B$ is measurable, and $M := \int_R \|f\|_{L^p(Q)}\,d\nu < \infty$, then $g(q) := \int_R f(q,\cdot)\,d\nu \in B$ is defined a.e., $g \in L^p(Q;B)$ and $\|g\|_p \le M$, i.e., (B.42) holds.*

If we did not make the assumption $\int_R \|f\|_{L^p(Q)}\,d\nu < \infty$ in (b), we would have to write

$$\|\chi_{f(q,\cdot)\in L^1(R,X)} \int_R f\,d\nu\|_{L^p(Q,X)} \le \int_R \|f\|_{L^p(Q,X)} \tag{B.43}$$

or use some other trick to make the function in the left well defined (here it is taken to be zero for those $q$, for which $f(q,\cdot) \notin L^1$). The set $\{q \in Q \,\big|\, \int_R \|f(q,\cdot)\|\,d\nu = \infty\}$ is measurable by the Fubini Theorem, hence so is $\chi_{f(q,\cdot)\in L^1(R,X)}$, and thus (B.43) can be proved in the same way as (b).

**Proof:** (a) By the Fubini Theorem, the inner integrals define measurable functions; in particular, $g(q) := \int_R f(q,r)\,dr$ defines a measurable function $Q \to [0,\infty]$.

If $s \geq 0$ is a simple function and $\|s\|_{p'} \leq 1$, where $1/p + 1/p' = 1$, then

$$\int_Q gs\, d\mu = \int_Q \int_R f(q,r)s(q)\, d\nu(r)\, d\mu(q) \tag{B.44}$$

$$= \int_R \int_Q f(q,r)s(q)\, d\mu(q)\, d\nu(r) \leq \int_R \|f\|_{\mathrm{L}^p(Q)}\, d\nu, \tag{B.45}$$

hence $\|\int_R f\, d\nu\|_{\mathrm{L}^p(Q)} = \|g\|_{\mathrm{L}^p(Q)} \leq \int_R \|f\|_{\mathrm{L}^p(Q)}$, by Theorem B.4.12 ($p \in [1,\infty]$).

(b) Find sets $Q_j \subset Q$ with $\mu(Q_j) < \infty$ ($j \in \mathbf{N}$) s.t. $Q_1 \subset Q_2 \subset \cdots$ and $Q = \cup_{j\in\mathbf{N}} Q_j$. The assumption $\int_R \|f\|_p\, d\nu < \infty$ implies that

$$\int_Q |\int_R \|f\|_B\, d\nu|^p\, d\mu \leq \big(\int_R \|f\|_{\mathrm{L}^p(Q)}\, d\nu\big)^p < \infty, \tag{B.46}$$

by (a) (applied to $\|f\|_B$) for $p < \infty$, and $\|\int_R \|f\|_B\, d\nu\|_\infty \leq \int_R \|f\|_\infty\, d\nu < \infty$ for $p = \infty$. In particular, $\int_R \|f\|_B\, d\nu < \infty$ for a.e. $q \in Q'$, where $\mu(Q \setminus Q') = 0$, and $g$ is defined on $Q'$, hence a.e.

Because $\mathrm{L}^p(Q_j;B) \subset \mathrm{L}^1(Q_j;B)$ for $j \in \mathbf{N}$, we have $\int_{Q_j} \int_R \|f\|_B\, d\nu\, d\mu < \infty$, hence $g_j := \int_R f\, d\nu : Q_j \to B$ is defined a.e. and measurable, by the Fubini Theorem.

Because $g_j = g$ on $Q_j$, also $g$ is measurable. By (B.46), $\|g\|_{\mathrm{L}^p(Q)} \leq \int_R \|f\|_{\mathrm{L}^p(Q)}\, d\nu < \infty$. □

The following result allows one to take Laplace and Fourier transforms of $\mathcal{C}(\mathbf{R};\mathrm{L}^p)$ functions componentwise (see Proposition D.1.13).

**Lemma B.4.17 ($\int_Q f(\cdot)(r)\, d\mu = (\int_Q f\, d\mu)(r)$)** *Assume that $\mu : \mathfrak{M} \to [0,\infty]$ and $\nu : \mathfrak{M}' \to [0,\infty]$ are σ-finite, complete, positive measures on $Q$ and $R$, respectively.*

*Let $f \in \mathrm{L}^1(Q;\mathrm{L}^p(R;B)) \cap \mathrm{L}(Q \times R;B)$, $p \in [1,\infty]$. Then $g(r) := \int_Q f(\cdot)(r)\, d\mu$ exists a.e., and $g = \int_Q f\, d\mu \in \mathrm{L}^p(R;B)$.*

Thus, then $g(r) = (\int_Q f\, d\mu)(r)$ for almost every $r \in R$. See also Example B.4.18.

**Proof:** (By using the Fubini Theorem, one easily verifies that $f = 0$ as an element of $\mathrm{L}^1(Q;\mathrm{L}^p(R;B))$ iff $f = 0$ as an element of $\mathrm{L}(Q \times R;B)$, hence $\mathrm{L}^1(Q;\mathrm{L}^p(R;B)) \cap \mathrm{L}(Q \times R;B)$ is well-defined (we used the assumption that $f \in \mathrm{L}(Q \times R;B)$; by Counter-example 8.9(c) of [Rud86], there is $f : [0,1] \times [0,1] \to \mathbf{R}$ s.t. $f$ is not measurable, but $[f] \in \mathrm{L}^1([0,1];\mathrm{L}^p([0,1];B))$ has a representative that is measurable, because $f(t) = 1$ a.e. for every $t \in [0,1]$, i.e., $[f] = [1]$.)

By Theorem B.4.16, $g(r)$ exists for a.e. $r \in R$, $g \in \mathrm{L}^p(R;B)$ and $\|g\|_p \leq \|f\|_1$. Set $F := \int_Q f\, d\mu \in \mathrm{L}^p(R;B)$.

1° *Case $f = \chi_E b$, $b \in \mathrm{L}^p$, $E \in \mathfrak{M}$:* Now $F = \mu(E)b$, $g(r) = \mu(E)b(r)$ ($r \in R$), hence $F = g$.

2° *Case $f$ is simple:* This follows easily from 1°, by linearity.

3° *General case:* Let $f_n \to f$ in $\mathrm{L}^1$, as $n \to \infty$, and let $f_n$ be simple and measurable for each $n \in \mathbf{N}$. Set $F_n := \int_Q f_n\, d\mu$, $g_n(r) := \int_Q f_n(\cdot)(r), d\mu$ ($r \in R$), so that $F_n = g_n$, by 2°.

Because $\int$ is continuous, we have $F_n \to F$ in $\mathrm{L}^p$, as $n \to \infty$. By Theorem B.4.16,

$$\|g_n - g\|_p = \|\int_Q (f_n - f)(\cdot)(r)\,d\mu\|_p \leq \int_Q \|f_n - f\|\,d\mu = \|f_n - f\|_1 \to 0, \quad \text{(B.47)}$$

as $n \to \infty$. Therefore, some subsequence of $\{g_n\}$ converges to $g$ and $F$ a.e., hence $g = F$ a.e. □

The assumption that $f \in \mathrm{L}(Q \times R; B)$ is necessary in Lemma B.4.17:

**Example B.4.18** [$\int_Q f(\cdot)(r)\,d\mu \neq (\int_Q f\,d\mu)(r)$] Define $f : [0,1] \times [0,1] \to \mathbf{R}$ as in Counter-example 8.9(c) of [Rud86] (which uses the Continuum Hypothesis), and define $h : [0,1] \times [0,1] \to \mathbf{R}$ by $h \equiv 1$. Let $p, a \in [1, \infty]$.

Then $f$ is not measurable $[0,1] \times [0,1] \to \mathbf{R}$, but for each $q \in Q := [0,1]$, we have $f(q) = 1 = h(q)$ a.e. on $R$, in particular, $f(q) \in \mathrm{L}^p([0,1]; B)$. Thus, $[f] = [h]$ as elements of $\mathrm{L}^a([0,1]; \mathrm{L}^p([0,1]; B))$, even as elements of $\mathcal{C}([0,1]; \mathrm{L}^p([0,1]; B))$.

Even worse, for each $r \in [0,1]$, we have $f(q,r) = 0$ for a.e. $q \in Q$. Thus, $g(r) := \int_Q f(q,r)\,dq = 0$ for each $r \in [0,1]$, although $\int_Q f\,dm = \int_Q 1\,dm = 1 \in \mathrm{L}^p([0,1]; B)$, hence $g \neq \int_Q f\,dm$ (cf. Lemma B.4.17). ◁

If $f \in \mathrm{L}^a(Q; L^p)$, then there is $h \in \mathrm{L}(Q \times R)$ s.t. $h(q) = f(q)$ a.e. on $R$ (hence as elements of $\mathrm{L}^p$) for a.e. $q \in Q$ (we omit the nontrivial proof). Thus, then Lemma B.4.17 can be applied to $h$, but not to $f$: the value of $\int_Q f(q)(r)\,d\mu(q)$ may differ everywhere from

$$(\int_Q f\,d\mu)(r) = (\int_Q h\,d\mu)(r) = \int_Q h(q)(r)\,d\mu(q) \quad \text{(B.48)}$$

(this equality holds for a.e. $r \in R$), as shown in the above example.

If a continuous function has a $\mathrm{L}^1$ limit function on the left boundary, then this function is $\mathrm{L}^1$ over the whole rectangle and also sideways:

**Lemma B.4.19** *Let $f \in \mathcal{C}((r,s] \times (a,b); B)$ and $f|_{\{r\} \times (a,b)} \in \mathrm{L}^1(\{r\} \times (a,b); B)$, $a, b, r, s \in \mathbf{R}$, $r < s$, $a < b$.*

*If $f(t,\cdot) \to f(r,\cdot)$ in $\mathrm{L}^1((a,b); B)$, as $t \to r+$, then $f \in \mathrm{L}^1((r,s] \times (a,b); B)$ and $f(\cdot, c) \in \mathrm{L}^1((r,s); B)$ for a.e. $c \in (a,b)$.*

**Proof:** Being continuous, $f$ is Borel-measurable on $Q := (r,s] \times (a,b)$, hence Lebesgue-measurable on $Q$. Due to the continuity and convergence, $M := \sup_{t \in [r,s]} \|f(t,\cdot)\|_{\mathrm{L}^1((a,b);B)} < \infty$. By The Fubini Theorem,

$$(s - r)M \geq \int_Q \|f\|_B\,dm = \int_r^s \int_a^b \|f\|_B\,dm = \int_a^b \int_r^s \|f\|_B\,dm; \quad \text{(B.49)}$$

in particular, $\int_r^s \|f\|_B\,dm < \infty$ for a.e. $c \in (a,b)$ and $f \in \mathrm{L}^1(Q; B)$. □

(See the notes on p. 947.)

## B.5 Differentiation of integrals ($\frac{\mathrm{d}}{\mathrm{dt}}\int$)

*"Chesire-Puss," she began, "would you tell me, please, which way I ought to go from here?"*
*"That depends a good deal on where you want to get to," said the Cat.*
*"I don't care much where–" said Alice.*
*"Then it doesn't matter which way you go," said the Cat.*
— Lewis Carroll (1832–98)

In this section we present Lebesgue points and a few results on differentiation of integrals.

Also for vector-valued functions, almost all points are Lebesgue points:

**Theorem B.5.1 (Lebesgue points)** *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$. Then, for almost all $t \in \mathbf{R}^n$, we have*

$$\lim_{r\to 0+} m(D_r)^{-1}\int_{D_r(t)} \|f(s)-f(t)\|_B\,ds = 0 \quad \textit{and} \quad f(t) = \lim_{r\to 0+} m(D_r)^{-1}\int_{D_r(t)} f(s)\,ds. \tag{B.50}$$

*Such $t$ are called the* Lebesgue points *of $f$, and* $\mathrm{Leb}(f) \subset \mathbf{R}^n$ *is the set of such $t$.*

(Here $D_r(t) := \{t' \in \mathbf{R}^n \bigm| |t'-t| < r$, $D_r := D_r(0)$.) From (B.50) it follows that $\|f(t)\| \le \|f\|_\infty \le \infty$ for all $t \in \mathrm{Leb}(f)$.

Note that if $f$ is continuous at $t$, then $t \in \mathrm{Leb}(f)$. Obviously, $\mathrm{Leb}(f) \cap \mathrm{Leb}(g) \subset \mathrm{Leb}(\alpha f + \beta g)$ for any $f,g \in \mathrm{L}^1_{\mathrm{loc}}$, $\alpha,\beta \in \mathbf{K}$.

**Proof:** The first claim follows from Theorem 3.8.5 of [HP] for $t \in D_k$, if we replace $f$ by $\chi_{D_k} f$ ($k \in \mathbf{N}$); the second claim follows from the first for (at least) same $t$:

$$\|f(t) - m(D_r)^{-1}\int_{D_r(t)} f(s)\,ds\|_B = \|m(D_r)^{-1}\int_{D_r(t)} (f(t)-f(s))\,ds\|_B \to 0. \tag{B.51}$$

Let $N_k \subset D_k$ be the null set where the first limit in (B.50) is nonzero or does not converge. Then (B.50) holds for $t \in \mathbf{R}^n \setminus \cup_{k\in\mathbf{N}} N_k$, hence a.e. (To be exact, [HP] gives the result for cubes, but it can be easily generalized to any nicely shrinking sets, as in Section 7 of [Rud86].) □

From the above theorem we obtain

**Corollary B.5.2** *Let $J \subset \mathbf{R}$ be an interval and $f \in \mathrm{L}^1_{\mathrm{loc}}(J;B)$. Then, for almost all $t \in J$,*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,dt = f(t) \quad \textit{and} \quad \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} \|f(s)-f(t)\|\,ds = 0. \tag{B.52}$$

*In particular, if $a \in J$ and $F(t) = \int_a^t f(s)\,ds$, then $F \in \mathcal{C}(J;B)$ and $F' = f$ a.e.*
□

(Of course, we have set $\int_b^a := -\int_a^b$.)

One easily verifies that a function $F$ of the above form is locally absolutely continuous (see (scalar) Definition 7.17 of [Rud86]). However, unlike in the finite-dimensional (or Hilbert) case, there are absolutely continuous functions that are nowhere differentiable (however, this is not the case for reflexive spaces, a fortiori not in Hilbert spaces; see, e.g., [DU] or p. 125 of [KOS] for details).

There is a method for uniquely choosing representatives for $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ "functions" so that these representative have all possible Lebesgue points:

**Lemma B.5.3 (Lebesgue representative)** *Let $[f] \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$. For each $t \in \mathbf{R}^n$ s.t.*

$$\lim_{r\to 0+} m(D_r)^{-1} \int_{D_r(t)} |f - x_t|\, dm = 0 \tag{B.53}$$

*for some $x_t \in B$, we set $(Lf)(t) := x_t$; for other values of $t$, we set $(Lf)(t) := 0$.*

*It follows that $(Lf)(t) = f(t)$ for all $t \in \mathrm{Leb}(f)$, hence $Lf = f$ a.e. and $\mathrm{Leb}(f) \subset \mathrm{Leb}(Lf)$. Moreover, $\|Lf\| \le \|f\|_\infty$ everywhere and $Lf$ depends on $[f]$ only.*

*Moreover, for all $[f],[g] \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};Y)$, $\Lambda \in Y^*$, $\psi \in \mathcal{GC}(\mathbf{R})$, and* every $t \in \mathbf{R}$ *we have*

$$(L\psi f)(t) = (\psi L f)(t), \qquad \mathrm{Leb}(\psi f) = \mathrm{Leb}(f), \tag{B.54}$$

$$\mathrm{Leb}(f) \subset \mathrm{Leb}(\Lambda f), \qquad \|(\Lambda L f)(t)\| \le \|(L\Lambda f)(t)\|, \tag{B.55}$$

$$t \in \mathrm{Leb}(f) \cap \mathrm{Leb}(g) \implies t \in \mathrm{Leb}(\alpha f + \beta g), \quad (L(\alpha f + \beta g))(t) = \alpha L f(t) + \beta L g(t). \tag{B.56}$$

*Finally, for each linear $F : B \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B_2)$ and $t \in \mathbf{R}^n$, the space*

$$B_t := \{x \in B \,\big|\, t \in \mathrm{Leb}(LFx)\} \tag{B.57}$$

*is a subspace of $B$.* □

(We omit the simple proof.) Note that $\mathrm{Leb}(Lf)$ becomes the union of the Lebesgue sets of all representatives of $[f]$, and that $f \mapsto Lf$ and $[f] \mapsto Lf$ are not linear, neither $f \mapsto (Lf)(t)$ for any $t \in \mathbf{R}^n$ (but $f \mapsto [Lf]$ and $[f] \mapsto [Lf]$ are linear, since $[Lf] = [f]$).

The standard differentiation formula for integrals can easily be extended:

**Lemma B.5.4** *Let $J, J' \subset \mathbf{R}$ be intervals, $a, b \in \mathcal{C}^1(J;J')$. Let $f : J \times J' \ni (t,s) \to f(t,s) \in B$ be s.t. $f, f_t \in \mathcal{C}(J \times J';B)$. Then $F(t) := \int_{a(t)}^{b(t)} f(t,s)\, ds$ is in $\mathcal{C}^1(J;B)$, and*

$$F'(t) := \int_{a(t)}^{b(t)} f_t(t,s)\, ds + b'(t) f(t,b(t)) - a'(t) f(t,a(t)) \text{ for all } t \in J. \tag{B.58}$$

Recall that we allow $J$ and $J'$ be non-open (the derivatives at the endpoints are, of course, one-sided).

**Proof:** Because $\int_{a(t)}^{b(t)} = \int_c^{b(t)} - \int_c^{a(t)}$, where $c \in J'$ is arbitrary, we may assume that $a$ is a constant. Now

$$F(t+h)-F(t)+\int_{b(t)}^{b(t+h)} \left(f(t,b(t))\right) ds \tag{B.59}$$

From the first term we compute

$$\begin{aligned} &\frac{1}{h}\left(\int_a^{b(t)} \left(f(t+h,s)-f(t,s)\right) ds - \int_a^{b(t)} f_t(t,s)\, ds\right) \\ &= \frac{1}{h}\int_a^{b(t)} \int_t^{t+h} \left(f_t(r,s)-f_t(t,s)\right) dr\, ds = \frac{1}{h}\int_t^{t+h} \int_a^{b(t)} \to 0, \end{aligned} \tag{B.60}$$

by the continuity of $f_t$. (By the Fubini Theorem, we were allowed to interchange the order of integration.)

One easily verifies that the second and third terms of (B.59) multiplied by $1/h$ converge to 0 and $b'(t)f(t,b(t))$, respectively. The continuity of $F'$ follows analogously. □

A smooth change of variable preserves Lebesgue points:

**Lemma B.5.5** *Let* $-\infty \le a < b \le +\infty$, $\phi \in \mathcal{C}^1((a,b);\mathbf{R})$, $\phi' > 0$ *and* $f \in \mathrm{L}^1_{\mathrm{loc}}((\alpha,\beta);B)$, *where* $\alpha := \phi(a)$, $\beta := \phi(b)$. *Then* $\mathrm{Leb}(f) = \phi[\mathrm{Leb}(f\circ\phi)]$.

Here Leb refers to the zero extensions (or any other $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};B)$ extensions) of $f$ and $f\circ\phi$.

**Proof:** Let $T \in (a,b)$. Set $g := f\circ\phi$, $t := \phi(T)$. Choose $R > 0$ s.t. $[t-R,t+R] \subset (\alpha,\beta)$. Set $M := \max_{[t-R,t+R]} \|\phi'\|$, $M' := \max_{\phi[t-R,t+R]} \|{\phi^{-1}}'\|$. By Lemma B.4.10, we have

$$\frac{1}{2r}\int_{t-r}^{t+r} \|f(s)-f(t)\|_B\, ds = \frac{1}{2r}\int_{\phi^{-1}(t-r)}^{\phi^{-1}(t+r)} \|g(s)-g(T)\|_B \phi'(s)\, ds \tag{B.61}$$

$$= \frac{MM'}{2rM'}\int_{T-M'r}^{T+M'r} \|g(s)-g(T)\|_B\, ds. \tag{B.62}$$

But this converges to zero whenever $T \in \mathrm{Leb}(g)$, hence then $t \in \mathrm{Leb}(f)$. Because $T \in (a,b)$ was arbitrary, we have $\phi[\mathrm{Leb}(f\circ\phi)] \subset \mathrm{Leb}(f)$. Exchange the roles of $f$ and $g$ (and $\phi$ and $\phi^{-1}$) to obtain that $\phi^{-1}[\mathrm{Leb}(f)] \subset \mathrm{Leb}(f\circ\phi)$, i.e., $\mathrm{Leb}(f) \subset \phi[\mathrm{Leb}(f\circ\phi)]$. □

The Mean Value Theorem is only true for **R**-valued functions (not even for **C**-valued or $\mathbf{R}^2$-valued; e.g., set $f(t) := e^{it}$, $[a,b] := [0,2\pi]$); in the multidimensional case it becomes a mere inequality:

**Lemma B.5.6 (Mean Value Inequality)** *Let* $a < b$, *and let* $f \in \mathcal{C}([a,b];B)$ *be differentiable on* $(a,b)$. *Then there is* $\xi \in (a,b)$ *s.t.*

$$\|f(b)-f(a)\|_B \le (b-a)\|f'(\xi)\| \le (b-a)\sup_{t\in(a,b)} \|f'(x)\|. \tag{B.63}$$

*If, in addition,* $f'(a)$ *exists, then there is* $\tilde{\xi} \in (a,b)$ *s.t.*

$$\|\frac{f(b)-f(a)}{b-a} - f'(a)\|_B \le \|f'(\tilde{\xi}) - f'(a)\|_B. \tag{B.64}$$

**Proof:** Let $\Lambda \in X^*$ be s.t. $\|\Lambda\| \le 1$ and $\Lambda(f(b) - f(a)) = \|f(b) - f(a)\|_B$. Obviously, $(\mathrm{Re}\,\Lambda f)' = \mathrm{Re}\,\Lambda f'$, hence

$$\|f(b) - f(a)\|_B = \mathrm{Re}\,\Lambda f(b) - \mathrm{Re}\,\Lambda f(a) = (b-a)(\mathrm{Re}\,\Lambda f')(\xi) \le (b-a)\|f'(\xi)\|_B, \tag{B.65}$$

by the classical Mean Value Theorem. The second inequality follows from the first applied to $F(t) := f(t) - f(a) - tf'(a)$ (note that $F'(t) = f'(t) - f'(a)$). □

**Lemma B.5.7** *Let $J \subset \mathbf{R}$ be an interval and $n \in \mathbf{N}$. Then the following are equivalent for $F : J \to B$:*

*(i) $F \in \mathcal{C}^{n+1}(J;B)$;*

*(ii) $F = \int f$ for some $f \in \mathcal{C}^n(J;B)$.*

*Moreover, if (ii) holds, then $F' = f$.*

By $F = \int f$ we mean that $F(t) = F(a) + \int_a^t f(s)\,ds$ for all $t \in J$ and some (hence all) $a \in J$. See Lemma B.7.6 for an analogous result for absolutely continuous functions.

**Proof:** One easily verifies that (ii) implies (i). Given (i), fix $a \in J$ and set $f := F'$, $G(t) := F(a) + \int_a^t f$. Then $(F - G)' = 0$ on $J$, hence $(\Lambda F - \Lambda G)' = 0$ on $J$ and $(\Lambda F - \Lambda G)(a) = 0$, hence $\Lambda F = \Lambda G$ on $J$; this holds for all $\Lambda \in B^*$, hence $F = G$. Therefore, (i) implies (ii). □

If $f(\cdot,q) \in \mathcal{C}^1(\Omega;B)$ for fixed $q \in Q$, and $f(z,\cdot)$ is measurable and has a common $\mathrm{L}^1$ majorant for all $z$, then $\int_Q f(z,\cdot)\,d\mu \in \mathcal{C}^1(\Omega;B)$ with derivative $\int_Q f_z(z,\cdot)\,d\mu \in \mathcal{C}(\Omega;B)$, by (c)&(d) below:

**Lemma B.5.8** *Let $\Omega$ be a metric space, let $Q$ be σ-finite, let $1 \le p \le \infty$, let $f : \Omega \times Q \to B$, and set $F(z) := f(z,\cdot)$. Then we have the following:*

*(a) We have $F \in \mathcal{C}(\Omega;\mathrm{L}^p(Q;B))$, if $p < \infty$ and (1.)–(3.) hold, where*

*(1.) $f(\cdot,q) \in \mathcal{C}(\Omega;B)$ for a.e. $q \in Q$;*
*(2.) $f(z,\cdot) \in \mathrm{L}(Q;B)$ for all $z \in \Omega$;*
*(3.) there is $g \in \mathrm{L}^p(Q;[0,\infty])$ s.t. $\|f(z,\cdot)\|_B \le g$ a.e. for all $z \in \Omega$.*

*(b) If $F \in \mathcal{C}(\Omega;\mathrm{L}^1(Q;B))$, then $\widetilde{F} \in \mathcal{C}(\Omega;B)$, where $\widetilde{F}(z) := \int_Q f(z,\cdot)\,d\mu$.*

*(c) Claims (a) and (b) also hold with $\mathcal{C}^1$ in place of $\mathcal{C}$ if $\Omega \subset \mathbf{R}$ is an interval and in (a) we also require that $\|f_z(z,\cdot)\|_B \le g$ a.e. for all $z \in \Omega$.*

*In case (a) we then also have $F'(z) = f_z(z,\cdot)$, hence then (d) applies.*

*(d) Assume that $\Omega \subset \mathbf{R}$ is an interval, $f(\cdot,q) \in \mathcal{C}^1(\Omega;B)$ for a.e. $q \in Q$, and $F, G \in \mathcal{C}(\Omega;\mathrm{L}^p)$, where $G(z) = f_z(z,\cdot)$.*

*Then $F \in \mathcal{C}^1(\Omega;\mathrm{L}^p)$ and $F' = G$ (and $\widetilde{F} \in \mathcal{C}^1(\Omega;B)$ and $\widetilde{F}'(z) = \int_Q f_z(z,\cdot)\,d\mu$ if $p = 1$) for all $z \in \Omega$.*

*(e) Claims (c) and (d) also hold with* H *in place of* $\mathcal{C}$ *if* $\Omega \subset \mathbf{C}$ *is open.*

See Definition D.1.3 for $\mathrm{H}(\Omega;B)$.

**Proof:** (a) If $z_n \to z$, then $F(z_n) \to F(z)$, by Theorem B.4.3. Thus, $F \in \mathcal{C}(\Omega;\mathrm{L}^p)$.

(b) This follows from (a) Lemma B.4.2.

(d) By Lemma B.5.7, we have $f(z,q) - f(a,q) = \int_a^z f_z(s,q)\,ds$ $(z,a \in \Omega)$.

Moreover, $\Omega$ is $\sigma$-finite, separable and metric, and open subsets of $\Omega$ are $m$-measurable. Thus, $f_z$ is $m \times \mu$-measurable, by Lemma B.4.8.

Fix $z \in \Omega$. Given now $h \in \Omega$, we have (here $\|\cdot\|_p$ refers to the $\mathrm{L}^p(Q;B)$ norm and $q \in Q$ is the corresponding dummy (i.e., dependent) variable)

$$\|h^{-1}\big(F(z+h) - F(z)\big) - G(z)\|_p = \|h^{-1}\int_z^{z+h}\big(f_z(s,q) - f_z(z,q)\big)\,ds\|_p \tag{B.66}$$

$$\leq h^{-1}\int_z^{z+h}\|f_z(s,q) - f_z(z,q)\|_p\,ds \to 0, \tag{B.67}$$

(the inequality is from Theorem B.4.16(b)) as $h \to 0$, by Theorem B.4.3, because $\|f_z(s,q) - f_z(z,q)\|_p = \|G(s) - G(z)\|_p \to 0$ as $s \to z$, and $\|G(s) - G(z)\|_p$ is bounded ($\leq M < \infty$) near $z$, by continuity, and $M \in \mathrm{L}^1([z,z+h])$.

Therefore, $F' = G$ exists in $\mathrm{L}^p$. Consequently, $F \in \mathcal{C}^1(\Omega;\mathrm{L}^p)$.

(c) By (a), we have $F, G \in \mathcal{C}(\Omega;\mathrm{L}^p)$, where $G(z) := f_z(z,\cdot)$ (note that $f_z(z,q) = \lim_{n\to\infty} n^{-1}(f(z+1/n,q) - f(z,q))$ for a.e. $q \in Q$, hence also $f_z$ satisfies condition (2.)). Therefore, we get the conclusions from (d).

(e) This is analogous to the $\mathcal{C}^1$ case (use a path integral and (b5) (and (b1) for $p = 1$) of Lemma D.1.2 instead of Lemma B.5.7). □

By (B.50), The averages of any $f \in \mathrm{L}^p(\mathbf{R};B)$ converge to $f$ pointwise a.e.; we also have the convergence in $\mathrm{L}^p$:

**Lemma B.5.9** *Let* $1 \leq p < \infty$ *and* $f \in \mathrm{L}^p(\mathbf{R};B)$*. Then*

$$\|\frac{1}{r}\int_0^r f(t+s)\,ds - f(t)\|_p \to 0, \quad \textit{as} \ \ r \to 0 \tag{B.68}$$

*(as functions of t).*

**Proof:** By the Minkovski Integral Inequality, we have (here the $\mathrm{L}^p$ norm refers to the variable $t$)

$$\frac{1}{|r|}\|\int_{s=0}^r [f(t+s) - f(t)]\,ds\|_p \leq \frac{1}{r}\int_{s=0}^r \|f(t+s) - f(t)\|_p\,ds \leq \sup_{s\in[0,r]}\|\tau^s f - f\|_p \to 0, \tag{B.69}$$

as $r \to 0$, by Lemma B.3.9. □

We finish this section by a technical lemma:

**Lemma B.5.10** *Let $T > 0$. If $0$ is a Lebesgue point of $f \in \mathrm{L}^1([0,T);B)$, then*

$$\int_0^T s\mathrm{e}^{-st} f(t)\,dt \to f(0), \quad \textit{as } s \to +\infty. \tag{B.70}$$

**Proof:** Because $\int_0^T s\mathrm{e}^{-st}\,dt \to 1$, we have (B.70) for constant functions $f$. Therefore, we may assume that $f(0) = 0$, i.e., that

$$g(t) := t\int_0^t \|f(r)\|_B\,dr \to 0 \quad \text{as } t \to 0+. \tag{B.71}$$

Set $F(t) := \int_0^t f\,dm \in \mathcal{C}([0,T];B)$. Then $F(0) = 0$, $\|\int_0^\varepsilon s^2\mathrm{e}^{-st}F(t)\,dt\|_B \leq g(\varepsilon)$ and $\int_\varepsilon^T s^2\mathrm{e}^{-st}F(t)\,dt \to 0$, as $s \to +\infty$, for any $\varepsilon \in (0,T)$. Using these three facts and partial integration, one easily obtains (B.70). □

## Notes for Sections B.1–B.5

As indicated in the proofs, many of the above results are known at least to some extent or in the scalar case. A further treatment on Bochner measurability, Bochner integral and vector-valued $\mathrm{L}^p$ spaces is given in, e.g., Sections 3.5–3.9 of [HP] and in [KOS], [DU] and [Yosida]; the monograph [Dinculeanu] treats same concepts from the Bourbaki point of view. The scalar case (the Lebesgue integral and measurability and $\mathrm{L}^p$ and $\mathcal{C}$ spaces) is contained in most books on real analysis, such as [Rud86].

## B.6 Vector-valued distributions $\mathcal{D}'(\Omega; B)$

*No, my friend, the way to have good and safe government, is not to trust it all to one, but to divide it among the many, distributing to every one exactly the functions he is competent to. It is by dividing and subdividing these republics from the national one down through all its subordinations, until it ends in the administration of every man's farm by himself; by placing under every one what his own eye may superintend, that all will be done for the best.*

— Thomas Jefferson (1743–1826), to Joseph Cabell, 1816

Here we briefly present straightforward vector-valued generalizations of some basic scalar distribution results. We have written the other sections so that the reader may skip this section, but its contents give a deeper insight to some concepts needed in, e.g., Section B.7.

Throughout this section, $B$ is a Banach space and $\Omega$ is an open subset of $\mathbf{R}^n$.

The space $\mathcal{D}(\Omega)$ is not equal to $C_c^\infty(\Omega)$, although rather close. One traditionally uses a rather complicated topology; fortunately, for most applications one does not need to know this topology, just some of its basic implications:

**Definition B.6.1** *The* test function space $\mathcal{D} := \mathcal{D}(\Omega)$ *is the set of functions* $\phi \in C^\infty(\mathbf{R})$*, whose support* $\operatorname{supp}\phi := \overline{\{x \in \mathbf{R}^n \mid \phi(x) \neq 0\}}$ *lies in* $\Omega$*, equipped with the standard (locally convex, complete, non metrizable) test function topology [Rud73, 6.3–6.5], where a sequence* $\{\phi_k\}$ *converges to* $\phi \in \mathcal{D}$ *iff there is a compact* $K \subset \Omega$ *s.t.* $\operatorname{supp}\phi_k \subset K$ *for all* $k$ *and* $D^\alpha\phi_k \to D^\alpha\phi$ *uniformly for all* $\alpha \in \mathbf{N}^n$.

*The elements of* $\mathcal{D}' := \mathcal{D}'(\Omega; B) := \mathcal{B}(\mathcal{D}(\Omega); B)$ *are called* ($B$-)distributions.

As in the proof of [Rud73, Theorem 6.6], one can show that a linear mapping $T : \mathcal{D}(\Omega) \to B$ is continuous iff it is sequentially continuous, i.e., iff $\phi_k \to \phi$ (in $\mathcal{D}(\Omega)$) implies $T\phi_k \to T\phi$ (it is enough to verify this for $\phi = 0$). Also most other results of [Rud73, Section 6] can easily be generalized.

We define the $\alpha$*th weak derivative* (or $\alpha$th distributional derivative) $\partial^\alpha T \in \mathcal{D}'$ of $T \in \mathcal{D}'$ by

$$\partial^\alpha T(\phi) := (-1)^{|\alpha|} T(D^\alpha \phi) \quad (\phi \in \mathcal{D}(\Omega)), \tag{B.72}$$

in particular, $\partial T(\phi) := -T(\phi')$ if $n = 1$. Here we have used the standard *multi-index* notation: $\alpha \in \mathbf{N}^n$, $|\alpha| := \sum_{j=1}^n \alpha_j$, $(x_1, \ldots, x_n)^\alpha := x_1^{\alpha_1} + \cdots + x_n^{\alpha_n}$, $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$; here $D_j := \frac{\mathrm{d}}{\mathrm{d}x_j}$ and $\partial_j$ is the corresponding weak derivative.

A function $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; B)$ (i.e., $f : \Omega \to B$ is s.t. $f \in \mathrm{L}^1(K; B)$ for each compact $K \subset \Omega$; note that $\mathrm{L}^p \subset \mathrm{L}^1_{\mathrm{loc}}$ for $p \in [1, \infty]$). is identified with the distribution $\phi \mapsto \int_\Omega f\phi\, dm$. in $\mathcal{D}'(\Omega; B)$. The inclusion $\mathrm{L}^1_{\mathrm{loc}}(\Omega; B) \subset \mathcal{D}'(\Omega; B)$ is linear and injective. Similarly, a constant (function) $b \in B$ is identified with the $\phi \mapsto \int_\Omega b\phi\, dm = b\int_\Omega \phi\, dm$.

A distribution with zero partial derivatives is a constant:

**Lemma B.6.2** *Let* $\Omega$ *be connected and* $T \in \mathcal{D}'(\Omega; B)$*. If* $\delta_j T = 0$ *for all* $j$*, then* $T \in B$.

**Proof:** If $\Lambda \in B^*$, then $\Lambda T \in \mathcal{D}'$ and $\partial_j \Lambda T = \Lambda \partial_j T = 0$ for all $j$, hence $\Lambda T = \alpha_\Lambda \in \mathbf{K}$, by [Rauch, p. 256]. Choose $\phi_1 \in \mathcal{D}$ s.t. $\int_\Omega \phi_1 \, dm = 1$, and set $b_T := T\phi_1$.

Then $\Lambda b_T = \Lambda T \phi_1 = \alpha_\Lambda \int_\Omega \phi_1 \, dm = \alpha_\Lambda$ for all $\Lambda \in B^*$, hence $\Lambda T \phi = \alpha_\Lambda \int_\Omega \phi \, dm = \Lambda b_T \int_\Omega \phi \, dm$ for all $\phi \in \mathcal{D}(\Omega)$, $\Lambda \in B^*$, i.e., $T\phi = b_T \int \phi = \int_\Omega b_T \phi$ for all $\phi$. Thus, $T = b_T \in B$. □

By induction, we see that if the $k$th partial derivatives of $T$ are zero, then $T$ is a polynomial of degree $k$:

**Corollary B.6.3** *Let $\Omega$ be connected and $T \in \mathcal{D}'(\Omega; B)$. If $|\alpha| = k \Rightarrow \partial^\alpha T = 0$, then $T = \sum_{|\beta| \le k-1} q^\beta b_\beta$, where $b_\beta \in B$ for all $\beta$.* □

**Corollary B.6.4** *If $J \subset \mathbf{R}$ is an open interval, $T \in \mathcal{D}'(J; B)$, and $\partial T = f \in \mathrm{L}^1_{\mathrm{loc}}(J; B)$, then there is a locally absolutely continuous function $F : J \to B$ s.t. $T = F$ and $F(t) = F(a) + \int_a^t f \, dm$ when $a \in J$; in particular $F' = f$ a.e.* □

(This follows by defining $G(t) := \int_a^t f \, dm$ and then noting that $\partial(T - G) = 0$.)

Similarly, for any $F, f \in \mathcal{C}(\Omega; B)$ s.t. $\partial_j F = f$, the derivative $D_j F$ exists and equals $f$, as one can easily show by using mollifiers.

## Notes

The contents of this section are well known, although it may be difficult to find any references; some other results on vector-valued distributions are given in [Treves]. See, e.g., [Rud73] or [Rauch] for the scalar case; most scalar results also hold in our setting with same proofs, mutatis mutandis (some existence results require the Radon–Nikodym property).

## B.7 Sobolev spaces $W^{k,p}(\Omega;B)$

*The reason that every major university maintains a department of mathematics is that it's cheaper than institutionalizing all those people.*

In this section, we briefly generalize some facts about scalar Sobolev spaces to their vector-valued counterparts. Most of the time we follow the scalar representations in [Adams]. A casual reader might skip the definitions and other results and just read Lemma B.7.6 and Theorem B.7.4, since they suffice for most applications. Also other readers might wish to read first Lemma B.7.6 to get some intuition to $W^{k,p}$ spaces.

Throughout this section, $B$ is a Banach space, $1 \le p \le \infty$, $k \in \mathbf{N}$, $n \in \mathbf{N}+1$, $\Omega \subset \mathbf{R}^n$ is open, and $m$ is the Lebesgue measure on $\mathbf{R}^n$.

**Definition B.7.1 ($\partial^\alpha g$)** *Let $f \in L^1_{loc}(\Omega;B)$ and $\alpha \in \mathbf{N}^n$. We call $g \in L^1_{loc}(\Omega;B)$ the* $\alpha$th weak derivative *of $f$ (on $\Omega$) and we write $\partial^\alpha f = g$ if*

$$\int_\Omega g\phi\, dm = (-1)^{|\alpha|}\int_\Omega f D^\alpha \phi\, dm \quad (\phi \in C_c^\infty(\Omega)). \tag{B.73}$$

(By using linearity, projections, mollifiers and a partition of unity, one could in fact show that (B.73) for $\phi \in C_c^\infty(\Omega)$ implies (B.73) for all $\phi \in C_c^{|\alpha|}(\Omega;X)$, where $X$ is as in Theorem B.4.12(d). We omit the proof.)

Here we have used the standard *multi-index* notation: $\alpha \subset \mathbf{N}^n$, $|\alpha| := \sum_{j=1}^n \alpha_j$, $(x_1,\ldots,x_n)^\alpha := x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}$, $D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$; here $D_j := \frac{d}{dx_j}$ is the (classical) $j$th partial derivative (use Definition B.3.3 with the other coordinates fixed). If $n=1$, we write $\partial := \partial^1$. (Outside this and previous section, we use the same notation for weak and ordinary derivatives.)

Recall that $f \in L^1_{loc}(\Omega;B)$ means that $f : \Omega \to B$ is s.t. $f \in L^1(K;B)$ for each compact $K \subset \Omega$. The weak derivate is unique (as an element of $L^1_{loc}$, that is, a.e.), by Theorem B.4.12(d).

If $f \in C^k(\Omega;B)$ and $|\alpha| \le k$, then $\partial^\alpha f = D^\alpha f$ on $\Omega$, by partial integration. See also Theorem B.7.4 and Lemma B.7.6.

It is obvious that if $\partial^\alpha f = g$ on $\Omega$, then $\partial^\alpha \pi_{\Omega'} f = \pi_{\Omega'} g$ on $\Omega'$ for any open $\Omega' \subset \Omega$.

**Definition B.7.2 ($W^{k,p}$)** *The* Sobolev space $W^{k,p}$ *is defined by*

$$W^{k,p}(\Omega;B) := \{ f \in L^p(\Omega;B) \,\big|\, \partial^\alpha f \in L^p \text{ when } |\alpha| \le k\} \tag{B.74}$$

*for $p \in [1,\infty]$ and $k \in \mathbf{N}$, with norm*

$$\|f\|_{k,p} := \Big[\sum_{|\alpha|\le k} \|\partial^\alpha f\|_p^p\Big]^{1/p} \ (1 \le p < \infty), \qquad \|f\|_{k,\infty} := \max_{|\alpha|\le k} \|\partial^\alpha f\|_\infty \tag{B.75}$$

*We denote closure of $C_c^\infty(\Omega;B)$ in $W^{k,p}(\Omega;B)$ by $W_0^{k,p}(\Omega;B)$. When $\Omega \subset \mathbf{R}^1$, we set $f \in W^{k,p}_{loc}(\Omega;B)$ if $f \in W^{k,p}(J;B)$ for each bounded open interval $J \subset \Omega$.*

In particular, $W^{0,p} = L^p$. Let $\Omega' \subset \Omega$ be open. Obviously, $f \in W^{k,p}(\Omega;B) \Rightarrow \pi_{\Omega'} f \in W^{k,p}(\Omega';B)$; in particular $W^{k,p} \subset W^{k,p}_{\mathrm{loc}}$.

Let $\Omega \subset \Omega''$, $f \in W^{k,p}_0(\Omega;B)$, $|\alpha| \le k$ and $\partial^\alpha f = g$. Then $\pi_{\Omega''} f \in W^{k,p}(\Omega'';B)$ with the same norm, and $\partial^\alpha \pi_{\Omega''} f = \pi_{\Omega''} g$ (for $f \in C^\infty_c(\Omega;B)$ the latter claim follows by integration by parts; for general $f \in W^{k,p}_0(\Omega;B)$ by continuity; the former claim follows from this).

**Theorem B.7.3** *The spaces* $W^{k,p}(\Omega;B)$ *and* $W^{k,p}_0$ *are Banach spaces.*

*If* $p < \infty$, *then* $C^\infty(\Omega;B) \cap W^{k,p}(\Omega;B)$ *is dense in* $W^{k,p}(\Omega;B)$, *and* $W^{k,p}_0(\mathbf{R}^n;B) = W^{k,p}(\mathbf{R}^n;B)$.

However, $C^1((0,1);B)$ is not dense in $W^{1,\infty}((0,1);B)$ when $B \ne \{0\}$ (take an absolutely continuous function whose derivative has a jump discontinuity).

The space $W^{k,p}$ is dense in $L^p$ ($p < \infty$), because $C^\infty_c \subset W^{k,p}$. However, $W^{1,\infty}(\mathbf{R};B)$-functions are continuous, by Lemma B.7.6, hence not dense in $L^\infty$.

**Proof:** 1° *The completeness of* $W^{k,p}$*:* Let $\{f_n\}$ be a Cauchy sequence in $W^{k,p}$ and let $|\alpha| \le n$. Then $\{\partial^\alpha f_n\}$ is a Cauchy sequence in $L^p$, hence there is $f_\alpha \in L^p$ s.t. $\partial^\alpha f_n \to f_\alpha$ in $L^p$. Let $\phi \in C^\infty_c(\Omega)$. Because $D^\alpha \phi \in C^\infty_c(\Omega) \subset L^q(\Omega)$, the Hölder inequality implies that

$$\langle f_\alpha, \phi\rangle \leftarrow \langle \partial^\alpha f_n, \phi\rangle := (-1)^{|\alpha|} \int D^\alpha \phi f_n \, dm \to (-1)^{|\alpha|} \int D^\alpha \phi f_0 \, dm =: \langle \partial^\alpha f, \phi\rangle. \tag{B.76}$$

Because $\alpha$ and $\phi$ were arbitrary, we have $f_0 \in W^{k,p}$ and $\partial^\alpha f_0 = f_\alpha$ for $|\alpha| \le k$. Clearly $f_n \to f_0$ in $W^{k,p}$.

2° Being a closed subspace of $W^{k,p}$, also $W^{k,p}_0$ is a Banach space. The last sentence of the theorem follows from the straightforward generalizations of [Adams, 3.15–3.19] (note that on p. 53 of [Adams] the condition ‘contained in “$U_k$”’ should be ’contained in $U_k$ but not in $U_{k-1}$’). □

One can interpret $W^{k,p}$ as a closed subspace of $\prod_{1 \le j \le N_{n,k}} L^p$, hence it is separable if $B$ is (cf. [Adams, 3.4]). Similarly, $W^{k,2}$ is a Hilbert space if $B$ is.

If $\psi \in C^k_b(\Omega)$, then $\psi \in \mathcal{B}(W^{k,p})$ and $\|\psi\|_{\mathcal{B}} \le M_k \|\psi\|_{C^k_b}$ (as a multiplication operator on $W^{k,p}$) and $\psi f$ can be weakly differentiated by the Leibniz’ rule. For $\Lambda \in \mathcal{B}(B,B_2)$ we have $\|\Lambda\|_{\mathcal{B}(W^{k,p}(\Omega,B);W^{k,p}(\Omega,B_2))} = \|\Lambda\|_{\mathcal{B}(B,B_2)}$ (unless $\Omega = \emptyset$), moreover, $\partial^\alpha \Lambda f = \Lambda \partial^\alpha f$ for $f \in W^{k,p}$.

An open $\Omega \subset \mathbf{R}$ has the *cone property*, if there is a finite cone $C$ s.t. each point $x \in \Omega$ is the vertex of a finite cone $C_x \subset \Omega$ congruent to $C$. In particular, any ball or cube or a product of such will do.

**Theorem B.7.4 (Sobolev Imbedding Theorem)** *Let $B$ be a Banach space, let the open set $\Omega \subset \mathbf{R}^n$ have the cone property, and let $1 \le p < \infty$, $mp > n$, and* $j \in \mathbf{N}$.

*Then* $W^{j+m,p}(\Omega;B) \subset C^j_b(\Omega;B)$, *and* $W^{j+m,p}_0(\Omega;B) \subset C^j_0(\Omega;B)$, *and these imbeddings are continuous.*

In particular, $W^{j+m,p}(\mathbf{R}^n;B) \subset \mathcal{C}_0^j(\mathbf{R}^n;B)$. Of course, $f \in \mathcal{C}_0$ means that there is $\widetilde{f} \in \mathcal{C}_0$ s.t. $f = \widetilde{f}$ a.e. See also Corollary B.7.7 (which allows $p = \infty$).

**Proof:** (We use the norm $\|f\|_{\mathcal{C}_b^j} := \sum_{|\alpha| \le j} \sup \|D^\alpha f\|$ in $\mathcal{C}_b^j$ and $\mathcal{C}_0^j$.) By [Adams, 5.4C], there is $c = c_{j,m,p,n,\Omega}$ s.t. for $F \in W^{j+m,p}(\Omega)$ we have $\|\partial^\alpha F\|_\infty \le c\|F\|_{j+m,p}$ when $|\alpha| \le j$.

1° Case $f \in \mathcal{C}^k \cap W^{j+m,p}$: Let $|\alpha| \le j$. If we had $\|(\partial^\alpha f)(t)\| > c\|f\|_{j+m,p} =: M$, then there were $\Lambda \in X^*$ s.t. $\|\Lambda\| \le 1$ and $\Lambda \partial^\alpha f(t) > M$. But $\|\partial^\alpha \Lambda f\|_\infty \le c\|\Lambda f\|_{j+m,p} \le c\|f\|_{j+m,p} < (\partial^\alpha \Lambda f)(t)$ were a contradiction, hence $\|(\partial^\alpha f)(t)\| \le c\|f\|_{j+m,p}$.

2° Case $f \in W^{j+m,p}$: By Theorem B.7.3, some sequence $\{f_n\} \subset \mathcal{C}^k \cap W^{j+m,p}$ converges to $f$. By 1°, $\{f_n\}$ is a $\mathcal{C}_b^j$ Cauchy sequence, hence it converges in $\mathcal{C}_b^j$ to a function $g$. Because a subsequence converges to $f$ a.e. (generalize the corresponding scalar result, e.g., [Rud86, Theorem 3.12]), we have $g = f$ a.e. Moreover (because of the convergence in $\mathcal{C}_b^j$), we have

$$\|\partial^\alpha f\|_\infty = \lim_{n\to\infty} \|D^\alpha f_n\|_\infty \le \lim_{n\to\infty} c\|f_n\|_{j+m,p} = c\|f\|_{j+m,p}. \tag{B.77}$$

3° $W_0^{j+m,p}$: Here we can take $\{f_n\} \subset \mathcal{C}_c^\infty(\mathbf{R}^n;X)$ and obtain that $g$ $(= f)$ belongs to the closure of $\mathcal{C}_c^\infty$ in $\mathcal{C}_b^j$, i.e., to $\mathcal{C}_0^j$.

4° The continuity of the imbeddings follows from the bound $c$. □

If first (resp. $k$th) weak partial derivatives of $f$ are zero, then $f$ is a constant (resp. a polynomial of order $< k$):

**Lemma B.7.5** *Let $\Omega$ be connected and $f \in W^{1,p}(\Omega;B)$. If $\delta_j f = 0$ for all $j$, then $f \in B$.*

*Let, in addition, $f \in W^{k,p}(\Omega;B)$. If $|\alpha| = k \Rightarrow \partial^\alpha f = 0$, then $f = \sum_{|\beta| \le k-1} x^\beta b_\beta$, where $b_\beta \in B$ for all $\beta$.*

**Proof:** If $\Lambda \in B^*$, then $\Lambda f \in \mathcal{D}'$ and $\partial_j \Lambda f = \Lambda \partial_j f = 0$ for all $j$, hence $\Lambda f = \alpha_\Lambda \in \mathbf{K}$, by corresponding scalar result (see p. 256 of [Rauch]). Choose $\phi_1 \in \mathcal{D}$ s.t. $\int_\Omega \phi_1\, dm = 1$, and set $b_f := f\phi_1$.

Then $\Lambda b_f = \Lambda f \phi_1 = \alpha_\Lambda \int_\Omega \phi_1\, dm = \alpha_\Lambda$ for all $\Lambda \in B^*$, hence $\Lambda f \phi = \alpha_\Lambda \int_\Omega \phi\, dm = \Lambda b_f \int_\Omega \phi\, dm$ for all $\phi$, $\Lambda \in B^*$, i.e., $f\phi = b_f \int \phi = \int_\Omega b_f \phi$ for all $\phi$. Thus, $f = b_f \in B$. The $W^{k,p}$ claim follows by induction. □

Being a $W^{1,1}$ function on an interval is equivalent to absolute continuity:

**Lemma B.7.6 ($W^{1,p} = \int L^p$)** *Let $J \subset \mathbf{R}$ be an open interval and $f \in L^p(J;B)$. Then the following are equivalent:*

*(i) $f \in W^{1,p}(J;B)$;*

*(ii) there is $g \in L^p$ s.t. $f(t) = \int_a^t g\, dm + f(a)$ $(t \in J)$ for some $a \in J$.*

*Assume (ii). Then $g = f'$ a.e., $\partial f = g$, $f$ is locally absolutely continuous, and (ii) holds for any $a \in J$. If $p = 1$, then $f$ is absolutely continuous and has one-sided limits at the endpoints of $J$.*

Thus, if $f, \partial f \in \mathrm{L}^p$, then the weak derivative $\partial f$ is also a pointwise derivative of $f$ a.e. However, if $f$ is the Cantor function of Theorem 7.16 of [Rud86], then $f'$ exists a.e. and $f, f' \in \mathrm{L}^p([0,1])$ for any $p \in [1,+\infty]$, but $\partial f$ does not exist (in $\mathrm{L}^1_{\mathrm{loc}}$), because $f' = 0$ and $f \neq f(0) + \int_0^t 0\,dm$.

**Proof:** 1° "(ii)⇒(i)": Assume (ii). By using the Fubini theorem, one easily verifies that $\partial f = g$, hence $f \in \mathrm{W}^{1,p}$.

2° "(i)⇒(ii)": Let $f \in \mathrm{W}^{1,p}$ and $g := \partial f \in \mathrm{L}^p$. Define $F \in \mathcal{C}$ by $F(t) := \int_a^t g\,dt + f(a)$ for some $a \in J$. By 1°, we have $\partial F = g$, hence $F = f + b$ for some $b \in B$, by Lemma B.7.5, and $b = F(a) - f(a) = 0$. Thus, $f = F$.

3° Assume (ii). By Corollary B.5.2, $f' = g$ a.e. and $f$ is locally absolutely continuous. The rest follows from 1°–2° (recall that $\partial f$ is unique, by Theorem B.4.12(d)), except the $p = 1$ claims, which follow by choosing for $\varepsilon > 0$ a simple function $s \in \mathrm{L}^1$ s.t. $0 \le s \le \|g\|$ and $\int_J \|g\| - s\,dm < \varepsilon/2$, and setting $\delta_\varepsilon := \varepsilon/2\max s$ (or $\delta_\varepsilon = 1$ if $s = 0$) — then $m(E) < \delta_\varepsilon$ implies that $\int_E \|g\|\,dm < \varepsilon$; because $\varepsilon > 0$ was arbitrary, $f$ is absolutely continuous (see, e.g., Definition 7.17 of [Rud86]), hence it obviously is continuous on $\overline{J}$. □

In case $n = 1$, we can slightly improve Theorem B.7.4:

**Corollary B.7.7 ($\mathrm{W}^{k+1,p}(J;B) \subset \mathcal{C}^k(J;B)$)** *Let $J \subset \mathbf{R}$ be open. Then $\mathrm{W}^{k+1,p}(J;B) \subset \mathcal{C}^k(J;B)$. The mapping $\mathrm{W}^{k+1,p}(J;B) \to \mathcal{C}^k(K;B)$ is continuous for each compact interval $K \subset J$.*

**Proof:** Let $f \in \mathrm{W}^{k+1,p}(J;B)$. Let $J' \subset J$ be an interval. Then $f \in \mathrm{W}^{k+1,p}(J';B)$, hence $f \in \mathcal{C}^k(J';B)$, by Lemma B.7.6 and induction. Because $J$ was arbitrary, we have $f \in \mathcal{C}^k(J;B)$.

By Lemma A.3.6, $\mathrm{W}^{k+1,p}(J;B) \to \mathcal{C}^k(K;B)$ is continuous (because $\mathcal{C}^k(K;B) \subset \mathrm{L}^p(K;B)$, continuously). □

For $J = \mathbf{R}$, the above weak derivatives are, in fact, $\mathrm{L}^p$ derivatives (and vice versa):

**Lemma B.7.8** *Let $f \in \mathrm{L}^p(\mathbf{R};B)$. Then the following are equivalent:*

*(i) $f \in \mathrm{W}^{1,p}(\mathbf{R};B)$;*

*(ii) there is $g \in \mathrm{L}^p$ s.t. $f(t) = \int_0^t g\,dm + f(0)$ $(t \in \mathbf{R})$.*

*(iii) there is $\widetilde{g} \in \mathrm{L}^p$ s.t. $h^{-1}[\tau(h) - I]f \to \widetilde{g}$ in $\mathrm{L}^p$, as $h \to 0$;*

*If (i)–(iii) hold, then $g = \widetilde{g} = f'$ a.e., $\partial f = g$, and $f$ is locally absolutely continuous.*

**Proof:** The equivalence (i)⇔(ii) follows from Lemma B.7.6.

1° "(iii)⇒(i)": Set $D_h := h^{-1}[\tau(h) - I]$. Let $\phi \in \mathcal{C}_c^\infty(\mathbf{R})$ be arbitrary. We have $D_h\phi \to \phi'$ uniformly, as $h \to 0$, hence $D_h\phi \to \phi'$ in $\mathrm{L}^q$, where $1/p + 1/q = 1$. Therefore, by the Hölder inequality, we have the two convergences

$$\int_{\mathbf{R}} \widetilde{g}\phi\,dm \leftarrow \int_{\mathbf{R}} (D_h f)\phi\,dm = -\int_{\mathbf{R}} f(D_{-h}\phi)\,dm \to -\int_{\mathbf{R}} f\phi'\,dm, \tag{B.78}$$

as $h \to 0$. Therefore $\partial f = \tilde{g}$; in particular, $f \in W^{1,p}$.

2° "(ii)⇒(iii)": Assume (ii). Set $F_h(x) := [f(x+h) - f(x)]/h = h^{-1}\int_0^h g(x+t)\,dt$, so that $F_h(x) \to g(x)$ for a.e. $x$. Using the Minkovski integral inequality (Theorem B.4.16), we get for $\varepsilon > 0$ that (here $L^p$ norm is taken w.r.t. $x$)

$$\|F_h(x) - g(x)\|_p := \|h^{-1}\int_0^h [g(x+t) - g(x)]\,dt\|_p \tag{B.79}$$

$$\leq h^{-1}\int_0^h \|g(x+t) - g(x)\|_p\,dt < \varepsilon, \tag{B.80}$$

when $|h| < \delta_\varepsilon$, by Lemma B.3.9. Therefore $F_h \to g$ in $L^p$, as required.

3° Assume (ii). The identity $g = \tilde{g} = f'$ follows from 1°–3° (recall that $\partial f$ is unique, by Theorem B.4.12(d)), the rest from Lemma B.7.6. □

The subset $W_0^{1,p}$ of $W^{1,p}$ refers to the elements that are "zero on the boundary" in some sense:

**Lemma B.7.9 ($\mathbf{W_0^{1,p}}$)** *Let $J \subset \mathbf{R}$ be an open interval and $p < \infty$. Then* $W^{1,p}(J;B) \subset \mathcal{C}_0(\overline{J};B)$, *and* $W_0^{1,p}(J;B) = \{f \in W^{1,p}(J;B) \mid f(\inf J) = 0 = f(\sup J)\}$.

Thus, a $W^{1,p}(J;B)$ function $f$ has limits at endpoints of $J$; we have $f \in W_0^{1,p}$ iff these limits are zero (they are necessarily zero at endpoints $\pm\infty$, if any).

**Proof:** 0° Let $\frac{1}{p} + \frac{1}{q} = 1$. Choose $a, b \in [-\infty, +\infty]$ s.t. $J = (a,b)$. We shall assume that $a = 0 < b$. By translation, this then extends to any $a \in (-\infty, b)$. Using reflection, we can then cover intervals of form $(a, +\infty)$. The case $J = \mathbf{R}$ follows from by Theorem B.7.4,

1° Let $f \in W^{1,p}(J;B)$. For $n \in \mathbf{N}+1$, choose $b_n \in J$ s.t. $L^p((b_n,b);B)$ norm of $f$ and $f'$ is less than $1/n$. By this, Lemma B.7.6 and the Hölder Inequality, we have

$$\|f(s) - f(t)\| \leq \int_s^t \|f'\|_B\,dm \leq n^{-1}|t-s|^{1/q} \quad (s,t \in (b_n,b)). \tag{B.81}$$

2° Assume that $b = \infty$. By (B.81) and the Hölder Inequality we have

$$\|f(t)\|_B - 1/n \leq \|\int_t^{t+1} f\|_B \leq n^{-1}\cdot 1, \tag{B.82}$$

i.e., $f(t) < 2/n$. Because $t > b_n$ was arbitrary, we have $f(t) \to 0$ as $t \to +\infty$.

3° Assume that $b < \infty$. By the Hölder Inequality, we have $f, f' \in L^1(J;B)$, in particular, $f$ has a limit at $b$ and $f$ is absolutely continuous on $\overline{J}$.

4° By 2° and 3°, we have $W^{1,p}(J;B) \subset \mathcal{C}_0(\overline{J};B)$.

5° Because $f \mapsto f(b)$ is continuous, by Theorem B.7.4, we have $0 = f(b)$ for $f \in W_0^{1,p}$ for $b < \infty$; for $b = \infty$ this was shown in 2°. Analogously, $f(0) = 0$.

6° Now only the converse for 5° remains: We assume that $f \in W^{1,p}(J;B)$ is s.t. $f(0) = 0 = f(b)$, and prove that $f \in W_0^{1,p}$. By Theorem B.7.3, we may assume that $f \in \mathcal{C}^\infty(\Omega;B) \cap W^{1,p}(J;B)$.

Let $\varepsilon \in (0,1)$. We shall construct $g \in W^{1,p}$ with $\operatorname{supp} g \subset \Omega$ and $\|g - f\|_{W^{1,p}} < \varepsilon$; then one can find $\phi \in C_c^\infty$ s.t. $\|\phi - g\|_{W^{1,p}} < \varepsilon$ by mollifying $g$ (see [Adams]).

$6\frac{1}{3}^\circ$ *The case where $f = 0$ on $[b', b)$ for some $b' < b$:* Choose $\delta \in (0, \min(\varepsilon/5, b/2))$ s.t. $\|f\|_p, \|f'\|_p, \|f\|_\infty < \varepsilon/5$ over $(0,\delta)$. It follows that $\|f(\delta)\| \le \delta^{1/q}\varepsilon/5 \le \varepsilon/5 < 1$ (cf. (B.81). Choose $r \in (0,1)$ s.t. $r^{-1/q} < 2$. Set

$$g(t) := \begin{cases} 0, & t \in [0, (1-r)\delta]; \\ (1 - \frac{1}{r} + \frac{t}{r\delta}) f(\delta), & (t \in ((1-r)\delta, \delta)); \\ f(t), & t \in [\delta, b). \end{cases} \tag{B.83}$$

It follows that $g \in W^{1,p}$ and $g' = f(\delta)/r\delta$ on $((1-r)\delta, \delta))$ and $g' = f'$ on $(\delta, b)$. Moreover,

$$\|g'\|^p_{L^p((0,\delta);B)} < r\delta(\delta^{1/q}\varepsilon/5r\delta)^p = r^{1-p}\varepsilon^p/5^p, \tag{B.84}$$

hence $\|g'\|_{L^p((0,\delta);B)} < 2\varepsilon/5$. Obviously, $\|g\| \le \|f(\delta)\| < \varepsilon/5$ on $(0,\delta)$, hence $\|g\|_{L^p((0,\delta);B)} < \varepsilon/5$. It follows that $\|f - g\|_{W^{1,p}} < \varepsilon/5 + \varepsilon/5 + 2\varepsilon/5 + \varepsilon/5 = \varepsilon$, as required.

$6\frac{2}{3}^\circ$ *The general case:* If $b < \infty$, we can define $g$ on $(b - \delta, b)$ in the same way as above, so that $\|f - g\|_{W^{1,p}} < 2\varepsilon$. Let $b = \infty$. Then we can take $\delta := \varepsilon/6$, replace "$b - \delta$" above by some $b' \in (1,\infty)$ s.t. $\|f\|_p, \|f'\|_p, \|f\|_\infty < \delta^{1/q}\varepsilon/5$ over $(b', \infty)$, and go on as above. □

We set $W^{1,p}_\omega := \{f \in L^p_\omega \,|\, \partial f \in L^p_\omega\}$ for $\omega \in \mathbf{R}$ (the meaning of $W_0$ is apparent from the context), and $W^{1,p}_{0,\omega}$ denotes the closure of $C_c^\infty$ in $W^{1,p}_\omega$.

**Lemma B.7.10 ($\mathbf{W^{1,p}_\omega}$)** *The mapping $T_\alpha : f \mapsto e^{\alpha\cdot} f$ is a Banach isorphism of $W^{k,p}_\omega$ onto $W^{k,p}_{\omega+\alpha}$ and of $W^{k,p}_{0,\omega}$ onto $W^{k,p}_{0,\omega+\alpha}$, and it is a bijection of $C_c^\infty$ onto $C_c^\infty$ and $C^\infty$ onto $C^\infty$.*

*Moreover, Theorem B.7.3, Corollary B.7.7 and Lemmas B.7.5, B.7.6, B.7.8 and B.7.9 hold with replacements $L^p \mapsto L^p_\omega$, $W^{1,p} \mapsto W^{1,p}_\omega$, and $W^{1,p}_0 \mapsto W^{1,p}_{0,\omega}$ (except that if $f \in W^{1,1}_\omega(J;B)$, then $T_{-\omega} f$ is absolutely continuous, not necessarily $f$).*

Naturally, if $f \in W^{j+m,p}_\omega$, then $e^{-\omega\cdot} f \in C_b^j$ as in Theorem B.7.4, etc. (but $f$ itself need not be bounded).

Recall that by a (Banach) isomorphism $T : X \to Y$ we mean that $T \in \mathcal{GB}(X,Y)$; the above mapping does not map the derivatives of $f$ to those of its image $T_\alpha f$.

**Proof:** 1° *Bijections:* Because $(e^{\omega\cdot} f)' = \omega e^{\omega\cdot} f + e^{\omega\cdot} f'$, we have $\|T_\alpha f\| \le (|\alpha| + 1)\|f\|$ and $(T_\alpha)^{-1} = T_{-\alpha}$, hence the $W^{1,p}_\omega$ claim holds. The claim on $C_c^\infty$ and $C^\infty$ is obvious, and the claim on $W^{1,p}_{0,\omega}$ follows from these two. Use induction for general $k$.

2° *Theorem B.7.3 and Lemma B.7.9:* These follow directly from 1°.

3° *Lemma B.7.5:* Let $f \in W^{k,p}_\omega(\Omega;B)$. Then $f \in W^{k,p}(\Omega';B)$ for each open, bounded $\Omega' \subset \Omega$, so the claim holds for $k = 1$; use induction for general $k$.

4° *Lemma B.7.6:* If $f \in L^p_\omega(J;B)$, then $f \in L^p((-T,T) \cap J;B)$ for each $T > 0$, hence Lemma B.7.6 holds (we first get $g \in L^p_{loc}$, but we must have $g = \partial f$, hence $f \in W^{1,p}$ iff $g \in L^p$).

5° *Lemma B.7.8 and Corollary B.7.7:* Modify the original proof accordingly. □

The shift is a continuous operation on $W^{n,p}_\omega$:

**Lemma B.7.11** *If $f \in W^{n,p}_\omega(\mathbf{R};B)$, then $\tau f \in \mathcal{C}^{n-j}(\mathbf{R};W^{j,p}_\omega)$ ($j = 0,1,\ldots,n$).* □

(This follows from Corollary B.3.8, Lemma B.7.8 and induction.)

In some examples, we shall use following semigroups:

**Proposition B.7.12** *Let $p < \infty$ and $-\infty \leq a < b$, and set $J = (a,b)$. If $b = \infty$ (resp. $b < \infty$), then $\pi_J \tau \pi_J$ is a bounded $C_0$-semigroup on $L^p_\omega(J;B)$, and its generator is the weak differentiation operator $\partial$ with domain $W^{1,p}_\omega(J;B)$ (resp. with domain $\{f \in W^{1,p}_\omega(J;B) \mid f(b) = 0\}$) and its resolvent $(\lambda - \partial)^{-1}$ ($\operatorname{Re}\lambda > \omega$) maps $f \in L^p_\omega(J;B)$ into the element $J \ni t \mapsto \int_t^b e^{\lambda(t-s)} f(s)\,ds$ of this domain.*

*In particular, for $J = \mathbf{R}_-$ the domain is $W^{1,p}_{0,\omega}(\mathbf{R}_-;B)$.* □

See, e.g., Examples 3.2.3 and 3.3.2 of [Sbook] for the proof (except for the last claim, which follows from Lemmas B.7.9 and B.7.10). By using **Я** one obtains the dual results for $\pi_J \tau^* \pi_J$.

## Notes

The contents of this section are well known, although it may be difficult to find any references, particularly for the vector-valued case.

Popular references for Sobolev spaces include [Adams] and [Ziemer] in the scalar case. Most of their results also hold in the vector-valued case with same proofs, mutatis mutandis.