Chapter 9

Riccati Equations and *J*-Critical Control

> *Jacopo Francesco, Count Riccati, born at Venice on May 28, 1676, and died at Trèves on April 15, 1754, did a great deal to disseminate a knowledge of the Newtonian philosophy in Italy. Besides the equation known by his name, certain cases of which he succeeded in integrating, he discussed the question of the possibility of lowering the order of a given differential equation.*
>
> — 'A Short Account of the History of Mathematics' (4th edition, 1908) by W. W. Rouse Ball.

In this chapter, we shall establish the connection between optimal control and stabilizing solutions of Riccati equations (read "optimal" as "J-critical"). A summary of the main results is given on pp. 28–31, so here we only list the contents of each section.

Throughout this chapter, we shall assume that Standing Hypothesis 9.0.1 holds. Moreover, Standing Hypothesis 9.1.2 is assumed in Section 9.1, Standing Hypothesis 9.4.1 in Section 9.4, and Standing Hypothesis 9.5.1 in Sections 9.5 and 9.6.

In Section 9.1, we establish the equivalence mentioned on p. 9 of (I) J-critical control, (II) spectral or coprime factorization, and (III) stabilizing solutions of Riccati equations. Under further (e.g., MTIC type) regularity, (IV) the standard coercivity assumption is shown to be a fourth equivalent condition. Some further results in this direction are given in Sections 9.2 and 9.9.

In general, the connection between optimization and Riccati equation is trickier than in the case of a bounded input operator B: a J-critical control (when it exists) need not be given by a regular state feedback operator (nor by any well-posed state feedback). In Section 9.1, this difficulty is overcome by using the special classes of Theorem 8.4.9 (with partial results for the general case). In Section 9.2, we list several additional cases where this difficulty disappears, namely smoothing semigroups, bounded input maps or smooth I/O maps (see Hypothesis 9.2.2). We also the Riccati equation and corresponding results. A summary of most of our sufficient conditions is given in Remark

9.9.14. Applications to parabolic-type problems (i.e., to systems with an analytic semigroup) are given in Section 9.5.

A casual reader might be satisfied with the three sections mentioned above and skip most of the rest of the chapter, which contains a more general and hence necessarily less satisfactory theory and results on which the above is based.

In Section 9.7, we treat the most general case, where a unique J-critical control for any WR (Weakly Regular) system is shown to correspond to a solution of a *generalized Riccati equation* given on $\mathrm{Dom}(A_{\mathrm{crit}})$, the domain of the generator of the "(state-feedback controlled) closed-loop semigroup". We do not require the optimal control to be well-posed nor regular. This generalized Riccati equation is a rigorous extension of the Riccati equation of F. Flandoli et al. [FLT]. An integral version of the equation is given for arbitrary (even irregular) systems.

In Section 9.9, we show that for a WR system, there is a (well-posed) WR J-critical state feedback operator iff there is a "stabilizing" solution to the *extended Continuous-time Algebraic Riccati Equation (eCARE)*. In the general (possibly irregular) case the eCARE has to be replaced by the *extended Integral Algebraic Riccati Equation (eIARE)*, which also allows us to reduced several results to the substantially simpler discrete-time theory. The word "extended" corresponds to possibly noninvertible signature operators and is redundant under standard coercivity (and regularity) assumptions. Further results on Riccati equations are given in Sections 9.8–9.12.

In Section 9.13, we present examples that illustrate various pathological cases, including those mentioned above. In Section 9.14, we show that the J-critical control is given by (well-posed) state feedback iff certain factorization condition is satisfied, and we use this to extend a part of the "H^2 (generalized) canonical factorization theory" of [CG81] and [LS] to maps with infinite-dimensional input and output spaces. Positive Riccati equations will be treated in Sections 10.6 and 10.7.

A reader interested mainly in results can read the sections linearly. A more technically oriented reader may wish to read Sections 9.7 and 9.8–9.11 before 9.2–9.6, 9.1, 9.12, 9.13 and 9.14, although a brief glance at Section 9.1 before the start might nevertheless be a good idea. The reader wishing to verify all proofs rigorously may follow the order described in the proof of Theorem 14.1.3.

The results of this chapter hold when we optimize under any decent restrictions on the stability of the input, state, output and/or additional output:

Standing Hypothesis 9.0.1 *Throughout this chapter and Chapter 10, we assume that* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$. *The letters* U, H, Y *and* Z *denote Hilbert spaces of arbitrary dimensions.*

We also assume that $\begin{bmatrix}\mathbb{Q} & \mathbb{R}\end{bmatrix}$, Z^{u} *and* Z^{s} *are as in Definition 8.3.2,* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{Q} & \mathbb{R}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,\widetilde{Y})$ *for some Hilbert space* $\widetilde{Y}$, *and that* $\pi_+\tau^t z \in Z^{\mathrm{s}} \Leftrightarrow z \in Z^{\mathrm{s}}$ $\quad (z \in Z^{\mathrm{u}},\ t > 0)$.

The reader may ignore the latter paragraph of the hypothesis and read $\mathcal{U}_*^*$ as any of $\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$ (see Definition 8.3.2), since these obviously satisfy the latter assumption (with $\widetilde{Y} = H$, because $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{A} & \mathbb{B}\tau\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,H)$ and $x :=$

$\mathbb{A}x_0 + \mathbb{B}\tau u \in \mathcal{C}(\mathbf{R}_+;H) \subset \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;H)$, by Theorem 6.2.13(a1)). Sometimes we also require that Z^{s} is reflexive; this is satisfied by $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$.

In fact, if $Z^{\mathrm{u}} \subset \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\widetilde{Y})$, then $Z^{\mathrm{s}} = \mathrm{L}^p_\omega$ will do for any $p \in [1,\infty]$, $\omega \in \mathbf{R}$; and if $Z^{\mathrm{u}} \subset \mathcal{C}(\mathbf{R}_+;\widetilde{Y})$, then $Z^{\mathrm{s}} = \mathrm{e}^{-\omega\cdot}\mathcal{C}_{\mathrm{b}}$ or $Z^{\mathrm{s}} = \mathrm{e}^{-\omega\cdot}\mathcal{C}_0$ will do for any $\omega \in \mathbf{R}$ (assuming that $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B} \\ \hline \mathbb{Q} & \mathbb{R}\end{array}\right] \in \mathrm{WPLS}(U,H,\widetilde{Y})$).

By capital letters we again denote the generators or feedthrough operators of integral maps (see Lemma 6.1.16 and Definition 6.2.3).

9.1 The Riccati Equation: A summary for $\mathcal{U}_{\text{out}}$ (r.c.f.↔CARE)

It is not your fault, it is the fault of the mathematics.
— George Weiss, about the CARE at MTNS'98

In this section, we show that, for a regular strongly q.r.c.-stabilizable systems, the conditions (I)–(III) below are equivalent:

(I) the existence of regular optimal state feedback,

(II) the existence of a regular $(J,*)$-inner coprime factorization,

(III) the existence of a stabilizing solution of the Riccati equation,

(IV) the J-coercivity of the I/O map

(see, e.g., Theorem 9.1.7 for details). Moreover, we show that from the solution of the Riccati equation one can compute the optimal cost, feedback and factorization ("(III)↦(I),(II)"). Similarly, we also give formulae (I)↦(II),(III) as well as (II)↦(I),(III). Furthermore, the solution of (III) and the optimal state feedback operator are shown to be unique.

The standard coercivity assumption (IV) is necessary for (I)–(III). As noted in Section 8.4, (IV) is not sufficient in general, but for sufficiently regular systems we have the equivalence (I)–(IV), as stated in Corollary 9.1.11(iv) and Corollary 9.1.12(iv').

Condition (I) refers to a J-critical state feedback for the generalized optimal control problem of Section 8.3 (or Definition 9.1.3); see Chapters 10–12 for applications, such as LQR, H^2 and H^∞ control problems.

Thus, this is a summary of one aspect of the continuous-time algebraic Riccati equation (CARE) theory of this chapter, featuring the equivalence "(I)–(III)" on page 9 under various assumptions. The reader may wish to delay the verification of the proofs till the end of this chapter.

For strongly stable systems, the above requirements become simpler, as shown in Corollary 9.1.9, and the factorization condition (II) holds iff the Popov function has a regular spectral factorization.

Conditions (I)–(III) are simplified also in Proposition 9.1.15, assuming that the semigroup is smoothing, the input operator is bounded or the I/O map is smooth. The Riccati equation is compared to those existing in the literature in Remark 9.1.14. In Corollary 9.1.13 we solve the classical problem of finding a $(J,*)$-inner coprime factorization for a given I/O map by solving the Riccati equation corresponding to a realization of the map.

All of the above refers to the rather complicated case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, which shall be the subject of this section, but we first take a look at the simpler case of $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, where a solution of the CARE is J-critical iff it is exponentially stabilizing:

Theorem 9.1.1 *Let Σ be WR. Then the following are equivalent:*

(i) There is a J-critical WR state feedback operator K for Σ over $\mathcal{U}_{\text{exp}}$.

(ii) The eCARE (9.110) has an exponentially stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, if (ii) holds, then $\mathcal{P}$ is the J-critical cost operator, hence unique, K is a WR state feedback operator K for Σ, the J-critical control is given by the state feedback $u = K_{L,s}x$, and Theorem 9.9.1 applies. □

(This follows from Corollary 9.9.2; in the coercive case the eCARE is reduced to the CARE. However, both (i) and (ii) may be false even for a coercive system.)

Recall also from Lemma 8.3.3 that $\mathcal{U}_{\rm exp} = \mathcal{U}_{\rm out}$ when the system is exponentially stable or exponentially detectable. In Sections 9.9 and 9.2, we give analogous results further results for $\mathcal{U}_{\rm exp}$ and other $\mathcal{U}_*^*$'s, but for a general treatment of the equivalence of "(I)–(IV)", we use $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$. Therefore, we make the following assumption:

Standing Hypothesis 9.1.2 *Throughout the rest of this section, we assume that* $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$.

This makes the identification of the J-critical solution of the CARE more complicated than in the case $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ illustrated above. Instead of exponential stabilization, we should now check that the controlled (closed-loop) semigroup is output-stable and satisfies the condition (PB) in order to know that the control corresponding to a solution of the CARE truely optimizes over all $u \in \mathcal{U}_{\rm out}$, i.e., over all stable controls ($u \in \mathrm{L}^2$) that make the output stable ($y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$). (In case $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ it suffices to verify that the solution is exponentially stabilizing, as noted above.)

To avoid the verification of (PB), we (mostly) assume that the system is strongly stable or strongly right-coprime stabilizable. But before we go into this, we recall some definitions by simplifying special cases of Definitions 8.3.2, 6.6.10 and 9.8.1.

A state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ is called J-critical (i.e., optimal) if the resulting (closed-loop) contorl $u = \mathbb{K}_\circlearrowleft x_0$ is J-critical for each initial state $x_0 \in H$:

Definition 9.1.3 (Critical control) *Set* $\mathcal{U}_{\rm out}(0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{D}u \in \mathrm{L}^2\}$.

We call a control $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ J-critical *for* $x_0 \in H$ *(and Σ) if* $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, *and* $\langle \mathbb{D}\eta, Jy\rangle_{\mathrm{L}^2} = 0$ *for all* $\eta \in \mathcal{U}_{\rm out}(0)$.

Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ *be an admissible state feedback pair for Σ, and set*[1]

$$\mathbb{X} := I - \mathbb{F}, \qquad \mathbb{M} := \mathbb{X}^{-1} = \mathbb{F}_\circlearrowleft + I, \qquad \mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_\circlearrowleft; \tag{9.1}$$

$$\Sigma_{\rm ext} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F}\end{array}\right], \quad \Sigma_\circlearrowleft := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft\end{array}\right] := \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I\end{array}\right] \tag{9.2}$$

(so that $\Sigma_\circlearrowleft = (\Sigma_{\rm ext})_{[0\ I]}$ *is the corresponding closed-loop system; cf. Figure 9.1).*

We call $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ J-critical *for Σ if $u = \mathbb{K}_\circlearrowleft x_0$ is J-critical for each $x_0 \in H$ (and Σ). In this case, we call the equation $u = \mathbb{K}_\circlearrowleft x_0$ the* J-critical control in the feedback form.

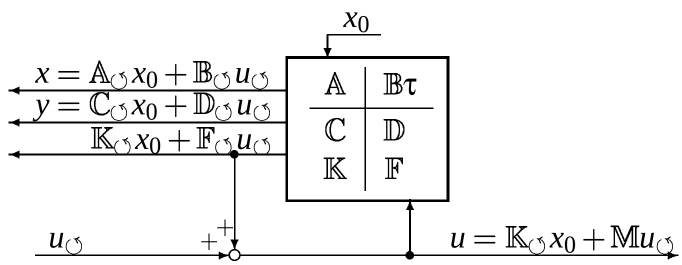

Figure 9.1: State feedback connection

(See Definition 8.3.2 for the general case.)

If $\mathbb{F}$ is weakly regular (WR), then feedback $\mathbb{K}x_0 + \mathbb{F}u$ can be written as $K_\mathrm{w} x(t) + Fu(t)$ for a.e. $t \geq 0$; we remind the reader that if the feedthrough is zero ($F = 0$), then K (or its weak Weiss extension K_w) is called a state feedback operator:

Definition 9.1.4 (J-critical K) *We call* $K \in \mathcal{B}(H_1, U)$ *a* (WR) admissible state feedback operator *if* $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ *generates a WR admissible state feedback pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *for* Σ. *We call* K stabilizing *(resp.* J-critical, r.c.-stabilizing, stable, UR, *...) if* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is stabilizing (resp. J-critical, r.c.-stabilizing, stable, UR, ...) (see Definition 6.6.10).*

(See Definition 8.3.2 for the general $\mathcal{U}_*^*$'s in place of $\mathcal{U}_\mathrm{out}$.) Thus, $K \in \mathcal{B}(H_1, U)$ is admissible iff $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a WR WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right]$ with $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$ (cf. Lemma 6.3.13), or equivalently, iff $H_B \subset \mathrm{Dom}(K_\mathrm{w})$ and the feedback $u = K_\mathrm{w} x + u_\circlearrowleft$ is admissible (see Proposition 6.2.8(a2), Theorem 6.2.13(a1) and Definition 6.6.10; for ULR K, admissibility is redundant, by Proposition 6.3.1(c)).

If this is the case, then $\left[\begin{smallmatrix} A & B \\ C & D \\ K & 0 \end{smallmatrix}\right]$ $\left[\left[\begin{smallmatrix} A+BK_\mathrm{w} & B \\ C+DK_\mathrm{w} & D \\ K_\mathrm{w} & 0 \end{smallmatrix}\right]\right]$ are [compatible] generating operators of Σ_ext $[\Sigma_\circlearrowleft]$ (see Proposition 6.6.18(d2) and recall that we denote the generating operators of maps by the corresponding capital letters, e.g., $M := \widehat{\mathbb{M}}(+\infty)$).

Now we can define the CAREs. As noted in Remark 9.1.6, their admissible solutions lead to WR state feedback operators, and such operators are J-critical iff they are $\mathcal{U}_\mathrm{out}$-stabilizing, or equivalently, q.r.c.-stabilizing, provided that Σ is strongly q.r.c.-stabilizable. These things will soon be clarified.

Definition 9.1.5 (CARE) *We call* $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ *(or* $(\mathcal{P}, S, K)$*) a* solution *(of the Continuous-time Algebraic Riccati Equation (CARE) induced by* Σ *and J) iff* Σ *is WR and* $\mathcal{P}$ *satisfies*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \textit{on } \mathrm{Dom}(A)\ , \\ S = D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_\mathrm{w}^*\mathcal{P}(s-A)^{-1}B & \textit{on } U\ , \\ K = -S^{-1}(B_\mathrm{w}^*\mathcal{P} + D^*JC) & \textit{on } \mathrm{Dom}(A)\ , \end{cases} \qquad (9.3)$$

[1]In this chapter, we usually denote the closed-loop system corresponding to $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ by $\Sigma_\circlearrowleft$ (instead of $\Sigma_\flat$), and leave $\Sigma_\flat$ for preliminarily closed systems (as in Theorem 9.1.10). In most applications, the second output of $\Sigma_\circlearrowleft$ equals the J-critical control (for $u_\circlearrowleft = 0$).

and $S = S^* \in \mathcal{GB}(U)$ (for some S and K).

A solution $\mathcal{P}$ of the CARE is called stabilizing *(resp. J-*critical, admissible, r.c.-stabilizing, stable, UR, ...*) if K is a stabilizing (resp. J-critical, admissible, r.c.-stabilizing, stable, UR, ...) state feedback operator for Σ.*

We call S the signature operator *corresponding to $\mathcal{P}$.*

(More general AREs will be formulated in Definition 9.8.1. See Proposition 9.1.15 for smooth cases where the w-lim term disappears, and recall that one often assumes that $D^*JD = I$ and $D^*JC = 0$ to obtain $S = I$, $K = -B_{\mathrm{w}}^*\mathcal{P}$.)

Note that, given $\mathcal{P}$, the operators S and K are uniquely determined and can be eliminated from the first equation. By Remark 9.1.6, we have $H_B \subset \mathrm{Dom}(K_{\mathrm{w}})$, hence any extended WPLS of Σ with generators $\left[\begin{array}{c|c} A & B \\ \hline C & \\ K & \end{array}\right]$ is WR, by Lemma 9.11.5(b).

Thus, a solution $\mathcal{P}$ is admissible iff the operators $\mathbb{K} : x_0 \mapsto K\mathbb{A}(\cdot)x_0$ and $\mathbb{F} := K_{\mathrm{w}}\mathbb{B}\tau$ extend Σ to another WPLS (see Lemma 6.3.13 and Remark 9.1.6) satisfying $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$ (if $\mathbb{F} \in \mathrm{ULR}$, then the latter condition is redundant, by Proposition 6.3.1(c)). In Remark 9.8.3 we give the necessary and sufficient conditions in details. However, in many applications K is bounded ($K \in \mathcal{B}(H,U)$), hence necessarily ULR and admissible, by Lemma 6.6.11 (see also Lemma 6.3.17).

In (9.3), as elsewhere, the weak limit of an operator function $s \mapsto \mathbb{T}(s) \in \mathcal{B}(U)$ means the map $u_0 \mapsto \text{w-lim}\,\mathbb{T}(s)u_0$ (see Lemma A.3.1(h)), whereas “lim” refers to limit in $\mathcal{B}(U)$, i.e., to a uniform limit. Note that we require this weak limit to exist (and be self-adjoint, equivalently, that $S = S^*$); such a weak limit is necessary bounded, by Lemma A.3.1(h3).

The equations of the CARE (9.3) are given in $\mathcal{B}(H_1, H_{-1}^*)$, $\mathcal{B}(U)$ and $\mathcal{B}(H_1, U)$, respectively; e.g., the first one is equivalent to

$$\langle Ax_0, \mathcal{P}x_1\rangle_H + \langle x_0, \mathcal{P}Ax_1\rangle_H = -\langle Cx_0, JCx_1\rangle_Y + \langle Kx_0, SKx_1\rangle_U \quad \text{for all} \quad x_0, x_1 \in \mathrm{Dom}(A) \tag{9.4}$$

(by Lemma A.3.1(g3), it is enough to verify this for $x_1 = x_0$).

Note that CAREs cover the Riccati equations presented earlier for WPLSs in [WW, Theorem 12.8], [S98b, Theorem 6.1(v)], [Mik97b] and [Mik98]. To get an example where $S \neq D^*JD$, apply Corollary 9.1.12 to $I = \tau(-1)^*I\tau(-1)$ (note that $\widehat{\tau(-1)}(+\infty) = 0$); see Example 9.8.15 for details.

The weak regularity of K (see Proposition 6.2.8(a2)) is inherent in the CARE:

Remark 9.1.6 (Implicit assumptions of the CARE) *The third equation of the CARE requires that $\mathcal{P}\,\mathrm{Dom}(A) \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and the second equation that $\mathcal{P}(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_{\mathrm{w}}^*)$.*

These hold iff $\mathcal{P}H_B \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$, equivalently, iff $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$.

It follows that $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U)$, $B_{\mathrm{w}}^*\mathcal{P}(\cdot - A)^{-1} \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(H,U))$, and $B_{\mathrm{w}}^*\mathcal{P}(\cdot - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$. This implies that $K_{\mathrm{w}} \in \mathcal{B}(H_B, U)$ and hence $K \in \mathcal{B}(H_1, U)$.

Note also that we have required S to be invertible (cf. Definition 9.8.1) and $\mathcal{P}$ and S to be self-adjoint.

If B is bounded, then any $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ is SR (see Definition 9.1.5) and $S = D^*JD = S^*$; in particular, the above implicit requirements mentioned in Theorem 9.1.6 are satisfied, see Proposition 9.1.15.

Proof: In order to have the term $B_w^*\mathcal{P}$ well-defined, we must require $\mathcal{P}\,\mathrm{Dom}(A) \subset \mathrm{Dom}(B_w^*)$, and for each $u_0 \in U$ there must exist $\alpha > \omega_A$ s.t. $\mathcal{P}(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_w^*)$. The latter implies that $\mathcal{P}(\alpha - A)^{-1}BU \subset \mathrm{Dom}(B_w^*)$ for all $\alpha \in \sigma(A)^c$, by the resolvent equation and the fact that $\mathcal{P}\,\mathrm{Dom}(A) \subset \mathrm{Dom}(B_w^*)$. Combining these two inclusions, we get $\mathcal{P}H_B \subset \mathrm{Dom}(B_w^*)$, hence $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_w^*))$, by Lemma A.3.6 and Proposition 6.2.8(b1). The converse implications are trivial.

Now $\widehat{\mathbb{H}} := B_w^*\mathcal{P}(\cdot - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$, because $\widehat{\mathbb{H}}(s) - \widehat{\mathbb{H}}(s_0) = (s_0 - s)B_w^*\mathcal{P}(s - A)^{-1}(s_0 - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$, by the resolvent equation, Lemma A.4.4(a).

Because $H_B \subset \mathrm{Dom}(C_w)$ and $H_B \subset \mathrm{Dom}(B_w^*\mathcal{P})$, we obtain $H_B \subset \mathrm{Dom}(K_w)$. This inclusion is continuous, by Lemma A.3.6, and $K_w \in \mathcal{B}(\mathrm{Dom}(K_w), U)$, by Lemma 6.2.8(b2). □

Thus, the CARE is well-defined only for $\mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(H_B, \mathrm{Dom}(B_w^*))$. This is not a restriction, because if a J-critical control can be given by a WR stabilizing state feedback operator K, then the *J-critical cost* $\mathcal{P} := \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ is in $\mathcal{B}(H) \cap \mathcal{B}(H_B, \mathrm{Dom}(B_w^*))$, and $\mathcal{P}$ satisfies the CARE, by Theorem 9.9.1(a1) and Proposition 9.8.10.

Conversely, any strongly stabilizing solution of the CARE is unique and corresponds to control $u = K_w x$ that is J-critical w.r.t. the closed-loop system; if we require the K to be strongly q.r.c.-stabilizing, then this control is J-critical w.r.t. the original system too. This is illustrated in the rest of this section; the remaining sections of this chapter study the (extended) CARE further.

Theorem 9.1.7 (r.c.f.⇔CARE) *Let Σ be WR and strongly q.r.c.-stabilizable. Then the following are equivalent:*

(i) There is a J-critical WR q.r.c.-stabilizing state feedback operator K for Σ.

(ii) The I/O-map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{X} := \mathbb{M}^{-1}$ WR and $X = I$.

(iii) The CARE (9.3) has a q.r.c.-stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, the following holds for the (possible) solutions of (i)–(iii):

(a1) The solutions of (i)–(iii) are unique and correspond to each other as follows: K is the state feedback operator "K" of the CARE, $\mathbb{X} = I - \mathbb{F}$ (and $\mathbb{N} = \mathbb{D}\mathbb{M}$), where $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is the pair generated by $\left[\begin{array}{c|c} K & 0 \end{array}\right]$; $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ are obtained from $\mathbb{N}$ and $\mathbb{M}$ as in Theorem 9.9.10(g).

(a2) The corresponding operators $\mathcal{P}$, $S = \mathbb{N}^* J \mathbb{N}$ and the pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ generated by $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ satisfy (Crit1)–(Crit4) Theorem 9.9.10; in particular, Theorem 9.9.10, Proposition 9.11.4 and Lemma 9.11.5 apply.

(a3) The J-critical SOS-stabilizing pairs for Σ are the ones generated by $\begin{bmatrix} EK \mid I-E \end{bmatrix}$ ($E \in \mathcal{GB}(U)$); in particular, they are WR and strongly q.r.c.-stabilizing.

(b) The operators K and $\mathcal{P}$ are strongly q.r.c.-stabilizing.

(c) The critical control for $x_0 \in H$ is given by $u_{\rm crit}(t) = \mathbb{K}_{\circlearrowleft} x_0 = K_{\rm w} x(t)$ (a.e.), and the closed-loop cost function $\mathcal{J}_{\circlearrowleft}(x_0, u_{\circlearrowleft})$ for $y = \mathbb{C}_{\circlearrowleft} x_0 + \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}$, $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+; U)$ is given by

$$\mathcal{J}_{\circlearrowleft}(x_0, u_{\circlearrowleft}) := \langle y, Jy \rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \langle x_0, \mathcal{P} x_0 \rangle_H + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle_{\mathrm{L}^2(\mathbf{R}_+;U)}. \tag{9.5}$$

(d1) If $\mathbb{X} \in \mathrm{SR}$, then $\mathbb{M} \in \mathrm{SR}$, $\mathbb{N} \in \mathrm{WR}$ and $u_{\rm crit}(t) = K_{\rm s} x(t)$ (a.e.).

(d2) Assume that $\mathbb{D} \in \mathrm{SR}$. Then we may replace "WR" by "SR" (resp. "UR") in (i) and (ii) if the weak limit in the CARE is replaced by a strong (resp. uniform) limit.

In the uniform case, the requirement $X = I$ can be removed if we allow multiple solutions in (ii) ($\mathbb{D} = (\mathbb{N}E)(\mathbb{M}E)^{-1}$ where $E \in \mathcal{GB}(U)$); cf. (a1). Note also the implications

$$\mathbb{D}, \mathbb{X} \in \mathrm{SR} \Rightarrow \mathbb{N}, \mathbb{M} \in \mathrm{SR}, \quad \mathbb{X} \in \mathrm{SR} \Leftrightarrow \mathbb{M} \in \mathrm{SR}; \tag{9.6}$$

$$\mathbb{D}, \mathbb{X} \in \mathrm{UR} \Rightarrow \mathbb{N}, \mathbb{M} \in \mathrm{UR}, \quad \mathbb{X} \in \mathrm{UR} \Leftrightarrow \mathbb{M} \in \mathrm{UR}. \tag{9.7}$$

(d3) We have $\widehat{\mathbb{X}}(s) = I - K_{\rm w}(s-A)^{-1}B$, and $\widehat{\mathbb{M}}(s) = I + K_{\rm w}(s - A - BK_{\rm w})^{-1}B$, and $\mathbb{K}_{\circlearrowleft} x_0 = K_{\rm w} \mathbb{A}_{\circlearrowleft}(\cdot) x_0$ a.e.

(e) If (i) has a solution [with $S \gg 0$], then $\mathbb{D}$ is [positively] J-coercive.

The condition $X = I$ (equivalently, $X \in \mathcal{GB}(U)$; cf. Lemma 6.4.5(e)) is not restrictive (at least) when $\dim U < \infty$ or $\mathbb{X} \in \mathrm{UR}$; cf. (d2), Lemma 9.9.7(d) and Corollary 9.1.11.

All theorems and corollaries of this section are special cases of Theorem 9.1.7 (with further details or simplifications). See Section 10.2 for the positive case.

Proof: The equivalence (i)⇔(ii)⇔(iii) is the one in Theorem 9.9.10(d1); in particular, (a2) holds.

(a1)-(a3),(c) These follow from (a1)–(b) and (e1)&(f1) of Theorem 9.9.10, respectively.

(b) By Theorem 6.7.15(a2), any q.r.c.-SOS-stabilizing K is strongly q.r.c.-stabilizing.

(d1) We have $\mathbb{M} \in \mathrm{SR}$, by Proposition 6.3.1(a3), and $\mathbb{N} \in \mathrm{WR}$, by Lemma 6.2.5.

Set $x = \mathbb{A}_{\circlearrowleft} x_0$, so that $u_{\rm crit} = \mathbb{K}_{\circlearrowleft} x_0 = (K_{\circlearrowleft})_{\rm s} x$ a.e., by (6.46), and $(K_{\circlearrowleft})_{\rm s} = K_{\rm s}$, by Lemma 6.6.18(d3). Therefore, $u_{\rm crit} = K_{\rm s} x$ a.e.

(The J-critical control corresponds to $x = \mathbb{A}_{\circlearrowleft} x_0 + \mathbb{B}_{\circlearrowleft} \tau u_{\circlearrowleft}$ with $u_{\circlearrowleft} = 0$ (see Definition 9.1.4), although the formula $u = K_{\rm s} x$ holds for arbitrary $u_{\circlearrowleft}$.)

(d2) The first claim follows from Lemma 9.11.5(e); the second from Proposition 6.3.1(b1) and Lemma 6.4.5(e) (the latter implies that the q.r.c.f. parametrization applies in WR and SR cases too if we replace "$X = I$" by "$X \in$

$\mathcal{GB}(U)$" in (ii)). The implications follow from (d1), the equation $\mathbb{N} = \mathbb{DM}$, and Lemma 6.2.5.

(d3) This follows from Proposition 6.6.18(d1).

(e) This follows from (Crit1) and Theorem 9.9.10(e2). □

From the above proof we note the following facts:

Remark 9.1.8 (Reduction of assumptions)

(a) We can remove the assumption in Theorem 9.1.7 that Σ is strongly q.r.c.-stabilizable if we add this condition as an additional requirement to condition (ii) and replace "q.r.c.-stabilizing" by "strongly q.r.c.-stabilizing" in (i) and (iii).

(b) Alternatively, we can use P-stabili[zabili]ty or P-SOS-stabili[zabili]ty (see Definition 9.8.1) instead of strong stabilizability (either in assumptions, as in the theorem, or in requirements, as in (a)) if we alter part (b) of the theorem accordingly.

Claims (a) and (b) applies also Theorem 9.1.10 and Corollary 9.1.11; claim (b) applies also Corollaries 9.1.9 and 9.1.12. □

On the other hand, Theorem 6.7.15 allows us often to reduce further the stabilization assumption; e.g., if Σ is exponentially q.r.c.-stabilizable, then any I/O-stabilizing (or input stabilizing) solution is exponentially q.r.c.-stabilizing. In the standard LQR problem for an exponentially q.r.c.-stabilizable or estimatable system, any nonnegative solution is a unique and minimizing, by Proposition 10.7.3(c3). See Sections 10.1 and 9.2 for further simplifications.

Theorem 9.9.1(a1) and Corollary 9.9.2 show the necessary and sufficient conditions for the existence of a J-critical state feedback pair in terms of solutions of Riccati equations, without additional stability or stabilizability assumptions.

If Σ is strongly stable, then an admissible solution $(\mathcal{P}, S, K)$ of the CARE is (strongly) r.c.-stabilizing if it is stable and stabilizing; in fact, it suffices that $\mathbb{K}$ is stable and $\mathbb{X} \in \mathcal{G}$TIC, by Proposition 9.8.11. Thus we get additional equivalent conditions in this case:

Corollary 9.1.9 (Stable Σ: SpF⇔CARE) *Let Σ be WR and strongly stable.*

Then the assumptions of Theorems 9.1.7 are satisfied, and each of the following conditions is equivalent to (i)–(iii) of Theorem 9.1.7

(i) There is a J-critical WR stable, stabilizing state feedback operator K for Σ.

(ii) The Popov operator $\mathbb{D}^* J \mathbb{D}$ has a WR spectral factorization $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$ with $X = I$.

(iii) The CARE (9.3) has a stable, stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, the solutions of (i)–(iii) solve the corresponding conditions of Theorem 9.1.7, and vice versa; in particular, (a)–(d) of that theorem apply, and "stable, stabilizing" is equivalent to "q.r.c.-stabilizing" (and to "stable and strongly r.c.-stabilizing") in (i) and (iii). Furthermore, *(a) If Σ is exponentially stable, then "stable, stabilizing" is equivalent to "exponentially stabilizing", and to "I/O-stabilizing", and to "$\mathbb{M}$-stabilizing".*

(b) If (iii) holds, then $\mathcal{P} = \mathbb{C}^*(J - J\mathbb{D}\pi_+(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1}\pi_+\mathbb{D}^*J)\mathbb{C}$.

Recall from Definition 6.4.4 that (ii) requires that $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ and $S \in \mathcal{GB}(U)$. See Theorem 10.6.3 for the positive case ($S \gg 0$, or equivalently, $\mathbb{D}^*J\mathbb{D} \gg 0$).

Proof: Because Σ is stable, now a stabilizing state feedback pair is stable iff it is q.r.c.-stabilizing, by Lemma 6.6.17(a). Therefore, (i) and (iii) are equivalent to those of Theorem 9.1.7, in particular, any solutions are stable and strongly stabilizing, by (b) of the theorem.

Finally, (ii) is equivalent to (ii) of Theorem 9.1.7, by Lemma 6.4.8(b) (use the r.c.f. $\mathbb{D} = \mathbb{D}I^{-1}$).

(a) This follows from Proposition 9.8.11.

(b) This follows from Theorem 9.9.10(g1). □

In Theorem 9.1.7 we assumed the existence of a preliminary strongly stabilizing feedback pair. Assuming that this pair is regular, we have three more equivalent conditions, namely conditions (i)–(iii) for the preliminarily stabilized system:

Theorem 9.1.10 (SR stabilized Σ: r.c.f.$\Leftrightarrow$CARE) *Let Σ be SR and have a SR strongly q.r.c.-stabilizing state feedback operator K' (i.e., pair $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ with $F' = 0$). Let $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ be the two top rows of the corresponding SR strongly stable closed-loop system. Let $J = J^* \in \mathcal{B}(Y)$.*

Then the following are equivalent:

(i) There is a J-critical SR q.r.c.-stabilizing state feedback operator K for Σ.

(ii) The I/O-map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{M}$ SR and $M = I$.

(iii) The CARE (9.3) has a q.r.c.-stabilizing solution $\mathcal{P} = \mathcal{P}^*$ admitting a strong limit in the CARE.

($\mathrm{i}_\flat$) The J-critical control for $\Sigma_\flat^1$ can be given by a SR stable, stabilizing state feedback operator $K_\natural$.

($\mathrm{ii}_\flat$) The Popov operator $\mathbb{D}_\flat^* J\mathbb{D}_\flat$ has a SR spectral factorization $\mathbb{D}_\flat^* J\mathbb{D}_\flat = \mathbb{X}_\natural^* S\mathbb{X}_\natural$ with $X_\natural = I$.

($\mathrm{iii}_\flat$) The CARE

$$\begin{cases} K_\natural^* S K_\natural = A_\flat^*\mathcal{P} + \mathcal{P}A_\flat + C_\flat^* J C_\flat, \\ S = D^*JD + \operatorname*{s-lim}_{s\to+\infty} B_w^*\mathcal{P}(s-A)^{-1}B \in \mathcal{GB}(U), \\ K_\natural = -S^{-1}(B_w^*\mathcal{P} + D^*JC_\flat). \end{cases} \tag{9.8}$$

(on $\mathrm{Dom}(A_\flat)$ for $\Sigma_\flat^1$ and J) has a stable, stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

(Note that the first and third equation of (9.8) are given on $\mathrm{Dom}(A_\flat)$*, and that* $A_\flat = A + BK'_s$ *and* $C_\flat = C + DK'_s$ *on* $H_B \supset \mathrm{Dom}(A_\flat)$*, by Proposition 6.6.18(d3).)*

If some (hence all) of (i)–(iii$_\flat$*) have a solution, then the following holds:*

(a) *The above assumptions and conditions (i)–(iii) are stronger than those of Theorem 9.1.7; in particular, (a1)–(d2) of that theorem apply for these solutions.*

In particular, the solutions are unique and correspond to each other as in that theorem. (The corresponding claim holds also for (i$_\flat$*)–(iii*$_\flat$*), by Corollary 9.1.9).*

(b) *The solutions of (i) and (i*$_\flat$*) are connected by* $K_s = K'_s + (K_\natural)_s$*, which holds on* $H_B = H_{B_\flat} = H_{B_\circlearrowleft}$

(The operators K_s *and* K'_s *are the strong Yosida extensions of* K *and* K' *w.r.t.* A*, and* $(K_\natural)_s$ *is the strong Yosida extension of* $K_\natural$ *w.r.t.* $A_\flat$*. However, if one extends the restriction (to* $\mathrm{Dom}(A)$*,* $\mathrm{Dom}(A_\flat)$ *or* $A_\circlearrowleft$*) of any of these three (extended) operators w.r.t. any of* A*,* $A_\flat$ *and* $A_\circlearrowleft$*, then the extension coincides with the original one on* H_B*.)*

The solutions of (ii) and (ii$_\flat$*) are connected by* $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$*,* $\mathbb{N} = \mathbb{D}\mathbb{M}$*, where* $\mathbb{M}' := (I - \mathbb{F}')^{-1}$*,* $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat$*. The solutions of (iii) and (iii*$_\flat$*) are equal.*

Recall from Corollary 9.1.9, that "stable, stabilizing" in (iii$_\flat$) is equivalent to "q.r.c.-stabilizing".

Proof: We shall again use the implication $\mathbb{X} \in \mathrm{SR}$ & $X \in \mathcal{GB} \Rightarrow \mathbb{X}^{-1} \in \mathrm{SR}$ (from Proposition 6.3.1(a3)) and the fact that SR is closed under compositions. E.g., we note that $\Sigma_\flat$ is SR.

Conditions (i)–(iii) are equivalent, by Theorem 9.1.7(d2); likewise, conditions (i$_\flat$)–(iii$_\flat$) are equivalent, by Corollary 9.1.9 and Theorem 9.1.7(d2). In particular, (a) holds. Equivalence (ii)⇔(ii$_\flat$) follows from Lemma 6.4.8(b) (by its proof, $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$).

(a) This was noted above.

(b) The formula $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$ was noted above. By Theorem 9.9.10(g2), the unique solution of (i$_\flat$) corresponds to the same $\mathcal{P}$ as that of (i), hence also the solutions of (iii) and (iii$_\flat$) are identical. The rest of the third paragraph follows from these. The first and second paragraphs follow from Proposition 6.6.18(a1)&(e)&(f). □

If, e.g., Σ is stable and $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, then we get further equivalent conditions:

Corollary 9.1.11 (MTIC Σ: r.c.f.⇔CARE) *Let Hypothesis 8.4.7 hold for* $\widetilde{\mathcal{A}}(U)$*. Let* Σ *be strongly q.r.c.-stabilizable in* $\widetilde{\mathcal{A}}$*, i.e., let it have a strongly stabilizing state feedback pair* $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ *with* $\mathbb{N}', \mathbb{M}' \in \widetilde{\mathcal{A}}$ *and q.r.c., where* $\mathbb{M}' := (I - \mathbb{F}')^{-1} = I + \mathbb{F}_\flat$*,* $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat$*.*

Then each of the following conditions if equivalent to (i)–(iii) of Theorem 9.1.7 as well as to (i)–(iii$_\flat$*) of Theorem 9.1.10:*

(ii') $\mathbb{D}$ *has a* $(J,*)$*-inner q.r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$*.*

(ii$'_\flat$) The Popov operator $\mathbb{D}_\flat^* J \mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\natural^* S \mathbb{X}_\natural$.

(iv) $\mathbb{D}$ is J-coercive.

(iv$_\flat$) $\mathbb{D}_\flat$ is J-coercive.

(iv$'_\flat$) The Popov Toeplitz operator $\pi_+ \mathbb{D}_\flat^* J \mathbb{D}_\flat \pi_+$ is invertible in $\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U))$.

Moreover, we have the following:

(a) The above equivalence holds even if the "SR" (resp. "s-lim") in any of (i)–(iii$_\flat$) of Theorem 9.1.10 are replaced by "WR" (resp. w-lim) or by "UR" (resp. lim).

(b) The assumptions of Theorems 9.1.7 and Theorem 9.1.10 are satisfied.

Assume that some (hence all) of (i)–(iv$_\flat$) holds. Then we have the following:

(c1) The solutions of (ii') (resp. (ii$'_\flat$)) are equal to the unique solution of (ii) (resp. (ii$_\flat$)) modulo a $\mathcal{GB}(U)$ operator.

Consequently, we necessarily have $\mathbb{D}, \mathbb{F}, \mathbb{X} \in$ ULR and $\mathbb{M}, \mathbb{N}, \mathbb{X}_\natural \in \widetilde{\mathcal{A}} \subset$ ULR.

(c2) Theorem 9.1.10 shows how the (unique) solutions of the other conditions relate to each other.

(d) If Hypothesis 8.4.8 holds for $\mathcal{A}(U)$, and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (J,S)-inner [q.]r.c.f., then $N^* J N = S$, i.e., $D^* J D = X^* S X$, where $\mathbb{X} := \mathbb{M}^{-1}$.

Moreover, any $(J,*)$-inner-right factorization of $\mathbb{D}$ is a $(J,*)$-inner [q.]r.c.f.

Proof: Obviously, (b) holds. By Corollary 8.4.14(b3), conditions (ii'), (ii$'_\flat$), (iv) and (iv$_\flat$) are equivalent to each other and to (ii)'s of both theorems, and the solutions are necessarily in $\widetilde{\mathcal{A}}$. By Lemma 8.4.11(a1), (iv$'_\flat$)$\Leftrightarrow$(iv$_\flat$). The whole equivalence follows from this.

(a) This follows, because the solutions are in $\widetilde{\mathcal{A}} \subset$ ULR $\subset$ UR.

(c1) This follows from Lemma 6.4.5(e) and Lemma 6.4.8(a).

(c2) This follows from the above.

(d) This follows from Corollary 8.4.14(b4) and Lemma 6.4.5(e) □

In the stable case, we get still more equivalent conditions:

Corollary 9.1.12 (Stable MTIC Σ: SpF⇔CARE) *Let Hypothesis 8.4.7 hold for $\widetilde{\mathcal{A}}(U)$. Let Σ be strongly stable, and let $\mathbb{D} \in \widetilde{\mathcal{A}}$.*

Then the assumptions of Corollaries 9.1.11 and 9.1.9 are satisfied, and we have two more equivalent conditions:

(ii") $\mathbb{D}^* J \mathbb{D}$ has a spectral factorization $\mathbb{X}^* S \mathbb{X}$.

(iv') The Popov Toeplitz operator $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+$ is invertible in $\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U))$.

(a) Assume (ii'). Then $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}$. If, in addition, Hypothesis 8.4.8 holds for $\mathcal{A}(U)$, then $D^* J D = X^* S X$.

(b) *The solutions of (ii") correspond to those of (ii') through* $\mathbb{M} = \mathbb{X}^{-1}$, $\mathbb{N} = \mathbb{D}\mathbb{M}$, $S = S$.

See Example 9.8.15 for an example with $X^*SX \neq D^*JD$.

Proof: Take $\left[\ \mathbb{K}' \mid \mathbb{F}'\ \right] = \left[\ 0 \mid 0\ \right]$ in order to have $\Sigma_\flat^1 = \Sigma$ in Theorem 9.1.10. □

In the classical theory, one is often given just an I/O map and uses some (e.g., minimal) realization to get a Riccati equation for solving the problem.

This can be done in the infinite-dimensional case too: if $\mathbb{D} \in \text{TIC}$ (resp. $\mathbb{D} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is as above), we can choose any strongly stable realization of $\mathbb{D}$ (e.g., the strongly stable exactly observable realization (6.11)) and apply Corollary 9.1.9 (resp. Corollary 9.1.12); if, instead, $\mathbb{D} \in \text{TIC}_\infty \setminus \text{TIC}$, we can proceed as follows:

Corollary 9.1.13 (I/O-result) *Let* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$ *have a q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$, *and let* $J = J^* \in \mathcal{B}(Y)$. *Take a strongly stable realization* $\Sigma_\flat$ *of* $\begin{bmatrix} \mathbb{N}' \\ \mathbb{M}'-I \end{bmatrix}$ *(e.g., the one of Lemma 6.6.29), and close it with the output feedback* $L = \begin{bmatrix} 0 & -I \end{bmatrix}$ *to obtain another system*

$$\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K}' & I - \mathbb{M}'^{-1} \end{array}\right] := (\Sigma_\flat)_L \in \text{WPLS}; \tag{9.9}$$

Then $\left[\ \mathbb{K}' \mid I - \mathbb{M}'^{-1}\ \right]$ *is strongly q.r.c.-stabilizing for* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mid & \mathbb{B} \\ \mathbb{C} & \mid & \mathbb{D} \end{smallmatrix}\right]$.

Therefore, we can apply Theorems 9.1.7 and 9.1.10 (and Corollary 9.1.11 if $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}$*) for the realization* Σ *of* $\mathbb{D}$ *in order to find a* $(J,*)$*-inner q.r.c.f. for* $\mathbb{D}$. □

(This is obvious. Here $\left[\begin{smallmatrix} \mathbb{A} & \mid & \mathbb{B} \\ \mathbb{C} & \mid & \mathbb{D} \end{smallmatrix}\right]$ and Σ refer to components of 9.9.)

We now take a look at cases where the CARE becomes simpler.

Remark 9.1.14 *The CARE takes the form*

$$A^*\mathcal{P} + \mathcal{P}A + C_1^*QC_1 = (B_{\rm w}^*\mathcal{P} + NC_1)^*(X^*X)^{-1}(B_{\rm w}^*\mathcal{P} + NC_1), \tag{9.10}$$

of M. Weiss and G. Weiss [WW, Theorem 12.8], if we make (some of) the assumptions of Section 2 of [WW], namely that $Y := Y_1 \times U$, $J := \left[\begin{smallmatrix} Q & N^* \\ N & R \end{smallmatrix}\right]$ $\mathbb{D} := \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$, $\mathbb{C} := \left[\begin{smallmatrix} \mathbb{C}_1 \\ 0 \end{smallmatrix}\right]$, *where* Y_1 *is a Hilbert space,* $\mathbb{D}_1 \in \text{TIC}(U,Y_1)$ *is the unique* TIC*-extension of "*$\mathbb{F}$*",* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] \in \mathcal{B}(Y_1 \times U)$, *and* $\mathbb{X}$ *and* S *are replaced by* $S^{1/2}\mathbb{X}$ *and* I*, respectively. In this case, the cost function takes the form*

$$\mathcal{J}(x_0,u) = \int_0^\infty \left\langle \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix} \begin{bmatrix} y_1(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} y_1(t) \\ u(t) \end{bmatrix} \right\rangle dt. \tag{9.11}$$

as in [WW, equation (2.8)]; here $\left[\begin{smallmatrix} y_1 \\ u \end{smallmatrix}\right] = \mathbb{C}x_0 + \mathbb{D}u$. □

When C is bounded (e.g., for finite-dimensional H), one often writes the above cost function in form $\mathcal{J} = \langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], J' \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \rangle$, where $J' = \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix}$, $\mathbb{C} = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ (note that here $y = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$).

In several special cases, the CARE can be simplified:

Proposition 9.1.15 (Special classes of systems) *If B is bounded (this includes the Pritchard–Salamon class), then each solution of the CARE is ULR, and we can formulate (iii) (the CARE) as follows:*

(iii') $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $D^*JD\in\mathcal{GB}$, *and* $K:=(D^*JD)^{-1}(B^*\mathcal{P}+D^*JC)$ *is q.r.c.-stabilizing.*

If $\mathbb{D}\in\mathrm{TIC}(U,Y)$ *and Hypothesis 9.2.2 is satisfied, then the can formulate (iii) (the CARE) as follows:*

(iii") $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $\mathcal{P}[H]\subset\mathrm{Dom}(B_{\mathrm{w}}^*)$, $D^*JD\in\mathcal{GB}$, *and* $K:=(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)$ *is q.r.c.-stabilizing.*

If Σ *is strongly stable and* $\mathbb{D}\in\widetilde{\mathcal{A}}(U,Y)$ *(resp.* Σ *is strongly q.r.c.-stabilizable in* $\widetilde{\mathcal{A}}$*), where* $\widetilde{\mathcal{A}}$ *satisfies Hypothesis 8.4.8 (e.g.,* $\widetilde{\mathcal{A}}=\mathrm{MTIC}^{\mathrm{L}^1}$*), then the can formulate (iii) (the CARE) in Corollary 9.1.12 (resp. in Corollary 9.1.11) as follows:*

(iii"') $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $D^*JD\in\mathcal{GB}$, $\text{w-lim}_{s\to+\infty}B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B=0$, *and* $K:=(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)$ *is q.r.c.-stabilizing.*

Moreover, in each of these cases, Σ_{ext}, $\Sigma_{\mathrm{ext}}^{\mathrm{d}}$, $\Sigma_{\circlearrowleft}$, $\Sigma_{\circlearrowleft}^{\mathrm{d}}$ *become ULR and we may use* B_{s}^* *and* s-lim *instead of* B_{w}^* *and* w-lim.

(In (iii') and (iii"), the operators K is necessarily admissible and ULR even if it were not q.r.c.-stabilizing.)

In the standard LQR (minimization) problem, we have $J=I$, hence then for these special classes the CARE (9.11) takes the familiar form

$$A^*\mathcal{P}+\mathcal{P}A+C_1^*C_1=(B_{\mathrm{w}}^*\mathcal{P})^*B_{\mathrm{w}}^*\mathcal{P} \tag{9.12}$$

and the corresponding (J-critical) control is given by $u(t)=K_{\mathrm{w}}x(t)=-B_{\mathrm{w}}^*\mathcal{P}x(t)$ a.e.; when B is bounded, then so is K, the CARE becomes $A^*\mathcal{P}+\mathcal{P}A+C_1^*C_1=\mathcal{P}BB^*\mathcal{P}$, and $u=-B^*\mathcal{P}x$, as in the finite-dimensional case. See also Theorem 9.2.14 and Corollary 9.2.15 for variants of (iii"), and also Corollary 10.2.3 and Theorem 9.9.6 for bounded B.

Proof: (As noted in Remark 6.9.3, a PS-system has a bounded B w.r.t. the larger of the two state spaces.)

1° $\widetilde{\mathcal{A}}$*: (iii"')⇔(iii):* Obviously, (iii"') holds iff (iii) holds and $S=D^*JD$. But if (iii) holds, then $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$ (resp. $\mathbb{D}_\flat^*J\mathbb{D}_\flat=\mathbb{X}_\natural^*S\mathbb{X}_\natural$) is a spectral factorization in $\widetilde{\mathcal{A}}$, by the corollary, hence then $D^*JD=I^*SI=S$, by the hypothesis.

2° *Assuming Hypothesis 9.2.2:* Use Theorem 9.2.9.

3° *Bounded B:* This follows from 2° and the fact that $B_{\mathrm{w}}^*=B$ with $\mathrm{Dom}(B_{\mathrm{w}}^*)=H$. □

Notes

Most of this section was contained in [Mik97b], partially also in [Mik98]. See the notes on p. 502 for earlier partial results for WPLSs (mainly special cases of implications "(ii)⇒(i)⇒(iii)"). Many analogous results are well-known for finite-dimensional systems (see [IOW]) and for Pritchard–Salamon systems (see [Weiss97]), particularly for exponentially stable systems (so that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$). Note that "Wiener–Hopf factorizations" and (J,S)-inner-outer factorizations are equivalent to spectral factorizations in the stable case, by Lemma 6.4.8.

We have defined the CAREs only for WR systems with WR optimal state feedback pairs. See Section 9.7 for "CAREs" (on $\text{Dom}(A_{\text{crit}})$ instead of $\text{Dom}(A)$) for arbitrary WR systems, and Remark 9.12.1 and Definition 9.8.4 (the IARE) for arbitrary WPLSs. The corresponding equivalences (for IAREs) for arbitrary WPLSs are given in Theorem 9.9.1 and Corollary 9.9.11.

In the finite-dimensional case, the theory of algebraic Riccati equations and inequalities is very mature, and an excellent reference on the theory is [LR], by Peter Lancaster and Leiba Rodman. Several aspects of the finite-dimensional theory still await generalizations.

9.2 Riccati equations when $\mathbb{A}Bu_0 \in \mathrm{L}^1$

Everything should be made as simple as possible, but not simpler.
— Albert Einstein (1879–1955)

In this section, we shall establish a simplified CARE theory for systems of the form studied in Section 6.8 (see Hypothesis 9.2.2). For them, a unique J-critical control is always of uniformly line-regular (ULR) state feedback form, i.e., it corresponds to a ULR $\mathcal{U}_*^*$-stabilizing solution of the CARE (we assume that $D^*JD \in \mathcal{GB}(U)$). Moreover, we can remove the limit term from the CARE (and hence $S = D^*JD$). We may, instead, require that $B_\mathrm{w}^*\mathcal{P} \in \mathcal{B}(U,H)$ (this is not the case for general CAREs); see Definition 9.2.6 for details. As a result, for this class we can and will formulate most results in this book to look like their finite-dimensional counterparts.

Main results of this section include Theorems 9.2.9–9.2.18 and 9.2.3. Several minimization results for these systems are given in Chapter 10 and H^∞ results in Chapters 11–12.

In practical applications of the theory of this section, one uses conditions such as those in Hypothesis 9.2.2, but to make room for future extensions of the theory, we often use the following, weaker and more abstract hypothesis in our results:

Hypothesis 9.2.1 (Σ is smooth) *The system $\Sigma \in \mathrm{WPLS}(U,H,Y)$ is ULR, $J = J^* \in \mathcal{B}(Y)$, and if there is a J-critical control for Σ over $\mathcal{U}_*^*$ in WPLS form, then $\mathcal{P}[H] \subset \mathrm{Dom}(B_\mathrm{w}^*)$.*

(Here $\mathcal{P}$ is the J-critical cost operator, it is defined in Theorem 8.3.9(b1). If we say that "$\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ satisfies Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_\mathrm{exp}$", we naturally mean that Hypothesis 9.2.1 holds with $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ replaced by $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ (also in the definition of $\mathcal{U}_\mathrm{exp}$) and B_w^* replaced by $(B_\flat^*)_\mathrm{w}$. Naturally, the last assumption above can be read as "either $\mathcal{P}[H] \subset \mathrm{Dom}(B_\mathrm{w}^*)$ or there is no J-critical control for Σ over $\mathcal{U}_*^*$ in WPLS form".)

Not even all stable ULR systems satisfy Hypothesis 9.2.1, by the counter-example given in Example 9.8.15 (with $\mathbb{D} = \tau^{-1} \in \mathrm{MTIC}$). However, Hypothesis 9.2.1 is satisfied in the following cases (and others):

Hypothesis 9.2.2 *At least one of (1.)–(7.) holds, where*

(1.) B is bounded (i.e., $B \in \mathcal{B}(U,H)$);

(2.) **(Analytic $\mathbb{A}$)** *Hypotheses 9.5.1 and 9.5.7 hold;*

(3.) $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$, $C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$;

(4.) $\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$, $C \in \mathcal{B}(H,Y)$, $D^*JD \in \mathcal{GB}(U)$;

(5.) $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ and $C_\mathrm{w}\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];Y)$ for all $u_0 \in U$;

(6.) **(Stable case)** *$C \in \mathcal{B}(H,Y)$, $D^*JC = 0$, $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^1(\mathbf{R}_+;Y))*$ and the assumptions in (b2) or (b2') of Theorem 8.3.9 hold;*

(7.) **(Stable case)** $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^2(\mathbf{R}_+;Y))*$ *and the assumptions in (b2) or (b2') of Theorem 8.3.9 hold.*

As in Section 6.8, by "$\mathbb{A}B \in \mathrm{L}^1([0,1],*)$" we mean that $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1],*)$, etc. See Lemma 6.8.1–6.8.3 for equivalent conditions for (1.)–(7.) (e.g., (5.) holds iff $(\widehat{\mathbb{D}} - D)u_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega;Y)$ and $(\cdot - A)^{-1}Bu_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega;H)$ for all $u_0 \in U$ and some (hence all) $\omega > \omega_A$, by Lemma 6.8.1(a)&(d1)). Recall from Theorem F.2.1(g), that $\mathrm{L}^p_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y)) \subset \mathcal{B}(U,\mathrm{L}^p(\mathbf{R}_+;Y))$.

Concerning (6.) and (7.), we note that for $\mathcal{U}^*_* = \mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$, $\mathcal{U}_{\text{exp}}$), the assumptions in Theorem 8.3.9(b2) hold if $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{C}\end{smallmatrix}\right]$ is output stable (resp. stable, strongly stable, exponentially stable). The convolutions in (6.) and (7.) are explained in Proposition 6.3.4(a3) and Lemma F.2.2(d1)–(d3).

Theorem 9.2.3 *If Hypothesis 9.2.2 holds, then Hypothesis 9.2.1 holds (even with* $\mathrm{Dom}(B^*_{\mathrm{L,w}})$ *in place of* $\mathrm{Dom}(B^*_{\mathrm{w}})$*).*

Note that most cases of Hypothesis 9.2.2 are independent of $\mathcal{U}^*_*$, whereas Hypothesis 9.2.1 depends on $\mathcal{U}^*_*$.

Moreover, we have $\mathcal{P} \in \mathcal{B}(U,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ in cases (1.), (2.) and (4.); this also holds in (3.)–(7.) whenever the conditions of Lemma 9.2.8(c1)&(c2) are satisfied, by Theorem 9.2.9 (and Lemma 9.6.2).

Proof of Theorem 9.2.3: This follows from Lemmas 9.3.2 and 9.3.4. □

All assumptions (1.)–(7.) guarantee certain regularity of $\mathbb{D}$. Roughly, (1.)–(5.) also require that B is bounded or $\mathbb{A}$ is smoothing (any of them implies that $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ for all $u_0 \in U$), and (6.) and (7.) require that Σ is somewhat stable. Indeed, under any of (1.)–(5.), we shall use (9.56) to obtain that $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$; under any of (6.)–(7.), we shall use (8.36) to obtain that $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$; from these formulae one observes why it is hard to weaken the above assumptions without giving up $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$ (and hence B^*_{w}-CARE's). See Lemma 9.3.4 for details.

Any of (2.)–(7.) allows B to be highly unbounded when $\mathbb{A}$ is highly smoothing (e.g., analytic), but (3.)–(5.) require B to be bounded when $\mathbb{A}$ is nonsmoothing (e.g., invertible).

An alternative approach is to give up "$\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$ and assume that $\mathbb{D} \in \mathrm{MTIC}$ (or that Σ is exponentially or q.r.c.-stabilizable in MTIC), and require J-coercivity to obtain a factorization of the Popov operator. See Section 9.1 and Remark 9.9.14 for details.

In the stable case with $\dim Y < \infty$, a bounded C is enough for Hypothesis 9.2.2:

Proposition 9.2.4 *If C is bounded,* $\dim Y < \infty$ *and $\mathbb{B}$ is stable, then* $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))$*; in particular, then* $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^2(\mathbf{R}_+;Y))*$ *(cf. (7.) above).*

If C is bounded, $\dim Y < \infty$, $\mathcal{U}^*_* = \mathcal{U}_{\text{exp}}$ (resp. $\mathcal{U}^*_* = \mathcal{U}_{\text{out}}$) and there is an exponentially (resp. q.r.c.-)stabilizing bounded state feedback operator $K \in$

$\mathcal{B}(H,U)$ for Σ, then we can reduce the problem to the stable case, which can be solved by Theorem 9.2.9, the above proposition, and (7.) of the hypothesis; indeed, the solution corresponds to the original one through the formulae of Proposition 6.6.18(f), by Theorem 8.4.5.

Naturally, if $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \in \mathcal{B}(U, Y_1 \times Y_2)$, then it suffices that $\dim Y_1 < \infty$, since then $\mathbb{D}_2 = D_2 \in \mathcal{B}(U, Y_2)$.

Proof: By Theorem 6.2.11(c2), we have $B^*(\cdot - A^*)^{-1}C^*y_0 \in \mathrm{H}^2$ for all $y_0 \in Y$, hence $\widehat{\mathbb{D}}^*(\bar{\cdot}) - D \in \mathrm{H}^2$, hence $\widehat{\mathbb{D}} - D \in \mathrm{H}^2$, hence $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^2(\mathbf{R}_+; Y))$, by Lemma F.3.4(d). □

The standing assumption that $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ is often implied by the other assumptions:

Remark 9.2.5 (Sufficient conditions for (3.) or (5.)) *Drop, for a moment, the standing assumption that* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$. *Assume, instead, that $\mathbb{A}$ is a C_0-semigroup on H, $B \in \mathcal{B}(U, H_{-1})$, $C \in \mathcal{B}(H_1, Y)$ and $D \in \mathcal{B}(U,Y)$ (recall that $H_1 :=$ Dom(A) with graph norm, and that $H_{-1} :=$ Dom$(A^*)^*$ w.r.t. the pivot space H).*

(a) If $B \in \mathcal{B}(U,H)$, and $C_{\mathrm{w}}\mathbb{A}x_0 \in \mathrm{L}^2([0,1);Y)$ for each $x_0 \in H$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (1.) and (5.) are satisfied.

(b) If $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B})$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (5.) is satisfied.

(c) If $B^*\mathbb{A}^* \in \mathrm{L}^2([0,1);\mathcal{B}(H,U))$, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ for all $u_0 \in U$, $C \in \mathcal{B}(H,Y)$, and $D^*JC = 0$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (3.) is satisfied.

(d) We have $C(\cdot - A)^{-1}, (\cdot - A)^{-1}B, \widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong},\infty}$ iff (5.) holds.

Analogously, $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U, \mathrm{L}^2(\mathbf{R}_+;Y))*$ iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U,Y))$, by Lemma F.3.4(d).

Proof: Claims (a)–(c) follow easily from, e.g., Definition 6.1.1, Lemma 6.8.1, Lemma 6.3.16(b)&(c) and duality.

By Lemma 6.8.1(a)&(d1)&(e1)&(e2), claim (d) holds (and we may use $C_{\mathrm{L,s}}$ in place of C_{w} everywhere in the proposition). □

Under Hypothesis 9.2.1 (with $D^*JD \in \mathcal{GB}(U)$), one can replace the CARE by the following simplified form, as shown in Theorem 9.2.9:

Definition 9.2.6 (B_{w}^*-CARE) *Let $\mathbb{D}$ be WR. An operator $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ is called a* solution of the B_{w}^*-CARE *if $D^*JD \in \mathcal{GB}(U)$ and $\mathcal{P}$ satisfies*

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \tag{9.13}$$

We call a solution $\mathcal{P}$ stabilizing (resp. ULR, ...) *if $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$ is stabilizing (resp. ULR, ...), and we call $\mathcal{P}$ $\mathcal{U}_{\mathrm{exp}}$*-stabilizing *iff K is exponentially stabilizing (see Definition 9.8.1 for other $\mathcal{U}_*^*$, $\Sigma_{\circlearrowleft}$, $\left[\begin{smallmatrix} \mathbb{K} & \mathbb{F} \end{smallmatrix}\right]$, $\mathbb{X}$, $\mathbb{N}$ and $\mathbb{M}$).*

We often call $(\mathcal{P},S,K)$ a *solution of the* B_w^**-CARE*, where $K := -(D^*JD)^{-1}(B_\mathrm{w}^*\mathcal{P}+D^*JC)$, $S := D^*JD$. In most standard forms of LQR and H^∞ problems, we have $D^*JC = 0$ and hence then (9.13) reduces even further and $K = -(D^*JD)^{-1}B_\mathrm{w}^*\mathcal{P}$.

The equation (9.13) is given in $\in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A)^*)$, i.e., it holds iff

$$\langle Kx_0,(D^*JD)Kx_1\rangle_U = \langle Ax_0,\mathcal{P}x_1\rangle_H + \langle \mathcal{P}x_0,A\mathcal{P}x_1\rangle_H + \langle Cx_0,JCx_1\rangle_Y \quad \text{for all } x_0,x_1 \in \mathrm{Dom}(A) \tag{9.14}$$

(equivalently, whenever $x_0 \in \mathrm{Dom}(A)$, $x_1 = x_0$, by Lemma A.3.1(g3)).

Note that the condition "$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H,\mathrm{Dom}(B_\mathrm{w}^*))$" may be replaced by "$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ & $\mathcal{P}[H] \subset \mathrm{Dom}(B_\mathrm{w}^*)$" (by Lemma A.3.6). Therefore, $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B_\mathrm{w}^*))$ iff $\mathcal{P} \in \mathcal{B}(H)$ and $\langle r(r-A)^{-1}Bu_0,\mathcal{P}x_0\rangle_H$ converges as $r \to +\infty$, for all $u_0 \in U$ and $x_0 \in H$. In most cases we may replace B_w^* by $B_\mathrm{L,w}^*$, often even by $B_\mathrm{L,s}^*$, as noted in Lemma 9.2.8(c).

We first note that the solutions of the B_w^*-CARE are admissible and ULR solutions of the CARE:

Proposition 9.2.7 (B_w^*-CARE⇒CARE&IARE) *Assume that* $\mathbb{D}$ *is* ULR *and that* $D^*JD \in \mathcal{GB}(U)$.

(a) An operator $\mathcal{P} = \mathcal{P}^*$ *is a solution of the* B_w^**-CARE iff it is a solution of the CARE and* $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B_\mathrm{w}^*))$.

(b) Any solution of the B_w^**-CARE is admissible and ULR, hence a solution of the IARE (with* $S := D^*JD$*).*

(c) If the B_w^**-CARE has a* $\mathcal{U}_*^*$*-stabilizing solution, then Hypothesis 9.2.1 holds.*

Note that the CARE only requires that $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H_B,\mathrm{Dom}(B_\mathrm{w}^*))$. Obviously, a solution of the CARE cannot be a solution of the B_w^*-CARE when $S \neq D^*JD$, and this can happen even if C is bounded, $D^*JC = 0$ and $\mathbb{D},\mathbb{F} \in$ ULR, even for the $\mathcal{U}_\mathrm{out}$-stabilizing solution as shown in Example 9.13.8.

Fortunately, under the assumptions of Hypothesis 9.2.2, the "optimizing (i.e., $\mathcal{U}_*^*$-stabilizing, or equivalently, J-critical) solution of the CARE is always a solution of the B_w^*-CARE too.

Proof of Proposition 9.2.7: (a) Either assumption implies that $B_\mathrm{w}^*\mathcal{P} \in \mathcal{B}(H,U)$, by Lemma A.3.6. It follows that $S = D^*JD$ in the CARE, by Proposition 9.11.4(b3). The rest of the equivalence follows directly from the definitions. (Claim (a) holds whenever $\mathbb{D}$ is WR and $D^*JD \in \mathcal{GB}(U)$.)

(b) By Lemma 6.3.17, K is an ULR admissible state feedback operator. By Proposition 9.8.10, any WR solution of the CARE is a solution of the IARE.

(c) By (b) and Theorem 9.9.1, a $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$ of the B_w^*-CARE is the J-critical cost operator; by assumption, $\mathcal{P}[H] \subset \mathrm{Dom}(B_\mathrm{w}^*)$. □

We usually have additional regularity for $\mathcal{P}$ and $\Sigma_\circlearrowleft$. Some of this is given in the lemma below, more can be observed from the proofs and Lemmas 6.8.1–6.8.4.

Lemma 9.2.8 *Let $\mathbb{D}$ be* ULR *and $D^*JD \in \mathcal{GB}(U)$, and let $\mathcal{P}$ be a solution of the $B^*_{\rm w}$-CARE. Then*

(a) If $C \in \mathcal{B}(H,Y)$ or $D^*JC = 0$, then $K \in \mathcal{B}(H,U)$.

(b1) If (5.) of Hypothesis 9.2.2 holds, then $\widehat{\mathbb{F}} := K_{\rm w}(s-A)^{-1}B \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for any $\omega > \omega_{\mathbb{A}}$, and $(s-A_{\circlearrowleft})^{-1}B, \widehat{\mathbb{D}_{\circlearrowleft}} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_\omega;*)$ for any $\omega > \omega_{\mathbb{A}_{\circlearrowleft}}$.

(b2) If $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ and $\dim U < \infty$, then $\widehat{\mathbb{F}} := K_{\rm w}(s-A)^{-1}B \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U))$ and $\widehat{\mathbb{D}_{\circlearrowleft}} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2(\mathbf{C}^+_\alpha;*)$, for $\alpha \ge \omega$ s.t. $\alpha > \max\{\omega_A, \omega_{A_{\circlearrowleft}}\}$.

(c1) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$ and $C \in \mathcal{B}(H,Y)$, or $C_{\rm L,w}\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$ and $\mathbb{A}B \in \mathcal{B}(U,H)$ a.e., then $\mathcal{P} \in \mathcal{B}(H;\mathrm{Dom}(B^*_{\rm L,s}))$.

(c2) If $\mathcal{P}$ is $\mathcal{U}^*_*$-stabilizing and either (6.) holds and $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}(U,Y)$, or (7.) holds and $\mathbb{D} - D \in \mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(U,Y))*$, then $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\rm L,s}))$.

Proof: (a) This follows from the fact that $B^*_{\rm w}\mathcal{P} \in \mathcal{B}(H,U)$.

(b1) By Lemma 6.3.17, we have

$$\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix} = \begin{bmatrix} K_{\rm w}\mathbb{A} \mid -S^{-1}(B^*_{\rm w}\mathcal{P}\mathbb{B}\tau + D^*J\mathbb{D}) \end{bmatrix}. \tag{9.15}$$

Because $C(s-A)^{-1}B, (s-A)^{-1}B \in \mathrm{H}^2_{\rm strong,\infty}$ and $B^*_{\rm w}\mathcal{P}$ is bounded, we have $K_{\rm w}(s-A)^{-1}B \in \mathrm{H}^2_{\rm strong,\infty}$, hence in $\mathrm{H}^2_{\rm strong,\omega}$ for any $\omega > \omega_A$, by Lemma 6.8.1(d1).

By Proposition 6.3.3(b1), we have $\widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\rm strong,\infty}$, where $\mathbb{M} := (I-\mathbb{F})^{-1}$. Consequently, $\widehat{\mathbb{B}_{\circlearrowleft}\tau} = \widehat{\mathbb{B}\tau}\widehat{\mathbb{M}} \in \mathrm{H}^2_{\rm strong,\infty}$ and $\widehat{\mathbb{D}_{\circlearrowleft}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\rm strong,\infty}$, by Proposition 6.3.3(c). The rest follows from this and Lemma 6.8.1(a)&(d1).

(b2) (Note that $\mathrm{H}^2_{\rm strong} = \mathrm{H}^2$, because $\dim U < \infty$. Note also that (b2) corresponds to assumption (7.).) Apply Lemma 6.3.16(b) to $\left[\begin{array}{c|c} A^* & T^* \\ \hline B^* & 0 \end{array}\right]$, where $T := -S^{-1}B^*_{\rm w}\mathcal{P} \in \mathcal{B}(H,U)$, to obtain (since $\dim U < \infty$ implies that weak=uniform) that

$$T\mathbb{B}\tau = -S^{-1}B^*_{\rm w}\mathcal{P}\mathbb{B}\tau \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U)), \tag{9.16}$$

because $\mathbb{B}$ is α-stable. Now the claim on $\widehat{\mathbb{F}}$ follows from (9.15).

Because $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\alpha(U)$, we have $\widehat{\mathbb{M}} - I \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U))$, by Proposition 6.3.3(b1). Because $\widehat{\mathbb{D}} - D \in (\mathrm{H}^2 \cap \mathrm{H}^\infty)(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$, we have $\widehat{\mathbb{D}}\widehat{\mathbb{M}} - D \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$.

(c1) This follows as in the proof of Lemma 9.3.4 (see its 2° and 4°) with $\Sigma_{\circlearrowleft}$ in place of $\Sigma_{\rm crit}$ except that in 6° we need to use the fact that $\mathcal{P}$ is also a solution of the $B^*_{\rm w}$-CARE for $\Sigma_\flat$, by Lemma 9.12.3 and Proposition 9.2.7(a). We need to use equation (9.156), and it follows from Lemma 9.10.1(b4) and Proposition 9.2.7(b).

(c2) This was shown in 5° of the proof of Lemma 9.3.4 (combined with, e.g., Theorem 9.2.9(i)&(iii)). $\square$

Now we are ready for the main result, the equivalence between the existence of a stabilizing solutions of the B_w^*-CARE and the existence of a unique optimal control:

Theorem 9.2.9 (B_w^*-CARE $\Leftrightarrow$ J-critical) *Assume that $S := D^*JD \in \mathcal{GB}(U)$ and that Hypothesis 9.2.1 holds. Then the following are equivalent:*

(i) there is a unique J-critical control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;

(ii) there is a J-critical state feedback pair over $\mathcal{U}_*^*$;

(iii) the B_w^*-CARE has a $\mathcal{U}_*^*$-stabilizing solution;

(iv) the CARE has a $\mathcal{U}_*^*$-stabilizing solution;

(v) the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.

Assume, in addition, that any (hence all) of (i)–(v) has a solution. Then

(a1) The solutions $\mathcal{P}$ of (iii)–(v) are unique and equal, with the same S, K and $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}$ (modulo (9.114) for (v)) $B_w^*\mathcal{P} \in \mathcal{B}(H,U)$, and the corresponding triple is given by $(\mathcal{P},S,K)$, where $K := -S^{-1}(B_w^*\mathcal{P} + D^*JC)$.

All J-critical state feedback pairs over $\mathcal{U}_*^*$ are generated by $\begin{bmatrix} EK & I-E \end{bmatrix}$ ($E \in \mathcal{GB}(U)$).

(a2) Σ and its closed-loop system $\Sigma_\circlearrowleft$ corresponding to the state feedback operator K are ULR, *and $\Sigma_\circlearrowleft$ has generators*

$$\left[\begin{array}{c|c} A+BK_s & B \\ \hline C+DK_s & D \\ K_s & 0 \end{array}\right] \tag{9.17}$$

(b1) Theorem 8.3.9 applies to the left column of $\Sigma_\circlearrowleft$.

(b2) If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, then $\Sigma_\circlearrowleft$ is exponentially stable and, for any initial state x_0 and closed-loop input $u_\circlearrowleft$, the corresponding (J-critical if $u_\circlearrowleft = 0$) closed-loop cost is given by (cf. Figure 9.1)

$$\mathcal{J}(x_0, \mathbb{K}_\circlearrowleft x_0 + (\mathbb{F}_\circlearrowleft + I)u_\circlearrowleft) = \langle x_0, \mathcal{P}x_0\rangle_H + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle \quad (x_0 \in H,\ u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;U)). \tag{9.18}$$

(c1) Assume that (2.) or (4.) of Hypothesis 9.2.2 holds (or that $\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H)), C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$).

Then $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset \mathrm{ULR} \cap \mathrm{UVR}$, in particular, $\mathbb{A}_\circlearrowleft B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, for any $\omega > \omega_{A_\circlearrowleft}$.

(c2) Assume that (1.) or (5.) of Hypothesis 9.2.2 holds. Then $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft - D, \mathbb{F}_\circlearrowleft, \mathbb{M} - I \in \mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}) \subset \mathrm{ULR} \cap \mathrm{SVR}$; in particular, $\mathbb{A}_\circlearrowleft Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$, for all $\omega > \omega_{A_\circlearrowleft}$ and $u_0 \in U$.

(c3) If Σ satisfies (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2, then so does $\Sigma_\circlearrowleft$.

(d1) If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ (or $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{C}\end{smallmatrix}\right]$ is estimatable and $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$), then $\omega_{A_\circlearrowleft} < 0$.

(d2) *If* $\omega_{A_\circlearrowleft} < 0$ *and the assumptions of (c1) or (c2) hold, then* $\mathbb{B}_\flat\tau, \mathbb{D}_\flat, \mathbb{F}_\flat \in$ SHPR$\cap$ULR.

Recall that $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ means that $\mathbb{D}u = D + f * u$ $(u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U))$ for some $f \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ and $D \in \mathcal{B}(U,Y)$.

See Theorem 9.9.1 for further details (e.g., for "(b2)" for $\mathcal{U}_*^* \neq \mathcal{U}_{\mathrm{exp}}$). Recall that our cost function is

$$\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt, \tag{9.19}$$

where $y := C_{\mathrm{w}}x + Du$ a.e., $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ (the latter is the strong solution of $x' = Ax_0 + Bu_0$).

Proof: "(iii)⇒(iv)" follows from Proposition 9.2.7(a), "(iv)⇒(v)" from Proposition 9.8.10, "(v)⇒(ii)" from Theorem 9.9.1(a1), "(ii)⇒(iii)⇐(i)" from Theorem 9.2.3 and Proposition 9.3.1, "(iv)⇒(i)" from Theorem 9.9.1(f2).

(a1) By Proposition 9.2.7(a), a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P},S,K)$ of the B_{w}^*-CARE solves also the CARE and hence the eIARE; by Theorem 9.8.12(e)&(s2)&(s3), there are no other $\mathcal{U}_*^*$-stabilizing solutions of the eIARE nor of the CARE (hence nor of the B_{w}^*-CARE). By Theorem 9.9.1(f2), all J-critical state feedback pairs over $\mathcal{U}_*^*$ are generated by $\begin{bmatrix} EK & I-E \end{bmatrix}$ $(E \in \mathcal{GB}(U))$.

(a2) See Proposition 6.6.18(d4) for the generators of $\Sigma_\circlearrowleft$. By Lemma 9.3.2, $\mathbb{D} \in$ ULR. By Proposition 9.2.7(b), $\mathbb{F}$ is ULR, hence so are $\mathbb{M}$ and $\mathbb{D}_\circlearrowleft = \mathbb{D}\mathbb{M}$, by Proposition 6.3.1(b2).

(b1) By (a1), $\mathbb{K}_\circlearrowleft x_0$ is J-critical for each $x_0 \in H$, hence (b1) holds.

(b2) This follows from Theorem 8.3.9(a2) and (9.139).

(c1) Note first that $\mathrm{MTIC}_\infty^{\mathrm{L}^1} \subset$ ULR$\cap$UVR, by Proposition 6.3.4(a1).

1° *Case (2.):* This follows from Lemma 9.6.1.

2° *Cases (4.):* By Lemma 6.8.4(a1) (note that (a2) would provide the sharp result $\omega = \omega_{A_\circlearrowleft}$ in some cases), we have $\mathbb{A}_\circlearrowleft B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, i.e., $\mathbb{B}_\circlearrowleft\tau \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$. Because K and $C+DK$ are bounded, we have $\mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ (recall that $\mathbb{F}_\circlearrowleft = \mathbb{M} - I$).

(c2) Note first that $\mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}} \subset$ ULR$\cap$SVR, by Proposition 6.3.3(a).

1° *Case (1.):* Combine (9.17) and Theorem 6.9.1(a).

2° *Case (5.):* This follows from Lemma 9.2.8(b1). (Also (7.) will do for $\omega > \max\{\omega_A, \omega_{A_\circlearrowleft}\}$ if $\omega \geq 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, by Lemma 9.2.8(b2).)

(c3) For (1.) this is obvious. For (2.) this follows from Lemma 9.6.1, for (4.) from Lemma 6.8.4(a1), and for (5.) from (c2).

(d1) This follows from Theorem 8.3.9(a2)&(a5) (cf. Lemma 8.3.3).

(d2) The ULR property was established in (a2). We have $\mathrm{MTIC}^{\mathrm{L}^1} \subset$ UHPR, by Theorem 2.6.4(f), and $\mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}} \in$ SHPR, by Proposition 6.3.3(a). □

In coercive minimization problems, we usually do not have the check whether a solution of the CARE is stabilizing:

Theorem 9.2.10 *Assume Hypothesis 9.2.1. Assume also that* $J \gg 0$ *and that there is* $\varepsilon > 0$ *s.t.* $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \geq \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ *on* $H_1 \times U$*. Then* *(a) A minimizing (see Definition 10.2.1) solution of the B_{w}^*-CARE is nonnegative.*

(b) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, then the following are equivalent:

(i) There is a minimizing control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;
(ii) There is a minimizing ULR state feedback operator over $\mathcal{U}_*^*$;
(iii) $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$
(iv) B_{w}^*-CARE has a nonnegative solution.

Moreover, if (i)–(iv) hold, then the smallest nonnegative solution $\mathcal{P}$ of the B_{w}^*-CARE is the unique $\mathcal{U}_{\mathrm{out}}$-stabilizing (and SOS-stabilizing) solution of the B_{w}^*-CARE (and of the CARE), and strictly minimizing over $\mathcal{U}_{\mathrm{out}}$.

(c) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. Then (i) and (ii) are equivalent, and they hold iff the B_{w}^*-CARE has an exponentially stabilizing solution.

Moreover, if such a solution exists, then it is the greatest solution of the B_{w}^*-CARE and strictly minimizing over $\mathcal{U}_{\mathrm{exp}}$.

(d) **($\mathcal{P}$ is unique)** *Assume that Σ is strongly stable (resp. estimatable; e.g., C is bounded and $C^*C \gg 0$). Then $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$ (resp. $= \mathcal{U}_{\mathrm{exp}}$), hence then (i)–(iv) are equivalent also for any of these.*

Moreover, then the B_{w}^*-CARE has at most one nonnegative solution, and such a solution is strongly (resp. exponentially) q.r.c.-stabilizing and strictly minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ (resp. and $\mathcal{U}_{\mathrm{exp}}$).

Note that $J \gg 0$, $D^*JC = 0$ and $D^*D \gg 0$ (or $J = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \gg 0$, $\mathbb{C} = \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$ and $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$) imply that the coercivity conditions of the theorem are satisfied.

Assume that $\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$. Then (b) and (c) show that for the standard LQR cost function, there is a smallest nonnegative solution of the B_{w}^*-CARE, and it corresponds to the minimizing control over $\mathcal{U}_{\mathrm{out}}$, and if there is a minimizing control over $\mathcal{U}_{\mathrm{exp}}$, then it corresponds to the greatest solution of the B_{w}^*-CARE (and such a solution exists). A more detailed treatment of this phenomenom is given in Theorem 3.0.5 of [Dumortier] (assuming that B and C are bounded), which couples the nonnegative solutions of the CARE with the non-observable poles of Σ.

Proof of Theorem 9.2.10: Note first that $\mathbb{D}$ is ULR, by Theorem 9.2.3, and that (10.87) holds (see the proof of Proposition 10.7.3(c2)).

(a) This follows from equation $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ (see Theorem 9.9.1(a2)&(g2)).

(b) 1° *(ii)⇒(i)⇒(iii)⇒(i)⇒(ii)&(iv) and (iv)⇒(i):* Directly from the definitions we obtain that (ii)⇒(i)⇒(iii). By (the proofs of) Proposition 10.7.3(c2)&(d), we have (iii)⇒(i) and any minimizing control over $\mathcal{U}_{\mathrm{out}}$ is unique.

By Theorem 9.2.9(i)&(ii)&(a1)&(a2), a unique minimizing control corresponds to a $\mathcal{U}_{\mathrm{out}}$-stabilizing ULR solution $\mathcal{P}$ of the B_{w}^*-CARE. Thus, (i)⇒(ii)&(iv) (because $\mathcal{P} = \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} \geq 0$).

Finally, assume (iv), so that there is an admissible nonnegative solution of the IARE, by Proposition 9.2.7(b). Then there is a unique minimizing control over $\mathcal{U}_{\mathrm{out}}(x_0)$ for each $x_0 \in H$, by Proposition 10.7.3(c2)&(d), so that (i) holds.

2° Let $\mathcal{P}$ be as in 1°. By Proposition 10.7.3(d1), $\mathcal{P}$ is the smallest nonnegative solution of the eIARE, hence it is the smallest nonnegative solution of the B_w^*-CARE(and of the CARE).

(c) The equivalence follows from Theorem 9.2.9(i)&(ii)&(iii)&(a1)&(a2), and the fact that $S = D^*JD \geq \varepsilon I \gg 0$ (which also implies that a minimizing control is necessarily unique, by Theorem 9.9.1(f2)).

Any exponentially stabilizing solution is the greatest solution of the B_w^*-CARE, by Corollary 9.2.11.

(d) (Bounded C with $C^*C \gg 0$ implies exponential detectability, by Lemma 6.6.25.) This follows from Proposition 10.7.3(d2)&(d3). □

For the B_w^*-CARE with $S \gg 0$, a strongly stabilizing solution the greatest of all (self-adjoint) solutions, not just of nonnegative ones (cf. Theorem 9.8.13 and Corollary 15.5.3):

Corollary 9.2.11 (Greatest solution $\mathcal{P}_+$ of the B_w^*-CARE) *If $D^*JD \gg 0$ and the B_w^*-CARE has a strongly ($\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$-)stabilizing solution, then this solution is the greatest solution of the B_w^*-CARE.* □

(This follows from Proposition 9.2.7(b) and Theorem 9.8.13. Recall that "greatest" is always "maximal".)

The example $A = i$, $B = 0 = C$, $D = 1 = J$, $\mathcal{P} \in \mathbf{R}$ shows that "strongly" is not redundant in the above corollary; by Example 9.13.12(b), "strongly" cannot be replaced by "weakly" (take, e.g., $B = 0 = C$, $\mathbb{A} = \tau$, $H := \mathrm{L}^2$, $D = 1 = J$, $\widetilde{\Sigma}$ as in the example; however, "strongly" and "weakly" coincide in the finite-dimensional case).

G. Weiss and R. Rebarber [WR97] [WR00] have posed the question whether optimizability is equivalent to exponential stabilizability. We give here a positive answer for a special case:

Theorem 9.2.12 (Optimizable ⇔ exp. stabilizable) *Assume that $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$. Then the following are equivalent:*

(i) $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ is optimizable;

(ii) Σ is exponentially stabilizable;

(iii) Σ has an exponentially stabilizing bounded state feedback operator $K \in \mathcal{B}(H,U)$;

(iv) There is $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_w^*))$ s.t. $\mathcal{P} \geq 0$ and

$$(B_w^*\mathcal{P})^* B_w^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + I. \tag{9.20}$$

By Lemma 6.8.4, it follows that if $\mathbb{D}, \mathbb{B}\tau \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and Σ is optimizable, then $\mathbb{D}$ is exponentially stabilizable in $\mathrm{MTIC}_\infty^{\mathrm{L}^1}$.

Analogously, if $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ for all $u_0 \in U$ and $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ is optimizable, $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_{-\varepsilon}(\mathbf{R}_+;H)$ for all $u_0 \in U$ and some $\varepsilon > 0$, by Lemma 6.8.4(b).

In the theorem, we showed that $\mathcal{U}_{\exp}(x_0) \neq \emptyset$ for each $x_0 \in H$ iff $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$ is exponentially stabilizable. We note that one can analogously show that Σ

is output-stabilizable iff $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$, by using substitutions $\mathbb{C} \mapsto \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}$, $\mathbb{D} \mapsto \begin{bmatrix} \mathbb{D} \\ I \end{bmatrix}$, $J := I$.

Proof of Theorem 9.2.12: *(iii)⇒(ii)⇒(i)⇒(iv)⇒(iii):* Obviously, (iii)⇒(ii)⇒(i). Assume (i). Set $C := \begin{bmatrix} I \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J = I$ (i.e., "$\mathbb{D} = \begin{bmatrix} x \\ u \end{bmatrix}$, $\mathcal{J} = \|x\|^2 + \|u\|^2$"). Because $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$, there is a unique J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$ iff (i) holds, by Theorem 8.4.3.

But this is the case iff (iv) holds, by Theorem 9.2.9(i)&(iii) and Theorem 9.2.10(a). Thus, (iv)⇔(i). Moreover, (iv) implies (iii), because the operator K in Theorem 9.2.9(a1) is $\mathcal{U}_{\text{exp}}$-stabilizing, i.e., exponentially stabilizing. Thus, (iv)⇒(iii), and equivalence is established. □

By combining the above with its dual, we observe that optimizability and estimatability are equivalent to exponential joint stabilizability and detectability:

Corollary 9.2.13 *Assume that* $\mathbb{A}^* C^* y_0 \in \mathrm{L}^1([0,1];H)$ *for all* $y_0 \in Y$. *Then*

(a) Σ *is estimatable iff* Σ *has an exponentially detecting (bounded) output injection operator* $\mathtt{H} \in \mathcal{B}(Y,H)$.

(b) If, in addition, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ *for all* $u_0 \in U$, Σ *is optimizable and estimatable, and* $\mathtt{H}$ *and* K *are as in (a) and Theorem 9.2.12, then* K *and* $\mathtt{H}$ *are exponentially (r.c.- and l.c.-) jointly stabilizing.*

(c) If $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);*)$, *and* Σ *is optimizable and estimatable, then* $\mathbb{D}$ *has a d.c.f. over* $\mathrm{MTIC}_{\text{exp}}^{\mathrm{L}^1}$.

Thus, if the two strong L^1 assumptions are satisfied, then Σ is exponentially [jointly stabilizable and] detectable iff Σ is [optimizable and] estimatable, by (b).

Note that Hypothesis 9.5.1 implies the assumptions of (c), by Lemma 9.5.2.

Proof: (a) This is the dual of Theorem 9.2.12.

(b) This follows from Lemma 6.6.26.

(c) Now both closed-loop systems of (6.169) have their I/O maps in $\mathrm{MTIC}_{-\varepsilon}^{\mathrm{L}^1}(Y \times U)$ for some $\varepsilon > 0$, by Lemma 6.8.4(c1) (and its dual). □

Next we show that the invertibility of the Popov Toeplitz operator (i.e., J-coercivity over $\mathcal{U}_{\text{out}}$) is now equivalent to the existence of a spectral factorization. We assume Σ to be exponentially stable to guarantee the stability of the spectral factor (this is not needed for most MTIC classes):

Theorem 9.2.14 (Popov⇔SpF)

(a) Let Σ *be exponentially stable. Assume that (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2 holds, or that* $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H, Y_1 \times Y_2)$ *and* $\dim Y_1 < \infty$. *Then the following are equivalent:*

(i) $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$ *(i.e.,* $\mathbb{D}$ *is J-coercive over* $\mathcal{U}_{\text{out}}(= \mathcal{U}_{\text{exp}})$*);*

(i') $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$ *and* $D^* J D \in \mathcal{GB}(U)$*;*

(ii) $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S \in \mathcal{GB}(U)$*;*

(ii') $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S = D^*JD \in \mathcal{GB}(U)$*, and* $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$ *and* $X = I$*;*

(iii) the B_w^**-CARE has an exponentially stabilizing solution.*

(iii') the B_w^**-CARE has an* $\mathbb{M}$*-stabilizing solution.*

(iv) the IARE (or CARE) has an $\mathbb{M}$*-stabilizing solution.*

(b) Assume that Σ *is exponentially stable and Hypothesis 9.2.1 holds for* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ $(= \mathcal{U}_{\mathrm{exp}})$.

Then (i)–(iv) are still equivalent provided that $\mathbb{D}$ *and* $\mathbb{D}^{\mathrm{d}}$ *are strongly half-plane-regular or that we assume that* $D^*JD \in \mathcal{GB}(U)$.

(c1) Conditions (i) and (ii) are equivalent (and imply that $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$*) if* $\mathbb{D} \in \widetilde{\mathcal{A}}$*.*

(c2) If $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ *for some* $\varepsilon > 0$ *or* $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*, then (i)–(ii') are equivalent.*

(c3) If $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*, and* Σ *is exponentially stable, then (i)–(iv') are equivalent once we replace the* B_w^**-CARE by the CARE.*

(d) Assume that Σ *is exponentially stable and ULR. Then (ii)* $\Leftarrow$ *(iv)* $\Leftarrow$ *(iii')* $\Leftrightarrow$ *(iii)* $\Rightarrow$ *(ii')* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i).*

As one observes from the results of this section, $\mathbb{X}$ shares most properties of $\mathbb{D}$. If $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+ \gg 0$, then we can allow for $\Sigma \in \mathrm{SOS}$ and weaken the assumptions as shown in Theorem 10.6.3 and Lemma 10.6.2(b)–(d) (in the indefinite case, Σ may be even strongly stable with $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ invertible and still $\mathbb{X}, \mathbb{X}^{-1}$ unstable, by Example 8.4.13).

Proof of Theorem 9.2.14: (Naturally, "$\mathbb{M}$-stabilizing" means "s.t. $\mathbb{M}$ is stable" (cf. Definition 9.8.1).)

Note that by (d), we only have to prove "(i)$\Rightarrow$(iii)" in (a) and (b).

(a) 1° *Cases (2.)&(4.):* These follow from Lemma 9.3.2 and Hypotheses 8.4.7 and 8.4.8.

2° *Case (1.):* There is a unique J-critical control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, for each $x_0 \in H$, by Proposition 8.3.10. By Theorem 9.9.6(d), it follows that the eCARE has an exponentially stabilizing solution with $S = D^*JD$ and $\widehat{\mathbb{D}} - D, \widehat{\mathbb{X}} - I \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{\omega}; \mathcal{B}(*,*))$.

Moreover, $S \in \mathcal{GB}(U)$, by Lemma 9.10.3 and Lemma 8.4.11(a1). By Theorem 9.9.1(g2), we have $\mathbb{N}^* J\mathbb{N} = S$, hence $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$.

3° *Case (5.):* This follows from (c2), since now $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$, by Lemma 6.8.1(d1).

4° *Bounded* $\left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$ *with* $\dim Y_1 < \infty$*:* As in the proof of Proposition 9.2.4, one can show that $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$. Thus, also this follows from (c2).

(b) 1° $D^*JD \in \mathcal{GB}(U)$*:* If $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in \mathrm{SHPR}$ and (i) (or (ii)) holds, then $D^*JD \in \mathcal{GB}(U)$, by Lemma 6.3.6(c1), hence we may assume that $D^*JD \in \mathcal{GB}(U)$ (since it is contained in the other conditions).

2° *(i)$\Rightarrow$(iii):* Assume (i). Then there is a unique J-critical control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, for each $x_0 \in H$, by Proposition 8.3.10. By Theorem

9.2.9(i)&(iii)&(a2), it follows that the B_w^*-CARE has an exponentially stabilizing solution with $X = I$, $S = D^*JD$ and $\mathbb{D}, \mathbb{X} \in$ ULR.

(c1) This follows from Hypothesis 8.4.7.

(c2) For $\mathbb{D} \in \mathrm{MTIC}^{L^1}$, this follows from Theorem 8.4.9(a)&(b)

Assume that $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$. Then $\widehat{\mathbb{D}}$ has a realization $\Sigma_{\mathbb{D}}$ of type (1.), by Theorem 6.9.1(a)&(d1), hence then this follows from (a).

Remark: If also $\widehat{\mathbb{D}}(\bar{\cdot})^* - D^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(Y,U))$, then (apply Lemma 6.3.17 with $R := -S^{-1}D^*J$ and $T := -S^{-1}B_w^*\mathcal{P}$; here we refer to $\Sigma_{\mathbb{D}}$, not to Σ)

$$\widehat{\mathbb{X}}(\bar{\cdot})^* - X^* = \widehat{\mathbb{F}}(\bar{\cdot})^* = (R(\widehat{\mathbb{D}} - D) + T\mathbb{B}\tau)(\bar{\cdot})^* \tag{9.21}$$

$$= (\widehat{\mathbb{D}}(\bar{\cdot})^* - D^*)R^* + B^*(\cdot - A^*)T^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon'}; \mathcal{B}(Y,U)), \tag{9.22}$$

by Lemma A.4.5(v)&(i)&(vi) (choose $\varepsilon' \in (0,\varepsilon]$ s.t. $-\varepsilon' > \omega_{A_\circlearrowleft}$).

(c3) By (c2), (i)–(ii') are equivalent. The rest follow from Corollary 9.1.12 and Proposition 9.8.11(d1).

(d) Trivially, (iii)⇒(iii'), and (ii')⇒(ii). By Proposition 9.2.7(b), we have (iii')⇒(iv). By Proposition 9.8.11(c)&(d), we have (iv)⇒(ii). By Proposition 9.2.7(b) and Proposition 9.8.11(c)&(d), we have (iii)⇒(ii'). By Theorem 8.4.12, we have (ii)⇒(i). □

In the unstable case, a corresponding result can be formulated in the following way:

Corollary 9.2.15 (*J*-coercive⇔RCF) *Assume that Σ is optimizable and estimatable. Assume that (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2 holds (or that Hypothesis 9.2.1 holds and $D^*JD \in \mathcal{GB}$). Then the following are equivalent:*

(i) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}} (= \mathcal{U}_{\text{exp}})$;

(ii) $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$;

(iii) the B_w^*-CARE has an exponentially stabilizing solution.

(iii') the B_w^*-CARE has an I/O-stabilizing solution.

Let K correspond to (iii). Then K is ULR, J-critical over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$ and exponentially stabilizing, and $\mathbb{N} = \mathbb{D}_\circlearrowleft$ and $\mathbb{M} = \mathbb{F}_\circlearrowleft + I$.

In particular, then $\mathbb{N}$ and $\mathbb{M}$ are exponentially stable, $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in$ ULR (here $\mathbb{X} := \mathbb{M}^{-1}$), $M = I = X$ and $S = D^*JD$.

(Note that $\mathbb{M}, \mathbb{N} \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B})$ for some $\varepsilon > 0$ in cases (1.) and (5.).)

Cf. Corollary 8.4.14 and Theorem 9.9.10.

Proof: Recall from Lemma 8.3.3 that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$. By Theorem 6.7.15(c1), we may replace "I/O-stabilizing" by "exponentially stabilizing", "input-stabilizing" or "output-stabilizing" in (iii').

1° *(ii)⇒(i):* This follows from Corollary 8.4.14(b1).

2° *(iii)⇒(ii):* This follows from Theorem 9.9.10(d2).

3° *(iii)⇔(iii'):* By Theorem 6.7.15(c1), "I/O-stabilizing" is equivalent to "exponentially q.r.c.-stabilizing".

4° *(i)⇒ $D^*JD \in \mathcal{GB}(U)$:* Assume (i) and any of (1.), (2.), (4.) and (5.). By Lemma 9.3.2, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$. By Theorem 9.2.12, Σ has a bounded exponentially stabilizing state feedback operator $K' \in \mathcal{B}(U,H)$.

By Lemma 9.3.3, also $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ (corresponding to K') satisfies (1.), (2.), (4.) or (5.). By Theorem 8.4.5(g1), $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\text{out}}^{\Sigma_\flat}$, so that Theorem 9.2.14(a)(ii') holds; in particular $D^*JD = D_\flat^*JD_\flat \in \mathcal{GB}(U)$.

5° *(i)⇒(iii):* Assume (i), so that $D^*JD \in \mathcal{GB}(U)$, by 4° (or by assumption). By Theorem 8.4.3, there is a unique J-critical control over $\mathcal{U}_{\text{exp}}$. Therefore, (iii) holds, by Theorem 9.2.9(iii),

5° The rest follows easily from Theorem 9.2.9.

6° *Remark:* In addition to (1.), (2.), (4.) or (5.) and optimizability, estimatability, assume that (ii) holds and that Σ is also input-detectable (by Corollary 9.2.13, the latter holds in case (2.)).

Then Σ is exponentially detectable, by the dual of Theorem 6.7.15(c1). Therefore, $\mathbb{D}_\flat$ and $\mathbb{F}_\flat + I$ are part of an exponential d.c.f., by Lemma 6.6.26, hence so are $\mathbb{N} = \mathbb{D}_\flat\mathbb{W}$ and $\mathbb{M} = (\mathbb{F}_\flat + I)\mathbb{W}$, by Lemma 6.5.9(d), where $\widetilde{\mathbb{X}}^*S\widetilde{\mathbb{X}} = \mathbb{D}_\flat^*J\mathbb{D}_\flat$ as in Theorem 9.2.14(a)(ii) and $\mathbb{W} := \widetilde{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{TIC}$. In particular, then $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (exponential) r.c.f. (We do not know whether the q.r.c.f. has to be a r.c.f. in general.) □

We can now show that for $\mathcal{U}_{\text{exp}}$ (assuming that $D^*JD \in \mathcal{GB}(U)$), J-coercivity is equivalent to the existence of a unique optimal control:

Theorem 9.2.16 ($\mathcal{U}_{\text{exp}}$: Unique optimum ⇔ B_{w}^*-CARE⇔ J-coercive) *Assume Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, and that $D^*JD \in \mathcal{GB}(U)$. Then conditions (i)–(iii) are equivalent.*

(i) There is a unique J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The B_{w}^*-CARE has an exponentially stabilizing solution.

(iii) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{exp}}$, and Σ is exponentially stabilizable. □

(The proof of Theorem 14.2.7 applies mutatis mutandis.)

The assumption $D^*JD \in \mathcal{GB}(U)$ is not superfluous neither redundant in general (in (i); probably neither in (iii)), by, e.g., Example 9.13.3. However, in (iii) it is often redundant:

Lemma 9.2.17 ($\mathcal{U}_{\text{exp}}$: J-coercive ⇒ $\exists(D^*JD)^{-1}$) *Assume that 1. $\mathbb{D}$ is ULR [or SLR] and that $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, (this is the case if (1.), (2.) or (4.) of Hypothesis 9.2.2 holds), or 2. $(\cdot - A)^{-1}B, C(\cdot - A)^{-1}B, (\cdot - A^*)^{-1}C^*, B_{\text{w}}^*(\cdot - A^*)^{-1}C^* \in \mathrm{H}^2_{\text{strong},\infty}$ [or 3. that $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1);H)$ for all $u_0 \in U$].*

If Σ is optimizable and $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\text{exp}}$, then $D^*JD \in \mathcal{GB}(U)$ [$\gg 0$].

See Corollary 9.2.19 for the "converse" (where $D^*JD \in \mathcal{GB}(U)$ is equivalent to J-coercivity when there is a J-critical control).

Proof: (Set $\mathcal{U}_*^* =: \mathcal{U}_{\text{exp}}$.)

1° *Case 1.* *($\mathbb{D} \in$ ULR etc.):* By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$. As noted in the proof of Lemma 6.8.4(d), we have $\mathbb{B}_\flat \tau, \mathbb{M} \in \mathrm{UHPR} \cap \mathrm{TIC}$, hence $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{ULR} \cap \mathrm{TIC}$.

But $\mathbb{D}_\flat$ is J-coercive, by Theorem 8.4.5(d), hence $D_\flat^* J D_\flat \in \mathcal{GB}(U)$, by Lemma 6.3.6(d1). Since $D = D_\flat$, we have $D^* J D \in \mathcal{GB}(U)$.

2° *Case 2.:* (Note that "2." holds iff Σ and Σ^{d} satisfy (5.) of Hypothesis 9.2.2, by Lemma 6.8.1(a)&(d1).)

By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$; choose some $\omega \in (\omega_{A_\flat}, 0)$. By Lemma 6.8.4(b)&(c3), we have $\mathbb{B}_\flat \tau, \mathbb{F}_\flat, \mathbb{D}_\flat - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,*))$ (since $\mathbb{F}_\flat = K\mathbb{B}_\flat \tau$). Since $\mathbb{C}^{\mathrm{d}}_\flat = \mathbb{C}^{\mathrm{d}} + \mathbb{K}^{\mathrm{d}}_{\circlearrowleft} \tau \mathbb{D}^{\mathrm{d}}$, we have

$$\widehat{\mathbb{C}^{\mathrm{d}}_\flat \tau} = \widehat{\mathbb{C}^{\mathrm{d}} \tau} + \widehat{\mathbb{K}^{\mathrm{d}}_{\circlearrowleft} \tau} \widehat{\mathbb{D}^{\mathrm{d}}} \in \mathrm{H}^2_{\mathrm{strong},\infty} + \mathrm{H}^\infty_{\mathrm{strong},\infty} \mathrm{H}^2_{\mathrm{strong},\infty} \subset \mathrm{H}^2_{\mathrm{strong},\infty}. \tag{9.23}$$

By Lemma 6.8.1(a)&(d1) (applied to Σ^{d}), it follows that $\widehat{\mathbb{C}^{\mathrm{d}}_\flat \tau}, \widehat{\mathbb{D}^{\mathrm{d}}_\flat} \subset \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,*))$. But $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,*)) \subset \mathrm{ULR} \cap \mathrm{SHPR}$, by Proposition 6.3.3(a). We conclude that the assumptions of Lemma 6.3.6(d1) are satisfied by $\Sigma_\flat$, hence $D^* J D \in \mathcal{GB}(U)$ (as in 1°).

3° *Positive case:* (Note that here we have allowed also assumption 3.) Replace "ULR" by "SLR", "UHPR" by "SHPR" and "(d1)" by "(d2)" in 1°.

□

We finish this section by presenting two "generalizations" of Theorem 9.2.16, based on B^*_{w}-CARE theory, which allow one to use weaker assumptions than in the above results, at the cost of having to use the CARE instead of the B^*_{w}-CARE:

Theorem 9.2.18 ($\mathcal{U}_{\mathrm{exp}}$: J-coercive $\Rightarrow$ CARE) *Assume that* $\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H,Y))$ *and* $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,Y))$, *and that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$. Then the following are equivalent:*

(i) there is a J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ *is optimizable.*

If $(\mathcal{P}, S, K)$ is as in (ii), then K is ULR and J-critical over $\mathcal{U}_{\mathrm{exp}}$, $S = D^* J D \in \mathcal{GB}(U)$, $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty$ *and* $\mathbb{B}_{\circlearrowleft}\tau, \mathbb{N}, \mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega \subset \mathrm{UHPR}$ *for some* $\omega < 0$.

Recall from Lemma 8.3.3 that if Σ is estimatable, then J-coercivity over $\mathcal{U}_{\mathrm{exp}}$ is equivalent to J-coercivity over $\mathcal{U}_{\mathrm{out}}$, and that then (i)–(iii) become equivalent to the existence of a (unique) J-critical control over $\mathcal{U}_{\mathrm{str}}$ (or $\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_{\mathrm{sta}}$ or $\mathcal{U}_{\mathrm{exp}}$).

If $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{exp}}$, then "$J$-critical" becomes equivalent to "minimizing", by Lemma 10.2.2.

Proof: We note first that $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$ for any $\omega > \omega_A$, by Lemma 6.8.3(c), hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y) \subset \mathrm{ULR}$, by Lemma 6.8.1(e1).

1° *(ii)⇒(i)⇒(iii):* Trivially, we have (i)⇒(iii). If $(\mathcal{P}, S, K)$ solves (ii), then K is J-critical, by Theorem 9.8.5, hence (ii) implies (i).

2° *(iii)⇒(ii):* Assume (iii). By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing state feedback operator $\widetilde{K} \in \mathcal{B}(H,U)$. Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ and $\Sigma_\flat$ be corresponding state feedback pair and closed-loop system, so that $\omega_{A_\flat} < 0$ and $D_\flat = D$.

By Lemma 6.8.4(c1)&(a1), we have $\mathbb{B}_\flat\tau, \mathbb{D}_\flat, \widetilde{\mathbb{M}} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for all $\omega > \omega_{A_\flat}$, where $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{MTIC}_\infty^{\mathrm{L}^1}(U)$, $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$. Thus, (ii) follows from Proposition 9.9.5.

3° *The rest:* By Proposition 9.9.5 (see 2°), the solution of (ii) is unique, K is J-critical over $\mathcal{U}_{\mathrm{exp}}$, and $S = D^*JD \in \mathcal{GB}(U)$.

By Theorem 8.4.9, we have $\mathbb{X}_\natural \in \mathcal{G}\mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega \in (\omega_{A_\flat}, 0)$ (since $\mathbb{D}_\flat \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for any such ω), where $\mathbb{X}_\natural^* S \mathbb{X}_\natural = \mathbb{D}_\flat^* J \mathbb{D}_\flat$ (cf. the proof of Proposition 9.9.5); fix such an ω.

Then, $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ (since they are equal to $\mathbb{B}_\flat\tau\mathbb{X}_\natural^{-1}, \mathbb{D}_\flat\mathbb{X}_\natural^{-1}, \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$), hence $\mathbb{F}, \mathbb{X} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1} \subset \mathrm{ULR}$; in particular, K is ULR. □

By strengthening the assumption on $\mathbb{B}\tau$, we can show that J-coercivity is also necessary when D^*JD is invertible:

Corollary 9.2.19 ($\mathcal{U}_{\mathrm{exp}}$: Unique optimum ⇔ CARE ⇔ J-coercive) *Assume that* $\mathbb{A}B \in \mathrm{L}^2([0,1); \mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H,Y))$, $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,Y))$. *Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$, and $D^*JD \in \mathcal{GB}(U)$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$.

Any solution of (ii) is as in Theorem 9.2.18.

Note that any solution of (i) or (ii) are unique. See Corollary 10.2.10 for case $D^*JD \gg 0$.

Proof: Set $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. By Theorem 9.2.18, we have (iii)⇒(ii).

1° *(ii)⇒(i):* Assume (ii). Then K is J-critical, by Theorem 9.8.5, and $\mathbb{D}$ is J-coercive, by Proposition 9.9.12(b). By Lemma 9.2.17, we have $D^*JD \in \mathcal{GB}(U)$. Thus, (i) holds.

2° *(i)⇒(iii):* Assume (i). Then $\mathcal{U}_{\mathrm{exp}}(x_0) \neq \{0\}$ for each $x_0 \in H$, i.e., $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable. By Lemma 9.3.7(4), there is at most one (hence exactly one) J-critical control for Σ over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$. By Proposition 9.9.12(c)(3.), $\mathbb{D}$ is J-coercive, hence (iii) holds.

3° If some triple $(\mathcal{P}, S, K)$ solves (ii), then the assumptions of Theorem 9.2.18 are satisfied (since (ii) implies (iii), by the above), hence its conclusions hold. □

Notes for Sections 9.2 and 9.3

In the case of bounded B and C, the Riccati equation theory for WPLSs is rather well-known (see, e.g., [CZ]). For Pritchard–Salamon systems (which are a special case of Hypothesis 9.2.2(1.)), many of the results of this section are known (see, e.g., [Keu] and [Weiss97]). See also the notes on pp. 465 and 520. In the generality of these two sections, our results seem to be new.

See the notes on p. 418 for Theorem 9.2.14 and Corollary 9.2.15, and the notes on p. 853 for Corollary 9.2.11. Most of Proposition 9.2.4 is contained in [Sal89].

9.3 Proofs for Section 9.2

A witty saying proves nothing.

— Voltaire (1694–1778)

Now we shall show that, under Hypothesis 9.2.1, a J-critical control over $\mathcal{U}_*^*$ in WPLS form (if any) is necessarily of state feedback form:

Proposition 9.3.1 (Case $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_\mathrm{w}^*))$) *Let $\Sigma_\mathrm{crit} \in \mathrm{WPLS}(\{0\}, H, Y)$ be a J-critical control for Σ in WPLS form. Assume that $\mathbb{D}$ is ULR and $D^*JD \in \mathcal{GB}(U)$, and that $\mathcal{P}[H] \subset \mathrm{Dom}(B_\mathrm{w}^*)$.*

Then $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_\mathrm{w}^*))$, $\mathcal{P}$ satisfies the B_w^*-CARE, and

$$K := -(D^*JD)^{-1}(B_\mathrm{w}^*\mathcal{P} + D^*JC) \tag{9.24}$$

is the unique ULR J-critical state feedback operator for Σ.

Thus, then Σ_crit is of state feedback form.

Proof: Set $S := D^*JD$.

1° *$B_\mathrm{w}^*\mathcal{P} \in \mathcal{B}(H, U)$:* By Lemma A.3.6, $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,w}}^*))$, hence $B_\mathrm{w}^*\mathcal{P} \in \mathcal{B}(H, U)$, because $B_\mathrm{w}^* \in \mathcal{B}(\mathrm{Dom}(B_\mathrm{w}^*), U)$, by Proposition 6.2.8(b1).

2° *K is ULR and admissible:* By 1°, we have $K := -S^{-1}(B_\mathrm{w}^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_1, U)$. By Lemma 6.3.17, K is an ULR admissible state feedback operator for Σ,

3° *K is J-critical and unique:* Obviously, $K_\mathrm{w} = -S^{-1}(B_\mathrm{w}^*\mathcal{P} + D^*JC_\mathrm{w}) \in \mathcal{B}(\mathrm{Dom}(C_\mathrm{w}), U)$. By regularity, $H_B \underset{c}{\subset} \mathrm{Dom}(C_\mathrm{w})$. Therefore, $K_\mathrm{w} = K_\mathrm{crit}$ on $\mathrm{Dom}(A_\mathrm{crit})$, by (9.66). By Lemma 8.3.17(b), it follows that $\left[\begin{array}{c|c} \Sigma_\mathrm{crit} & * \end{array}\right]$ is the corresponding closed-loop system; in particular, K is J-critical. From Lemma 8.3.17(b) we also obtain that K_w is the unique J-critical compatible state feedback operator.

4° *$\mathcal{P}$ is a solution of the B_w^*-CARE:* By 3° and Corollary 9.9.2, $\mathcal{P}$ is a $\mathcal{U}_*^*$-stabilizing solution of the CARE, hence of the B_w^*-CARE, by Proposition 9.2.7(a),

5° *Remark: Case $S := D^*JD \notin \mathcal{GB}(U)$?* If $S^*S \gg 0$ on $\mathrm{Ker}(S)^\perp$ (this is the case whenever $\dim U < \infty$), then the above procedure produces an ULR (hence admissible) K s.t. $SK_\mathrm{w} = SK_\mathrm{crit}$ on $\mathrm{Dom}(A_\mathrm{crit})$ (set, e.g., $K_2 := K|_{\mathrm{Ker}(S)} := 0$).

However, we would in general have $\mathrm{Dom}(A_\circlearrowleft) \neq \mathrm{Dom}(A_\mathrm{crit})$, (it is not clear whether we could even find a new definition for K_2 s.t. K were admissible (it is for any bounded K_2) and $K_2 = (K_\mathrm{crit})_2$ on $\mathrm{Dom}(A_\mathrm{crit})$). Consequently, we should somehow prove that the WPLS $\Sigma_\circlearrowleft$ is still $\mathcal{U}_*^*$-stabilizing and that (9.65) holds (alternatively, we could more directly verify the equations of Theorem 9.7.1). □

We list here some basic consequences of Hypothesis 9.2.2:

Lemma 9.3.2 *Assume Hypothesis 9.2.2. Then $\mathbb{D}$ is ULR and SVR; in cases (1.)–(5.) we also have $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1]; H)$ for all $u_0 \in U$, hence then also $\mathbb{B}\tau$ is ULR and SVR.*

If Σ is exponentially stable, then $\mathbb{D}$ is strongly half-plane-regular except possibly in case (7.), $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega < 0$ in cases (2.) and (4.). and $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega < 0$ in cases (1.)–(5.).

Proof: (The claim on $\mathbb{B}\tau$ follows from Lemma 6.3.16(c) and Lemma 6.8.1(e1).)

For (5.) and (7.), this follows from Proposition 6.3.3(a) and Lemma 6.8.1(d1); for (1.), from Lemma 6.3.16(b); for (2.), from Lemma 9.5.2.

In case (3.), (4.) or (6.), $\mathbb{D}$ is ULR, by Lemma 6.3.16(c), and SVR (and SHPR if Σ is exponentially stable, by Lemma 6.8.1(d1)), by Proposition 6.3.4(a3) or Lemma 6.8.1(a). □

Most classes are closed w.r.t. bounded state feedback:

Lemma 9.3.3 *If(f) Σ satisfies (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2, then the closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ corresponding to any bounded state feedback operator ($K \in \mathcal{B}(H,U)$) satisfies the same condition.* □

(This follows from Lemma 6.8.4(a1)&(c3) and Lemma 9.5.4.)

Now we establish the sufficiency of Hypothesis 9.2.2:

Lemma 9.3.4 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.2.2 holds. Then $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B^*_{\mathrm{L,w}})) \subset \mathcal{B}(H, \mathrm{Dom}(B^*_{\mathrm{w}}))$.*

Thus, Proposition 9.3.1 applies if, in addition, $D^*JD \in \mathcal{GB}(U)$.

The key to the theory behind Section 9.2 is the method used in 1°–2° below:

Proof: In case (1.) we have $\mathrm{Dom}(B^*) = H$, hence then trivially $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B^*))$ (and $B^* = B^*_{\mathrm{w}} = B^*_{\mathrm{L,s}}$). For case (2.), this will be shown in Lemma 9.6.2. For the others, we go on as follows:

Let $x_0 \in H$ be arbitrary. Choose some $t > 0$. We shall use (9.56) to show that $\mathcal{P}x_0 \in \mathrm{Dom}(B_{\mathrm{L,w}})$.

1° *We have $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t_{\mathrm{crit}}x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}) \subset \mathrm{Dom}(B_{\mathrm{L,w}})$ in cases (1.)–(5.):* By Lemma 6.8.1(b2)&(b1), we have $\mathbb{A}^{t\,*} \in \mathcal{B}(H, \mathrm{Dom}(B^*_{\mathrm{L,s}}))$. Therefore, (set $z_0 := \mathcal{P}\mathbb{A}^t_{\mathrm{crit}}x_0$)

$$\frac{1}{r}B^* \int_0^r (\mathbb{A}^q)^* \mathbb{A}^{t\,*} z_0 \, dq = \frac{1}{r} B^*_{\mathrm{L,s}} \mathbb{A}^{t\,*} \int_0^r (\mathbb{A}^q)^* z_0 \, dq \to B^*_{\mathrm{L,s}} \mathbb{A}^{t\,*} z_0, \tag{9.25}$$

by continuity. Thus, $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t_{\mathrm{crit}}x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}) \subset \mathrm{Dom}(B_{\mathrm{L,w}})$, by Proposition 6.2.8(c1)&(c4)&(d).

2° *Assume (3.):* Set $f := JC\mathbb{A}_{\mathrm{crit}}x_0 \in \mathcal{C}(\mathbf{R}_+;Y)$, $F := C\mathbb{A}Bu_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;Y)$, so that $\mathbb{D}\chi_{[-r,0)}u_0 = F * \chi_{[-r,0)} = \int_0^r \tau F \, dm$, by Lemma 6.8.1(f). Then

$$\frac{1}{r}\langle \mathbb{B}\chi_{[-r,0)}u_0, \mathbb{C}^{t\,*}J\mathbb{C}^t_{\mathrm{crit}}x_0\rangle_H = \frac{1}{r}\langle \mathbb{B}\chi_{[-r,0)}u_0, \mathbb{C}^*\pi_{[0,t)}f\rangle_H \tag{9.26}$$

$$= \frac{1}{r}\langle \mathbb{C}\mathbb{B}\chi_{[-r,0)}u_0, \pi_{[0,t)}f\rangle_Y = \frac{1}{r}\int_0^t \left\langle \int_0^r (\tau^s F)(q)\,dq, f(s)\right\rangle_Y ds \tag{9.27}$$

$$= \frac{1}{r}\int_0^r \int_0^t \langle (\tau^q F)(s), f(s)\rangle_Y \, ds\, dq = \frac{1}{r}\int_0^r \langle \tau^q F, \pi_{[0,t)} f(s)\rangle_{\mathrm{L}^2} dq \tag{9.28}$$

$$\to \langle F, \pi_{[0,t)} f(s)\rangle_{\mathrm{L}^2}, \tag{9.29}$$

as $r \to 0+$, by continuity, because $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^1, as $q \to 0+$, and $\pi_{[0,t)} f \in \mathcal{C} \subset L^\infty([0,t);Y)$ (hence the last integrand in (9.28) is continuous in q). (Obviously, the use of the Fubini Theorem in the beginning of (9.28) was justified.)

Because $(\mathbb{B}^{\mathrm{d}})^*\chi_{[0,r)}u_0 = \mathbb{B}\chi_{[-r,0)}u_0$, and $u_0 \in U$ was arbitrary, it follows from Proposition 6.2.8(c2) that $\mathbb{C}^{t\,*}J\mathbb{C}^t_{\mathrm{crit}}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ ($\in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $\mathbb{A}B \in L^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,H))$ (e.g., if (4.) holds), because then $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^1 independently of u_0 (as long as $\|u_0\|_U \le 1$)).

Combine this with 1° to observe that $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ (even $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $\mathbb{A}B \in L^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,H))$), by (9.56).

3° *Assume (4.):* With the additional assumption that $D^*JC = 0$, this is contained in case (3.). In 6°, we shall remove this assumption.

4° *Assume (5.):* We can work as in case (3.), except that we have to set $f := J\mathbb{C}_{\mathrm{crit}}x_0 \in L^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$, but we have $F := C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in L^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$. Now $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^2, so we again get a convergence as $r \to 0+$, by the Hölder Inequality.

Thus, we again have $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ (even $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $C_{\mathrm{L,s}}\mathbb{A}B \in L^2_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,Y))$, as in 2°).

5° *Assume (6.) or (7.)* Now $\mathbb{C}$ is stable and $\mathcal{P} = \mathbb{C}^*J\mathbb{C}_{\mathrm{crit}}$, by Theorem 8.3.9(b2), so that we may take $t = +\infty$ and skip 1°. In (6.), we have $f \in \mathcal{C}_{\mathrm{b}} \subset L^\infty$ (by Theorem 8.3.9(a2)&(a3)) and we may work as in 2° (and replace $B^*_{\mathrm{L,w}}$ by $B^*_{\mathrm{L,s}}$ if $\mathbb{D} - D \in L^1(\mathbf{R}_+;\mathcal{B}(U,Y))*$, i.e., if $\mathbb{D} \in \mathrm{MTIC}^{L^1}(U,Y)$). In (7.), we have $f := J\mathbb{C}_{\mathrm{crit}}x_0 \in L^2(\mathbf{R}_+;Y)$, $F \in L^2(\mathbf{R}_+;Y)$, and we may work as in 4° (and replace $B^*_{\mathrm{L,w}}$ by $B^*_{\mathrm{L,s}}$ if $\mathbb{D} - D \in L^2(\mathbf{R}_+;\mathcal{B}(U,Y))*$).

(N.B. We could replace the assumptions of Theorem 8.3.9(b2)&(b2') in (6.) and (7.) by the slightly weaker assumptions that $\mathbb{C}$ is stable and $\mathcal{P} = \mathbb{C}^*J\mathbb{C}_{\mathrm{crit}}$ for any J-critical control in WPLS form, since that assumption is never used for anything else in this monograph.)

6° *Case (4.) when $D^*JC \neq 0$:* Set $K' := -S^{-1}D^*JC \in \mathcal{B}(H,U)$, and let $\Sigma_\flat$ be the corresponding closed-loop system as in Lemma 6.8.4. Then both (3.) and (4.) of Hypothesis 9.2.2 are satisfied with $\Sigma_\flat$ in place of Σ, because $D_\flat = D$ and $C_\flat = C + DK' \in \mathcal{B}(H,Y)$, hence $D^*JC_\flat = D^*J(C+DK') = 0$.

By Theorem 8.4.5(f)&(b), there is a J-critical control over $\mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat\ \mathbb{R}_\flat]}$ in WPLS form for $\Sigma_\flat$ (and Standing Hypothesis 9.0.1 is obviously satisfied for $\Sigma_\flat$, J and $\begin{bmatrix}\mathbb{Q}_\flat & \mathbb{R}_\flat\end{bmatrix}$ too).

Thus, we obtain a J-critical state feedback operator $K_\natural$ for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ over $\mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat\ \mathbb{R}_\flat]}$ from case (3.) of this lemma and Proposition 9.3.1 (which is already known to hold in case (3.)), and then $K := K' + K_\natural \in \mathcal{B}(H,U)$ is J-critical for Σ, by Theorem 8.4.5(f)&(b) and Proposition 6.6.18(f).

Because $\mathcal{P} = \mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft$ is the same for both systems, $\mathcal{P}[H] \subset \mathrm{Dom}((B_\flat)^*_{\mathrm{L,s}})$, and $B^*_{\mathrm{L,s}} = (B_\flat)^*_{\mathrm{L,s}}$ (with same domains), by Proposition 6.6.18(c6) (since $\widetilde{\mathbb{X}} := I - \mathbf{K}'\mathbb{B}\tau = I - K'\mathbb{A}B* \in \mathrm{MTIC}_\infty$), we have $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})$. □

One might be tempted to try to remove the $(\cdot - A)^{-1}B \in \mathrm{H}^2_{\text{strong},\infty}$ assumption from (5.) in case $\mathcal{U}^*_* = \mathcal{U}_{\text{out}}$ by replacing Σ by a suitable realization of $\mathbb{D}$. However, if we choose, e.g., the shift realization (6.11), and ω is big enough to allow that $\mathbb{B}u := \pi_+\mathbb{D}\pi_- u \in H := \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ for each $u \in \mathrm{L}^2_\alpha$ (for some $\alpha \in \mathbf{R}$), then we do no longer know whether $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$ for each $x_0 \in H$ (unless $\omega < 0$, in which case this reduces to (7.)). In general, if we weaken $\|\cdot\|_H$ enough to get $\|Bu_0\|_H < \infty$ (or $\mathbb{A}^t Bu_0 \in H$ for some $t = t_{u_0}$) for all $u_0 \in U$, the closure of $B[U]$ (or $\mathrm{Ran}(\mathbb{B})$) in H grows, and we do no longer know whether $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$ for each $x_0 \in H$.

In case $\mathcal{U}^*_* = \mathcal{U}_{\text{exp}}$, such a removal seems even harder. Analogous problems are faced when one tries to weaken (3.) to the assumption that $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^1_\omega(\mathbf{R}_+;Y))*, C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$.

Next we show how a B^*_{w}-CARE can be reduced to a stabilized one:

Proposition 9.3.5 (Σ-B^*_{w}-CARE $\cong$ $\Sigma_\flat$-B^*_{w}-CARE) *Let K' be an ULR state feedback operator for Σ. Then the solutions $(\mathcal{P}_\natural, S_\natural, K_\natural)$ of the B^*_{w}-CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to the solutions $(\mathcal{P}, S, K)$ of the B^*_{w}-CARE for Σ through*

$$K = K' + K_\natural, \ \ S = S_\natural, \ \ \mathcal{P} = \mathcal{P}_\natural. \tag{9.30}$$

Let K' and $(\mathcal{P}, S, K)$ be as above and $K_\natural = K - K'$. Then

(a) The two top rows ($\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$) of the corresponding closed-loop systems are equal, and Lemma 6.7.11(a') and Lemma 9.12.3(a)–(d2) apply.

(b) If K' is [q.]r.c.-SOS-stabilizing, then K is [q.]r.c.-SOS-stabilizing for Σ iff $K_\natural$ is q.r.c.-SOS-stabilizing (equivalently, stable and r.c.-SOS-stabilizing) for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

Proof: This follows from Proposition 9.12.4:

0° Let $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ be the pair generated by K', $\mathbb{M}' := (I - \mathbb{F}')^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$, so that $\mathbb{M}' \in \mathcal{G}\mathrm{ULR}$. Then $\mathbb{D}$ is ULR iff $\mathbb{D}_\flat := \mathbb{D}\mathbb{M}'$ is ULR, and $D^*JD = D^*_\flat JD_\flat$; in particular the B^*_{w}-CAREs are well-defined (if either is).

1° Let $(\mathcal{P}, S, K)$ be a solution of the B^*_{w}-CARE for Σ. Then $(\mathcal{P}, S, K_\natural)$ is a solution of the CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, Proposition 9.12.4. But $\mathrm{Dom}((B^*_\flat)_{\text{w}}) = \mathrm{Dom}(B^*_{\text{w}})$, by Proposition 6.6.18(c5), hence $(\mathcal{P}, S, K_\natural)$ is a solution of the B^*_{w}-CARE too, by Proposition 9.2.7(a).

2° Conversely, by Proposition 9.12.4, all solutions of the B^*_{w}-CARE for $\Sigma_\flat$ are of this form.

3° Exchange the roles of Σ and $\Sigma_\flat$ for the converse.

(a)&(b) These follow from Lemma 9.12.3. □

Also Hypothesis 9.2.1 can be reduced to the stable case:

Lemma 9.3.6

(a) Assume that Σ has an exponentially stabilizing ULR state feedback operator K', $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\rm exp}^{\Sigma_\flat}$, and $D^*JD \in \mathcal{GB}(U)$. Then Σ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\rm exp}$.

(b) Part (a) also holds with the replacements of Theorem 8.4.5(f).

In particular, then the $B_{\rm w}^*$-CARE for Σ has an exponentially stabilizing solution iff the $B_{\rm w}^*$-CARE for $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ does (with same $\mathcal{P}$, whereas $K = K' + K_\natural$, where $K_\natural$ corresponds to $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ and K' is the preliminary state feedback operator).

Proof: (a) Since K' and the corresponding closed-loop system $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ are ULR, by the assumption, Σ is ULR.

By Theorem 8.4.5(c4), $\mathcal{P}$ (if any) is common for $\mathcal{U}_{\rm exp}$ and $\mathcal{U}_{\rm exp}^{\Sigma_\flat}$. But $\mathrm{Dom}((B_\flat^*)_{\rm w}) = \mathrm{Dom}(B_{\rm w}^*)$, by Proposition 6.6.18(c5), hence also Σ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\rm exp}$.

(b) This follows as in 1°. □

The operator D^*JD does not necessarily contain any information on the signature properties of a problem (see Example 9.13.7), but under sufficient regularity it does:

Lemma 9.3.7 ($\exists (D^*JD)^{-1} \Rightarrow J$-critical control is unique) *If any of (1+)–(4) holds and $D^*JD \in \mathcal{GB}(U)$, then there is at most one J-critical control for each $x_0 \in H$.*

(1+) $J \geq 0$ *and* $\mathbb{D} \in \mathrm{UR}$.

(2+) $\mathcal{J} \geq 0$, $\mathbb{D} \in \mathrm{MTIC}_\infty$ *and* $D^*JD \gg 0$.

(3) $\Sigma \in \mathrm{SOS}$, $\mathbb{D} \in \mathrm{MTIC}_\infty$, $\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$ *and* $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$.

(4) $\mathbb{D} \in \mathrm{MTIC}_\infty$, $\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$, Σ *is optimizable and* $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$.

Proof: Assume that $u \in \mathcal{U}_*^*(0) \setminus \{0\}$. We shall below construct $v \in \mathcal{U}_*^*(0)$ (the construction depends on the additional assumption; in (1+) and (2+) we shall take $v = u$) s.t. $\langle \mathbb{D}v, J\mathbb{D}u\rangle \neq 0$. Since this shows that u is not J-critical (by definition), it follows 0 is the unique J-critical control for $x_0 = 0$. By Lemma 8.3.8, it follows that there is at most one J-critical control for any $x_0 \in H$. W.l.o.g., we assume that

$$T := \inf\{t \geq 0 \,|\, \|\pi_{[0,t)}u\|_2 = 0\} = 0, \quad \text{i.e., } \|\pi_{[0,t)}u\|_2 > 0 \text{ for all } t > 0. \quad (9.31)$$

(by Lemma 9.7.9, $\tau^T u = \pi_+\tau^T u \in \mathcal{U}_*^*(0)$, and $\tau^T u$ is J-critical (since $\langle \mathbb{D}\tau^T u, J\mathbb{D}\eta\rangle = \langle \mathbb{D}u, J\mathbb{D}\tau^{-T}\eta\rangle = 0$ for all $\eta \in \mathcal{U}_*^*(0)$), hence u can be replaced by $\tau^T u$ (and later v by $\tau^{-T}v$) to satisfy (9.31)).

(1+) Choose $\varepsilon > 0$ s.t. $D^*JD \gg \varepsilon^2 I$. Then $\|J^{1/2}Du_0\| \geq \varepsilon\|u_0\|$ for all $u_0 \in U$. Choose $\omega > 0$ s.t. $\|J^{1/2}\widehat{\mathbb{D}}(s) - J^{1/2}D\|_{\mathcal{B}} < \varepsilon/2$ for all $s \in \mathbf{C}_\omega^+$. Choose $s \in \mathbf{C}_\omega^+$ s.t. $\widehat{u}(s) \neq 0$. Then $\|J^{1/2}\widehat{\mathbb{D}}(s)\widehat{u}(s)\|_Y > \varepsilon\|\widehat{u}(s)\| - \varepsilon\|\widehat{u}(s)\|/2 > 0$, hence $J^{1/2}\mathbb{D}u \neq 0$, hence $0 < \|J^{1/2}\mathbb{D}u\|_2^2 = \langle \mathbb{D}u, J\mathbb{D}u\rangle$.

(2+) Set $\varepsilon := \|(D^*JD)^{-1/2}\|^{-1} > 0$, so that $\langle u_0, D^*JDu_0\rangle = \|(D^*JD)^{1/2}u_0\|_U^2 \geq \varepsilon^2\|u_0\|_U^2$ for all $u_0 \in U$. By Theorem 2.6.4(i1), there is $t > 0$ s.t.

$$\|\pi_{[0,t)}(\mathbb{D}^*\pi_{[0,t)}J\mathbb{D} - D^*JD)\pi_{[0,t)}\| < \frac{\varepsilon^2}{2}. \tag{9.32}$$

Consequently,

$$\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \geq \langle Du, \pi_{[0,t)}JDu\rangle - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 \geq \varepsilon^2\|\pi_{[0,t)}u\|_2^2 - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 > 0. \tag{9.33}$$

Thus, $\langle \mathbb{D}u, J\mathbb{D}u\rangle = \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle + \mathcal{J}(\mathbb{B}^t u, \pi_+\tau^t u) \geq \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle > 0$, by (8.52), hence u is not J-critical.

(3) Set $\varepsilon := \|(D^*JD)^{-1}\|^{-1} > 0$, $M := \|D^*JD\|$, $M' := \|\mathbb{D}^*J\|\|\mathbb{C}\|$, so that $\|D^*JDu_0\| \geq \varepsilon\|u_0\|$ for all $u_0 \in U$. By Theorem 2.6.4(i1), there is $t > 0$ s.t.

$$\|\pi_{[0,t)}D^*JD(\mathbb{D}^*\pi_{[0,t)}J\mathbb{D} - D^*JD)\pi_{[0,t)}\| < \varepsilon^2/2 \quad \text{and} \quad \|\mathbb{B}^t\| < \varepsilon^2/3MM' \tag{9.34}$$

(take $\|\pi_{[0,t)}(J\mathbb{D} - JD)\pi_{[0,t)}\|$ and $\|\pi_{[0,t)}(\mathbb{D} - D)D^*JD\pi_{[0,t)}\|$ small enough). Set $v := \pi_{[0,t)}D^*JDu$. Then

$$\operatorname{Re}\langle \mathbb{D}v, J\mathbb{D}u\rangle_{L^2} = \operatorname{Re}\langle \pi_{[0,t)}\mathbb{D}D^*JDu, J\mathbb{D}u\rangle + \operatorname{Re}\langle \tau^{-t}\mathbb{C}\mathbb{B}^t D^*JDu, J\mathbb{D}u\rangle \tag{9.35}$$

$$\geq \langle DD^*JDu, \pi_{[0,t)}JDu\rangle - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 - \frac{\varepsilon}{3}\|\pi_{[0,t)}u\|_2^2 \tag{9.36}$$

$$\geq \|\pi_{[0,t)}D^*JDu\|_2^2 - \frac{5\varepsilon^2}{6}\|\pi_{[0,t)}u\|_2^2 \geq \varepsilon^2\|\pi_{[0,t)}u\|_2^2 - \frac{5\varepsilon^2}{6}\|\pi_{[0,t)}u\|_2^2, \tag{9.37}$$

by (9.31).

(4) By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$. By Lemma 6.8.4(b), we have $\mathbb{A}_\flat B \in L^2(\mathbf{R}_+; \mathcal{B}(U,H))$ and $\mathbb{D}_\flat \in \mathrm{MTIC}_\infty$.

Since $D_\flat^*JD_\flat = D^*JD \in \mathcal{GB}(U)$, there is at most one J-critical control for $\Sigma_\flat$ over $\mathcal{U}_{\exp}^{\Sigma_\flat}(x_0) = \mathcal{U}_{\text{out}}^{\Sigma_\flat}(x_0)$ for each $x_0 \in H$, by (3).

By Theorem 8.4.5(c2)&(c1) (or the "iff" in (c3)), it follows that there is at most one J-critical control for Σ over $\mathcal{U}_{\exp}(x_0)$ for each $x_0 \in H$. □

(See the notes on p. 434.)

9.4 Analytic semigroups

A modern revolutionary group heads for the television station.
— Abbie Hoffman (1936–)

In this section we shall list the basic properties of analytic semigroups. These will be applied in Section 9.5 to WPLSs with an analytic semigroup.

Standing Hypothesis 9.4.1 *Throughout this section, $\mathbb{A}$ is assumed to be an analytic semigroup. We also assume that $\zeta > \omega_A$.*

An *analytic semigroup* is a C_0-semigroup having numbers $\omega > \omega_A$ and $M_{A,\omega} < \infty$ s.t. $\|(s-A)^{-1}\| \le M_{A,\omega}/|s-\omega|$ for all $s \in \mathbf{C}^+_\omega$ (see, e.g., Section 2.5 of [Pazy] for equivalent definitions). (Note from Lemma A.4.4(c1) that any semigroup satisfies $\|(s-A)^{-1}\| \le M_{A,\omega}/\operatorname{Re}(s-\omega)$.)

For all $\beta > 0$, we define the fractional powers of $(\cdot - A)$ by setting

$$(\zeta - A)^{-\beta} := \Gamma(\beta)^{-1} \int_0^\infty t^{\beta-1} e^{-\zeta t} \mathbb{A}(t)\, dt \in \mathcal{B}(H) \tag{9.38}$$

$$(\zeta - A)^{\beta} := ((\zeta - A)^{-\beta})^{-1}, \quad H_\beta := \operatorname{Dom}((\zeta - A)^{\beta}) := \operatorname{Ran}((\zeta - A)^{-\beta}). \tag{9.39}$$

We let H_β be the closure of H w.r.t. $\|\cdot\|_{H_\beta} := \|(\zeta - A)^\beta \cdot\|_H$ for $\beta < 0$, and set $\|x\|_{H_\beta} := \|(\zeta - A)^\beta x\|_H$ for all $\beta \in \mathbf{R}$ (these definitions coinside with spaces H_n ($n \in \mathbf{Z}$) defined in Lemma 6.1.16; in particular, $H_0 = H$). These spaces are independent of ζ up to an equivalent norm, by Lemma 9.4.2(f2).

Lemma 9.4.2 (Properties of analytic semigroups)

(a) For any $\omega > \omega_A$, there are $\theta \in (\pi/2, \pi]$ and $M = M_{A,\omega,\theta} < \infty$ s.t. $\|(s-A)^{-1}\| \le M/|s-\omega|$ for all s in

$$\Sigma_{\theta,\omega} := \{s \in \mathbf{C} \mid s \ne \omega,\ |\arg(s-\omega)| < \theta\}. \tag{9.40}$$

(Note that $\mathbf{C}^+_{\omega'} \subset \Sigma_{\theta,\omega}$ for any $\omega' > \omega$.)

(b) $H_\gamma \underset{c}{\subset} H_\beta$ densely ($\gamma \ge \beta$), the map $(\zeta - A)^\beta \in \mathcal{B}(H_{\gamma+\beta}, H_\gamma)$ is an isometric isomorphism, and $(\zeta - A)^\beta (\zeta - A)^\gamma = (\zeta - A)^{\beta+\gamma}$ ($\beta, \gamma \in \mathbf{R}$).

(c1) Also $\mathbb{A}^*$ is analytic, and $(H_\beta)^* = (H^*)_{-\beta} =: H^*_{-\beta}$ ($\beta \in \mathbf{R}$).

(c2) Also $e^{s\cdot}\mathbb{A}$ (with generator $s + A$) is analytic for any $s \in \mathbf{C}$.

(d) The semigroup $\mathbb{A}_\beta := (\zeta - A)^{-\beta} \mathbb{A} (\zeta - A)^\beta$ on H_β is isometrically isomorphic to $\mathbb{A}$ ($\beta \in \mathbf{R}$); we denote all these semigroups on $\mathbb{A}$ and their generators by A.

(e) $(\omega - A)^\beta \mathbb{A}^t = \mathbb{A}^t (\omega - A)^\beta$ and $(\omega - A)^\beta (s-A)^{-1} = (s-A)^{-1} (\omega - A)^\beta$ ($t \ge 0$, $\beta \in \mathbf{R}$, $\omega > \omega_A$, $s \in \sigma(A)^c$).

(f1) $(s-A)^{-1} \in \mathcal{GB}(H_\beta, H_{\beta+1})$ for any $s \in \sigma(A)^c$, $\beta \in \mathbf{R}$, by the resolvent equation.

(f2) $(\omega - A)^{-\beta} \in \mathcal{GB}(H_\alpha, H_{\alpha+\beta})$ *$(\alpha, \beta \in \mathbf{R}, \omega > \omega_A)$.*

(f3) $(\cdot - A)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_\zeta; \mathcal{B}(H_\beta, H_{\beta+1}))$, $(s \mapsto s(s-A)^{-1}) \in \mathrm{H}^\infty(\mathbf{C}^+_\zeta; \mathcal{B}(H_\beta))$ *$(\beta \in \mathbf{R})$.*

(f4) $A \in \mathcal{GB}(H_\beta, H_{\beta-1})$, $(\beta \in \mathbf{R})$.

(g) $(s-A)^{-1} \to 0$ *strongly in* $\mathcal{B}(H_\beta, H_{\beta+1})$*, as* $s \in \Sigma_{\theta,\omega}$, $|s| \to +\infty$*, for any* ω *and* θ *as in (a).*

(h1) For each $\omega > \omega_A$*, there is* $M' < \infty$ *s.t.*

$$\|(\zeta - A)^\beta \mathbb{A}^t\|_{\mathcal{B}(H)} \le M'(1 + t^{-\beta})e^{\omega t} \quad (t > 0,\ \beta \in [0,1]). \tag{9.41}$$

(h2) We have $\mathbb{A}^t \in \mathcal{B}(H_\alpha, H_\beta)$ *and* $\|\mathbb{A}^t\|_{\mathcal{B}(H_\alpha, H_{\alpha+\beta})} \le M_{\beta,\zeta} t^{-\beta} e^{\zeta t}$ $(\alpha, \beta \in \mathbf{R},\ t > 0)$.

(i) $\mathbb{A} \in C^\infty((0,+\infty); \mathcal{B}(H_\beta))$ *$(\beta \in \mathbf{R})$.*

(j) $(\zeta - A)^\beta \mathbb{A} \in \mathrm{L}^p_\omega(\mathbf{R}_+; \mathcal{B}(H_\alpha))$ *$(\beta p < 1$, $\omega > \omega_A$, $p \in [1,\infty]$, $\beta \le 1$, $\alpha \in \mathbf{R})$.*

(k) $\|(\omega - A)^\alpha (s-A)^{-1}\|_{\mathcal{B}(H)} \le M(1 + |s - \omega_0|^\alpha)/|s - \omega_0|$ *$(s \in \Sigma_{\theta,\omega_0}$, $\omega > \omega_0 > \omega_A$, $0 \le \alpha \le 1)$ for* $\theta \in (\pi/2, \pi]$ *as in (a), where* $M < \infty$ *depends only on* A*,* ω_0 *and* θ*.*

(l) $(\omega - A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{-\alpha}(s + \omega - A)^{-1}\, ds$ *$(\omega > \omega_A$, $\alpha \in (0,1))$.*

Proof: (a) One obtains this from Theorem 2.5.2 of [Pazy] (move the sector to the left and decrease the angle simultaneously, so that the sector is contained in the union of the old sector plus a compact subset of $\mathbf{C}^+_{\omega_A}$).

(c1) By, e.g., Theorem 5.2 of [Pazy] (shifted by ζ_A), $\mathbb{A}^*$ is analytic with uniformly bounded growth bound ζ_A.

Define the spaces $(H^*)_\beta =: H^*_\beta$ with A^* in place of A. For all $x \in H_\beta$, $z \in H$, we have

$$|\langle x, z\rangle_H| = |\langle (\zeta - A)^\beta x, (\zeta - A^*)^{-\beta} z\rangle_H| \le \|x\|_{H_\beta} \|z\|_{H^*_{-\beta}}, \tag{9.42}$$

hence $H^*_{-\beta}$ is the closure of H w.r.t. $\|\cdot\|_{(H_\beta)^*}$, hence it can be identified to $(H_\beta)^*$.

(f2) Use, e.g., the sine formula (2.6.4) of [Pazy] to show that the ranges of $(\zeta - A)^{-\beta}$ and $(r - A)^{-\beta}$ are equal for $\beta \in (0,1)$. Integral powers agree by the resolvent equation (i.e., by (f1)), and negative powers follow from the positive ones. The topologies coincide by Lemma A.3.6.

(f3) Use the resolvent equation (boundedness follows from the definition of an analytic semigroup).

(g) This was shown in Lemma A.4.4(e2) (which provides us several convergence results) in case $\beta = 0$; the other cases follow from (d).

(i) This follows from Corollary 2.4.4 and Theorem 2.5.2(d) of [Pazy].

(j) By (i), $(\zeta - A)^\beta \mathbb{A} \in \mathrm{L}(\mathbf{R}_+; \mathcal{B}(H_\beta))$. By (h) (with some $\alpha \in (\omega_A, \omega)$ in place of ω), we have $\|(\zeta - A)^\beta \mathbb{A}\| \in \mathrm{L}^p_\omega$.

(b)&(c2)–(f4)&(h1)&(h2)&(k)&(l) All this is well-known or almost obvious; see, e.g., [Pazy], [Sbook], [Lunardi] and/or [HP]. □

If the "discontinuity" of a perturbation (e.g., "feedback operator") T is less than one, the resulting ("closed-loop") semigroup is also analytic:

Lemma 9.4.3 (Properties of interpolation spaces H_α $(\alpha \in \mathbf{R})$)

(a1) Let $T \in \mathcal{B}(H_\gamma, H_\alpha)$, $\gamma - \alpha < 1$. Then the operator $A+T$ with domain $H_{\beta+1}$ generates an analytic semigroup on H_β, for any $\beta \in [\gamma-1, \alpha+1]$.

In particular, if we define the spaces $\widetilde{H}_\beta$ ($r \in \mathbf{R}$) as H_β with $A+T$ in place of A, then $H_\beta = \widetilde{H}_\beta$ (with equivalent norms), i.e., $(\omega - A)^{-\beta}[H] = (\omega - A - T)^{-\beta}[H]$, for all $\beta \in [\gamma-1, \alpha+1]$.

(a2) Even if we shifted the indices by replacing H by H_{β_0} (i.e., by considering A as a semigroup on H_{β_0}, not on H), for any $\beta_0 \in [\gamma-1, \alpha+1]$), before defining the spaces H_β and $\widetilde{H}_\beta$ (in (a1)), the results and the spaces H_β and $\widetilde{H}_\beta$ would be unaltered (except for the shift in the index).

(b1) Let A and $\widetilde{A}$ generate analytic semigroups on H, and $\mathrm{Dom}(A) = \mathrm{Dom}(\widetilde{A})$. Define the spaces $\widetilde{H}_\alpha$ ($\alpha \in \mathbf{R}$) as H_α with $\widetilde{A}$ in place of A. Then

$$\widetilde{H}_1 = H_1 \underset{c}{\subset} H_\alpha \underset{c}{\subset} \widetilde{H}_\beta \underset{c}{\subset} H_\gamma \underset{c}{\subset} H = \widetilde{H}_0 \ \ (1 \geq \alpha \geq \beta \geq \gamma \geq 0). \tag{9.43}$$

(b2) We have $(\omega - A)^{-\alpha} \in \mathcal{B}(\widetilde{H}_\beta, \widetilde{H}_\gamma)$ and $(r - \widetilde{A})^{-\alpha} \in \mathcal{B}(H_\beta, H_\gamma)$ in (b1) when $\alpha > \gamma - \beta$, $\beta, \gamma \in [0,1]$, $\omega > \omega_A$, $r > \omega_{\widetilde{A}}$.

Note that if $0 \in [\gamma-1, \alpha+1]$, then $A+T$ generates an analytic semigroup on H, and the definition of $\widetilde{H}_\alpha$ can be based on H (instead of some H_β).

It seems that if we would replace the spaces H_r ($r \in (0,1)$) by the interpolation spaces $\mathcal{D}(r,p)$ or $\mathcal{D}(p)$ of [Lunardi], we would obtain analogous results even more easily. However, we have chosen the spaces H_r to make the comparison to earlier results easier. Because of the interpolation properties, the difference in smoothness is less than ε for any $\varepsilon > 0$.

Proof of Lemma 9.4.3: (a1) *Part I: Assume that $T \in \mathcal{B}(H_1, H_\alpha)$, $\alpha \in (0,1]$.*

1° *We have $(\omega - A - T)^{-\beta} \in \mathcal{B}(H, H_\beta)$ for $\beta \in [0,1)$:* Assume, w.l.o.g., that $\beta > 0$. By Propositions 2.4.1(ii) and 2.2.13 of [Lunardi], $A+T$ with $\mathrm{Dom}(A+T) := \mathrm{Dom}(A)$ generates an analytic semigroup on H. Obviously,

$$(s-A)^{-1}(I + T(s-A-T)^{-1}) = (s-A-T)^{-1} \quad (s \in \sigma(A)^c \cap \sigma(A+T)^c). \tag{9.44}$$

Let $\omega > \omega_1 := \max\{\omega_A, \omega_{A+T}\}$ and $\beta \in (0,1)$. Then, by (9.44), we have (here $c := \pi^{-1} \sin \pi\alpha$)

$$(\omega - A - T)^{-\beta} - (\omega - A)^{-\beta} = c \int_0^\infty s^{-\beta} (\omega + s - A)^{-1} T (\omega + s - A - T)^{-1} \, ds. \tag{9.45}$$

By Lemma A.4.4(c2), $\|T(\omega + s - A - T)^{-1}\|_{\mathcal{B}(H, H_\alpha)} \leq M_\omega$ and $\|(\omega + s - A)^{-1} T (\omega + s - A - T)^{-1}\|_{\mathcal{B}(H, H_{\alpha+1})} \leq M'_\omega$ for $s \geq 0$. Thus, part $\int_0^1$ of the integral (9.45) converges in $H_{\alpha+1}$, so we only need to show that also $(\omega - A)^\beta \int_1^\infty$

belongs to $\mathcal{B}(H)$ in order to establish that $(\omega - A - T)^{-\beta} \in \mathcal{B}(H, H_\beta)$. By Lemma 9.4.2(k), this follows from the fact that (choose any $\omega_0 \in (\omega_A, \omega)$)

$$\int_1^\infty s^{-\beta} M(1 + |s + \omega - \omega_0|^{\beta-\alpha})/|s + \omega - \omega_0| \, ds < \infty. \tag{9.46}$$

2° *We have* $H_\beta = \widetilde{H}_\beta$ *for* $\beta \in [0, 1)$*:* By 1°, we have $\widetilde{H}_\beta := \mathrm{Ran}((\omega - A - T)^{-\beta}) \subset H_\beta$. Exchange the roles of A and $A + T$ to obtain that $H_\beta \subset \widetilde{H}_\beta$. By Lemma A.3.6, also the topologies coincide (i.e., the norms are equivalent).

Part II: Assume that $T \in \mathcal{B}(H_\alpha, H)$, $\alpha < 1$. W.l.o.g., we assume that $\alpha \geq 0$, because $H \underset{c}{\subset} H_\alpha$, hence $\mathcal{B}(H_\alpha, H) \subset \mathcal{B}(H_0, H)$ for $\alpha < 0$.

3° *We have* $H_\beta = \widetilde{H}_\beta$ *for* $\beta \in [\alpha - 1, \alpha]$*:* Apply 2° to the state space $H_{\alpha-1}$.

4° *We have* $H_\beta = \widetilde{H}_\beta$ *for* $\beta \in [\alpha - 1, 1]$*:* Let $\beta \in [\alpha - 1, 0]$. Then $(I + T(s - A - T)^{-1}) \in \mathcal{GB}(H_\beta)$ for s big enough, hence the two resolvents have same range, i.e., $H_{\beta+1} = \widetilde{H}_{\beta+1}$, by (9.44). Thus, we have covered the values in $[\alpha, 1]$; the others were covered in 3°.

Part III: Assume that $T \in \mathcal{B}(H_\gamma, H_\alpha)$, $\gamma - \alpha < 1$*:* Just apply Part II with H_α in place of H.

5° *Remarks:* Note that we again identify the analytic semigroup $\widetilde{\mathbb{A}}$ on $H_{\alpha-1}$ generated by $A + T$ with domain $\mathrm{Dom}(A + T) := H_\alpha$ and its restriction onto H_β (which is generated by $A + T$ with domain $\mathrm{Dom}(A + T) := H_{\beta+1}$) for any $\beta \in [\alpha - 1, 1]$.

If $\beta > 1$*, then we may have* $\widetilde{H}_\beta \neq H_\beta$*:* Let $\beta \in (1, 2]$. Unless A is bounded, we can choose $x_0 \in H_2$, $\alpha \in [0, 1)$, $z \in H \setminus H_{\beta-1}$ and $\Lambda \in H_\alpha^*$ s.t. $\Lambda x_0 = 1$, and set $Tx := (Lax)z$ ($x \in H_\alpha^*$), so that $T \in \mathcal{B}(H_\alpha, H)$ and $Tx_0 = z \notin H_{\beta-1}$. It follows that $(A + T)x_0 \notin H_{\beta-1}$, hence $x_0 \in H_\beta \setminus \widetilde{H}_\beta$ (since $(A + T)[\widetilde{H}_\beta] \subset \widetilde{H}_{\beta-1} = H_{\beta-1}$).

(a2) (You can consider A (or its restriction or extension) as an analytic semigroup on any H_β, and it is not always obvious, which one you should use. Here we stated that you may use any of them and still get the same results and spaces.)

Set $G := H_{\beta_0}$, and define the spaces G_β and $\widetilde{G}_\beta$ as in (a1) (by (a1), this is possible and $G_\beta = \widetilde{G}_\beta$ for all $\beta \in [\gamma - 1 - \beta_0, \alpha + 1 - \beta_0]$).

By Lemma 9.4.2(f2), we have $G_\beta = H_{\beta_0+\beta}$ and $\widetilde{G}_\beta = \widetilde{H}_{\beta_0+\beta}$ ($\beta \in \mathbf{R}$). Thus, we obtained again the same fractional power spaces, even though we had the starting point G in place of H.

(b1) By Lemma A.3.6, the spaces $H_1 := \mathrm{Dom}(A)$ and $\widetilde{H}_1 := \mathrm{Dom}(\widetilde{A})$ have equivalent norms. If $\alpha = 1$, then $H_\alpha = \widetilde{H}_1 \underset{c}{\subset} \widetilde{H}_\beta$, so we assume that $\alpha < 1$ (and $\beta > 0$). By Propositions 2.2.13 and 1.2.3 of [Lunardi], we have

$$H_\alpha \underset{c}{\subset} (H, H_1)_{\alpha,\infty} \underset{c}{\subset} (H, H_1)_{\beta,1} \underset{c}{\subset} \widetilde{H}_\beta. \tag{9.47}$$

Analogously, $\widetilde{H}_\beta \underset{c}{\subset} H_\gamma$.

(b2) This follows from (9.43) and Lemma 9.4.2(b)&(f2). □

Notes

As obvious from the proofs, most of the above is well known. Classical references on semigroups include [Pazy] and [HP]; see [Lunardi] and [Sbook] for further results on analytic semigroups.

9.5 Parabolic problems and CAREs

All animals are equal, but some animals are more equal than others
— George Orwell (1903–1950), "Animal Farm", 1945

In this section, we apply our results for systems having analytic (see Section 9.4) semigroups. The proofs only use the fact that the smoothness of $\mathbb{A}$ compensates the unboundedness of B and C, so that the results of this section (with same proofs, mutatis mutandis) can be applied whenever $\mathbb{A}$ is smoothing.

In Theorem 9.5.9, we show that a smooth Riccati equation has a stabilizing solution iff there is a unique optimal control. The resulting closed-loop system is also analytic. A corollary of this for minimization problems is given in In Corollary 9.5.10 we interpret this for minimization problems, and in Corollaries 9.5.11–9.5.12 for H^∞ problems. E.g., there is a suboptimal H^∞ state feedback pair for the system iff the Riccati equation (11.15) has a nonnegative solution s.t. $A+BK$ is exponentially stable (assuming the standard signature and nonsingularity conditions therein); moreover, the Riccati equation can be replaced by the smoother on in Theorem 9.5.9 and the closed-loop system is analytic.

We assume that $\mathbb{A}$ is analytic and that the unboundedness the input and output operators is less than $1/2$ each:

Standing Hypothesis 9.5.1 *Throughout this section and Section 9.6, letters U, H and Y denote Hilbert spaces of arbitrary dimensions, $\mathbb{A}$ is an analytic semigroup, $\zeta > \omega_A$, $\beta > -1/2$, $\gamma < 1/2$, $B \in \mathcal{B}(U, H_\beta)$, $C \in \mathcal{B}(H_\gamma, Y)$, $D \in \mathcal{B}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$.*

(By Lemma 9.5.2, it follows that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate a WPLS (also without Standing Hypothesis 9.0.1).)

By spaces H_r we denote the spaces $(\zeta - A)^{-r} H$ ($r \in \mathbf{R}$), as in Section 9.4. Note that we may assume that $\gamma \ge 0 \ge \beta$, w.l.o.g. (i.e., if the hypothesis holds and we replace γ by $\max\{0,\gamma\}$ and β by $\min\{0,\beta\}$, then the hypothesis still holds, by Lemma 9.4.2(b)).

Drop the assumptions $\beta > -1/2$, $\gamma < 1/2$ for a moment. Whenever $\gamma, \beta \in \mathbf{R}$ and $\gamma - \beta < 1$, we can replace H by $H_{(\gamma+\beta)/2}$ above to obtain a WPLS, by Lemma 9.5.2. The only difference between different state spaces is that in applications we must require that the finite cost condition is satisfied in H (not in some H_r), hence one might wish to take H as small as possible. Therefore, in some applications it might be more suitable to realize $\mathbb{D}$ on, e.g., H_γ, in which case Σ need no longer be a WPLS (unless $\gamma - \beta < 1/2$); cf. Section 8.6.

The system $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a smooth Pritchard–Salamon system w.r.t. $\mathcal{W} := H_\gamma$ and $\mathcal{V} := H_\beta$ whenever $\gamma - \beta < 1/2$, as one can deduce from the following:

Lemma 9.5.2 *Let $\omega > \omega_A$. Then the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ $\mathrm{WPLS}_\omega(U,H,Y)$. Moreover, $\mathbb{A}B \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,H))$, $C\mathbb{A} \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,H))$, and $C\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$, hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega \subset \mathrm{ULR} \cap \mathrm{UVR}$. Finally, $C \subset (C_{|H_1})_{\mathrm{L,s}}$.*

Note that the "standard" C (that of Chapter 6) is given by $C_{|H_1}$. Analogously, the "standard" B is given by $B \in \mathcal{B}(U, H_{-1})$ with adjoint $B^* \in \mathcal{B}(H_1^*, U)$, but we shall write B^* for the adjoint $B^* \in \mathcal{B}(H_{-\beta}^*, U)$ of $B \in \mathcal{B}(U, H_\beta)$; we still have $B^* \subset B^*_{\mathrm{L,s}}$, where $B^*_{\mathrm{L,s}}$ can be computed from either B^*.

Proof: (This holds also without Standing Hypothesis 9.0.1.) The "moreover" claims follow from Lemma 9.4.2(j), and they imply that $\Sigma \in$ WPLS, by Lemmas D.1.7 and 6.3.13. We have $C \subset (C_{|H_1})_{\mathrm{L,s}}$, because $\frac{1}{t}\int_0^t \mathbb{A}^r x_0\, dr \to x_0$ in H_γ, for any $x_0 \in H_\gamma$, by Lemma 9.4.2(d), □

Remark 9.5.3 (Optimizable iff exponentially stabilizable) *The* L^1 *assumptions of Theorem 9.2.12 and Corollary 9.2.13 are satisfied (under Standing Hypothesis 9.5.1). In particular,* Σ *is optimizable [and estimatable] iff* Σ *is exponentially [jointly] stabilizable [and detectable].* □

The class of analytic WPLSs is invariant under smooth feedback, and the spaces H_α remain unaffected (for α's sufficiently close to zero):

Lemma 9.5.4 *If* $K \in \mathcal{B}(H_r, U)$, $r < 1/2$, *then* K *is an ULR admissible state feedback operator for* Σ. *Moreover, if* $\Sigma_\flat$ *is the closed-loop system corresponding to* $\begin{pmatrix} K \mid I - X \end{pmatrix}$ *for some* $X \in \mathcal{GB}(U)$, *then* $\mathbb{A}_\flat$ *is analytic on* $H_\alpha = (H_\flat)_\alpha$ *for* $\alpha \in [r-1, \beta+1]$, *and also* $\Sigma_\flat$ *satisfies Hypothesis 9.5.1 (with* $\gamma_{\flat'} := \max\{\gamma, r\} < 1/2$ *in place of* γ*).*

From Proposition 6.6.18 it follows that the closed-loop generators are given by $\left[\begin{array}{c|c} A+BMK & BM \\ \hline C+DMK & DM \\ K & M-I \end{array}\right]$, where $M := X^{-1}$.

Proof: 1° *Case* $X = I$*:* By Lemma 9.5.2, $\left[\begin{array}{c|c} A & B \\ \hline K & 0 \end{array}\right]$ generate a WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right]$ s.t. $\mathbb{F} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset$ ULR and $F = 0$, hence $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (even $\mathcal{G}\mathrm{MTIC}_\infty^{\mathrm{L}^1}(U)$), by Proposition 6.3.1(c). The last claim follows from Lemma 9.4.3(a) (recall that $C_\flat = C + DK$).

2° *General case:* First apply 1° to $\begin{pmatrix} MK \mid 0 \end{pmatrix}$ (note that $MK \in \mathcal{B}(H_r, U)$), and then apply Lemma 6.6.12 to replace $\begin{pmatrix} MK \mid 0 \end{pmatrix}$ by $\begin{pmatrix} XMK \mid I - X \end{pmatrix}$. □

Lemma 9.5.5 *If* $\mathcal{P} \in \mathcal{B}(H)$, *then* $\|sB^*(s-A)^{-*}\mathcal{P}(s-A)^{-1}B\| \to 0$, *as* $s \in \Sigma_{\theta,\omega_0}$, $|s| \to \infty$. □

(This follows from Lemma 9.4.2(k) with $\alpha \mapsto -\beta < 1/2$.)

Corollary 9.5.6 *If* $(\mathcal{P}, S, \begin{bmatrix} K \mid F \end{bmatrix})$ *is a SR solution of the eCARE, then* $X^*SX = D^*JD$. □

(This follows from Lemma 9.5.5 and Propositions 9.11.3(b) and 9.8.10.)

We shall often make the following, stronger hypothesis:

Hypothesis 9.5.7 *Let at least one of (1.)–(3.) hold, where*

(1.) $\gamma < 1/4$, $\beta > -1/2$ *and* $D^*JC = 0$*;*

(2.) $\gamma < 1/4$, $\beta > -1/2$ *and* $D^*JD \in \mathcal{GB}(U)$*;*

(3.) $\gamma - \beta < 1/2$*;*

By Theorem 9.2.3, Hypothesis 9.5.7 implies Hypothesis 9.2.1, hence it implies that Theorems 9.2.9, 9.2.10 and 9.2.14, Corollary 9.2.15 etc. apply (under the additional assumptions of those results).

Naturally, whenever $\gamma, \beta \in \mathbf{R}$, $\gamma - \beta < 1/2$, we can replace H by H_r for any $r \in (\gamma - 1/2, \beta + 1/2)$ to satisfy (3.). Analogously, whenever $\gamma, \beta \in \mathbf{R}$, $\gamma - \beta < 3/4$ and $D^*JC = 0$ or $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and $D^*JD \in \mathcal{GB}(U)$, we can replace H by H_r for any $r \in (\gamma - 1/4, \beta + 1/2)$ to satisfy (1.) or (2.). However, we need additional assumptions for guaranteeing that this does not change the problem (the most important case is the stable case for $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{exp}}$):

Lemma 9.5.8 (Shifting $H \mapsto H_r$) *Let Σ' be the system Σ with state space H_r, where $r \in (\gamma - 1/2, \beta + 1/2)$, and that $J = J^* \in \mathcal{B}(Y)$. Assume that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$ (so that $\mathbb{D}'$ is J-coercive over $\mathcal{U}'_{\text{out}}$). Assume that $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$ for all $x_0 \in H$ and $\mathcal{U}'_{\text{out}}(x_0) \neq \emptyset$ for all $x_0 \in H_r$ (e.g., that $\mathbb{A}$ is exponentially stable).*

If $r > 0$, then $\mathcal{P}' = \mathcal{P}|_{H_r}$ and $\Sigma'_{\text{crit}} = \Sigma_{\text{crit}}|_{H_r}$ (if K' is as in Lemma 9.6.1, then K' is the unique J-critical state feedback operator for Σ too). Moreover, $\mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ for all $x_0 \in H \cap H_r$.

(Exchange the roles of H and H_r for the case $r < 0$.)

Proof: (Here $\mathcal{U}'_{\text{out}}$ means $\mathcal{U}_{\text{out}}$ for Σ' etc., and $\mathcal{P}$ (resp. $\mathcal{P}'$) is the J-critical cost operator over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}'_{\text{out}}$). Note that $r \in (\gamma - 1/2, \beta + 1/2)$ is equivalent to the condition that Standing Hypothesis 9.5.1 holds for Σ'.)

Note that $\mathcal{U}_{\text{out}}(x_0)$ depends on x_0, $\mathbb{D}$ and J only (not on H, nor on the rest of Σ).

Recall from Lemma 8.4.2 that J-coercivity over $\mathcal{U}_{\text{out}}$ depends only on J and $\mathbb{D}$, hence $\mathbb{D}'$ is J-coercive over $\mathcal{U}'_{\text{out}}$.

Under the assumptions, there is a unique J-critical control over $\mathcal{U}_{\text{out}}$ for each $x_0 \in H$ and over $\mathcal{U}'_{\text{out}}$ for each $x_0 \in H_r$, by Theorem 8.4.3, thus $\mathcal{P}$, Σ_{crit} and Σ'_{crit} are well-defined, by Theorem 8.3.9.

Assume that $r > 0$. Since ($\mathbb{A}' = \mathbb{A}|_{H_r}$ and) $\mathbb{C}' = \mathbb{C}|_{H_r}$, we have $\mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ for all $x_0 \in H_r = H_r \cap H$ (because $\mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}'x_0 + \mathbb{D}u$ for all $u \in \mathrm{L}^2(\mathbf{R}_+; U)$).

Given $x_0 \in H_r$, a vector $u \in \mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ is J-critical over $\mathcal{U}_{\text{out}}$ iff it is J-critical over $\mathcal{U}'_{\text{out}}$ (iff $0 = \langle \mathbb{C}'x_0 + \mathbb{D}u, J\mathbb{D}\eta \rangle = \langle \mathbb{C}x_0 + \mathbb{D}u, J\mathbb{D}\eta \rangle$ for all $\eta \in \mathcal{U}_{\text{out}}(0) = \mathcal{U}'_{\text{out}}(0)$). It follows that $\Sigma'_{\text{crit}} = \Sigma_{\text{crit}}|_{H_r}$, hence $\mathcal{P}' := (\mathbb{C}'_{\text{crit}})^* J \mathbb{C}'_{\text{crit}} = \mathcal{P}|_{H_r}$.

If K' is as in Lemma 9.6.1, then $K' \in \mathcal{B}(H_\gamma, U)$, hence then K' is admissible for Σ too; let $\Sigma_\circlearrowleft$ be the corresponding closed-loop system. The restriction of

K_{crit} to H_{r+1} is equal to $K'_{\text{crit}} = K'$, hence $K_{\text{crit}} = K'$ on H_1, by continuity. We conclude that $\mathbb{K}_{\circlearrowleft} = \mathbb{K}_{\text{crit}}$, i.e., K' is J-critical for Σ over $\mathcal{U}_{\text{out}}$. Uniqueness follows from Lemma 8.3.17(b). □

Under Hypothesis 9.5.7 we have more smoothness than in the B_{w}^*-CARE theory:

Theorem 9.5.9 *Assume that* $S := D^*JD \in \mathcal{GB}(U)$, $\gamma < 1/4$ *and* $\beta > -1/2$. *Then the assumptions of Theorem 9.2.9 are satisfied and we have one more equivalent condition:*

(vi) There is $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, (H_\beta)^*)$ *s.t.*

$$(B^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \qquad (9.48)$$

on $H_{\gamma+\varepsilon}$ *for some (equivalently, all)* $\varepsilon > 0$, *and* $K := -(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC)$ *is* $\mathcal{U}_*^*$*-stabilizing.*

Moreover, also Lemmas 9.6.1 and 9.5.4 are applicable for $\mathcal{P}$ *and* K.

Proof: 1° *The proof:* The assumptions of Theorem 9.2.9 are now satisfied, by Theorem 9.2.3 (use Hypothesis 9.2.2(2.)). One easily verifies that (vi) implies (iii). Conversely, (i) implies (vi), by Lemma 9.6.1.

2° *Remarks:* Recall that $(H_\beta)^* = H^*_{-\beta}$. Condition $\mathcal{P} \in \mathcal{B}(H, (H_\beta)^*)$ implies that $B^*\mathcal{P} \in \mathcal{B}(H, U)$, hence necessarily $K \in \mathcal{B}(H_\gamma, U)$, so that Lemma 9.5.4 applies for the closed-loop system (in particular, K is necessarily admissible and $\mathbb{A}_{\circlearrowleft}$ is analytic).

"On $H_{\gamma+\varepsilon}$" means that $\langle Kx_0, D^*JDKx_1\rangle = \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle Cx_0, JCx_1\rangle$ for all $x_0, x_1 \in H_{\gamma+\varepsilon}$, equivalently, that $K^*D^*JDK = A^*\mathcal{P} + \mathcal{P}A + C^*JC$ in $\mathcal{B}(H_{\gamma+\varepsilon}, (H_{\gamma+\varepsilon})^*)$.

For $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, the last condition means that the (analytic) semigroup generated by $A + BK$ is exponentially stable. □

The above remarks lead to applications of the theory in other sections, such as the ones below:

Corollary 9.5.10 (LQR — minimization)

(a) The assumptions of Corollary 10.2.10 are satisfied.

(b) Assume that Hypothesis 9.5.7 holds and that $D^*JD \gg 0$. *Then the assumptions of Corollary 10.2.9 are satisfied, and Lemma 9.6.1 applies; in particular, the* B_{w}^**-CARE in (ii) is satisfied on* $H_{\gamma+\varepsilon}$ *too (iff it is satisfied on* H_1*), for any* $\varepsilon > 0$, *and* $\mathbb{A}_{\circlearrowleft}$ *is analytic.*

A similar comment applies to Theorem 10.1.4(b3)&(b4)&(b6).

(c) Assume that Σ *is estimatable. Then there is a nonnegative solution of the LQR-CARE iff* Σ *is optimizable. If this is the case, then this solution is unique and strictly minimizing over* $\mathcal{U}_{\text{exp}}$ *(and* $\mathcal{U}_{\text{out}}$*).*

If $\gamma < 1/4$ *and* $\beta > -1/2$, *then Theorem 9.5.9 applies.*

Note that the proof and the remarks below Hypothesis 9.5.7 lead to further simplifications of the results mentioned in the corollary and several others.

See the comments below Corollary 10.4.4 for the parabolic H^2 problem.

Proof: (a) This follows from Lemma 9.5.2.

(b) The assumptions follow from Theorem 9.2.3; the rest follows from Lemma 9.6.1.

(c) See Theorem 10.1.4(c1) and the above, and note that now $D^*JD \gg 0$. □

Under Standing Hypothesis 9.5.1, we can strengthen our H^∞ Full-Information Control Problem results:

Corollary 9.5.11 (H^∞ FICP)

(a) Assumption (2.) of Theorem 11.1.4 is satisfied.

(b) Assume Hypothesis 9.5.7.

Then assumption (2.) of Theorems 11.1.3 and 11.1.6 is satisfied, and Lemma 9.6.1 applies; in particular, the B_w^*-CARE in (iii) is satisfied on $H_{\gamma+\varepsilon}$ too (iff it is satisfied on H_1), for any $\varepsilon > 0$, and $\mathbb{A}_\circlearrowleft$ is necessarily analytic.

(c) The assumptions of Theorem 11.2.7 are satisfied for $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ iff (A, B_1) is optimizable and (1.) of Theorem 11.1.4 holds.

(Do not mix the γ ("γ_C") of this section (and (b) above) with that ("challenge number") of Chapter 11. This applies Corollary 9.5.12 too.)

Proof: (a) This follows from Lemma 9.5.2.

(b) Assumption (2.) follows from Theorem 9.2.3. The rest follows from Lemma 9.6.1.

(c) Now there is an exponentially stabilizing state feedback operator $\widetilde{K} \in \mathcal{B}(H, U)$ iff (A, B_1) is optimizable, by Remark 9.5.3. By Lemma 9.5.4, corresponding (exponentially stable) closed-loop system $\Sigma_\flat$ (and $\Sigma_\circlearrowleft$ if any) also satisfies Hypothesis 9.5.1, in particular, $\mathbb{D}_\flat \in \mathrm{MTIC}(U, Y)$, by Lemma 9.5.2. See 0.1° of the proof of Theorem 11.1.4 for (1.). □

Our results for the standard H^∞ problem (the H^∞ "measurement-feedback" or "four-block" problem) can also be strengthened:

Corollary 9.5.12 (H^∞ 4BP) *Assume Hypothesis 12.1.1.*

(a) Assumption (A1) of Theorem 12.1.5 is satisfied.

(b) Assume (A1)(I) and (A2) of Theorem 12.1.4. Then Lemma 9.6.1 applies for the three CAREs. In particular, the $\mathcal{P}_X$-CARE (1.) is satisfied on $H_{\gamma+\varepsilon}$ too (iff it is satisfied on H_1), for any $\varepsilon > 0$, and $\mathbb{A}_\circlearrowleft$ is necessarily analytic. Analogously, the $\mathcal{P}_Y$-CARE (2.) (and the $\mathcal{P}_Z$-CARE (4.) if (1.) is satisfied) is satisfied on $H^*_{-\beta+\varepsilon}$ (iff it is satisfied on H_1^*), for any $\varepsilon > 0$.

Proof: (Note from Lemma 12.5.4 that Hypothesis 12.5.1 is satisfied iff (A,B_1) is optimizable, (A,C_2) is estimatable and (A2) of Theorem 12.1.5 is satisfied, by (a).)

(a) This follows from Lemma 9.5.2.

(b) It was shown in 4° of the proof of Theorem 12.1.4 that (2.) or (3.) of Hypothesis 9.5.7 is satisfied by Σ_X and Σ_Y; the same was observed for Σ_Z (assuming that Σ_X is satisfied) close to the end of the proof. Therefore, Lemma 9.6.1 applies to these CAREs. □

With our (weaker) standing hypothesis that $\gamma < 1/2$ and $\beta > -1/2$, we obtain the following:

Theorem 9.5.13 ($\mathcal{U}_{\mathbf{exp}}$: Unique optimum ⇔ CARE ⇔ J-coercive) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$, and $D^*JD \in \mathcal{GB}(U)$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P},S,K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{exp}}$.

If $(\mathcal{P},S,K)$ is as above, then K is J-critical over $\mathcal{U}_{\text{exp}}$, $S = D^*JD \in \mathcal{GB}(U)$, $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \text{MTIC}_\infty^{\text{L}^1}$ and $\mathbb{B}_\circlearrowleft\tau, \mathbb{N}, \mathbb{M} \in \text{MTIC}_{-\varepsilon}^{\text{L}^1}$ for some $\varepsilon > 0$.

Note that any solutions of (i) or (ii) are unique. If $D^*JD \gg 0$, then "J-critical" becomes equivalent to "minimizing" and "J-coercive" equivalent to "positively J-coercive", by Proposition 9.9.12. Thus, then Corollary 10.2.10 (and Theorem 10.1.4(b6)&(b4)) applies.

Proof: This follows from Lemma 9.5.2 and Corollary 9.2.19. □

Notes for Sections 9.5 and 9.6

A state-of-art treatment on parabolic systems is given in [LT00a] by Irena Lasiecka and Roberto Triggiani, by using a p.d.e. approach. They only treat the LQR and H^∞ FICP problems, in the case of $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ with a standard cost function (whose form is a special case of that of Proposition 9.7.6), but they allow for any $\beta > -1$ (for $\gamma = 0$, i.e., they take $H = H_\gamma$, so that C becomes bounded), hence their results also apply to systems with very unbounded input operators, thus covering a wide range of important applications. Since their proofs are based on the convergence of the finite-horizon solutions, they might be inappropriate for more general cost functions, as explained in the notes on p. 465.

The history and status quo of research in the p.d.e. approach to analytic systems is well documented in the notes of [LT00a], with references to works of the authors, G. Da Prato, F. Flandoli, X. Li, C. McMillan and others, including articles on singular control.

The first application of WPLS theory to parabolic systems seems to be [St97a] (and [S98e]), which uses the theory of [S97b] to establish a solution to the positively J-coercive stable quadratic minimization problem and show that its J-critical cost operator satisfies the Riccati equation. Its proof leans heavily on the results of Lasiecka and Triggiani, so that its contribution is mainly the WPLS formulation and the characterization of the closed-loop sensibility to external input (i.e., the second column of $\Sigma_{\circlearrowleft}$). That article and [Sbook] essentially contain Lemmas 9.5.2, 9.5.8 and 9.6.3.

9.6 Parabolic problems: proofs

The voice of the majority is no proof of justice.

— Friedrich von Schiller (1759–1805)

In this section, we prove three lemmas that were used for the results of the above section. We still assume Standing Hypothesis 9.5.1.

We first show that a J-critical control in WPLS form (e.g., a unique J-critical control, by Lemma 8.3.16(a1)), is always of state feedback form and corresponds to a solution of the B_{w}^*-CARE:

Lemma 9.6.1 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.5.7 holds and that $S := D^*JD \in \mathcal{GB}(U)$.*

Then $\mathcal{P} \in \mathcal{B}(H,(H_r)^*)$ for any $r < 1-2\gamma$; in particular, $\mathcal{P}[H] \subset (H_\beta)^* =$ $\mathrm{Dom}(B^*) \subset \mathrm{Dom}(B_{\mathrm{L,s}}^*) \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$. Thus, $B^*\mathcal{P} \in \mathcal{B}(H,U)$, and

$$K := -S^{-1}(B^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_\gamma, U) \tag{9.49}$$

is a unique ULR J-critical state feedback operator for Σ. Moreover $\mathcal{P}$ satisfies the B_{w}^*-CARE

$$A^*\mathcal{P} + \mathcal{P}A + C^*JC = K^*SK \in \mathcal{B}(H_1,(H_1)^*) \tag{9.50}$$

(even $\in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$ for all $\varepsilon > 0$).

Finally, the corresponding closed-loop system $\Sigma_\circlearrowleft$ is generated by $A+BK$, and $\Sigma_\circlearrowleft$ is analytic on $H_\alpha = (H_\circlearrowleft)_\alpha$ for $\alpha \in [\gamma-1, \beta+1]$. Consequently, $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft \in$ $\mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for all $\omega > \omega_{A_\circlearrowleft}$.

See the remark for B^* below Lemma 9.5.4. Note that K is bounded (i.e., $K \in \mathcal{B}(H,U)$) if $D^*JC = 0$.

Proof: 1° *K is ULR and J-critical:* By Lemma 9.6.2, we have $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$. By Proposition 9.3.1, $K := -S^{-1}(B_{\mathrm{L,s}}^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_\gamma, U)$ is a unique ULR J-critical state feedback operator for Σ (in particular, $\mathbb{K}_\circlearrowleft = \mathbb{K}_{\mathrm{crit}}$, where $\Sigma_\circlearrowleft$ is the corresponding closed-loop system). We obtain "$\mathcal{P}(H,H_r)$" from 2° and the rest of the claim from Lemmas 9.5.4 and 9.5.2 and 3°.

2° *We have $\mathcal{P} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$ when $r,s \ge 0$, $r+s < 1$:* (For $\gamma - r = 0$ this becomes $\gamma - s > 2\gamma - 1$, hence $\gamma - s = \beta$ is allowed, so that $\mathcal{P} \in \mathcal{B}(H,(H_\beta)^*)$.) W.l.o.g., we assume that $\gamma \ge 0$.

Choose $p,q \in [1,\infty]$ s.t. $p^{-1} > r$, $q^{-1} > s + 2\gamma$ and $p^{-1} + q^{-1} = 1$. Then $C_{\mathrm{crit}}\mathbb{A}_{\mathrm{crit}} \in \mathrm{L}_\omega^p(\mathbf{R}_+; \mathcal{B}(H_{\gamma-r}, Y))$, by Lemma 9.4.2(j) (since $C_{\mathrm{crit}} = C + DK_{\mathrm{crit}} \in \mathcal{B}(H_\gamma, Y)$), and $C\mathbb{A} \in \mathrm{L}_\omega^q(\mathbf{R}_+; \mathcal{B}(H_{\gamma-s}, Y))$, hence $\mathbb{C}^*\pi_{[0,t)}J\mathbb{C}_{\mathrm{crit}} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$.

But $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}_{\mathrm{crit}}^t \in \mathcal{B}(H_{\gamma-1},(H_v)^*)$, for any $v \in \mathbf{R}$, by Lemma 9.5.4 and Lemma 9.4.2(h2). Therefore, $\mathcal{P} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$.

3° *The B_{w}^*-CARE on $H_{\gamma+\varepsilon}$:* Let $\varepsilon > 0$. We have $A \in \mathcal{B}(H_{\gamma+\varepsilon}, H_{\gamma+\varepsilon-1})$, $\mathcal{P} \in \mathcal{B}(H_{\gamma+\varepsilon-1},(H_\gamma)^*)$ (take $s = 0$, $r := 1-\varepsilon$), and $(H_\gamma)^* \underset{c}{\subset} (H_{\gamma+\varepsilon})^*$, hence $\mathcal{P}A \in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$.

By taking adjoints, we obtain that $A^*\mathcal{P} \in \mathcal{B}(H_{\gamma+\varepsilon}, (H_{\gamma+\varepsilon})^*)$. Since obviously $C^*JC, K^*SK \in \mathcal{B}(H_{\gamma+\varepsilon}, (H_{\gamma+\varepsilon})^*)$, and H_1 is dense in $H_{\gamma+\varepsilon}$, the B_{w}^*-CARE holds also on $H_{\gamma+\varepsilon}$, i.e.,

$$\langle Ax_0, \mathcal{P}x_1\rangle_{\langle H_{\gamma+\varepsilon-1}, (H_{\gamma+\varepsilon-1})^*\rangle} + \langle \mathcal{P}x_0, Ax_1\rangle_{\langle (H_{\gamma+\varepsilon-1})^*, H_{\gamma+\varepsilon-1}\rangle} + \langle Cx_0, JCx_1\rangle_Y = \langle Kx_0, SKx_1\rangle. \tag{9.51}$$

□

The above proof was based on the following result:

Lemma 9.6.2 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.5.7 holds. Then $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$.*

Proof: 1° *Assumption (1.):* Now we have to alter 2° of the proof of Lemma 9.3.4 as follows:

Let $\omega > \omega_A$. By Lemma 9.4.2(j), we have $C\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$ for any $p \in [1, (\beta+\gamma)^{-1})$; set $p := 4/3$. Analogously, $\mathbb{C} = C\mathbb{A} \in \mathrm{L}^4_\omega(\mathbf{R}_+; \mathcal{B}(H,Y))$.

Fix $x_0 \in H$ and $u_0 \in U$. Then $\mathbb{K}_{\mathrm{crit}}x_0 \in \mathrm{L}^2(\mathbf{R}_+; Y)$, hence $C\mathbb{B}\tau\mathbb{K}_{\mathrm{crit}}x_0 = C\mathbb{A}B * \mathbb{K}_{\mathrm{crit}}x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+; Y)$, because $p^{-1} + 2^{-1} = 1 + 4^{-1}$ (see Lemma D.1.7). Consequently, $f := C\mathbb{A}_{\mathrm{crit}}x_0 = C(\mathbb{A} + \mathbb{B}\tau\mathbb{K}_{\mathrm{crit}})x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+; Y)$. Thus,

$$\mathbb{C}_{\mathrm{crit}}x_0 - D\mathbb{K}_{\mathrm{crit}}x_0 = C\mathbb{A}_{\mathrm{crit}}x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+; Y). \tag{9.52}$$

Set $F := C\mathbb{A}Bu_0 \in \mathrm{L}^p_\omega$. Because $p^{-1} + 4^{-1} = 1$ (and $C^*JD = 0$), we can again work as in 2° of the proof of Lemma 9.3.4 to obtain that $\mathcal{P}x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}^*)$.

(The moral of the proof: the "p" in "L^p" does not always mean 1, 2 or ∞.)

2° *Assumption (2.):* Part 6° of the proof of Lemma 9.3.4 applies here too, mutatis mutandis; we sketch this below:

By Lemma 9.5.4, $K' := -(D^*JD)^{-1}D^*JC \in \mathcal{B}(H_\gamma, U)$ is admissible and ULR for Σ, and assumption (1.) is satisfied by the corresponding closed-loop system (which has the input operator B and the output operator $C + DK' \in \mathcal{B}(H_\gamma, U)$; the spaces H_β, H_γ and $\mathrm{Dom}(B_{\mathrm{L,s}}^*)$ remain unchanged).

3° *Assumption (3.):* This follows from Lemma 9.3.4, since Hypothesis 9.2.2(5.) is satisfied, by Lemma 9.4.2(j). □

Lemma 9.6.3 *If $\mathbb{D} \in \mathrm{TIC}$, then $\mathbb{D}$ is uniformly half-plane-regular.*

(Naturally, this is not true without Standing Hypothesis 9.5.1.)

Proof: We shall show that we have $\widehat{\mathbb{D}}(s) \to D$, as $s \in \mathbf{C}^+$, $|s| \to \infty$ regardless of the stability of $\mathbb{D}$. By Definition 6.2.3, it follows that $\mathbb{D}$ is UHPR if $\mathbb{D} \in \mathrm{TIC}$ (or $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2$).

Let $\omega > \omega_0 > \omega_A$. Choose θ and M for ω_0 as in Lemma 9.4.2(a). Then

$$\widehat{\mathbb{D}}(s) - D = C(\omega - A)^{-\gamma} \cdot (\omega - A)^{\gamma+\beta}(s - A)^{-1} \cdot (\omega - A)^{-\gamma}B \to 0, \tag{9.53}$$

as $s \in \Sigma_{\theta,\omega_0}$, $|s| \to \infty$, by Lemma 9.4.2(k). □

(See the notes on p. 451.)

9.7 Riccati equations on $\mathrm{Dom}(A_{crit})$

Just because they are called 'forbidden' transitions does not mean that they are forbidden. They are less allowed than allowed transitions, if you see what I mean.

— From a Part 2 Quantum Mechanics lecture.

In Theorem 8.3.9, we observed that a unique J-critical control is always of WPLS form. In most earlier theory (e.g., for bounded B), such a control is necessarily of state feedback form, and corresponding cost, signature and state feedback operators $(\mathcal{P},S,K)$ satisfy the CARE. Conversely, the K-operator of any "stabilizing" solution of the CARE is a J-critical state feedback operator (also in our setting). We shall extend this equivalence to sufficiently regular systems with unbounded B in Theorem 9.2.9, based on the results of this section (see also Remark 9.9.14).

In this section, we shall study the situation for an arbitrary WPLS with a unique J-critical control (or more generally, with a control in WPLS form) without assuming the J-critical control to be of (well-posed) state feedback form (cf. Remark 9.7.7(a1)–(a3)), and we shall derive certain Riccati-like equations.

Thus, given, e.g., a J-coercive ($\mathcal{U}_*^*$-) stabilizable regular system, we can obtain the optimal state feedback by solving the generalized Riccati equation (9.67). However, since these equations are given on $\mathrm{Dom}(A_{\mathrm{crit}})$ (for WR systems), which is unknown a priori, it seems very hard to solve such equations and thus find the optimal control. The integral version given below (for general WPLSs) seems even less applicable.

Our results rigourously extend the equation in [FLT] (contained in Proposition 9.7.6), where similar equations are given in a coercive, positive setting with a bounded output operator. A more general setting, still with bounded C, is treated in Lemma 9.7.5, and for general WR systems the equations are given in Theorem 9.7.3.

However, for the above reasons and others explained in this chapter (e.g., D^*JD cannot serve as the signature operator, as explained in the notes to Section 9.8), we consider the "regular Riccati equations" due to M. Weiss, G. Weiss and O. Staffans more applicable than these "closed-loop domain Riccati equations", and prefer developing their theory to cover standard control problems. Neverthless, the latter can be build on the former, as we partially do, and in some cases the two theories coincide (see the parabolic theory of Chapter 9.5).

If there is a J-critical control in WPLS form (e.g., a unique J-critical control, see Lemma 8.3.16(a1)), then Σ_{crit} is "$\mathcal{U}_*^*$-stable" and (9.55)–(9.57) hold:

Theorem 9.7.1 (Generalized IARE) *Assume that* $\Sigma_0 := \begin{bmatrix} \mathbb{A}_0 \\ \hline \mathbb{C}_0 \\ \mathbb{K}_0 \end{bmatrix}$ *is a control in WPLS form, and that* $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.

Then $\mathbb{K}_0 x_0$ *is J-critical for* x_0 *for each* $x_0 \in H$ *and* $\mathcal{P} = \mathbb{C}_0^* J \mathbb{C}_0$ *iff* $\mathbb{K}_0 x_0 \in$

$\mathcal{U}_*^*(x_0)$ *for all* $x_0 \in H$ *and the following hold:*

$$\langle \mathbb{B}^t u + \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{A}_0^t x_0 \rangle \to 0, \text{ as } t \to +\infty \quad (x_0 \in H,\ u \in \mathcal{U}_*^*(0)), \tag{9.54}$$

$$0 = (\mathbb{D}^t)^* J \mathbb{C}_0^t + (\mathbb{B}^t)^* \mathcal{P}\mathbb{A}_0^t \in \mathcal{B}(H, \mathrm{L}^2([0,t];U)), \tag{9.55}$$

$$\mathcal{P} = \mathbb{A}^{t\,*} \mathcal{P}\mathbb{A}_0^t + \mathbb{C}^{t\,*} J \mathbb{C}_0^t \in \mathcal{B}(H). \tag{9.56}$$

We can make the following enhancements above:

(a) We may replace (9.56) above by

$$\mathcal{P} = \mathbb{A}_0^{t\,*} \mathcal{P}\mathbb{A}_0^t + \mathbb{C}_0^{t\,*} J \mathbb{C}_0^t \in \mathcal{B}(H). \tag{9.57}$$

(b) Condition (9.54) is redundant if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *or* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$*.*

(c) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *(resp.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$*), then* Σ_0 *is* J*-critical iff* Σ_0 *is exponentially (resp. strongly) stable and (9.55)–(9.56) hold.*

Thus, we can consider (9.55)–(9.56) as a generalized IARE, whose solution $\mathcal{P}$ is "stabilizing" iff $\mathbb{K}_\circlearrowleft x_0 \in \mathcal{U}_*^*(x_0)$ and (9.54) holds. When applied to the left column of a (state feedback) closed-loop system, these conditions determine whether the corresponding state feedback pair is J-critical.

Since the above conditions are hard to verify, we go on to develop further conditions, but first we make an observation from part (c) above:

Corollary 9.7.2 ($\mathcal{U}_{\mathrm{exp}} \Rightarrow \mathcal{U}_{\mathrm{str}}$) *If there is a unique* J*-critical control over* $\mathcal{U}_{\mathrm{exp}}$*, then this control is* J*-critical over* $\mathcal{U}_{\mathrm{str}}$*.* □

(The converse does not hold even for exponentially stabilizable systems, by Example 9.13.14, which also illustrates that a J-critical state feedback operator over $\mathcal{U}_{\mathrm{str}}$ need not be strongly stabilizing, although it must stabilize the left column strongly).

Proof of Theorem 9.7.1: Trivially, condition $\mathbb{K}_\circlearrowleft x_0 \in \mathcal{U}_*^*(x_0)$ $(x_0 \in H)$ is necessary. For the rest of the proof, we assume that this condition holds. Consequently, $\mathbb{C}_\circlearrowleft$ is stable and Theorem 8.3.9(a2) holds.

1° *"Only if":* Given $\widetilde{\eta} \in \mathrm{L}^2([0,t);U)$, we have for $\eta := \widetilde{\eta} + \tau^{-t}\mathbb{K}_0\mathbb{B}^t\widetilde{\eta}$ $(\in \mathcal{U}_*^*(0)$, by Lemma 9.7.10) and any $x_0 \in H$ that

$$0 = \langle J\mathbb{C}x_0 + \mathbb{D}\mathbb{K}_0 x_0, \mathbb{D}\eta\rangle = \langle (\pi_{[0,t)} + \tau^{-t}\tau^t\pi_{[t,\infty)}) J\mathbb{C}_0 x_0, \mathbb{D}\eta\rangle \tag{9.58}$$

$$= \langle \pi_{[0,t)} J\mathbb{C}_0 x_0, \mathbb{D}\eta\rangle + \langle J\pi_+\tau^t\mathbb{C}_0 x_0, \mathbb{D}\tau^t\eta\rangle \tag{9.59}$$

$$= \langle \pi_{[0,t)} J\mathbb{C}_0 x_0, \mathbb{D}\eta\rangle + \langle J\mathbb{C}_0\mathbb{A}_0^t x_0, \mathbb{C}_0\mathbb{B}^t\widetilde{\eta}\rangle \tag{9.60}$$

$$= \langle J\mathbb{C}_0^t x_0, \mathbb{D}^t\widetilde{\eta}\rangle + \langle \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{B}^t\widetilde{\eta}\rangle. \tag{9.61}$$

Thus, (9.55) holds. Equation (9.57) and the convergence $(\mathbb{A}_0^t)^*\mathcal{P}\mathbb{A}_0^t x_0 \to 0$ are obtained as in the proof of Lemma 9.10.1(d1).

Let now $x_0 \in H$ and $\eta \in \mathcal{U}_*^*(0)$ be arbitrary. Because $\langle \pi_{[0,t)}\mathbb{D}\eta, J\mathbb{C}_0 x_0\rangle \to \langle \mathbb{D}\eta, J\mathbb{C}_0 x_0\rangle$, as $t \to \infty$, by Corollary B.3.8, equation (9.55) implies that

$$\langle \mathbb{B}\tau^t\eta, \mathcal{P}\mathbb{A}_0^t x_0\rangle \to -\langle \mathbb{D}\eta, J\mathbb{C}_0 x_0\rangle, \quad \text{as } t \to +\infty. \tag{9.62}$$

Because $\eta \in \mathcal{U}_*^*(0)$ was arbitrary, J-criticality implies that (9.54) holds.

2° *"If":* Assume that Σ_0 is $\mathcal{U}_*^*$-stable and that (9.54)–(9.57) hold.

Identity $\mathcal{P} = \mathbb{C}_0^* J \mathbb{C}_0$ is obtained as in the proof of Lemma 9.10.1(d1). From (9.62) and (9.54) we obtain that $\langle \mathbb{D}\eta, J\mathbb{C}_0 x_0\rangle = 0$.

3° *Remark:* If the equivalent conditions hold, then $\mathbb{A}_0^t \mathcal{P} \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{A}_0^t x_0 \to 0$, as $t \to \infty$, as one observes from the proof of Lemma 9.10.1(d1).

(a) Equation (9.57) is equivalent to (9.56), because $(9.56)^* - (9.57) = \mathbb{K}_0^*((\mathbb{B}^t)^* \mathcal{P} \mathbb{A}_0^t + (\mathbb{D}^t)^* J \mathbb{C}_0^t) = 0$ when (9.55) holds.

(b) Let $x_0 \in H$ and $\eta \in \mathcal{U}_*^*(0)$. If $\mathcal{U}_*^* = \mathcal{U}_{\text{str}}$, then $\mathbb{A}_0 x_0, \mathbb{B}\tau\eta \in C_0(\mathbf{R}_+;H)$, by Theorem 8.3.9(a2), hence then (9.54) obviously holds.

If $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, then $\mathbb{A}_0 x_0, \mathbb{B}\tau\eta \in \mathrm{L}^2(\mathbf{R}_+;H)$, hence then the limit in (9.62) cannot be nonzero in any case, so it must be zero (since it exists, by (9.62)).

(c) "Only if" follows from Theorem 8.3.9(a2), and "if" from (b). □

If $\mathbb{D}$ is regular, then it follows that certain Riccati equation is satisfied on $\mathrm{Dom}(A_{\text{crit}})$:

Theorem 9.7.3 ($\mathrm{Dom}(A_{\text{crit}})$-CARE) *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Then*

$$-A_{\text{crit}}^* \mathcal{P} = \mathcal{P} A_{\text{crit}} + C_{\text{crit}}^* J C_{\text{crit}} \quad \in \mathcal{B}(\mathrm{Dom}(A_{\text{crit}}), \mathrm{Dom}(A_{\text{crit}})^*), \tag{9.63}$$
$$-A_{\text{crit}}^* \mathcal{P} = \mathcal{P} A + C_{\text{crit}}^* J C \quad \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A_{\text{crit}})^*), \tag{9.64}$$
$$-A^* \mathcal{P} = \mathcal{P} A_{\text{crit}} + C^* J C_{\text{crit}} \quad \in \mathcal{B}(\mathrm{Dom}(A_{\text{crit}}), \mathrm{Dom}(A)^*). \tag{9.65}$$

Recall that $C_{\text{crit}} = C_{\text{c}} + D_{\text{c}} K_{\text{crit}}$ and $A_{\text{crit}} = A + BK_{\text{crit}}$ on $\mathrm{Dom}(A_{\text{crit}}) \subset H_B$.

Assume, in addition, that $\mathbb{D}$ is WR. Then

(a) **("$K_{\text{crit}} = -B^*\mathcal{P}$")** $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\text{crit}}), \mathrm{Dom}(B_{\text{L,w}}^*))$, $B_{\text{L,w}}^* \mathcal{P} = -D^* J C_{\text{crit}}$ *on* $\mathrm{Dom}(A_{\text{crit}})$, *and*

$$(D^*JD)K_{\text{crit}} = -B_{\text{L,w}}^* \mathcal{P} - D^* J C_{\text{L,w}} \in \mathcal{B}(\mathrm{Dom}(A_{\text{crit}}), U). \tag{9.66}$$

(b) **("$u_{\text{crit}} = -B^*\mathcal{P}x_{\text{crit}}$")** $(K_{\text{crit}})_{\text{L,s}} x(t) = -(D^*JD)^{-1}(B_{\text{L,w}}^* \mathcal{P} + D^* J C_{\text{L,w}}) x(t) = (\mathbb{K}_{\text{crit}} x_0)(t)$ *for a.e. $t \ge 0$ and all $x_0 \in H$, where $x := x_{\text{crit}}(x_0) := \mathbb{A}_{\text{crit}} x_0$, if $D^*JD \in \mathcal{GB}(U)$. In particular, $\mathcal{P}x(t) \in \mathrm{Dom}(B_{\text{L,w}}^*)$ a.e.*

(c) **("CARE" on $\mathrm{Dom}(A_{\text{crit}})$)** *If $\mathbb{D}$ and $\mathbb{D}^{\text{d}}$ are SR and $D^*JD \in \mathcal{GB}(U)$, then we can replace $B_{\text{L,w}}^*$ by $B_{\text{L,s}}^*$ and $C_{\text{L,w}}$ by $C_{\text{L,s}}$ in (a) and (b), and we have*

$$A^*\mathcal{P} + \mathcal{P}A + C^*JC_{\text{L,s}} = (\mathcal{P}B + C^*JD)(D^*JD)^{-1}(D^*JC_{\text{L,s}} + B_{\text{L,s}}^*\mathcal{P}) \tag{9.67}$$

in $\mathcal{B}(\mathrm{Dom}(A_{\text{crit}}), \mathrm{Dom}(A_{\text{crit}})^*)$.

(d) **("CARE" ⇔ *J*-critical)** *Assume, instead, that Σ_{crit} is a control in WPLS form, $\mathbb{D} \in \mathrm{WR}$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.*

Then Σ_{crit} is J-critical and $\mathcal{P} = \mathbb{C}_{\text{crit}}^* J \mathbb{C}_{\text{crit}}$ iff (9.65) and (9.66) hold and Σ_{crit} is "$\mathcal{U}_*^*$-stabilizing" (i.e., $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$ and (9.54) holds).

It would be more useful to obtain a Riccati equation on $\mathrm{Dom}(A)$, and we would like to know that K_{crit} extends to a (regular) state feedback operator for Σ (e.g., that K_{crit} would have an extension to H_B, and that this extension would equal $K_{\text{w}}|_{H_B}$ for some WR state feedback operator K for Σ). Unfortunately, both of these require additional assumptions, by Example 11.3.7.

Proof: (In fact, $\Sigma_{\rm crit}$ need not be J-critical, it suffices that it is a control in WPLS form for Σ (see Definition 8.3.15) s.t. (9.55) and (9.56) hold. Thus, $\Sigma_{\rm crit}$ need not be "$\mathcal{U}_*^*$-stable" provided that it is otherwise as in Theorem 9.7.1.)

Apply Lemma 9.7.8 to (9.57), (9.56) and (9.56)* to obtain (9.63), (9.64) and (9.65). Formulae for $A_{\rm crit}$ and $C_{\rm crit}$ are from Lemma 8.3.17(a).

(b) Let $x_0 \in H$ and $T > 0$. We "connect $\Sigma_{\rm crit}$ to $\Sigma^{\rm d}$ through $\mathcal{P}$, J and time-inversion", as in [S98b, Section 5], [WW, Section 8] or Lemma 9.11.1: Set

$$x_0^* := \mathcal{P}\mathbb{A}_{\rm crit}^T x_0 \in H, \;\; y^* := \pi_{[0,T)} \boldsymbol{Я} \tau^T J \mathbb{C}_{\rm crit} x_0 \in \mathrm{L}^2([0,T);Y). \tag{9.68}$$

Then, for any $s \in [0,T]$, we have

$$x^*(s) := \mathbb{A}^*(s) x_0^* + \mathbb{C}^{\rm d} \tau^s y^* = \mathbb{A}^{s*} \mathcal{P} \mathbb{A}_{\rm crit}^s \mathbb{A}_{\rm crit}^{T-s} x_0 + \mathbb{C}^* J \pi_{[0,s)} \tau^{T-s} \mathbb{C}_{\rm crit} x_0 \tag{9.69}$$

$$= \mathbb{A}^{s*} \mathcal{P} \mathbb{A}_{\rm crit}^s \mathbb{A}_{\rm crit}^{T-s} x_0 + \mathbb{C}^* J \pi_{[0,s)} \mathbb{C}_{\rm crit} \mathbb{A}_{\rm crit}^{T-s} x_0 = \mathcal{P} \mathbb{A}_{\rm crit}^{T-s} x_0, \tag{9.70}$$

by (9.56). Trivially, $\pi_{[0,T)} \mathbb{B}^{\rm d} x_0^* = \boldsymbol{Я} \tau^T (\mathbb{B}^T)^* x_0^*$, and $\pi_{[0,T)} \mathbb{D}^{\rm d} y^* = \boldsymbol{Я} \tau^T (\mathbb{D}^T)^* J \mathbb{C}_{\rm crit}^T x_0$. Therefore, (9.55) implies that

$$0 = \pi_{[0,T)} \mathbb{B}^{\rm d} x_0^* + \pi_{[0,T)} \mathbb{D}^{\rm d} y^*. \tag{9.71}$$

Since $\mathbb{D}$ is WR, we have for a.e. $t \in [0,T]$ that $x^*(T-t) = \mathcal{P} \mathbb{A}_{\rm crit}^t x_0 \in \mathrm{Dom}(B_{\rm L,w}^*)$, by (9.70) and Theorem 6.2.13(a2), and

$$0 = B_{\rm L,w}^* x^*(T-t) + D^* y^*(T-t) = B_{\rm L,w}^* \mathcal{P} \mathbb{A}_{\rm crit}^t x_0 + D^* J (\mathbb{C}_{\rm crit} x_0)(t) \tag{9.72}$$

$$= B_{\rm L,w}^* \mathcal{P} x(t) + D^* J (C_{\rm L,w} x(t) + D u(t)), \tag{9.73}$$

by (9.71) and (6.46) where $x(t) := \mathbb{A}_{\rm crit}^t x_0$, $u := \mathbb{K}_{\rm crit} x_0$. (recall that $\mathbb{C}_{\rm crit} x_0 = \mathbb{C} x_0 + \mathbb{D} \mathbb{K}_{\rm crit} x_0 = \mathbb{C} x_0 + \mathbb{D} u$). Because $u = (K_{\rm crit})_{\rm L,s} x(t)$ a.e., by Lemma 6.2.12(a), and $T > 0$, $t \in [0,T]$ and $x_0 \in H$ were arbitrary, (b) follows.

(a) Let $x_0 \in \mathrm{Dom}(A_{\rm crit})$. Then $x := \mathbb{A}_{\rm crit} x_0 \in \mathcal{C}(\mathbf{R}_+; \mathrm{Dom}(A_{\rm crit})) \cap \mathrm{W}_{\rm loc}^{1,2}(\mathbf{R}_+; H)$, hence $\mathbb{C}_{\rm crit} x_0 = C_{\rm crit} \mathbb{A}_{\rm crit} x_0$ and

$$A^* x_0^* + C^* y^*(0) = A^* \mathcal{P} x(T) + C^* J C_{\rm crit} x(T) = -\mathcal{P} A_{\rm crit} x(T) \in H, \tag{9.74}$$

by (9.65) and (9.68). Therefore, y^* and $\mathbb{B}^{\rm d} x_0 + \mathbb{D}^{\rm d} y^*$ are $\mathrm{W}_{\rm loc}^{1,2}$, hence continuous, by Theorem 6.2.13(b1), so that (9.72)–(9.73) hold everywhere on $[0,T]$.

Since $(C_{\rm L,w}, D)$ is a compatible pair for Σ, by Lemma 6.3.10(e), we have $C_{\rm crit} = C_{\rm L,w} + D K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$, by Lemma 8.3.17(a). Thus, (a) follows from (9.72) at $t = 0$.

(c) 1° *(a) and (b):* From the proofs of (a) and (b) we see that we can replace $B_{\rm L,w}^*$ by $B_{\rm L,s}^*$ if $\mathbb{D}^{\rm d} \in$ SR and $C_{\rm L,w}$ by $C_{\rm L,s}$ if $\mathbb{D}$ is SR.

2° *The "CARE":* By (a), we have $\mathcal{P} \in \mathcal{B}(Z^*, \mathrm{Dom}(A_{\rm crit})^*)$, where $Z := \mathrm{Dom}(B_{\rm L,s}^*)$.

Let $x_0, x_1 \in \mathrm{Dom}(A_{\rm crit})$. Then $\mathcal{P} x_k \in Z$ and $A x_k, B K_{\rm crit} x_k \in Z^*$, by Proposition 6.2.8(e)&(f), because $\mathrm{Dom}(A_{\rm crit}) \subset H_B$, by Lemma 8.3.17(a). (Here we needed the strong regularity of $\mathbb{D}^{\rm d}$; that of $\mathbb{D}$ could be dropped without essential changes.) Therefore,

$$\langle A_{\rm crit}x_0, \mathcal{P}x_1\rangle_H = \langle Ax_0 + BK_{\rm crit}x_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} = \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle BK_{\rm crit}x_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} \tag{9.75}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle K_{\rm crit}x_0, B^*_{\rm L,s}\mathcal{P}x_1\rangle_U, \tag{9.76}$$

by Proposition 6.2.8(e). Now we obtain from (9.64) and the formulae $A_{\rm crit} = A + BK_{\rm crit}$ and $C_{\rm crit} = C_{\rm w} + DK_{\rm crit}$ that

$$0 = \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle K_{\rm crit}x_0, B^*_{\rm L,s}\mathcal{P}x_1\rangle + \langle B^*_{\rm L,s}\mathcal{P}x_0, K_{\rm crit}x_1\rangle \tag{9.77}$$

$$+ \langle C_{\rm w}x_0, JC_{\rm w}x_1\rangle + \langle C_{\rm w}x_0, JDK_{\rm crit}x_1\rangle + \langle DK_{\rm crit}x_0, JC_{\rm w}x_1\rangle + \langle DK_{\rm crit}x_0, JDK_{\rm crit}x_1\rangle \tag{9.78}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle C_{\rm w}x_0, JC_{\rm w}x_1\rangle + \langle K_{\rm crit}x_0, (B^*_{\rm L,s}\mathcal{P} + D^*JC_{\rm w})x_1\rangle \tag{9.79}$$

$$+ \langle (B^*_{\rm L,s}\mathcal{P} + D^*JC_{\rm w})x_0, K_{\rm crit}x_1\rangle + \langle B^*_{\rm L,s}\mathcal{P}x_0, K_{\rm crit}x_1\rangle + \langle K_{\rm crit}x_0, D^*JDK_{\rm crit}x_1\rangle \tag{9.80}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle \mathcal{P}x_0, Ax_1\rangle_{\langle Z,Z^*\rangle} + \langle C_{\rm w}x_0, JC_{\rm w}x_1\rangle_Y \tag{9.81}$$

$$- \langle (B^*_{\rm L,s}\mathcal{P} + D^*JC_{\rm w})x_0, (D^*JD)^{-1}(B^*_{\rm L,s}\mathcal{P} + D^*JC_{\rm w})x_1\rangle_U. \tag{9.82}$$

Assume now that also $\mathbb{D}$ is SR. Then $H_B \subset \mathrm{Dom}(C_{\rm L,s})$, so that we can replace $C_{\rm w}$ by $C_{\rm L,s}$ above, by Proposition 6.2.8(c1)&(c4)&(d1). Apply Proposition 6.2.8(e) to B and C^* to obtain that

$$K^*_{\rm crit} = -(\mathcal{P}B + C^*JD)(D^*JD)^{-1} \in \mathcal{B}(U, \mathrm{Dom}(C_{\rm L,s})^*). \tag{9.83}$$

(actually, the right-hand-side is an element of $\mathcal{B}(U, \mathrm{Dom}(C_{\rm L,s})^*)$, but $\mathrm{Dom}(A_{\rm crit}) \underset{c}{\subset} \mathrm{Dom}(C_{\rm L,s})$ so that $K^*_{\rm crit}$ is the restriction of the right-hand-side onto $\mathrm{Dom}(A_{\rm crit})$).

(d) By Lemma 9.7.8, we obtain (9.56). By going backwards the proofs of (a) and (b), we obtain that (9.71) holds for any T and x_0, hence (9.55) holds. The rest follows from Theorem 9.7.1. □

If D^*JD is one-to-one, then we obtain the following uniqueness results:

Corollary 9.7.4 *Let $\mathbb{D}$ be WR and* $\mathrm{Ker}(D^*JD) = \{0\}$.

(a) There is at most one J-critical control in WPLS form.

(b) There is at most one J-critical SR state feedback operator for Σ.

Although sufficient, the condition $\mathrm{Ker}(D^*JD) = \{0\}$ is not necessary for uniqueness, by, e.g., Example 9.13.10 in which $D = 0$ but the J-critical control over $\mathcal{U}_{\rm out}$ is nevertheless unique, by J-coercivity. (This might at first seem strange, but (9.66) is not the only condition on $K_{\rm crit}$ in Theorem 9.7.3, also $A_{\rm crit}$ and $C_{\rm crit}$ restrict $K_{\rm crit}$, hence so does also $\mathcal{U}^*_*$-stabilization.)

Whenever there is a J-critical control over $\mathcal{U}^*_*$ in state feedback form, then the uniqueness of the J-critical control is equivalent to $\mathrm{Ker}(S) = \{0\}$, where S is the signature operator of the problem (which is equal to D^*JD under stronger regularity assumptions), by Theorem 9.9.1(a1)&(f2). See also the notes to Section 9.8.

Proof of Corollary 9.7.4: (a) Now $K_{\rm crit}$ is uniquely defined by the unique (by Lemma 8.3.8) J-critical cost operator $\mathcal{P}$, through (9.66). Now $A_{\rm crit} = A + BK_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit}) = \{x_0 \in H \mid (A+BK_{\rm crit})x_0 \in H\}$, by Lemma A.4.6, and $C_{\rm crit} = C + DK_{\rm crit}$, hence $\mathcal{P}$ determines $\Sigma_{\rm crit}$ uniquely.

(b) Let K and $\widetilde{K}$ be J-critical SR state feedback operators for Σ. Let $\Sigma_{\circlearrowleft}$ and $\Sigma_{\flat}$ be corresponding closed-loop systems. By (6.145) and Proposition 6.6.18(d4), $K_{\rm crit}$ is SR and $K_{\rm s} = (K_{\rm crit})_{\rm s} = \widetilde{K}_{\rm s}$, hence $K = \widetilde{K}$. □

For bounded C, the above "generalized Riccati equations" can be simplified:

Lemma 9.7.5 (Bounded C) *Let $\Sigma_{\rm crit}$ be a J-critical control for Σ in WPLS form. Assume that $C \in \mathcal{B}(H,Y)$. Then $(\mathcal{P}, D^*JD, K_{\rm crit})$ satisfy*

$$K_{\rm crit}^* D^*JDK_{\rm crit} = A^*\mathcal{P} + \mathcal{P}A + C^*JC \quad \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), \mathrm{Dom}(A_{\rm crit})^*), \tag{9.84}$$

$$(D^*JD)K_{\rm crit} = -B^*\mathcal{P} - D^*JC \quad \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), U), \tag{9.85}$$

$$\mathcal{P}^* = \mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), \mathrm{Dom}(A^*)) \cap \mathcal{B}(\mathrm{Dom}(A^*)^*, \mathrm{Dom}(A_{\rm crit})^*). \tag{9.86}$$

Proof: 1° $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), \mathrm{Dom}(A^*))$*:* From (9.65) we obtain that (here $\alpha \in \sigma(A)^c$ is as in Definition 6.1.17)

$$(\bar{\alpha} - A^*)\mathcal{P} = \bar{\alpha}\mathcal{P} + \mathcal{P}A_{\rm crit} + C^*JC_{\rm crit} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), H), \quad \text{hence} \tag{9.87}$$

$$\mathcal{P} = (\bar{\alpha} - A^*)^{-1}(\bar{\alpha}\mathcal{P} + \mathcal{P}A_{\rm crit} + C^*JC_{\rm crit}) \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), \mathrm{Dom}(A^*)). \tag{9.88}$$

2° *The other claims:* Because $B \in \mathcal{B}(H_1^*, U)$, we have $B^*\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), U)$. Consequently, (9.85) follows from (9.66). Now we can get (9.84) from (9.67), but we give a direct proof below:

Part 1° implies that $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A^*)^*, \mathrm{Dom}(A_{\rm crit})^*)$, by Lemma A.3.24, and $H_{-1} = \mathrm{Dom}(A^*)^*$, $A \in \mathcal{B}(H, H_{-1})$ and $B(U, H_{-1})$. Therefore, we can substitute identities $A_{\rm crit} = A + BK_{\rm crit}$ and $C_{\rm crit} = C + DK_{\rm crit}$ (from Lemma 8.3.17(a)) into (9.65) and separete the terms to obtain

$$0 = A^*\mathcal{P} + \mathcal{P}A + \mathcal{P}BK_{\rm crit} + C^*JC + C^*JDK_{\rm crit}. \tag{9.89}$$

Combine this with (9.85) to obtain (9.84). □

The proposition below corresponds to standard control problems, where D and C have been simplified. One example of such is the stabilization or LQR problem, where $\mathcal{J}(x_0,u) := \|x\|_2^2 + \|u\|_2^2$, i.e., $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = I$, $Y = H \times U$. Also in simplified H^∞ problems, one often has $D^*JC = 0$ and $D^*JD \in \mathcal{GB}(U)$ (e.g., $C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, $\gamma > 0$). In these settings, we can simplify the above "$\mathrm{Dom}(A_{\rm crit})$ CARE" even more to obtain the one in [FLT]:

Proposition 9.7.6 (Bounded C, standard cost) *Let there be a unique J-critical control $u_{\mathrm{crit}}(x_0)$ over $\mathcal{U}_*^*$ for each $x_0 \in H$. Assume that $C \in \mathcal{B}(H,Y)$, $Q := C^*JC \in \mathcal{B}(H)$, $D^*JC = 0$, $R := D^*JD \in \mathcal{GB}(U)$. Then*

$$K_{\mathrm{crit}} = -R^{-1}B^*\mathcal{P} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),U), \tag{9.90}$$

$$-A^*\mathcal{P} = \mathcal{P}A - \mathcal{P}BR^{-1}B^*\mathcal{P} + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),H), \tag{9.91}$$

$$-(A - BR^{-1}B^*\mathcal{P})^*\mathcal{P} = \mathcal{P}A + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A),H), \tag{9.92}$$

$$\mathcal{P}^* = \mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A_{\mathrm{crit}}^*)) \cap \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*)) \tag{9.93}$$

$$\cap\, \mathcal{B}(\mathrm{Dom}(A^*)^*,\mathrm{Dom}(A_{\mathrm{crit}})^*) \cap \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}^*)^*,\mathrm{Dom}(A)^*). \tag{9.94}$$

Therefore, $B^*\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),U)$ and $K_{\mathrm{crit}}^* = -\mathcal{P}BR^{-1} \in \mathcal{B}(U,\mathrm{Dom}(A_{\mathrm{crit}})^*)$.

Unfortunately, also this “CARE” is given on $\mathrm{Dom}(A_{\mathrm{crit}})$ instead of $\mathrm{Dom}(A)$. We would like to remove the parenthesis on the left-hand-side of (9.92), but this cannot be done in general, since it would require, e.g., that $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A))$, which is not the case. By replacing B^* by B_{w}^* we could slightly relax the requirement, but one still need an additional condition to get further (cf. Section 9.2).

Even when C is bounded, $D^*JD = I$, $D^*JC = 0$ and $J = I$, the operator $K_{\mathrm{crit}} = -B^*\mathcal{P}$ need not be bounded (nor $B_{\mathrm{w}}^*\mathcal{P}$), by Example 9.13.8. This shows that while the boundedness of B makes things easy (see Theorem 9.9.6), that of C is not as helpful.

Under the assumptions of the proposition, one easily obtains from (9.55) that $\mathbb{K}_{\mathrm{crit}}^t + \mathbb{B}^{t\,*}(Q+\mathcal{P})\mathbb{A}_{\mathrm{crit}}^t = 0$ (use the fact that $\mathbb{D}^t = \pi_{[0,t)}(D + C\mathbb{B}^t)$, by Theorem 6.2.13), or equivalently, that $u_{\mathrm{crit}} = -\mathbb{B}^{t\,*}(Q+\mathcal{P})x_{\mathrm{crit}}$ a.e., hence this case is essentially simpler than the general case.

Proof of Proposition 9.7.6: 1° By direct substitutions to Theorem 9.7.3, we obtain that

$$-A_{\mathrm{crit}}^*\mathcal{P} = \mathcal{P}A_{\mathrm{crit}} + Q + K_{\mathrm{crit}}^*RK_{\mathrm{crit}} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A_{\mathrm{crit}})^*), \tag{9.95}$$

$$-A_{\mathrm{crit}}^*\mathcal{P} = \mathcal{P}A + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A),H), \tag{9.96}$$

$$-A^*\mathcal{P} = \mathcal{P}A_{\mathrm{crit}} + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),H). \tag{9.97}$$

From (9.96) we obtain that (here $\alpha \in \sigma(A)^c$ and $\beta \in \sigma(A_{\mathrm{crit}})^c$ are as in Definition 6.1.17)

$$(\bar{\beta} - A_{\mathrm{crit}}^*)\mathcal{P} = \bar{\beta}\mathcal{P} + \mathcal{P}A + Q \in \mathcal{B}(\mathrm{Dom}(A),H), \quad \text{hence} \tag{9.98}$$

$$\mathcal{P} = (\bar{\beta} - A_{\mathrm{crit}}^*)^{-1}(\bar{\beta}\mathcal{P} + \mathcal{P}A + Q) \in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A_{\mathrm{crit}}^*)). \tag{9.99}$$

Analogously, from (9.97) (or (9.86)) we obtain that $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*))$. Consequently, $B_{\mathrm{w}}^*\mathcal{P} = B^*\mathcal{P}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$, and hence (9.90) follows from (9.66).

2° Because $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ and hence $\mathcal{P} = \mathcal{P}^*$ on any subset of H (e.g., $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*))$), also the adjoints $\mathcal{P},\mathcal{P}^* \in \mathcal{B}(\mathrm{Dom}(A^*)^*,\mathrm{Dom}(A_{\mathrm{crit}})^*)$ become equal (recall that $\mathrm{Dom}(A^*)^* = H_{-1}$), cf. Lemma A.3.24. Thus, one easily verifies (9.93)–(9.94).

3° Now we obtain (9.91) and (9.92) by direct substitutions. Obviously, $K_{\text{crit}}^* = -\mathcal{P}BR^{-1} \in \mathcal{B}(U, \text{Dom}(A_{\text{crit}})^*)$. □

Remark 9.7.7 *Let Σ_{crit} be a J-critical control for Σ in WPLS form.*

(a1) Given $x_0 \in \text{Dom}(A_{\text{crit}})$, the J-critical control $u = u_{\text{crit}}(x_0)$ and state $x = x_{\text{crit}}(x_0)$ are given in the "state feedback form" in the weak sense that

$$u(t) = K_{\text{crit}}x(t) \qquad (t \in \mathbf{R}_+); \tag{9.100}$$

$$x'(t) = Ax(t) + Bu(t) = (A + BK_{\text{crit}})x \qquad (t \in \mathbf{R}_+) \tag{9.101}$$

(note that u and x are continuous in this case). Furthermore, for an arbitrary $x_0 \in H$, we have $u(t) = (K_{\text{crit}})_s x(t)$ for almost every $t \in \mathbf{R}_+$

(a2) The operator $A_{\text{crit}} := A + BK_{\text{crit}}$ generates the C_0-semigroup $\mathbb{A}_{\text{crit}}$, and, for any $x_0 \in H$, the critical control $x = x_{\text{crit}}(x_0)$ is the strong solution of $x' = A_{\text{crit}}x,\ x(0) = x_0$ (here A_{crit} is the extension of the original operator, as in Lemma 6.1.16).

(a3) The "state feedback" of (a1)–(a2) is not well posed if Σ_{crit} is not of state feedback form (equivalently, does not correspond to a solution of the eIARE).

In fact, such "non-well-posed state feedback" might "explode" under any external input ("$u_\circlearrowleft$"; e.g., disturbance or modelling error); cf. Figure 9.1 (p. 408) and Example 11.3.7.

(b1) **(CARE)** *The "Riccati" equations of Theorem 9.7.3 are satisfied; in particular, for WR $\mathbb{D}$ with $D^*JD \in \mathcal{GB}(U)$, $\mathcal{P}$ determines K_{crit} uniquely on $\text{Dom}(A_{\text{crit}})$ and $u_{\text{crit}} = -(D^*JD)^{-1}(B_w^*\mathcal{P} + D^*JC_{\text{L,w}})x_{\text{crit}}$ a.e., for any $x_0 \in H$.*

Moreover, $\mathcal{J}(x_0, u_{\text{crit}}) = \langle x_0, \mathcal{P}x_0\rangle$.

(b2) **(Uniqueness)** *A $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$ (if any) of the "eIARE" of Theorem 9.7.1 is equal to the J-critical cost operator, hence it is unique, by Lemma 8.3.8.*

If $\mathbb{D}$ is WR, then we can use the "$\text{Dom}(A_{\text{crit}})$-CARE" (9.65) and (9.66) instead of the "eIARE", by Theorem 9.7.3(d). In particular, a "$\mathcal{U}_*^*$-stabilizing" solution $\mathcal{P}$ of this "$\text{Dom}(A_{\text{crit}})$-CARE" is unique (and so is K_{crit} if $D^*JD \in \mathcal{GB}(U)$).

Thus, the ("$\mathcal{U}_*^*$-stabilizing") solution $\mathcal{P}$ of the "CARE" leads to the "state feedback" formula of (b1) and to the J-critical cost $\langle x_0, \mathcal{P}x_0\rangle$.

If Σ is stable (or suitably (well-posedly) stabilizable) and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{exp}}$, then minimization corresponds to a well-posed (and stable) state feedback, as shown in Section 10.2. However, in, e.g., the stable J-coercive indefinite case, the J-critical state feedback pair might be unstable, or even non-well-posed (i.e., $\widehat{\mathbb{F}}, \widehat{\mathbb{F}_\circlearrowleft} \notin \text{H}_\infty^\infty$), although $\mathbb{K}_\circlearrowleft$ is always well-posed and stable (cf. (a3)); see Example 11.3.7.

Recall that by state feedback we mean the (well-posed) state feedback of Definition 6.6.10, not that of (a3).

Proof of Remark 9.7.7: (a1) This follows from (a2), Lemma A.4.2(c1) and Lemma 6.2.12.

(a2) By Lemma 8.3.17(a), $A_{\rm crit} = A + BK_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$. The rest follows from Lemma 6.1.16(b).

(a3) 1° This is intuitively rather obvious. We shall show in 2° that if the map $u_\circlearrowleft \to u$ is well posed, where $u_\circlearrowleft$ is an external input and u is the effective input corresponding to $u_\circlearrowleft$ and initial state $x_0 = 0$, as in Figure 9.1, then $\Sigma_{\rm crit}$ is the left column of the corresponding closed-loop system; this proves our first claim. If $u \mapsto y$ is coercive (e.g., $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, as in the standard setting), then this means that also the map $u_\circlearrowleft \mapsto y$ becomes ill-posed; the same applies to $u_\circlearrowleft \mapsto x$.

2° If $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, where $\mathbb{M}u_\circlearrowleft := u$ (a time-invariant operator $\mathbb{M} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R}_+;U))$ can be extended to $\mathcal{G}\mathrm{TIC}_\omega(U)$, by Lemma 2.1.3), then $\mathbb{M}$ defines an admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ for Σ, and $\Sigma_{\rm crit}$ is the left column of the corresponding closed-loop system.

Indeed, if $u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_-;U)$ and $u = \mathbb{M}u_\circlearrowleft$, then, obviously, $\pi_+ u = \mathbb{K}_{\rm crit}\mathbb{B}u = \mathbb{K}_{\rm crit}\mathbb{B}\mathbb{M}u_\circlearrowleft$ (this is the control corresponding to $\pi_+ u_\circlearrowleft = 0$ and $x_0 = \mathbb{B}u$). Now we obtain $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ from Theorem 8.3.13(b1) (see Lemma 8.3.16(b)).

(b1) This follows from Theorem 9.7.3.

(b2) This follows from Theorems 9.7.1 and 9.7.3.

Note that the definition of a "$\mathcal{U}_*^*$-stabilizing" solution $\mathcal{P} \in \mathcal{B}(H)$ is rather clumsy; e.g., we require $K_{\rm crit}$ to generate a WPLS ("$\begin{bmatrix} \mathbb{A}_0 \\ \mathbb{C}_0 \\ \mathbb{K}_0 \end{bmatrix}$") with $A + BK_{\rm crit}$ and $C_{\rm w} + DK_{\rm crit}$, and pose the conditions $\mathbb{K}_0 x_0 \subset \mathcal{U}_*^*(x_0)$ and (9.54); the latter two conditions become equivalent to $\mathbb{A}_0$ being exponentially stable if $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$, by Theorem 9.7.1(c). This can be simplified in the positively coercive case (as done for the CARE in Section 10.7, in particular, Proposition 10.7.3(d3)).

However, a CARE defined on $\mathrm{Dom}(A_{\rm crit})$ does not seem to be useful, therefore we will not go into these simplifications; we just mention that in Theorem 5.3 of [FLT] it is shown that if $C = \begin{bmatrix} R \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J = I$ and $R \gg 0$ (thus $\mathcal{J}(x_0,u) = \|u\|_2^2 + \langle x, Rx\rangle_{\mathrm{L}^2}$), then (9.67) has at most one self-adjoint solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ s.t. $B^*\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), H)$. □

The rest of this section consists of auxiliary lemmas only. Some algebraic (instantaneous) Riccati-like equations can be equivalently written in integral forms and vice versa, as described below:

Lemma 9.7.8 *(a) Let* $\begin{bmatrix} \mathbb{A}_k \\ \mathbb{C}_k \end{bmatrix} \in \mathrm{WPLS}(0,H,Y)$ $(k = 1,2)$, $P \in \mathcal{B}(H)$ *and* $\widetilde{J} \in \mathcal{B}(Y)$. *Then*

$$\langle A_1 x_1, P x_2\rangle_{H_1} + \langle x_1, P A_2 x_2\rangle_{H_1} + \langle C_1 x_1, \widetilde{J} C_2 x_2\rangle_{Y_1} \ge 0 \qquad (x_1 \in \mathrm{Dom}(A_1),\ x_2 \in \mathrm{Dom}(A_2)) \tag{9.102}$$

$$\iff \qquad (\mathbb{A}_1^t)^* P \mathbb{A}_2^t + (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t \ge P \qquad (t \in [0,+\infty)). \tag{9.103}$$

Moreover, we can, equivalently, replace "$(t \in [0,+\infty))$*" by "*$(t \in (0,\varepsilon))$*", for* *any* $\varepsilon > 0$, *or require (9.102) only for* $x_k \in \cap_{n\in\mathbf{N}} \mathrm{Dom}(A_k^n)$. *All this also holds with* "$=$" *or* "$\leq$" *in place of* "$\geq$".

(b) Part (a) also holds if we replace "$\begin{bmatrix} \mathbb{A}_k \\ \mathbb{C}_k \end{bmatrix} \in \mathrm{WPLS}(0,H,Y)$" *by* "$\mathbb{A}_k$ *is a* C_0*-semigroup on* H *and* $C_k \in \mathcal{B}(H_1,Y)$" *and set* $(\mathbb{C}_k x_0)(t) = C_k \mathbb{A}_k(t) x_0$ $(t \geq 0)$ *to define* $\mathbb{C}_k : H_1 \to \mathcal{C}(\mathbf{R}_+;Y)$, *except that (9.103) must be applied to elements of* $\mathrm{Dom}(A)$ *only.*

Equation (9.102) is equivalent to

$$A_1^* P + PA_2 + C_1^* \widetilde{J} C_2 \geq 0 \quad (\text{in } \mathcal{B}(\mathrm{Dom}(A_2), \mathrm{Dom}(A_1)^*)) \,, \tag{9.104}$$

where $\mathrm{Dom}(A_k)$ is equipped with the graph topology and $\mathrm{Dom}(A_k)^*$ is its dual w.r.t. the pivot space H as in Lemma A.3.24 (or in Lemma 6.1.16 and Definition 6.1.17). However, we encourage the reader to always write such formulae into the longer form (9.102) when he or she has problems to verify them in their short forms.

For equations containing I/O maps ("$\mathbb{D}$"), we need some regularity assumption (for "D" to exist), and for equations containing input maps ("$\mathbb{B}$"), the equivalence becomes rather complicated, and they need special extensions of B^* and $\overline{B^*}P$, as illustrated in Section 9.8 and corresponding proofs in Section 9.11.

Proof: 1° "$\Leftarrow$"*:* Let $x_k \in \mathrm{Dom}(A_k)$ $(k=1,2)$. By Lemma A.4.2(c1), we have

$$(\mathbb{A}_k x_k)' = A_k \mathbb{A}_k x_k = \mathbb{A}_k A_k x_k \in \mathcal{C}(\mathbf{R}_+;H_k) \quad (k=1,2), \tag{9.105}$$

in particular, $\mathbb{A}_k x_k \in \mathcal{C}(\mathbf{R}_+;\mathrm{Dom}(A_k))$. Consequently, $\mathbb{C}_k x_k = C_k \mathbb{A}_k x_k \in \mathcal{C}(\mathbf{R}_+;Y)$ $(k=1,2)$, by Lemma 6.2.12.

Since $f := \langle x_1, g x_2\rangle$, where $g := (\mathbb{A}_1^t)^* P \mathbb{A}_2^t + (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t - P$, satisfies $f(0) = 0$, $f \geq 0$ and $f \in \mathcal{C}^1(\mathbf{R}_+)$, we have $f'(0) \geq 0$, which implies that (9.102) holds.

2° "$\Rightarrow$"*:* Assume that (9.102) holds on $\mathrm{Dom}(A_1^\infty) \times \mathrm{Dom}(A_2^\infty)$. Let $a_k \in \mathrm{Dom}(A_k^\infty) := \cap_{n\in\mathbf{N}} \mathrm{Dom}(A_k^n)$ and $t \geq 0$. Set $x_k := \mathbb{A}_k^t a_k \in \mathrm{Dom}(A_k^\infty)$, so that $\mathbb{C}_k a_k = C_k x_k$ and $(\mathbb{A}_k a_k)'(t) = A_k x_k$ $(k=1,2)$, as in 1°. By substituting these into (9.102), we obtain

$$\begin{aligned} 0 \,&\leq\, \langle \mathbb{A}_1'(t) a_1, P\mathbb{A}_2(t) a_2\rangle + \langle \mathbb{A}_1(t) a_1, P\mathbb{A}_2'(t) a_2\rangle + \langle C_1 \mathbb{A}_1(t) a_1, \widetilde{J} C_1 \mathbb{A}_2(t) a_2\rangle \\ &= \frac{d}{dt}\left[\langle \mathbb{A}_1(t) a_1, P\mathbb{A}_2(t) a_2\rangle_H + \int_0^t \langle C_1 \mathbb{A}_1(t) a_1, \widetilde{J} C_2 \mathbb{A}_2(t) a_2\rangle_Y \, dt\right] \\ &= \frac{d}{dt}\left[\langle a_1, \mathbb{A}_1(t)^* P \mathbb{A}_2(t) a_2\rangle_H + \langle a_1, (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t a_2\rangle_H\right]. \end{aligned}$$

Thus, the expression in brackets must be increasing, hence for any $t > 0$, we have

$$\begin{aligned} &\langle a_1, \mathbb{A}_1(t)^* P \mathbb{A}_2(t) a_2\rangle + \langle a_1, \mathbb{C}_1^* \widetilde{J} \pi_{[0,t]} \mathbb{C}_2 a_2\rangle \\ &\qquad \geq \langle a_1, \mathbb{A}_1(0)^* P \mathbb{A}_2(0) a_2\rangle + \langle a_1, \mathbb{C}_1^* \widetilde{J} \pi_{[0,0]} \mathbb{C}_2 a_2\rangle = \langle a_1, P a_2\rangle - 0. \end{aligned}$$

The same holds for $a_1, a_2 \in H \times H$ too, because $\mathrm{Dom}(A_k^\infty)$ is dense in H, by Lemma A.4.2(b).

3° The "moreover" claim can be observed from the above proofs; the claim on "$\le$" follows by replacing P by $-P$ and $\widetilde{J}$ by $-\widetilde{J}$; the claim on "$=$" follows "$\le$" and "$\ge$".

(b) The proof of (a) applies here too. Note that here (9.103) means that $\langle x_0, \Big((\mathbb{A}_1^t)^* P \mathbb{A}_2^t + (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t - P\Big) x_0\rangle_H \ge 0$ for all $x_0 \in \mathrm{Dom}(A) =: H_1$ and all $t \in [0,+\infty)$. □

We shall sometimes need the following lemma:

Lemma 9.7.9 *Let $x_0 \in H$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$. Then $u \in \mathcal{U}_*^*(x_0)$ iff $\pi_+\tau^t u \in \mathcal{U}_*^*(\mathbb{A}^t x_0 + \mathbb{B}^t u)$ for some (equivalently, all) $t \ge 0$.*

This says that u is admissible for some initial state $x(0)$ iff at some (hence any) moment t the rest of u is admissible for the current state $x(t)$.

Proof: Given $t \ge 0$, set $u' := \pi_{[0,t)} u$, $u'' := \pi_+ \tau^t u$, so that $u = u' \diamondsuit_t u''$ (see p. 158) and $x_t := \mathbb{A}^t x_0 + \mathbb{B}^t u = \mathbb{A}^t x_0 + \mathbb{B}^t u'$, Obviously, $u \in \mathrm{L}^2_\vartheta \Leftrightarrow u'' \in \mathrm{L}^2_\vartheta$. We have (recall that $\tau^t u'' = \pi_- \tau^t u$)

$$(\mathbb{C}x_t) + \mathbb{D}u'' = (\pi_+\tau^t \mathbb{C}x_0 + \pi_+\mathbb{D}\tau^t u'') + \pi_+\mathbb{D}\pi_+\tau^t u = \pi_+\tau^t(\mathbb{C}x_0 + \mathbb{D}u) \quad (9.106)$$

hence $\mathbb{C}x_t + \mathbb{D}u'' \in \mathrm{L}^2$ iff $\mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$.

Analogously, we can show that $\mathbb{Q}x_t + \mathbb{R}u'' \in Z^{\mathrm{s}}$ iff $\pi_+\tau^t(\mathbb{Q}x_0 + \mathbb{R}u) \in Z^{\mathrm{s}}$, i.e., iff $\mathbb{Q}x_0 + \mathbb{R}u \in Z^{\mathrm{s}}$ (by Standing Hypothesis 9.0.1). Thus, we have shown that $u \in \mathcal{U}_*^*(x_0) \Leftrightarrow u'' \in \mathcal{U}_*^*(x_t)$. Since $t \ge 0$ was arbitrary, this establishes the claim. □

We finish this section by a result that was used in the proof of Theorem 9.7.1:

Lemma 9.7.10 *Assume that $\begin{bmatrix} \mathbb{A}_0 \\ \hline \mathbb{C}_0 \\ \mathbb{K}_0 \end{bmatrix}$ is a control in WPLS form s.t. $\mathbb{K}_0 x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$.*

Then, for any $t \ge 0$ and $\widetilde{\eta} \in \mathrm{L}^2([0,t);U)$, we have $\eta := \pi_{[0,t)}\widetilde{\eta} + \tau^{-t}\mathbb{K}_0\mathbb{B}^t\widetilde{\eta} \in \mathcal{U}_*^*(0)$ and $\pi_{[0,t)}\eta = \widetilde{\eta}$. □

(This follows from Lemma 9.7.9 by setting $x_0 = 0$, since $\mathbb{K}_0\mathbb{B}^t\widetilde{\eta} \in \mathcal{U}_*^*(\mathbb{B}^t\widetilde{\eta})$.)

Notes

As necessary conditions, equations (9.55)–(9.57) and (9.63)–(9.65) are well-known for some special cases; see, e.g., [S97b] for the case of well-posed minimizing state feedback for WPLSs.

Most of Proposition 9.7.6 was established in [FLT] by solving first the finite-time interval problem and then obtaining the infinite-time results as a limit. This method requires a rather coercive cost function (see the notes below the proposition) and is hence not applicable to the more general results of Theorem 9.7.3 and Lemma 9.7.5, whose formulae are well-known at least for systems with bounded B and C [CZ, Exercise 6.21c]. The integrations and differentiations used in Lemma 9.7.8 are well-known [WW] [S97b].

We shall now treat two questions on [FLT] (or on [LT00b]) that have caused some controversy. One observes from (6.52) (with $C = I$) that $(\pi_{[0,t)}\mathbb{B}\tau\pi_{[0,t)})^* v =$

$B^*_{\mathrm{L,s}} \int_t^T \mathbb{A}(s)^* v(s)\, ds$ for all $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;H)$. In (1.3b) of [FLT], the same formula is given with "B^*" in place of "$B^*_{\mathrm{L,s}}$", and the results of that article are derived using this formula. According to the authors, "B^*" refers to a "maximal, possibly nonunique extension of the original $B^* \in \mathcal{B}(\mathrm{Dom}(A^*),U)$", whose existence should follow (nonconstructively) directly from the (standard) assumptions that $A^{-1}B \in \mathcal{B}(U,H)$ (use $(s-A)^{-1}$ instead if $0 \in \sigma(A)$) and that $B^*\mathbb{A}^* : \mathrm{Dom}(A^*) \to \mathcal{C}$ extends to an operator $\mathbb{B}^{\mathrm{d}} : H \to \mathrm{L}^2([0,T);U)$ $(T > 0)$. We cannot follow this argument, nor can the experts that we have contacted. However, the constructive, highly nontrivial proof of [W89a] (see (6.52)) can be used. Nevertheless, in several applications, such as in the parabolic setting of Section 9.5, the proof is rather simple.

The second controversial thing is the following: If $D^*JD = I$, then we obtain from Proposition 9.7.6 that

$$\langle B^*\mathcal{P}x_0, B^*\mathcal{P}z_0\rangle_U = \langle \mathcal{P}Ax_0, z_0\rangle_{\langle \mathrm{Dom}(A_{\mathrm{crit}})^*, \mathrm{Dom}(A_{\mathrm{crit}})\rangle} - \langle \mathcal{P}A_{\mathrm{crit}}x_0, z_0\rangle_H \quad (x_0, z_0 \in \mathrm{Dom}(A_{\mathrm{crit}})). \tag{9.107}$$

By (9.94), the expression

$$\langle \mathcal{P}Ax_0, z_0\rangle_H - \langle \mathcal{P}A_{\mathrm{crit}}x_0, z_0\rangle_{\langle \mathrm{Dom}(A)^*, \mathrm{Dom}(A)\rangle} \quad (x_0, z_0 \in \mathrm{Dom}(A)), \tag{9.108}$$

is continuous $\mathrm{Dom}(A) \times \mathrm{Dom}(A) \to \mathbf{C}$. In [FLT, Corollary 4.9], Flandoli et al. define $\overline{\langle B^*\mathcal{P}x_0, B^*\mathcal{P}z_0\rangle_U}$ for $x_0, z_0 \in \mathrm{Dom}(A)$ by (9.108) (even if $\mathrm{Dom}(A) \cap \mathrm{Dom}(A_{\mathrm{crit}}) = \{0\}$), so that $A_{\mathrm{crit}}\mathcal{P} = A^*\mathcal{P} - (\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}} \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*)$ and hence (9.96) becomes

$$(\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}} = A^*\mathcal{P} + \mathcal{P}A + Q \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*), \tag{9.109}$$

by the definition of $(\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}}$. Thus, (9.109) is actually an equivalent way of writing (9.96).

G. Weiss and H. Zwart (Section 8 of [WZ]) have constructed a simple example (our Example 9.13.8) satisfying the assumptions of Proposition 9.7.6 and having a ULR unique minimizing (hence J-critical) state feedback operator K that differs from $\overline{B^*\mathcal{P}}$ (as defined by (9.108); see [WZ] for details). In fact, since $\mathrm{Dom}(A) \cap \mathrm{Dom}(A_{\mathrm{crit}}) = \{0\} = \mathrm{Ker}(\mathcal{P})$ in [WZ], one could extend $B^*\mathcal{P}$ to $\mathrm{Dom}(A)$ arbitrarily. The extension $B^*_{\mathrm{L,s}}\mathcal{P} = B^*_{\mathrm{w}}\mathcal{P}$ is the only one leading to the minimizing K.

Therefore, the extension of $B^*\mathcal{P}$ used for (9.109) would seem artificial and not having a connection to the minimizing state feedback operator $-K = B^*_{\mathrm{w}}\mathcal{P} = B^*_{\mathrm{L,s}}\mathcal{P}$ (this formula is valid on $H_B \supset \mathrm{Dom}(A) \cup \mathrm{Dom}(A_{\mathrm{crit}})$, see Section 9.9).

It is not obvious whether this $\overline{B^*\mathcal{P}}$ always corresponds to $\overline{B^*}\mathcal{P}$ for any extension $\overline{B^*}$ of B^*. However, after finishing this chapter, we learned abouve [BLT], in which the authors succeed in settling this last problem for several special cases by proving the existence of an extension $\overline{B^*}$ of B^* on $\mathcal{P}[\mathrm{Dom}(A)]$ s.t. $\overline{B^*}\mathcal{P}$ equal "$\overline{B^*\mathcal{P}}$". In these constructions, their methods resemble the definition of B_{s}.

However, in several applications, such as the parabolic problems of [LT00a], equation (9.109) coincides with the CARE (9.3) and hence becomes useful for the computation of the Riccati operator and the optimizing state feedback operator. In particular, in these cases the signature operator S of the problem equals D^*JD

(See the notes to Section 9.8 for a comparison between D^*JD and the signature operator.) One of the merits of [FLT] is the results that if $Q \gg 0$, then $\mathcal{P}$ is unique (see the proof of Remark 9.7.7(c)).

9.8 Algebraic and integral Riccati equations (CARE↔IARE)

For every complex problem, there is a solution that is simple, neat, and wrong.

— H. L. Mencken

In this section, we shall divide the correspondence between optimal (i.e., J-critical) control and CAREs into two parts, by introducing "Integral Algebraic Riccati Equation (IARE)" in between these two concepts.

The IARE is essentially an integral of the CARE, but it can be formulated regardless of the regularity of the system and J-critical control (as long as it can be given in the state feedback form). Therefore, this new concept allows us to extend the classical one-to-one correspondence between the optimal control and the stabilizing solution of the Riccati equation to general WPLSs in Section 9.9. In this section, we shall establish the equivalence of IAREs and CAREs under weak regularity assumptions (Proposition 9.8.10), and show the uniqueness of their stabilizing solutions (Theorem 9.8.12).

In addition, we shall study the extended forms of these Riccati equations, "eIAREs" and "eCAREs", where the signature operators need not be invertible, since a J-critical control need not correspond to an invertible signature operator unless we assume a coercive cost function (J-coercivity).

We shall also observe that the eIARE is a reformulation of the *extended Discrete-time Algebraic Riccati Equation (eDARE)*, see Proposition 9.8.7 and Theorem 13.4.4. Thus, we may use the eDARE theory from Section 14.1 to solve continuous-time problems in discrete time. In particular, if one wishes to verify the proofs, one should read first Section 14.1. The equivalence of eCAREs and eIAREs then extends the discrete-time results also to (e)CAREs.

Because of the appearance of the feedthrough operators D and $X = I - F$ in the eCAREs, we can define them only for weakly regular $\mathbb{D}$ and $\mathbb{F}$ (their feedthrough operators are often assumed to be zero in classical theory; this cannot be done in general). Nevertheless, also part of the "CARE" theory can be applied to more general systems, as shown in Remark 9.12.1 and Section 9.7.

Further results on Riccati equations will be given in the following sections, and on positive Riccati equations (roughly, with a nonnegative signature operator S) in Section 10.6. The proofs of Proposition 9.8.11(ii) and Theorem 9.8.12(d)&(e) depend on Section 9.9.

Recall our standing assumptions that $\Sigma = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ and that $J = J^* \in \mathcal{B}(Y)$, and the fact that we denote generators by same letters as corresponding operators ($\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right) = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$).

We start by extending Definition 9.1.5 to cover the case where $S \notin \mathcal{GB}(U)$ and $F \neq 0$. Solutions of the eCARE will be called "$\mathcal{U}_*^*$-stabilizing" if the corresponding control $u := \mathbb{K}_\circlearrowleft x_0$ belongs to the class over which we are optimizing ($u \in \mathcal{U}_*^*(x_0)$) and the residual cost condition (PB) is satisfied (the latter is redundant for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ and for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$, by Theorem 9.8.5). In Corollary 9.9.2 we shall show that such solutions correspond one-to-one to the optimal state feedback operators.

Definition 9.8.1 (eCARE) *We call $\mathcal{P}$ (or $(\mathcal{P},S,\left[\begin{array}{c|c} K & F \end{array}\right])$) a* solution of the extended Continuous-time Algebraic Riccati Equation (eCARE) *(induced by Σ and J) iff Σ is WR and $\mathcal{P}$ satisfies*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*) \\ X^*SX = D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ X^*SK = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\mathrm{Dom}(A), U) \end{cases} \tag{9.110}$$

(here $X := I - F$) and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ $F, S \in \mathcal{B}(U)$, $S = S^*$ and $K \in \mathcal{B}(H_1, U)$.

A solution $\mathcal{P}$ of the eCARE is called WR *(resp. SR, UR, admissible, stabilizing, stable, ...) if generators $\left[\begin{array}{c|c} K & F \end{array}\right]$ extend Σ to another WR WPLS Σ_{ext} and the resulting pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is WR (resp. SR, UR, admissible, stabilizing, stable, ...). See Definition 6.6.10 for further prefices and suffices.*

If $\mathcal{P}$ is admissible, then we denote the corresponding closed-loop system by $\Sigma_{\circlearrowleft}$ and set $\mathbb{X} := I - \mathbb{F} \in \mathrm{TIC}_\infty(U)$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathrm{TIC}_\infty(U)$, $\mathbb{N} := \mathbb{D}_{\circlearrowleft} := \mathbb{D}\mathbb{M} \in \mathrm{TIC}_\infty(U,Y)$.

We add the prefix "P-" or "PB-" if $\mathcal{P}$ is admissible and satisfies the corresponding residual cost condition *below:*

(P) $\langle \mathbb{A}_{\circlearrowleft}^t x_0, \mathcal{P}\mathbb{A}_{\circlearrowleft}^t x_0\rangle \to 0$*, as $t \to +\infty$, for all $x_0 \in H$.*

(PB) $\langle \mathbb{B}^t u, \mathcal{P}\mathbb{A}_{\circlearrowleft}^t x_0\rangle \to 0$*, as $t \to +\infty$, for all $x_0 \in H$ and $u \in \mathcal{U}_*^*(0)$, and (P) holds.*

*We call $\mathcal{P}$ $\mathcal{U}_*^*$-*stabilizing *if (PB) is satisfied and $\mathbb{K}_{\circlearrowleft}x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$.*

The eCARE with additional requirements $X = I$ and $S \in \mathcal{GB}(U)$ is called CARE.

(Note that $\left[\begin{array}{c|c} K & F \end{array}\right]$ extends Σ to a WR WPLS iff $\left[\begin{array}{c|c} A & B \\ \hline K & F \end{array}\right]$ generate a WR WPLS. Note also that when we say that the eCARE has a solution, we tacitly say that $\mathbb{D}$ is WR. Admissible pairs are described in Definition 6.6.10; also here we set $\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right]$, so that $\Sigma_{\circlearrowleft} = (\Sigma_{\mathrm{ext}})_{[0\ I]}$.)

A less abstract formulation of the first of the equations in (9.110) is given in (9.4) (cf. Lemma 9.7.8 and inequality (9.104)).

We shall show in Theorem 9.9.1(a1)&(e1)&(e2) that $\mathcal{U}_*^*$-stabilizing solutions of the eCARE correspond to WR J-critical state feedback pairs over $\mathcal{U}_*^*$. We are mainly interested in WR J-critical state feedback operators, equivalently, in the case $F = 0$ (or $X = I$).

The prefix "P-" (resp. "PB-", "$\mathcal{U}_*^*$-") does not affect other prefices and vice versa, it just adds the requirement (P) (resp. (PB), $\mathcal{U}_*^*$-stability of $\Sigma_{\circlearrowleft}$) (e.g., by "exponentially PB-stabilizing" we mean "exponentially stabilizing and satisfying (PB)").

(For analogy with the traditional concept "stabilizing solution", we say "$\mathcal{U}_*^*$-stabilizing", not "$\mathcal{U}_*^*$-admissible", although this term does not imply that $\Sigma_{\circlearrowleft}$ is stable (similarly, $\mathbb{C}$-stabilizing means that $\mathbb{C}_{\circlearrowleft}$ is stable but the whole $\Sigma_{\circlearrowleft}$ need not be, by Definition 6.6.10).)

Remark 9.1.6 applies also to eCAREs, except that K need not be WR unless $X,S \in \mathcal{GB}(U)$; cf. Lemma 9.9.7.

Thus, a solution $\mathcal{P}$ is WR if $\left[\begin{smallmatrix} A & B \\ C & D \\ K & F \end{smallmatrix}\right]$ generate a WR WPLS, i.e., if the operators $\mathbb{K} : x_0 \mapsto K\mathbb{A}(\cdot)x_0$ and $\mathbb{F} := I - X + K_{\mathrm{w}}\mathbb{B}\tau$ extend Σ to another WR WPLS (see Lemma 6.3.13 and Remark 9.1.6). A WR solution is admissible iff $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$, by Definition 6.6.10; if $\mathbb{F} \in$ ULR, then this holds iff $X \in \mathcal{GB}$, by Proposition 6.3.1(c). By Lemma 6.6.11, any solution leading to a bounded K is ULR (and admissible iff $I - F \in \mathcal{GB}(U)$).

Condition (P) is satisfied by any strongly stabilizing solution. It follows from Proposition 9.8.10, that for a $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right]$-stabilizing solution of the eCARE, condition (P) is equivalent to any of (P1)–(P4) of Lemma 9.10.1(d1).

Remark 9.8.2 (eCARE vs. CARE) *A WR solution $\mathcal{P}$ of eCARE is a WR solution of the CARE (with $S' = X^*SX$ and $K' = X^{-1}K$ in place of S and K) iff $S,X \in \mathcal{GB}(U)$.* □

(This is obvious.) Moreover, the admissibility [stabilizability] of $\left[\begin{array}{c|c} K' & 0 \end{array}\right]$ is the same as that of $\left[\begin{array}{c|c} K & I - X \end{array}\right]$ (see the formula for $\Sigma_{\circlearrowleft}$ in Theorem 9.8.12(s1)) and the same applies for other attributes of $\mathcal{P}$, hence we can consider $\mathcal{P}$ alone a solution of the eCARE when $S,X \in \mathcal{GB}(U)$ (because then $\mathcal{P}$ determines the S and K corresponding to $X = I$ uniquely).

Moderate regularity and coercivity assumptions force X and S to be invertible, by Lemma 9.9.7, hence the eCARE is equivalent to the CARE in most applications.

Next we give necessary and sufficient conditions for a solution of the CARE to be admissible, both in state-space and frequency-domain terms. Fortunately, we only rarely need these conditions.

Remark 9.8.3 (Admissibility of a solution of the CARE) *Let $(\mathcal{P},S,K)$ be a solution of the CARE.*

- *(a1) By Lemma 6.3.13 and Remark 9.1.6, $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a WR WPLS iff for some $\varepsilon > 0$ and $\omega > \omega_A$, $B_{\mathrm{w}}^*\mathcal{P}\mathbb{A}(\cdot) : \mathrm{Dom}(A) \to \mathcal{C}(\mathbf{R}_+;U)$ extends to a continuous map $H \to \mathrm{L}^2([0,\varepsilon);Y)$, and $B_{\mathrm{w}}^*\mathcal{P}\mathbb{B}\tau : \mathcal{C}_{\mathrm{c}}^\infty([0,\varepsilon);U) \to \mathcal{C}_{\mathrm{b}}([0,\varepsilon);Y)$ extends to a continuous map $\mathrm{L}^2_\omega([0,\varepsilon);U) \to \mathrm{L}^2_\omega([0,\varepsilon);Y)$ for some $\varepsilon > 0$ and $\omega > \omega_A$.*
- *(a2) The corresponding frequency-domain conditions are as follows: There are $\varepsilon > 0$ and $\omega > \omega_A$ s.t. for each $x_0 \in H$ we have $B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}_\omega^+;U)$ and $B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}_\omega^+;\mathcal{B}(U,Y))$ (as functions of s).*
- *(b) Assume (a1) (equivalently, (a2)). If $\mathbb{F}$ is ULR, then $\mathcal{P}$ is admissible (i.e., $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$).*
- *(c) The solution $\mathcal{P}$ is admissible iff $\mathbb{X} := I - \mathbb{F} = I - K_{\mathrm{w}}\mathbb{B}\tau$ is invertible on $\mathrm{TIC}_\infty(U)$, or equivalently, iff $I - K_{\mathrm{w}}(\cdot - A)^{-1}B \in \mathcal{G}\mathrm{H}^\infty_\infty$, or equivalently, iff $\pi_{[0,\varepsilon)}\mathbb{X}\pi_{[0,\varepsilon)}$ is invertible on $\mathrm{L}^2([0,\varepsilon);U)$ for some $\varepsilon > 0$.*

For the eCARE, corresponding conditions can be given, e.g., as follows: $H_B \subset \mathrm{Dom}(K_w)$, $K_w\mathbb{A}(\cdot) : \mathrm{Dom}(A) \to \mathcal{C}(\mathbf{R}_+;U)$ extends to a continuous map $H \to \mathrm{L}^2([0,\varepsilon);Y)$ for some $\varepsilon > 0$, $K_w\mathbb{B}\tau : \mathcal{C}_c^\infty([0,\varepsilon);U) \to \mathcal{C}_b([0,\varepsilon);Y)$ extends to a continuous map $\mathrm{L}^2_\omega([0,\varepsilon);U) \to \mathrm{L}^2_\omega([0,\varepsilon);Y)$ for some $\varepsilon > 0$ and $\omega > \omega_A$, $A_\flat := A + BK_w$ generates a (C_0-)semigroup on H and $K_w(s - A_\flat)^{-1}B \in \mathcal{G}\mathrm{H}^\infty_\infty$.

Proof: (a1) This follows from the definition of K, Remark 9.1.6 and Lemma 6.3.13 (use the fact that $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ is a WPLS).

(a2) Use Lemmas 6.3.15 instead of Lemma 6.3.13 in (a1).

(b) Use Proposition 6.3.1(c).

(c) Use Theorem 6.2.1 and Lemma 2.2.8. □

We define the solutions of the eIARE analogously to those of the eCARE (recall the notation from (6.5)):

Definition 9.8.4 (A P-stabilizing solution of the eIARE) *We call $\mathcal{P}$ (or $(\mathcal{P}, S, \left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right])$) a* solution of the extended Integral Algebraic Riccati Equation (eIARE) *(induced by Σ and J) iff the eIARE*

$$\begin{cases} \mathbb{K}^{t\,*}S\mathbb{K}^t = \mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*}J\mathbb{C}^t \quad (\in \mathcal{B}(H)),\\ \mathbb{X}^{t\,*}S\mathbb{X}^t = \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t,\\ \mathbb{X}^{t\,*}S\mathbb{K}^t = -\left(\mathbb{D}^{t\,*}J\mathbb{C}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{A}^t\right)\end{cases} \tag{9.111}$$

(here $\mathbb{X} := I - \mathbb{F}$) is satisfied for all $t > 0$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S = S^* \in \mathcal{B}(U)$, $\mathbb{K} \in \mathcal{B}(H, \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U))$, and $\mathbb{F} \in \mathrm{TIC}_\infty(U)$.

A solution $\mathcal{P}$ is called well-posed *(resp. WR, SR, UR, admissible, stabilizing, stable, ...) if $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ extends Σ to another WPLS Σ_{ext} (resp. -"- and the pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is WR, SR, UR, admissible, stabilizing, stable, ...).*

We use prefices "$\mathcal{U}_*^*$-", "P-" and "PB-" as in Definition 9.8.1.

The eIARE with additional requirement $S \in \mathcal{G}\mathcal{B}(U)$ is called IARE.

If $\mathcal{P}$ is admissible, then we denote the corresponding closed-loop system by $\Sigma_\circlearrowleft$ and set $\mathbb{X} := I - \mathbb{F} \in \mathrm{TIC}_\infty(U)$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathrm{TIC}_\infty(U)$, $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}\mathbb{M} \in \mathrm{TIC}_\infty(U,Y)$.

(Note that $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ extends Σ iff $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in$ WPLS.) One explanation of the eIARE is that it specifies the value of the terms in $\langle u_\circlearrowleft, S\pi_{[0,t)}u_\circlearrowleft\rangle$, where $u_\circlearrowleft := -\mathbb{K}x_0 + \mathbb{X}u$ is the closed-loop input (the disturbance to the J-critical state feedback) corresponding to open-loop input u and initial state x_0 (indeed, $\mathbb{K}x_0 + \mathbb{F}u = u - u_\circlearrowleft$ in Figure 9.1 (p. 408)).

The IARE should not be mistaken for "IRE", the integral of the "Differential Riccati Equation", which both correspond to finite-time interval problems (see Section 8.5).

By Lemma 9.10.1(b4), equations (9.153)–(9.161) are satisfied by any admissible solution of the eIARE.

In case of $\mathcal{U}_{\mathrm{exp}}$ or $\mathcal{U}_{\mathrm{str}}$, the attribute "$\mathcal{U}_*^*$-stabilizing" can be reduced substantially:

Theorem 9.8.5 ($\mathcal{U}_*^*$-stabilizing solution) *Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ be an admissible solution of the eIARE, with closed-loop system $\Sigma_\circlearrowleft$. Then $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is J-critical over $\mathcal{U}_*^*$.*

Moreover, $\mathcal{P}$ is $\mathcal{U}_{\text{exp}}$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is exponentially stabilizing; and $\mathcal{P}$ is $\mathcal{U}_{\text{str}}$-stabilizing iff $\begin{bmatrix} \mathbb{A}_\circlearrowleft^{\mathrm{T}} \mid \mathbb{C}_\circlearrowleft^{\mathrm{T}} \quad \mathbb{K}_\circlearrowleft^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is strongly stable.

Finally, $\mathcal{P}$ is $\mathcal{U}_{\text{out}}$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is PB-output-stabilizing; and $\mathcal{P}$ is $\mathcal{U}_{\text{sta}}$-stabilizing iff $\begin{bmatrix} \mathbb{A}_\circlearrowleft^{\mathrm{T}} \mid \mathbb{C}_\circlearrowleft^{\mathrm{T}} \quad \mathbb{K}_\circlearrowleft^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is stable and (PB) holds.

(Theorem 9.9.1 develops this further.) In particular, if $\mathcal{P}$ is $\mathcal{U}_{\text{exp}}$-stabilizing, then $\mathcal{P}$ is $\mathcal{U}_{\text{str}}$-stabilizing; the converse is not true, by Example 9.13.14. However, a feedback being $\mathcal{U}_{\text{str}}$-stabilizing does not mean that $\mathbb{B}_\circlearrowleft$, $\mathbb{D}_\circlearrowleft$ or $\mathbb{F}_\circlearrowleft$ is stable, hence it need not be strongly stabilizing in general; in Theorem 9.9.1 we use q.r.c.-stabilizability to provide strongly stabilizing optimal controls.

Proof: The first claim follows from Proposition 9.10.2(i)&(ii)&(c). The other claims follow from (d), (e1) and (f1) of Proposition 9.10.2 (use also Theorem 8.3.9(a1)&(a2) for the “only if” part). □

Condition (PB) is, unfortunately, not redundant for $\mathcal{U}_{\text{out}}$ (nor $\mathcal{U}_{\text{sta}}$) in general, by Example 9.13.2 (or Example 9.13.9). However, in several special cases (PB) can be relaxed also for $\mathcal{U}_{\text{out}}$ and $\mathcal{U}_{\text{sta}}$; see, e.g., Theorem 9.9.1(c1)–(c3) and Theorem 9.2.10.

Even in the general case, the conditions can be slightly weakened:

Lemma 9.8.6 (Simplifications) *The following are equivalent:*

(i) $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE;

(ii) $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{B}(U)$, $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is an admissible state feedback pair for Σ, the eIARE (9.111) has a solution for one fixed $T := t > 0$, $\mathbb{K}_\circlearrowleft x_0 := (I - \mathbb{F})^{-1}\mathbb{K} \in \mathcal{U}_*^*(x_0)$ ($x_0 \in H$) and the limits in (P) and (PB) are zero when we restrict t to $T\mathbf{N}$ (or to some of its unbounded subsets).

(Thus, (P) and (PB) may be replaced by the condition that $\langle \mathbb{A}_\circlearrowleft^{n_k T} x_0 + \mathbb{B}^{n_k T} u, \mathcal{P}\mathbb{A}_\circlearrowleft^{n_k T} x_0 \rangle \to 0$, as $k \to +\infty$, for some sequence $\{n_k\} \subset \mathbf{N}$ s.t. $\lim_{k\to+\infty} n_k = +\infty$).

Proof: This follows from “(ii)⇔(iii)” and (h) Proposition 9.10.2. □

The name IARE instead of “IRE” reflects the fact that we can and (most often) will treat the IARE as an algebraic equation of the integrated terms $\mathbb{A}^t$, $\mathbb{B}^t$, $\mathbb{C}^t$, $\mathbb{D}^t$ and of the operators J and $\mathcal{P}$ (see Remark 9.8.8 for how to eliminate S and $\mathbb{K}^t$). In fact, it equals the Discrete-time Algebraic Riccati Equation (DARE) (for $t = 1$; see Definition 14.1.1 and Definition 13.4.2 for the DARE and for the discretization operator Δ^{S}):

Proposition 9.8.7 (eIARE⇔eDARE) *Let $t = 1$ (or let $t > 0$ be arbitrary and use Remark 13.4.6). Then*

(a) If $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is an [admissible [stabilizing]] solution of the eIARE (for Σ and J), then $(\mathcal{P}, S, \Delta^{\mathrm{S}} [\ \mathbb{K} \mid \mathbb{F}\ ])$ is an [admissible [stabilizing]] solution of the (discrete) eIARE for $\Delta^{\mathrm{S}}\Sigma$ and J. All prefices and suffices apply.

(b) A triple $(\mathcal{P}, S, \Delta^{\mathrm{S}} [\ \mathbb{K} \mid \mathbb{F}\ ])$ is an admissible [stabilizing] solution of the (discrete) eIARE for $\Delta^{\mathrm{S}}\Sigma$ and J iff $\mathbb{X}^t := I - \mathbb{F}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U))$ and $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is an admissible [stabilizing] solution of the eDARE for $\Delta^{\mathrm{S}}\Sigma$ and J, where $S_{\mathrm{d}} := (\mathbb{X}^t)^* S \mathbb{X}^t$, $K_{\mathrm{d}} = (\mathbb{X}^t)^{-1}\mathbb{K}^t$. All prefices and suffices apply.

(c1) Let $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ be an admissible [stabilizing] solution of the eIARE. Then $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is an admissible [stabilizing] solution of the eDARE, by (a) and (b). All prefices and suffices apply (including "$\mathcal{U}_*^*$-").

(c2) Conversely, let $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ be a $\mathbb{C}$-P-stabilizing solution of the eDARE for $\Delta^{\mathrm{S}}\Sigma$ and J; let $[\ \mathbb{K}_{\mathrm{d}} \mid \mathbb{F}_{\mathrm{d}}\ ]$ be the corresponding state feedback pair, and set $[\ \mathbb{K} \mid \mathbb{F}\ ] := (\Delta^{\mathrm{S}})^{-1} [\ \mathbb{K}_{\mathrm{d}} \mid \mathbb{F}_{\mathrm{d}}\ ]$.

Then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right] \in \mathrm{WPLS}$ iff $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is a $\mathbb{C}$-P-stabilizing solution of the eIARE, where $S := (\mathbb{X}^{t*})^{-1} S_{\mathrm{d}} (\mathbb{X}^t)^{-1}$.

(c3) Let $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ be an admissible solution of the eIARE. Then $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing (resp. $\mathcal{U}_{\mathrm{exp}}$-stabilizing) for Σ iff $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is $\mathcal{U}_*^*$-stabilizing (resp. $\mathcal{U}_{\mathrm{exp}}$-stabilizing) for $\Delta^{\mathrm{S}}\Sigma$.

Also most other attributes are invariant (e.g., "J-critical over $\mathcal{U}_{\mathrm{out}}$ (or over $\mathcal{U}_{\mathrm{exp}}$)", "[strongly] internally stabilizing", "P-output-stabilizing", "P-SOS-q.r.c.-stabilizing", "stable", and "exponentially q.r.c.-stabilizing"); see Theorem 13.4.4(d2)&(e)&(f2) for further attributes and details.

(Here continuous-time and discrete-time $\mathcal{U}_*^*$'s corresponds to each other as in (13.73); in particular, $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$ are "invariant".)

Thus, for admissible $[\ \mathbb{K} \mid \mathbb{F}\ ]$, the triple $(\mathcal{P}, S, \Delta^{\mathrm{S}} [\ \mathbb{K} \mid \mathbb{F}\ ])$ is a $\mathcal{U}_*^*$-stabilizing solution of the discrete-time eIARE iff $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE.

We have stated (a)–(c1) in one direction only, and the converse (c2)–(c3) contains only a partial result (sufficient for most applications). The reader might wish to consult the discretization theory of Section 13.4 for details and for tools for further results.

Note from (b) that S_{d} is invertible (resp. one-to-one) iff S is invertible (resp. one-to-one); in particular, the equivalence of IARE and DARE is analogous to that of eIARE and eDARE.

Proof of Proposition 9.8.7: (a)&(b) These are obvious (see also Theorem 13.4.4(d)&(e1)); we can obviously include the prefix "$\mathcal{U}_*^*$-".

(c1) This follows from (a) and (b).

(c2) By inverse discretization we obtain from the discrete-time eIARE the continuous-time eIARE for time values in $t\mathbf{N}$. By definition, $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right] \in \mathrm{WPLS}$ is necessary for $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ to be admissible, hence we may assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right] \in$ WPLS.

Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is P-$\mathbb{C}$-stabilizing iff $(\mathcal{P}, S_{\mathrm{d}}, \left[\begin{array}{c|c} \mathbb{K}_{\mathrm{d}} & \mathbb{F}_{\mathrm{d}} \end{array}\right])$ is P-$\mathbb{C}$-stabilizing, by Proposition 9.10.2(a1)(i)&(ii) (which is valid in discrete time too, by Theorem 14.1.3).

(c3) See Theorem 13.4.4(d2)&(e)&(f2) for the properties. In particular, "J-critical over $\mathcal{U}_*^*$" is preserved, hence also "$\mathcal{U}_*^*$-stabilizing", by Theorem 9.8.5.

E.g., "[strongly] internally", "output-" etc. are preserved by (d2) (or (e1)) of Theorem 13.4.4(d2), and "q.r.c.-" etc. are preserved by (e1).

By (c2), "P-output-stabilizing" is preserved (and "P-" in connection to anything stronger than "$\mathbb{C}$-stabilizing").

We do not know whether "P-admissible" is preserved (from discrete to continuous time), but usually even the discrete form of (P) is enough (e.g., the discrete form of "internally P-stabilizing" is enough to guarantee uniqueness, by Theorem 14.1.4(b)).

(Although we can use (f1) of Theorem 13.4.4 for $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$, note that we have to use (f2) for $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$; in particular, we cannot write explicitly "$\mathcal{P}$ is $\mathcal{U}_{\mathrm{sta}}^{\Sigma}$-stabilizing" iff "$\mathcal{P}$ is $\mathcal{U}_{\mathrm{sta}}^{\Delta^{\mathrm{S}}\Sigma}$-stabilizing" due to the reasons explained in the proof. Fortunately, this does not hinder us from applying certain discrete-time results for $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ in the same way as for general $\mathcal{U}_*^*$'s.) □

One can speak of a solution $\mathcal{P}$ of the eIARE without mentioning S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, because the S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ can be eliminated:

Remark 9.8.8 *The maps $\mathbb{K}^t$, $\mathbb{X}^t$ and S can be eliminated from the eIARE as follows (this is trivial for $S \in \mathcal{GB}$):*

Set $S' := \mathbb{X}^{t*} S \mathbb{X}^t$, $\mathbb{K}^{t\prime} := \mathbb{M}^t \mathbb{K}^t$, so that the second and third equation of the eIARE determine $S' = (S')^* \in \mathcal{B}(\mathrm{L}^2([0,t);U))$ and $S'\mathbb{K}^{t\prime}$. Let $P \in \mathcal{B}(U)$ be the orthogonal projection onto $\mathrm{Ker}(S')^{\perp}$. Then $S'\mathbb{K}^{t\prime}x_0 = S'P\mathbb{K}^{t\prime}x_0$ determines $P\mathbb{K}^{t\prime}x_0$ uniquely a.e., for any $x_0 \in H$. Consequently,

$$\langle \mathbb{K}^t x_0', S\mathbb{K}^t x_0\rangle = \langle \mathbb{K}^{t\prime} x_0', S'\mathbb{K}^{t\prime} x_0\rangle = \langle P\mathbb{K}^{t\prime} x_0', S'P\mathbb{K}^{t\prime} x_0\rangle \tag{9.112}$$

is uniquely determined by S' and $S'\mathbb{K}^{t\prime}$, for any $x_0, x_0' \in H$.

To establish the equivalence between the eIARE and the eCARE, we first have to show that the latter is well defined. Since $B_{\mathrm{w}}^* \in \mathcal{B}(H_{C,K}^*, U)$ if(f) Σ_{ext} is WR, by Proposition 6.2.8(a1), the following shows that the term $B_{\mathrm{w}}^*\mathcal{P}$ in the eCARE is defined on H_B (cf. Remark 9.1.6):

Lemma 9.8.9 ($\mathcal{P} \in \mathcal{B}(H_B, H_{C,K}^*)$) *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ and $J = J^* \in \mathcal{B}(Y)$. Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ be a solution of the eIARE s.t. $\Sigma_{\mathrm{ext}} \in$ WPLS. Then $\mathcal{P} \in \mathcal{B}(H_B, H_{C,K}^*)$.*

Proof: Let $x_0 \in H$, $u_0 \in U$ be s.t. $Ax_0 + Bu_0 \in H$. Choose $\omega \in \mathbf{R}$ s.t. $\Sigma, \Sigma_{\circlearrowleft} \in \mathrm{WPLS}_\omega$, and choose $a > \omega$. Set $u := \pi_+ e^{-a\cdot} u_0 \in \mathrm{W}_\omega^{1,2}$.

By Theorem 6.2.13(b), $x' = Ax + Bu \in \mathcal{C}(\mathbf{R}_+; H)$ and $y, z \in \mathrm{W}_\omega^{1,2}$, where $z := -\mathbb{K}x_0 + \mathbb{X}u$. Therefore, $y^{\mathrm{d}}, z^{\mathrm{d}} \in \mathrm{W}^{1,2}$ in Lemma 9.11.1, so that also x^{d} is

$\mathcal{C}^1$ on $[0,t]$, by Lemma 9.11.1. By Theorem 6.2.13(b1), we have

$$H_{-1}^* \ni A^*x^{\rm d} + C^*y^{\rm d} + K^*z^{\rm d} = -x^{{\rm d}\prime}(t-\cdot) = -\mathcal{P}x' \in H. \tag{9.113}$$

But $-\mathcal{P}x' \in \mathcal{C}([0,t];H)$ implies at 0 that $A^*\mathcal{P}x_0 + C^*Jy(0) + K^*z(0) \in H$, hence $\mathcal{P}x_0 \in H_{C,K}^*$ (see Definition 6.1.17). Because $x_0 \in H_B$ was arbitrary, we have $\mathcal{P}H_B \subset H_{C,K}^*$; the boundedness follows from Lemma A.3.6. □

Equations eIARE and eCARE are equivalent if(f) $\mathbb{D}, \mathbb{X} \in \text{WR}$:

Proposition 9.8.10 (eIARE⇔eCARE) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \text{WPLS}(U,H,Y)$ *be WR, and let* $J = J^* \in \mathcal{B}(Y)$. *Then the following problems are equivalent:*

(i) The eIARE has a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$.

(ii) The eCARE has a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & F \end{array}\right])$.

Moreover, every solution of (i) is a solution of (ii) and vice versa (here $\left[\begin{array}{c|c} K & F \end{array}\right]$ *generate* $\left[\begin{array}{c|c} \mathbb{K} & I-\mathbb{F} \end{array}\right]$*).*

Thus, the WR solutions of the CARE are exactly the WR solutions of the IARE having $F = 0$.

Proof: 1° (i)⇒(ii): The eCARE holds by Lemma 9.11.2, and Proposition 9.11.4(b1)&(d) (where K and $I-X$ are the generators of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$); in particular, the weak limit converges.

2° (ii)⇒(i): Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be the pair generated by $\left[\begin{array}{c|c} K & I-X \end{array}\right]$. The eIARE holds by Lemmas 9.11.2, 9.11.7 and 9.11.6.

(a) This follows from 1° and 2°. □

The usual q.r.c.-SOS-P-stabilizability requirement (cf. Theorem 9.9.10) becomes simple for $\Sigma \in \text{SOS}$:

Proposition 9.8.11 (Stable CARE/IARE) *If* $\Sigma \in \text{SOS}$, $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$, *and* $\mathcal{P}$ *is an admissible solution of the [e]IARE, then the following are equivalent:*

(i) $\mathcal{P}$ *is q.r.c.-SOS-P-stabilizing;*

(ii) $\mathcal{P}$ *is r.c.-SOS-PB-stabilizing and* $\mathcal{U}_*^*$*-stabilizing;*

(iii) $\mathbb{K}$ *is stable,* $\mathbb{X} := I - \mathbb{F} \in \mathcal{G}\text{TIC}$, *and (P) holds.*

Moreover:

(a) In (i)–(iii), we may replace (P) by

(P') $\langle \mathbb{A}^t x_0, \mathcal{P}\mathbb{A}^t x_0 \rangle \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$.

(b) If $\mathcal{P}$ *satisfies (P'),* $S \gg 0$, $\mathbb{M} \in \text{TIC}$ *and* Σ *is stable, then (i)–(iii) hold.*

(c) If (i)–(iii) hold, then $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$.

(d1) If Σ *is exponentially stable, then we have three more equivalent conditions:*

(iv) $\mathcal{P}$ *is exponentially stabilizing;*

(v) $\mathcal{P}$ is exponentially stable and exponentially r.c.-stabilizing;
(vi) $\mathbb{M}$ is stable.

(d2) If Σ is exponentially stable and the IARE has a $\mathcal{U}_{\text{out}}$-stabilizing solution, then we have one more equivalent condition:

(viii) $\mathcal{P}$ is I/O-, input-, output- or internally stabilizing.

(d3) If Σ is strongly stable and the IARE has a q.r.c.-SOS-P-stabilizing solution, then each of (iv') and (v') is equivalent to (i)–(iii):

(iv') $\mathcal{P}$ is internally stabilizing (i.e., $\mathbb{A}_{\circlearrowleft}$ is stable);
(v') $\mathcal{P}$ is stable and strongly r.c.-stabilizing;

For the CARE or IARE (i.e., when $S \in \mathcal{GB}(U)$) it follows from (c) that $\mathbb{X}^* S \mathbb{X}$ is a spectral factorization of $\mathbb{D}^* J \mathbb{D}$. By Lemma 9.10.1(b5)&(b6), this holds whenever $\mathcal{P}$ is a P-admissible solution and $\mathbb{D}, \mathbb{X}, \mathbb{X}^{-1} \in \text{TIC}$. See also Corollary 9.9.11.

However, even for a strongly stable system (with $\mathcal{U}_*^* = \mathcal{U}_{\text{out}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{str}}$), a $\mathcal{U}_*^*$-stabilizing solution need not satisfy any of (i)–(iii), by Example 11.3.7 (since $\mathbb{X}$ and $\mathbb{M}$ may be unstable), and a J-critical control can exist even if there is no $\mathcal{U}_*^*$-stabilizing solution (and hence no J-critical state feedback pair over $\mathcal{U}_*^*$), by the same example. Moreover, even if (i)–(iii) hold, there may also be other (non PB-) r.c.-stabilizing solutions, even if Σ is weakly stable and minimal, by Example 9.13.9.

Proof of Proposition 9.8.11: (See Definition 9.8.4 for the eIARE.)

By Lemma 6.6.17(a), $\mathcal{P}$ is [q.]r.c.-SOS-stabilizing iff it is stable and I/O-P-stabilizing (as in (iii)). By Theorem 9.9.1(b)&(g), (i) implies (PB), hence also "$\mathcal{U}_*^*$-".

(a) This follows from Proposition 9.10.2(a2) (and Lemma 9.10.1(d2)).

(b) This follows from Proposition 10.7.1 and (iii).

(c) This follows from Lemma 9.10.1(f2).

(d1) Now $\mathcal{P}$ is necessarily exponentially stable, by Lemma 6.1.10. If $\mathbb{M}$ is stable, $\Sigma_{\circlearrowleft}$ is exponentially stable, by Corollary 6.6.9, hence $\mathbb{M}$ and $\mathbb{X}$ are then exponentially stable, so that $\mathcal{P}$ is exponentially r.c.-stabilizing. Thus, (vi)$\Rightarrow$(v). Obviously, (vi)$\Leftarrow$(i)$\Leftarrow$(v)$\Rightarrow$(iv)$\Rightarrow$(vi).

(d2) This follows from (c1), (c3)(iv') and Theorem 6.7.15(c1), since a $\mathcal{U}_{\text{out}}$-stabilizing solution is (exponentially stable and) exponentially (q.)r.c.-stabilizing, by Theorem 6.7.15(c1).

(d3) Obviously, (v')$\Rightarrow$(i)$\Rightarrow$(iv'). Let $\widetilde{\mathcal{P}}$ be a q.r.c.-SOS-P-stabilizing solution, hence stable and strongly stabilizing, by (ii) and Theorem 6.7.15(a2). Assume (iv'). Then $\mathcal{P} = \widetilde{\mathcal{P}}$, by Theorem 9.8.12(a), hence (v') holds. □

A strongly internally stabilizing (i.e., s.t. $\mathbb{A}_{\circlearrowleft}^t x_0 \to 0$ as $t \to +\infty$, for all $x_0 \in H$) solution of the eIARE, eCARE or CARE is unique:

Theorem 9.8.12 ($\mathcal{P}$ is unique) *We have the following uniqueness results for a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ (not for S and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$) of the eIARE:*

(a) If the eIARE has a strongly internally stabilizing solution, then that solution is unique among internally stabilizing solutions.

(b) The eIARE has at most one internally P-stabilizing solution.

(c) If the eIARE has an internally ω*-stabilizing solution for some* $\omega < 0$*, then any other solution is (internally) at most* $-\omega$*-stabilizing.*

(d) The eIARE has at most one P-q.r.c.-SOS-stabilizing solution.

(e) The eIARE has at most one $\mathcal{U}_*^*$*-stabilizing solution.*

In the case of an IARE, the corresponding S *and* $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ *unique modulo an invertible operator:*

(s1) Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *be an admissible solution of the eIARE. Then so are the triples*

$$(\mathcal{P}, E^{-*}SE^{-1}, \begin{bmatrix} E\mathbb{K} & | & E\mathbb{F}+I-E \end{bmatrix}). \quad (E \in \mathcal{GB}(U)); \tag{9.114}$$

The corresponding closed-loop systems are given by $\Sigma_{\circlearrowleft E} = \begin{bmatrix} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft E^{-1} \\ \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft E^{-1} \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft E^{-1} \end{bmatrix}$.

All admissible solutions of form $(\mathcal{P}, *, *)$ *are given by (9.114) iff* $\operatorname{Ker}(S) = \{0\}$.

(s2) Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *be an admissible solution of the eIARE. Then* $(\mathcal{P}, \widetilde{S}, \begin{bmatrix} \widetilde{\mathbb{K}} & | & \widetilde{\mathbb{F}} \end{bmatrix})$ *is an admissible solution of the eIARE iff* $\begin{bmatrix} \widetilde{\mathbb{K}} & | & \widetilde{\mathbb{F}} \end{bmatrix}$ *is admissible for* Σ *and there is* $E \in \mathcal{GB}(U)$ *s.t.* $\widetilde{S} = E^{-*}SE^{-1}$*,* $\widetilde{S}\widetilde{\mathbb{K}} = \widetilde{S}E\mathbb{K}$ *and* $\widetilde{S}\widetilde{\mathbb{X}} = \widetilde{S}E\mathbb{X}$.[2]

(s3) Let $(\mathcal{P}, S, K)$ *and* $(\mathcal{P}, \widetilde{S}, \widetilde{K})$ *be solutions of the CARE. Then* $\widetilde{S} = S$ *and* $SK = S\widetilde{K}$*; in particular,* K *is unique if* $\operatorname{Ker}(S) = \{0\}$.

The results (a)–(s2) apply to solutions of eCARE and CARE too, by Proposition 9.8.10.

(s4) Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *be a* $\mathcal{U}_*^*$*-stabilizing solution of the eIARE and* $\operatorname{Ker}(S) = \{0\}$*. Then all* $\mathcal{U}_*^*$*-stabilizing solutions of the eIARE are given by (9.114).*

(In particular, $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}} \in \mathcal{C}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$ *is independent of the solution.)*

Note that when we call the solution unique, we mean that $\mathcal{P}$ is unique; see (s1)–(s4) for the uniqueness of S and $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ (corresponding to a fixed $\mathcal{P}$).

By Theorem 9.9.1(b)&(g), we have "(d)⊂(e)" in the sense that if $\mathcal{P}$ is as in (d), then it is as in (e). Obviously, we also have "(c)⊂(a)⊂(b)". However, in 1° of Example 9.13.2, the solution $(0,1,0)$ is as in (d)–(e) (it is the J-critical cost over $\mathcal{U}_{\text{out}}$, by Theorem 9.9.1(e2)), whereas the $\mathcal{U}_{\text{exp}}$-P-stabilizing solution $(2,1,-2)$ is as in (a)–(c) (the J-critical cost over $\mathcal{U}_{\text{exp}}$; note that "P-" is here redundant) — thus, we can have two "unique" solutions.

Therefore, the condition of (e) is the one to be watched (and that of (d) is sufficient) for $\mathcal{U}_{\text{out}}$, whereas for $\mathcal{U}_{\text{exp}}$ we can use either (e) (for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$)

[2] This formula is dedicated to Sari.

or exponential stabilization. By Lemma 8.3.3, the $\mathcal{U}_{\text{out}}$-stabilizing and $\mathcal{U}_{\text{exp}}$-stabilizing solutions (if either exists) coincide when Σ is estimatable, which is often the case in classical problems.

An intuitive explanation for the uniqueness is the following: a J-critical control need not be unique (see also Theorem 9.9.1(f2) and Example 9.13.6 on uniqueness of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ (or K)), but the J-critical cost is always unique, by Lemma 8.3.8, hence the J-critical cost operator $\mathcal{P} := \mathbb{C}_{\text{crit}}^* J \mathbb{C}_{\text{crit}}$ is unique.

In the positive case with suitable assumptions, there is at most one solution of the eIARE (and of the CARE), see Theorem 10.1.4(c1)&(c2) and Section 10.7.

Proof of Theorem 9.8.12: (a)–(e) This follows from Theorem 14.1.4. and Proposition 9.8.7(c1). (Alternatively, we could write the same proofs for $\mathcal{P}$, S, $\mathbb{K}^t$ and $\mathbb{F}^t$ in continuous time, even though the "no-feedthrough state feedback pair" need not be "well-posed".) Solutions of the eCARE and the CARE are solutions of the eIARE, hence the uniqueness result applies them too.

(s2) 1° *"If"*: This follows by a direct computation.

2° *"Only if"*: Let $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ be an admissible solution of the eIARE. From (9.160) it follows that $\mathbb{X}^{t\,*} S \mathbb{X}^t = \widetilde{\mathbb{X}}_+ \widetilde{S} \widetilde{\widetilde{\mathbb{X}}}_+$ for $t > 0$. By Lemma 2.3.5, we have $\widetilde{S} = E^{-*} S E^{-1}$ and $\widetilde{S}\widetilde{\widetilde{\mathbb{X}}} = \widetilde{S} E \mathbb{X}$ for some $E \in \mathcal{GB}(U)$. From the eIARE it then follows that $\widetilde{S}\widetilde{\widetilde{\mathbb{K}}} = \widetilde{S} E \mathbb{K}$.

(Note that if we split U as $U = U_1 \times U_2$, where $U_1 := \operatorname{Ker}(\widetilde{S})$, $U_2 := U_1^{\perp}$, and P_k is the orthogonal projection of U onto U_k ($k = 1, 2$), then $\mathbb{K} = \begin{bmatrix} \widetilde{\mathbb{K}}_1 \\ \widetilde{\mathbb{K}}_2 \end{bmatrix}$, $\mathbb{X} = \begin{bmatrix} \widetilde{\mathbb{X}}_1 \\ \widetilde{\mathbb{X}}_2 \end{bmatrix}$, where $\widetilde{\mathbb{K}}_2 = P_2 E \mathbb{K}$, $\widetilde{\mathbb{X}}_2 = P_2 E \mathbb{X}$, but $\widetilde{\mathbb{K}}_1$ and $\widetilde{\mathbb{X}}_1$ are arbitrary as long as $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is admissible for Σ. Equivalently, $K_2 = P_2 E K$ and $\widehat{\widetilde{\mathbb{X}}}_2 = P_2 E \widehat{\mathbb{X}}$, but K_1 is arbitrary as long as $\left[\begin{array}{c|c} A & B \\ \hline -K & * \end{array}\right]$ generate WPLSs, some of which have an I/O map in $\mathcal{G}\mathrm{TIC}_\infty$.)

(s1) Obviously, (9.114) defines a solution for all $E \in \mathcal{GB}$. If $\operatorname{Ker}(S) = \{0\}$, then there are no others, because then the E in (s2) determines $\widetilde{S}$ and $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ uniquely.

For the converse, let $\operatorname{Ker}(S) \neq \{0\}$. Choose $T \in \mathcal{B}(\operatorname{Ker}(S)) \setminus \{0\}$ so small that $\widetilde{\mathbb{X}} := \mathbb{X} + T \in \mathcal{GB}(U)$. Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & I - \widetilde{\mathbb{X}} \end{array}\right])$ is an admissible solution of the eIARE, by (s2) (with $E = I$).

(s3) This is obvious from the CARE.

(s4) This follows from (e) and (s1). □

We now note the continuous-time counterpart of Corollary 15.5.3:

Theorem 9.8.13 (Greatest solution $\mathcal{P}_+$ of the CARE/IARE) *If the CARE (resp. IARE) has a strongly ($\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$-)stabilizing solution s.t. $S \gg 0$, then this solution is the greatest admissible solution of the eCARE (resp. eIARE) having $S \geq 0$.* □

(This follows from Corollary 15.5.3 and Propositions 9.8.10 and 9.8.7. Note that it suffices that $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$ is strongly stable; $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ need not be.)

Analogously, one can deduce from Theorem 15.5.2 that if Σ is strongly $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$-stabilizable and the IARE has a solution s.t. $S \gg 0$, then there is an upper bound $\mathcal{P}_+$ for all solutions $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ of the eIARE having $S \geq 0$. However, we do not know whether $\mathcal{P}_+$ itself is a solution of the IARE, i.e., whether the corresponding dediscretized $[\ \mathbb{K}_+ \mid \mathbb{F}_+\ ]$ is well-posed in continuous time (though $(\mathcal{P}_+, S_+, [\ \mathbb{K}_+ \mid \mathbb{F}_+\ ])$ solves the discretized IARE).

Recall that "the greatest admissible" means that if $\mathcal{P}'$ is an admissible solution of the eIARE or eCARE (or of the corresponding (extended) Riccati inequality "eIARI", see Theorem 15.5.2) s.t. $S' \geq 0$, then $\mathcal{P}' \leq \mathcal{P}$. Recall that we require any solution of any ARE to be self-adjoint.

The CARE of Example 9.13.9 has several stabilizing solutions s.t. $S \gg 0$, but it does not have a maximal (hence not a greatest) solution; therefore, the system cannot be strongly stabilizable (by Theorem 15.5.2 and discretization). On the other hand, Example 9.13.12(b) shows that "strongly" cannot be replaced by "weakly" in the theorem.

In connection with H^∞ control problems, it is common to speak of lossless factorizations instead of Riccati equations. This is due to the fact that if $\mathcal{P} \geq 0$ is a $\mathcal{U}_*^*$-stabilizing solution with $\mathbb{D}_\circlearrowleft \in \text{TIC}$, then $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless:

Lemma 9.8.14 ($\mathcal{P} \geq 0 \Longrightarrow \mathbb{D}_\circlearrowleft$ is (J,S)-lossless) *Let $\mathcal{P} \geq 0$ be an admissible solution of the IARE s.t. $\mathbb{C}_\circlearrowleft$ and $\mathbb{D}_\circlearrowleft$ are stable and (P) holds. Then $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless.*

Thus, we obtain a (J,S)-lossless right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ when $\mathcal{P} \geq 0$ is P-SOS-stabilizing.

Proof: Indeed, if $\mathcal{P} \geq 0$ is admissible and $\mathbb{N} := \mathbb{D}_\circlearrowleft$ is stable, then $S - \mathbb{N}^{t\,*} J \mathbb{N}^t = \mathbb{B}_\circlearrowleft^{t\,*} \mathcal{P} \mathbb{B}_\circlearrowleft^t \geq 0$ for all $t > 0$, hence $\mathbb{N}^* \pi_- \mathbb{N} \leq \pi_- S$, by Lemma 2.2.4(b1). When also $\mathbb{C}_\circlearrowleft$ is stable and (P) holds, we have $\mathbb{N}^* J \mathbb{N} = S$, by Lemma 9.10.1(f2), hence then $\mathbb{N}$ is (J,S)-lossless.

(In fact, $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless iff $\mathcal{P} \geq 0$ on the reachable subspace $H_\mathbb{B}$ of Σ.)

□

Since we have let B be highly unbounded, we meet several phenomena that are not present in classical results. The generality of regular WPLSs allows a wide range of discontinuities, in particular, all discrete systems can be written in the form of a WPLS. Thus is feels somewhat natural that we must add the "$B^*\mathcal{P}B$-term" to the formula for S as in the (classical) discrete case (see, e.g., Section 14.1 or equation (B.2.27) of [GL]). Of course, with certain additional regularity assumptions one can guarantee that $S = D^*JD$ (see Remark 9.9.14(b)).

We give below an example, where $S \neq D^*JD$; see [S96], [WZ], the notes below and Section 9.13 for more examples and a further discussion on this phenomenon.

Example 9.8.15 **($S \neq D^*JD$)** Let $U = \mathbf{C} = Y$, $\mathbb{D} = \tau(-1) \in \text{MTIC}_\text{d} \subset \text{ULR}$, $J = I$, as in Examples 6.2.14, 6.3.7 and 8.3.12. Then $\mathbb{D}^* J \mathbb{D} = I = \mathbb{X}^* S \mathbb{X}$ with $S = I = \mathbb{X} \in \mathcal{G}\text{TIC}(U)$; $D = 0$, $X = I$, $D^*JD \neq X^*SX$. In particular, $\mathbb{D}$ is J-coercive over $\mathcal{U}_\text{out}$.

Let Σ be any strongly stable realization of $\mathbb{D}$. Then the CARE has a unique stable, stabilizing solution $(\mathcal{P}, S, K)$ (which is $\mathcal{U}_\text{out}$-stabilizing), by Corollary

9.1.9, and $\mathbb{X}$ is the operator corresponding to this solution, i.e., $\widehat{\mathbb{X}}(s) = I - K_{\mathrm{w}}(s-A)^{-1}B$. The corresponding control $\mathbb{K}_{\circlearrowleft} x_0$ is the unique J-critical control over $\mathcal{U}_{\mathrm{out}}$ for each $x_0 \in H$.

Since $\mathbb{X}$ and $\mathbb{D}$ are SR, we have $K_{\mathrm{w}} = K_{\mathrm{L,w}} = K_{\mathrm{s}} = K_{\mathrm{L,s}}$ on H_B and $B_{\mathrm{w}}^* = B_{\mathrm{L,w}}^* = B_{\mathrm{s}}^* = B_{\mathrm{L,s}}^*$ on $H_{C,K}^*$, by Proposition 6.2.8. By Proposition 9.11.4(a), we have (take $x_0 := (s-A)^{-1}Bu_0$ so that $Ax_0 + Bu_0 \in H$))

$$B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0 = (X^*SX - D^*JD - X^*SK_{\mathrm{w}}(s-A)^{-1}B)u_0 \tag{9.115}$$

$$= (I - 0 - X^*S(\widehat{\mathbb{X}} - X))u_0 = u_0 \to u_0 \tag{9.116}$$

for all $u_0 \in U$, as s goes to $+\infty$ (trivially). By the CARE, we have $K = -B_{\mathrm{w}}^*\mathcal{P}$. Thus, again by Proposition 9.11.4(a), we have

$$-(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}x_0 = K_{\mathrm{w}}x_0 = -B_{\mathrm{w}}^*\mathcal{P}x_0 + u_0 \quad \text{for } x_0, u \text{ s.t. } Ax_0 + Bu_0 \in H. \tag{9.117}$$

Let us write out the Riccati equation for the strongly stable realization

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_{[0,1]}\tau(-1) \\ \hline \pi_+ & \tau(-1) \end{array}\right] \in \mathrm{WPLS}_0(U, H, Y) \tag{9.118}$$

of Example 6.2.14; here $H := \mathrm{L}^2(\mathbf{R}_+; Y)$, $U = \mathbf{C} = Y$.

By Example 8.3.12, we have $\mathcal{P} = \pi_{[0,1)} \in \mathcal{B}(H)$ and $-\pi_+\tau^1 = \mathbb{K}_{\circlearrowleft} = \mathbb{K}$ (although $\mathbb{A}_{\circlearrowleft} \neq \mathbb{A}$), hence

$$Kx_0(t+\cdot) = K\mathbb{A}^t x_0 = (\mathbb{K}x_0)(t) = -x_0(t+1) = -\delta_1^* x_0(t+\cdot) \quad (x_0 \in H_1,\ t \geq 0); \tag{9.119}$$

consequently, $K = -\delta_1^*$. Using the results of Example 6.2.14, we get that $H_1 = \mathrm{W}^{1,2}((0,\infty))$ and

$$(s-A^*)^{-1}C^* = \mathrm{e}^{-s\cdot} \in \mathcal{B}(Y, H_C^*), \quad (s-A^*)^{-1}K^* = \mathrm{e}^{-s(\cdot-1)}\pi_{[1,\infty)} \in \mathcal{B}(Y, H_K^*), \tag{9.120}$$

$$\text{hence } H_C^* = \mathrm{W}^{1,2}(\mathbf{R}) \text{ and} \tag{9.121}$$

$$H_{C,K}^* = \mathrm{W}_0^{1,2} + \mathbf{C}\mathrm{e}^{-\cdot} + \mathbf{C}\mathrm{e}^{-(\cdot-1)}\pi_{[1,\infty)} = \{x_0 \in H \,\big|\, x_0' \in H + \mathbf{C}\delta_1\} \tag{9.122}$$

$$= \mathrm{W}^{1,2}((0,1)) + \mathrm{W}^{1,2}([1,\infty)) = H_B. \tag{9.123}$$

Thus, $\mathcal{P}H_B = \mathrm{W}^{1,2}((0,1)) \subset H_{C,K}^*$ as expected, and (recall that $B_{\mathrm{w}}^* = \delta_{1-}^*$)

$$Su_0 = B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0 = \delta_{1-}^*\pi_{[0,1)}\mathrm{e}^{-s(1-\cdot)}u_0 = u_0 \quad (s \in \mathbf{C}^+), \tag{9.124}$$

by (6.61, as proved above for any strongly stable realization of $\mathbb{D}$. By Proposition 6.2.8(c3)&(c1)&(c4)&(d1), we have

$$K_{\mathrm{w}}x_0 = K_{\mathrm{L,s}}x_0 = \lim_{s\to+\infty} \frac{1}{t}\int_0^t -x_0(1+r)\,dr = -x_0(1+) =: -\delta_{1+}^* x_0 \quad (x_0 \in H_B), \tag{9.125}$$

$$\text{hence} \quad K_{\mathrm{w}}(s-A)^{-1}B = 0 \quad \text{for all} \quad s \in \mathbf{C}^+, \tag{9.126}$$

as expected. This agrees with the CARE, since $B_{\mathrm{w}}^*\mathcal{P} = \delta_{1-}^*$ on H_B, so that

$-B_{\rm w}^*\mathcal{P} = -\delta_1^* = K$ on H_1. Thus we obtain from (9.125) that

$$(B_{\rm w}^*\mathcal{P})_{\rm w} = \delta_{1+}^* = -K_{\rm w} \text{ and } B_{\rm w}^*\mathcal{P} = \delta_{1-}^* \quad \text{on } H_B \tag{9.127}$$

(and $-K = B_{\rm w}^*\mathcal{P} = \delta_1^*$ on H_B). By combining (9.124), (9.127) and (9.126) we obtain for $z_0 \in H_1$, $u_0 \in U$, $x_0 = z_0 + (s-A)^{-1}Bu_0$ (i.e., for arbitrary $x_0 \in H_B$) that $K_{\rm w}x_0 = K_{\rm w}z_0$ and

$$B_{\rm w}^*\mathcal{P}x_0 = \delta_{1-}^* z_0 + u_0 = -K_{\rm w}x_0 + u_0 = (B_{\rm w}^*\mathcal{P})x_0 + u_0, \tag{9.128}$$

again as shown above.

Above, we have derived $\mathcal{P}$, S and K from the solution of the J-critical control problem (minimization problem) over $\mathcal{U}_{\rm out}$. By Corollary 9.1.9, $(\mathcal{P}, S, K)$ is the unique stable, stabilizing solution of the CARE. To verify this, we note that the second and third equations of the CARE hold by (9.124) and (9.127), respectively, and the first one given by

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} + \mathcal{P}\frac{\mathrm{d}}{\mathrm{d}t} + \delta_0\delta_0^* = (\delta_{1-}^*\mathcal{P})^*\delta_{1-}^*\mathcal{P}, \quad \text{equivalently,} \tag{9.129}$$

$$\int_{\mathbf{R}_+} x_0'\mathcal{P}\overline{x_1} + \int_{\mathbf{R}_+} x_0\mathcal{P}\overline{x_1'} = \delta_{1-}^*\mathcal{P}x_0\overline{\delta_{1-}^*\mathcal{P}x_1} - x_0(0)\overline{x_1(0)} \qquad (x_0, x_1 \in H_1). \tag{9.130}$$

With our $\mathcal{P} = \pi_{[0,1)}$, this becomes $\int_0^1 (x_0'\overline{x_1} + x_0\overline{x_1'}) = \big|_0^1 x_0\overline{x_1}$, which confirms that $(\mathcal{P}, S, K)$ indeed solves the CARE.

Obviously, the integral (6.67) does not converges for every $x_0 \in \mathrm{L}^2([0,1);Y) = \mathcal{P}[H]$, hence $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_{\rm w}^*)$ (although $\mathcal{P}[H_B] \subset H_{C,K}^* \subset \mathrm{Dom}(B_{\rm w}^*)$, as shown above), so that Hypothesis 9.2.1 is not satisfied. $\triangleleft$

Notes for Section 9.8

The CARE (9.3) is only a slightly extended version of the CARE presented independently by M. Weiss and G. Weiss [WW] and O. Staffans [S97b] (which contained the first and third equations in the setting of Proposition 8.3.10; the formula for S was published in [S98b]).

Our contributions to the theory contain the converse direction — the fact that a stabilizing solution of the Riccati equation leads to the optimal state feedback pair — and the generalization of these results to general cost functions (instead of J-coercive ones), for general regular WPLSs (instead of stable or jointly stabilizable and detectable ones), to general $\mathcal{U}_*^*$'s (instead of $\mathcal{U}_{\rm out}$), to nonunique optimal control (and the eCARE), and to WR state feedback pairs (instead of SR operators); in fact, the IARE theory also allows for arbitrary (irregular) WPLSs. These will be applied to further control problems in Chapters 10–12.

The equations that constitute the IARE have appeared among the equations in Section 5 of [S98b] and in Sections 7–11 of [WW]; at least some of them can be found in the older literature (e.g., a variant of (9.155) for a standard LQR cost function is contained in Section 5 of [Sal87] for WPLSs and in Lemma 4.3 of [CP78] and Corollary 4.1 of [Gibson] for systems with bounded B and C).

We have not seen such equations treated in the literature as sufficient (and necessary) conditions for optimal control, nor as a discrete-time Riccati equation

(DARE).

Lemma 9.8.9 and Proposition 9.8.10 are based on the methods used in Sections 5–7 of [S98b] (partially also in [WW]). We published an early version of the results of this chapter in [Mik97b] (the stable case); it also contained some of Proposition 9.8.11.

The proof of (a)–(c) Theorem 9.8.12 (in the proof of Theorem 14.1.4) is a generalization of the classical proof for the uniqueness of the exponentially stabilizing solution of DARE (see, e.g., Proposition 13.5.1 of [LR]). See the notes to Section 15.5 for Theorem 9.8.13.

Our proof of Lemma 9.8.14 follows that of Theorem 6.5 of [S98c]. Part of Example 9.8.15 is contained in [WZ] and [S95].

Much attention has been paid to systems with bounded input and output operators (B and C) and to Pritchard–Salamon systems, both of which have the signature operator S equal to $S = D^*JD$ (which is often taken to be the identity in the positive case and to $\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ in the indefinite case). Indeed, whenever, Hypothesis 9.2.2 holds (e.g., $B \in \mathcal{B}(U,H)$) we have $S = D^*JD$, by Section 9.2; other sufficient conditions are given in Remark 9.9.14(b).

However, in general *our signature operator "S" takes the role of D^*JD exactly as in discrete-time*. Indeed, S is the signature operator of the control problem corresponding to the CARE, by Theorem 9.9.1(h) and (9.139), whereas D^*JD need not contain any information on the signature properties of the problem: in Example 9.13.7 $\mathbb{D}$ and $\mathbb{X}$ are ULR (even MTIC) but the operator D^*JD may have any signature (as long as its norm is less that 4) and still the unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the CARE is maximizing over $\mathcal{U}_{\text{out}}$ (because $S \ll 0$).

In a strictly definite problem, as in Example 9.13.7, the operator S tells us all signature properties (definiteness) of the problem. However, S is not unique, but E^*SE is also a signature operator for Σ and J for any $E \in \mathcal{GB}(U)$, by Theorem 9.9.1(f2). Therefore, S only contains the information on the nature of the problem, not on corresponding directions (for which we need also the corresponding $\mathbb{X} := I - \mathbb{F}$ or K).

Nevertheless, when we consider solutions of the CARE, S becomes unique (since it fixes $X = I$), and then S contains all information on the signature properties of the problem (also on an infinitesimal moment). Thus, then the situation is analogous to the discrete time (see Chapter 14 or some classical textbook), where the signature operator $S := D^*JD + B^*\mathcal{P}B$ takes the role of D^*JD even for finite-dimensional systems. These facts are illustrated in Proposition 11.2.19, where in (b2) we can only report the dimensions of the positive and negative eigenspaces of S (of the IARE), whereas in (d1)&(d2) we can also tell the directions.

A notable special case of the signature properties of S is that $\text{Ker}(S) = \{0\}$ is necessary and sufficient for the J-critical control to be unique (for general WPLSs), whereas $\text{Ker}(D^*JD) = \{0\}$ is sufficient (for WR ones) but not necessary, as noted below Corollary 9.7.4.

Finally, in general (for irregular systems) we do not even have the operator D, whereas any WPLS having a J-critical state feedback pair has a signature operator S, by Theorem 9.9.1. For stable J-coercive systems this equals the signature operator of the spectral factorization, by Corollary 9.9.11.

If we do not require the existence of (well-posed) J-critical state feedback, i.e., if we use the setting of Section 9.7, then we could still define another signature operator, namely $\mathbb{S}^t := \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ (for some $t > 0$). The map $\mathbb{S}^t$ is the signature operator of the eDARE obtained by discretizing the eIARE as in Proposition 9.8.7, hence it tells us about the signature properties of the problem even if the eIARE would have no solutions; see also Proposition 9.9.12 for the properties of $\mathbb{S}^t$. If there is a J-critical control in state feedback form, then $\mathbb{S}^t = \mathbb{X}^{t*}S\mathbb{X}^t$ and hence then $\mathbb{S}^t$ tells us practically the same information on the problem that S does, by Lemma 2.3.5.

It remains an important open problem to find a decent formula for S in terms of the generating operators when $\mathbb{D}$ and $\mathbb{X}$ are not known to be regular (the one in the (e)IARE is rather complicated and the one in the (e)CARE is not applicable in the irregular case).

Another open problem is the exact connection between S and the signature properties of the problem in the general case. Usually (e.g., when $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, or when $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\mathcal{P}$ is q.r.c.-SOS-stabilizing), the set $\mathcal{U}_*^*(x_0)$ of admissible inputs corresponds exactly to the closed-loop inputs $u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;U)$, by Theorem 9.9.1(k), so that the dimensions of the positive and negative eigenspaces and kernel of S exactly describe the definiteness of the problem, through equation (9.139).

Even for general $\mathcal{U}_*^*$, the equation holds for (compactly supported) $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$, by Theorem 9.9.1(i3). Thus, e.g., S is [strictly] nonnegative if the problem has a [strict] minimum, but it remains an open problem whether the converse holds for general $\mathcal{U}_*^*$; naturally, a similar situation is met also in the indefinite case (Chapter 11). Fortunately, for most of the time, we only have to treat the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ or the quasi-coprime setting.

In applications to certain kinds of systems, one may wish to write the CARE on some larger space than $\mathrm{Dom}(A)$ and/or avoid the Weiss extensions ("B_{w}^*"). One example of this is given in Theorem 9.9.6, for systems with bounded input operator (B), and another in Section 9.5, for parabolic systems. In both examples, the proofs were based in establishing the equivalence of this "smoother CARE" to the original one. In cases where this cannot be done, one may, alternatively, rewrite our proofs of the equivalence CARE$\Leftrightarrow$IARE in Section 9.11 for this setting; most other results of this monograph are based on IAREs and are hence directly applicable also for such "modified CAREs" after this equivalence has been verified.

9.9 J-Critical control $\leftrightarrow$ Riccati Equation

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

— Douglas Adams, "The Hitchhiker's Guide to the Galaxy"

This section provides most of the Riccati equation theory needed for solving the different control problems in Chapters 10–12; in some special cases the results of Sections 9.1–9.2 will suffice, and in some cases we need additional results from the other sections.

We establish the equivalence of the existence of a J-critical state feedback pair and the existence of a $\mathcal{U}_*^*$-stabilizing solution to the Riccati equation (Theorem 9.9.1), as mentioned in preceding sections. We also develop some further results and simplifications under different stabilizability or regularity assumptions.

Most of the latter part of this section (Remark 9.9.9–Corollary 9.9.11) corresponds to quasi-coprime stabilization, which allows us to considerable simplifications when finding optimal [SOS/strongly] stabilizing state feeback; the case with exponentially stabilizing state feedback ($\mathcal{U}_{\text{exp}}$) is originally simpler and described in Corollary 9.9.3–Proposition 9.9.5.

In Proposition 9.9.12 we treat the “signature” operator in the general case of possibly ill-posed optimal “state feedback”. In Remark 9.9.14 we summarize several cases in which a unique J-critical control corresponds to a (well-posed) regular state feedback operator.

Recall from Section 8.3 that a J-critical control is one that makes the (Fréchet) derivative of the cost function vanish, any optimal control is usually J-critical and usually also the converse holds.

We start by the equivalence. We first give equivalent conditions under different stabilizability assumptions ((a1)–(d)), and then we note that that the solutions correspond to each other as in classical results ((e1)–(e2)). Parts (f1)–(i) list some facts that will be needed later.

Theorem 9.9.1 (J-Critical control$\Leftrightarrow$eIARE) *The following statements hold:*

(a1) **(J-critical)** *There is a J-critical state feedback pair over $\mathcal{U}_*^*$ for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.*

(a2) **(Min)** *There is a minimizing state feedback pair $\begin{bmatrix} \mathbb{K}_{\text{min}} \mid \mathbb{F}_{\text{min}} \end{bmatrix}$ over $\mathcal{U}_*^*$ for Σ iff $\mathcal{J}(0,\cdot) \geq 0$ and the eIARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$.*

Assume that $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is as above. Then $S \geq 0$, and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is minimizing (the control $\mathbb{K}_\circlearrowleft := (I - \mathbb{F})^{-1}\mathbb{K}$ is strictly minimizing iff $S > 0$).

If $\mathcal{P} \geq 0$ (e.g., $J \geq 0$) and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ then $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE.

If $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, $J \geq 0$ and $S \gg 0$, then $\mathcal{P}$ is the greatest nonnegative admissible solution of the eIARE.

(b) There is a J-critical q.r.c.-SOS-stabilizing state feedback pair over $\mathcal{U}_{\text{out}}$ *for* Σ *iff the eIARE has a P-q.r.c.-SOS-stabilizing solution.*

(c1) Let $\mathbb{B}$ *be stable. Then there is a J-critical strongly stabilizing state feedback pair for* Σ *over* $\mathcal{U}_{\text{out}}$ *[and* $\mathcal{U}_{\text{sta}}$ *and* $\mathcal{U}_{\text{str}}$*] iff the eIARE has a strongly stabilizing solution.*

(c2) Let Σ *be strongly stable. Then there is a J-critical state feedback pair for* Σ *over* $\mathcal{U}_{\text{out}}$ *[and* $\mathcal{U}_{\text{sta}}$ *and* $\mathcal{U}_{\text{str}}$*] iff the eIARE has an output-stabilizing solution.*

(c3) Let Σ *be strongly q.r.c.-stabilizable. Then there is a J-critical state feedback pair for* Σ *over* $\mathcal{U}_{\text{out}}$ *[and* $\mathcal{U}_{\text{sta}}$ *and* $\mathcal{U}_{\text{str}}$*] iff the eIARE has a* $\mathcal{U}_{\text{str}}$*-stabilizing solution.*

(d) Let Σ *be estimatable. Then* $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$*. Moreover, there is a J-critical state feedback pair over* $\mathcal{U}_{\text{out}}$ *for* Σ *iff the eIARE has an output-stabilizing solution.*

Such a solution is exponentially P-q.r.c.-stabilizing and it is the unique internally stabilizing solution.

(e1) If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *is a solution of the eIARE of the form required in some of (a1)–(d), then the corresponding state feedback pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is of the required form (i.e., J-critical or minimizing in the required sense).*

(e2) Conversely, if $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is of the required form in some of (a1)–(d), then so is the solution* $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *of eIARE, where* $\mathcal{P} := \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$, $\mathbb{X} = I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{C}_\circlearrowleft := \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K}$, $\mathbb{B}_\circlearrowleft := \mathbb{B}\mathbb{M}$, $\pi_{[0,t)} S = \mathbb{N}^{t\,*} J \mathbb{N}^t + \mathbb{B}_\circlearrowleft^{t\,*} \mathcal{P} \mathbb{B}_\circlearrowleft^t$*, and* $\mathbb{N} := \mathbb{D}\mathbb{M}$.

Assume that (at least) one of (a1)–(d) is satisfied by $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *(by* $\Sigma_\circlearrowleft$ *we denote the corresponding closed-loop system). Then the following statements hold:*

(f1) **(Uniqueness)** $\mathcal{P}$ *is unique,* $\mathcal{U}_*^*$*-stabilizing and equal to the J-critical cost operator (naturally,* $\mathcal{P}$ *may depend on the choice of* $\mathcal{U}_*^*$*).*

(f2) The J-critical control is unique (for each $x_0 \in H$*) iff S is one-to-one. If S is one-to-one, then all J-critical feedback pairs* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *are given by (9.114); the converse is not true:*

Assume that S is not one-to-one. Then the pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *solving the eIARE with* $\mathcal{P}$ *and S is not unique modulo (9.114) (but it is unique modulo Theorem 9.8.12(s2)). However, it may still be that only one* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *(modulo (9.114)) is* $\mathcal{U}_*^*$*-stabilizing (see Example 9.13.6 for details).*

(g1) $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *is* $\mathcal{U}_*^*$*-stabilizing (cf. (a1)) and satisfies (9.153)–(9.163), (P1)–(P4), (P) and (PB) (see Lemma 9.10.1(d1) and Definition 9.8.1).*

(g2) $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$*. If* $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}(I - \mathbb{F})^{-1}$ *is stable, then* $S = \mathbb{N}^* J \mathbb{N}$ *and* $\pi_+ \mathbb{N}^* J \mathbb{C} = 0$.

(h) Equation (9.139) holds for all $x_0 \in H$ *and all* $u_\circlearrowleft \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U)$ *(all* $u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+; U)$ *if* $\mathbb{D}_\circlearrowleft$ *is stable).*

(i1) If $\mathbb{X}$ is WR and $\exists X_{\text{left}}^{-1}$, then the J-critical control $u_{\text{crit}}(x_0) := (\mathbb{K}_{\circlearrowleft} x_0)$ is given by

$$u_{\text{crit}}(x_0)(t) := (\mathbb{K}_{\circlearrowleft} x_0)(t) = X_{\text{left}}^{-1} K_{\text{w}} x_{\text{crit}}(t) \text{ a.e.} \tag{9.131}$$

(i2) We have $\mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(H_B, H_{C,K}^*)$, and $\mathcal{P}\mathbb{A}_{\circlearrowleft}(t)x_0 \to 0$ as $t \to +\infty$, for all $x_0 \in H$.

(i3) $\mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M}\mathrm{L}_{\mathrm{c}}^2(\mathbf{R}_+;U) \subset \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$ (see also (k)).

(i4) $\langle \mathbb{D}u, J\mathbb{D}\mathbb{M}\eta\rangle = \langle \mathbb{M}^{-1}u, S\eta\rangle$ for all $u \in \mathcal{U}_*^*(0)$ and $\eta \in \mathrm{L}_{\mathrm{c}}^2(\mathbf{R};U)$.

(j) Theorem 8.3.9 applies for $\Sigma_{\text{crit}} := \left[\begin{array}{c} \mathbb{A}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} \end{array}\right]$.

(k) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, or that $\mathcal{P}$ is q.r.c.-stabilizing (resp. strongly-q.r.c.-stabilizing, SOS-q.r.c.-stabilizing) and $\mathcal{U}_*^*$ equals $\mathcal{U}_{\text{sta}}$ (resp. $\mathcal{U}_{\text{str}}$, $\mathcal{U}_{\text{out}}$).

Then $\mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U) = \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$, and (9.139) holds for all $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+;U)$. In particular, $\mathbb{K}_{\circlearrowleft}$ is [strictly] minimizing iff $S \geq 0$ [$S > 0$].

Also Lemma 9.8.6 and Theorem 9.8.5 apply to (a1)–(d); note that a solution of any of (a1)–(d) is a solution of (a1). We remind that a solution of the eIARE is required to be self-adjoint, by Definition 9.8.4.

Further simplifications in the positive case are given in Section 10.7.

To give a better understanding of criteria (P) and (PB), we note from Lemma 9.10.1(d) that if the eIARE has a output-stabilizing solution $\mathcal{P}$, then (P) holds iff $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ (which is the J-critical cost iff (PB) holds); (P) is also needed to get "$\pi_+ \mathbb{D}_{\circlearrowleft}^* J\mathbb{C} = 0$", which roughly says that the closed-loop system is "J-critical w.r.t. stable closed-loop inputs", and the second condition in (PB) then extends this to "$\pi_+ \mathbb{D}^* J\mathbb{C} = 0$", i.e., it makes $\mathbb{K}_{\circlearrowleft}$ J-critical (w.r.t. to open-loop inputs in $\mathcal{U}_*^*$).

Proof of Theorem 9.9.1: (a1) This follows from "(i)⇔(ii)" and (c) of Proposition 9.10.2.

(a2) 1° Now $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$, as in (a1) (see also (d)&(e)), and $\mathbb{K}_{\circlearrowleft}$ is cost-minimizing, by Lemma 10.2.2. By (f) and (9.139) we have $S \geq 0$. (A control is minimizing iff it is J-critical, hence $\mathbb{K}_{\circlearrowleft} x_0$ is strictly minimizing iff S is one-to-one, by (e2).)

2° Obviously, the minimal cost $\langle x_0, \mathcal{P}x_0\rangle$ is ≥ 0 iff $\mathcal{J}(x_0,\cdot) \geq 0$, and $J \geq 0$ suffices for this.

3° If $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and $J \geq 0$, then $\widetilde{\mathbb{X}}_+^* \widetilde{S} \widetilde{\mathbb{X}}_+ \geq 0$ (by (9.160)) and hence $\widetilde{S} \geq 0$ for any nonnegative admissible solution $(\widetilde{\mathcal{P}}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ of the eIARE. Since $\mathcal{P} \geq 0$ and $S \gg 0$ (the latter is redundant if $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$, by Lemma 9.10.3), $\mathcal{P}$ is the greatest nonnegative admissible solution, by Theorem 9.8.13.

(Even without the assumption that $J \geq 0$, we would know that $\mathcal{P}$ were the greatest admissible solution having $S \geq 0$, by Theorem 9.8.13.)

4° Let $\mathcal{P} \geq 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. Assume that also $\mathcal{P}' \geq 0$ is output-stabilizing, so that $\mathcal{P}' \geq \mathbb{C}_{\circlearrowleft}'^{\,*} J \mathbb{C}_{\circlearrowleft}'$, by (9.155). If $x_0 \in H$, then $\mathbb{K}_{\circlearrowleft}' x_0 \in \mathcal{U}_{\text{out}}(x_0)$, hence then

$$\langle x_0, \mathcal{P}' x_0\rangle \geq \langle y, Jy\rangle = \mathcal{J}(x_0, \mathbb{K}_{\circlearrowleft}' x_0) \geq \mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P}x_0\rangle, \tag{9.132}$$

where $u_{\min} := \mathbb{K}_\circlearrowleft x_0$, $y := \mathbb{C}x_0 + \mathbb{D}\mathbb{K}'_\circlearrowleft x_0 = \mathbb{C}'_\circlearrowleft x_0 \in \mathrm{L}^2$. Because $x_0 \in H$ was arbitrary, we have $\mathcal{P}' \geq \mathcal{P}$. Because $\mathcal{P}'$ was arbitrary, $\mathcal{P}$ is the smallest one.

(b) This follows from "(i)⇔(ii)" and (f2) [(f2)/(e2)] of Proposition 9.10.2.

(c1) If $\mathbb{B}$ is stable and $\mathbb{A}_\circlearrowleft$ is strongly stable, then (P) and (PB) obviously hold. Therefore, this follows from (a1) (see also Theorem 9.8.5).

(c2) The equivalence follows from (a1), because an output-stabilizing solution makes $\mathbb{A}_\circlearrowleft$ strongly stable, by Theorem 8.3.9(a3), hence (P) and (PB) hold (as in (c1)).

(c3) For $\mathcal{U}_{\mathrm{str}}$, this follows from (a1). For $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{sta}}$, this follows from Lemma 8.3.3. (We do not know whether a $\mathcal{U}_{\mathrm{str}}$-stabilizing solution has to be q.r.c.-stabilizing or even stabilizing.)

(d) By Lemma 8.3.3, we have $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$. By Theorem 6.7.15, a solution of the eIARE is output-stabilizing iff it is exponentially q.r.c.-stabilizing, hence iff it is $\mathcal{U}_{\exp}$-stabilizing (see Theorem 9.8.5). Thus, the equivalence follows from (a). Condition (P) follows from exponential stability of $\mathbb{A}_\circlearrowleft$,

(Note from Theorem 9.8.12(a) that a solution of (c1), (c2) or (d) is a unique internally stabilizing solution.)

(e1)&(e2)&(g1)&(g2) These follow from the above proofs, Proposition 9.10.2(a1)(iii)&(b2) and Lemma 9.10.1(f2).

(f1) The J-critical cost $\langle x_0, \mathcal{P}x_0\rangle$ is independent of the J-critical control $\mathbb{K}_\circlearrowleft x_0$, by Lemma 8.3.8, hence $\mathcal{P}$ is unique.

(f2) 1° If $\mathrm{Ker}(S) \neq \{0\}$, then the J-critical control is not unique, by Proposition 9.10.2(g). Assume then that $\mathrm{Ker}(S) = \{0\}$. Let u be J-critical for $x_0 = 0$. By (9.175), we then have $S\mathbb{M}^{-1}u \equiv 0$, hence $u \equiv 0$. By Lemma 8.3.8, the J-critical control is unique for each $x_0 \in H$.

2° If $\mathrm{Ker}(S) = \{0\}$, then all J-critical pairs are given by (9.114), by (a1)&(e2) and Theorem 9.8.12(s1).

3° By Theorem 9.8.12(s1), the pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is not unique when $\mathrm{Ker}(S) \neq \{0\}$. See Example 9.13.6 for the example.

(h) Now $y = \mathbb{C}x_0 + \mathbb{D}\mathbb{K}_\circlearrowleft x_0 + \mathbb{D}\mathbb{M}u_\circlearrowleft = \mathbb{C}_\circlearrowleft x_0 + \mathbb{D}_\circlearrowleft u_\circlearrowleft$, hence (9.139) follows from (9.162) and (9.163). The proofs of parts (f1) and (f2) are valid in this case too.

(i1) (Recall that $\mathbb{K}_\circlearrowleft x_0 = (K_{\mathrm{crit}})_{\mathrm{L,s}} x_{\mathrm{crit}}$ a.e., where $x_{\mathrm{crit}} := \mathbb{A}_\circlearrowleft x_0$.) By Proposition 6.6.18(d1), we have

$$u_{\mathrm{crit}}(x_0) = (\mathbb{K}_\circlearrowleft x_0) = X_{\mathrm{left}}^{-1} K_{\mathrm{w}} x_{\mathrm{crit}}(x_0) \text{ a.e.} \tag{9.133}$$

(i2) This follows from Lemma 9.8.9 and Lemma 9.10.1(d1).

(i3) This follows from Proposition 9.10.2(b1).

(i4) This follows from (9.175).

(j) See Theorem 8.3.9.

(k) The first claim follows from Proposition 9.10.2(d)&(e2)&(f2). The rest follows from the first and (h) (we do not know whether the same holds for general $\mathcal{U}_*^*$, as explained in the notes the Section 9.8). □

By Proposition 9.8.10, eIARE is equivalent to eCARE, hence we obtain the following corollary:

Corollary 9.9.2 (Critical control⇔eCARE) *Let $\mathbb{D}$ be WR. Then there is a J-critical WR state feedback operator for Σ iff the eCARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} K \mid 0 \end{bmatrix})$.*

A similar remark applies to (a2)–(d) of Theorem 9.9.1 too; in particular, there is a cost-minimizing WR state feedback operator for Σ iff $\mathcal{J}(0,\cdot) \geq 0$ and the eCARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} K \mid 0 \end{bmatrix})$.

Assume that such a solution exists. Then K is a J-critical state feedback operator for Σ, $\mathcal{P}$ is unique, and the J-critical control $u_{\text{crit}}(x_0)$ is given by $u_{\text{crit}}(x_0)(t) = K_{\text{L,s}}x(t)$, where $x = \mathbb{A}x_0 + \mathbb{B}\tau u_{\text{crit}}(x_0)$ is the corresponding state. Moreover, (e1)–(k) of Theorem 9.9.1 apply. □

(This follows directly from Theorem 9.9.1, Proposition 9.8.10 and Lemma 6.2.12(a).)

Here the zero in $\begin{bmatrix} K \mid 0 \end{bmatrix}$ refers to $X = I$ (i.e., $F = 0$), i.e., the eCARE becomes a CARE except that S need not be invertible (it is if, e.g., $\dim U < \infty$ and the J-critical control is unique). Of course, we could allow above an arbitrary WR $\mathcal{U}_*^*$-stabilizing $\begin{bmatrix} K \mid F \end{bmatrix}$, though it would not significantly increase generality, cf. Lemma 9.9.7.

In the exponentially stable case, we obtain the following formulae for $\mathcal{P}$:

Corollary 9.9.3 (Exponentially stable Σ) *Let Σ be exponentially stable. Then there is a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{exp}}$ iff the eIARE has an exponentially stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$. If this is the case, then*

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C} = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}. \tag{9.134}$$

Moreover, Theorem 9.9.10(e1) and Theorem 9.9.1 apply.

Recall from Lemma 8.3.3 that here $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$.

(An analogous result (except for Theorem 9.9.10(e1)) holds for (left-column-)strongly stable systems and $\mathcal{U}_{\text{str}}$ too, whereas Theorem 8.4.5 and hence Corollary 9.9.4 cannot be generalized to $\mathcal{U}_{\text{str}}$ (nor for $\mathcal{U}_{\text{sta}}$ or $\mathcal{U}_{\text{out}}$), by Example 9.13.2.)

Proof: The equivalence follows from Theorem 9.9.1(a1). Since $\mathbb{C}$ and $\mathbb{K}$ are necessarily exponentially stable, we obtain (9.134) from (8.36) and Lemma 9.10.1(d2). □

For exponentially stabilizable systems, optimization over $\mathcal{U}_{\text{exp}}$ can be reduced to the stable case:

Corollary 9.9.4 (Exponentially stabilizable Σ) *Let $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ be exponentially stabilizing for Σ, with closed-loop system $\Sigma_\flat$. Then $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix})$ is an exponentially stabilizing (equivalently, output-stabilizing) solution of the eIARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ iff $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K}' \mid \mathbb{F}' \end{bmatrix})$ is a $\mathcal{U}_{\text{exp}}$-stabilizing solution of the eIARE (for Σ), i.e., J-critical for Σ over $\mathcal{U}_{\text{exp}}$, where*

$$\begin{bmatrix} \mathbb{K}' \mid \mathbb{F}' \end{bmatrix} = \begin{bmatrix} \mathbb{K}_\natural + \mathbb{X}_\natural \mathbb{K} \mid \mathbb{F} + \mathbb{F}_\natural - \mathbb{F}_\natural \mathbb{F} \end{bmatrix} = \begin{bmatrix} \mathbb{X}' \mathbb{K}_\flat + \mathbb{K}_\natural \mid \mathbb{F}' \end{bmatrix}. \tag{9.135}$$

The corresponding closed-loop systems relate as in (6.194), $S = \mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft}$, and

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\flat}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\flat} = \mathbb{C}_{\flat}^* J \mathbb{C}_{\flat} - \mathbb{K}_{\natural}^* S \mathbb{K}_{\natural}. \tag{9.136}$$

Moreover, Theorem 9.9.10(e1) and Theorem 9.9.1 apply.

Proof: The equivalence follows from Lemma 6.7.9 and Theorem 8.4.5(a). By Theorem 9.9.1(g), $S = \mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft}$. By Lemma 6.7.12, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{D}_{\circlearrowleft}$ are the same for both systems, hence so are $\mathcal{P} := \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ and S. We obtain (9.136) from Corollary 9.9.3. □

If Σ is smoothly exponentially stabilizable and J-coercive, then the optimal control is given by a CARE:

Proposition 9.9.5 ($\mathcal{U}_{\text{exp}}$: $\mathbb{D}_{\flat} \in \widetilde{\mathcal{A}} \Rightarrow$ CARE) *Assume that Σ has a SR exponentially stabilizing state feedback operator K' s.t. $\mathbb{D}_{\flat} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7. Assume that $\mathbb{D}$ or $\mathbb{D}_{\flat}$ is J-coercive over $\mathcal{U}_{\text{exp}}$.*

Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE, K is J-critical over $\mathcal{U}_{\text{exp}}$, and $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in \text{SR}$.

Note from Proposition 9.12.4 that any strong or uniform regularity property of K' and $\widetilde{\mathcal{A}}$ is shared by K. An analogous result for $\mathcal{U}_{\text{out}}$ is given in Theorem 9.9.10(d3).

Proof: By Theorem 8.4.5(d), also $\mathbb{D}_{\flat}$ is J-coercive over $\mathcal{U}_{\text{exp}}^{\Sigma_{\flat}}$. Therefore, the CARE for $\left[\begin{array}{c|c} \mathbb{A}_{\flat} & \mathbb{B}_{\flat} \\ \hline \mathbb{C}_{\flat} & \mathbb{D}_{\flat} \end{array}\right]$ has an ULR SOS-stabilizing (hence exponentially stabilizing, by Theorem 6.7.15(b1)) solution $(\mathcal{P}, S, K_{\natural})$ with $\mathbb{F}_{\natural} \in \widetilde{\mathcal{A}}$, by Corollary 9.1.12.

By Proposition 9.12.4, $(\mathcal{P}, S, K' + K_{\natural})$ is an exponentially stabilizing solution of the CARE for Σ, hence J-critical over $\mathcal{U}_{\text{exp}}$, by Theorem 9.8.5. By Theorem 9.8.12(e)&(s3), $\mathcal{P}$, S and K are unique. By Proposition 6.6.18(f), we have $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in \text{SR}$. □

In discrete-time, a unique minimizing control is always of state feedback form, and it corresponds to the unique $\mathcal{U}_*^*$-stabilizing solution of the DARE, by Theorem 14.1.6. If B is bounded, then the same holds in continuous time too:

Theorem 9.9.6 (Bounded B) *Let B be bounded. Then there is a unique J-critical control for each $x_0 \in H$ iff the eCARE*

$$\begin{cases} K^* S K = A^* \mathcal{P} + \mathcal{P} A + C^* J C, \\ \quad S = D^* J D, \\ \quad S K = -(D^* J C + B^* \mathcal{P}). \end{cases} \tag{9.137}$$

has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ with S one-to-one.

Assume that this is the case. Then the following hold:

(a) The $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ is unique and ULR.

(b1) The J-critical control is determined by $u_{\rm crit}(x_0) = \mathbb{K}_\circlearrowleft x_0$, *i.e., by* $u_{\rm crit}(x_0)(t) = K_{\rm L,s}x(t)$ *for almost all* $t \ge 0$, *where* $x = \mathbb{A}x_0 + \mathbb{B}u_{\rm crit}(x_0)$ *and* $\Sigma_\circlearrowleft$ *is the closed-loop system corresponding to* $\begin{bmatrix} K & | & 0 \end{bmatrix}$; *in particular, the left column of* $\Sigma_\circlearrowleft$ *is equal to* $\Sigma_{\rm crit}$.

(b2) Conversely, $Kx_0 = u_{\rm crit}(x_0)(0)$ *for* $x_0 \in \mathrm{Dom}(A) = \mathrm{Dom}(A_{\rm crit}) = H_B$, $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$, *and* $S = D^*JD$ *is the corresponding signature operator (see (9.139)).*

(c) Theorem 9.9.1(d1)&(f) and Theorem 8.3.9 apply. If $S \in \mathcal{GB}(U)$, *then also the results of Section 9.2 apply.*

(d) We have $X = I$ *and* $\mathbb{D}, \mathbb{X}, \mathbb{M}, \mathbb{N} \in$ ULR.

In fact, $\widehat{\mathbb{D}} - D, \widehat{\mathbb{F}} \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_\omega; \mathcal{B})$ *for any* $\omega > \omega_A$, *and* $\widehat{\mathbb{N}} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+; \mathcal{B})$ $(\in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B})$ *for some* $\varepsilon > 0$, *hence strongly half-plane-regular, if* $\mathcal{P}$ *is exponentially stabilizing).*

(e1) Any J-critical control in WPLS form ($\Sigma_{\rm crit}$) *is actually of (ULR) state feedback form (even if S is not one-to-one).*

(e2) The J-critical state feedback operators correspond to $\mathcal{U}^*_*$*-stabilizing solutions of the eCARE (9.137) and conversely, as in (b1)–(b2), and (a)–(d) hold for such solutions.*

(Parts (e1) and (e2) holds even if the CARE does not have a $\mathcal{U}^*_*$*-stabilizing solution with S one-to-one.)*

Recall that $\mathcal{P}$ is exponentially stabilizing if $\mathcal{U}^*_* = \mathcal{U}_{\rm exp}$. Condition $B \in \mathcal{B}(U, H)$ can be relaxed to Hypothesis 9.2.2 when $S \in \mathcal{GB}(U)$, by Theorem 9.2.9.

For $x_0 \in H_1$, we have $x(t) \in \mathrm{Dom}(\mathbb{A}_\circlearrowleft) = H_1$ for all $t \ge 0$, hence $u(t) \equiv Kx(t)$ for such initial states.

Proof: 1° *The equivalence of the eCARE and a unique J-critical control:* The equivalence follows from Theorem 9.9.1(a1)&(g2), because any unique (for each x_0) J-critical control $u_{\rm crit}$ corresponds to an ULR J-critical state feedback operator $K = K_{\rm crit}$, by Lemma 8.3.18 and Theorem 8.3.9.

2° *The eCARE becomes (9.137):* By 1°, one choice of $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is given by $K = K_{\rm crit}$, $F = 0$ (i.e., $X = I$), which corresponds to (9.137), since $X^*SX = D^*JD$, as shown below.

Recall that "$B \in \mathcal{B}(U, H)$" means that $B \in \mathcal{B}(U, H_{-1})$ is such that $Bu_0 = B_0u_0$ for all $u_0 \in U$ for some $B_0 \in \mathcal{B}(U, H)$. Clearly $B^* = B_0|_{\mathrm{Dom}(A^*)}$, hence $B^*_{\rm w} = B^*_0 \in \mathcal{B}(H, U)$. Thus we may write $B = B_{\rm w} = B_0 \in \mathcal{B}(U, H)$ and $B^* = B^*_{\rm w} = B^*_0 \in \mathcal{B}(H, U)$ without misconceptions. The boundedness of B and Lemma A.4.4(d3) imply that $S = D^*JD$.

(a) The uniqueness follows from Theorem 9.9.1(f1); K is ULR by Lemma 6.3.16(b).

(b1) This follows from the formula $u_{\rm crit}(x_0) = \mathbb{K}_{\rm crit}x_0 = \mathbb{K}_\circlearrowleft x_0$ and 2° (see also Lemma 8.3.18).

(b2) If $x_0 \in \mathrm{Dom}(A_{\rm crit})$, then $u_{\rm crit}(x_0) = K_{\rm L,s}\mathbb{A}_{\rm crit}x_0 = K\mathbb{A}_{\rm crit}x_0 \in \mathcal{C}(\mathbf{R}_+; U)$, because $\mathbb{A}_{\rm crit}$ is a C_0-semigroup on $\mathrm{Dom}(A_{\rm crit})$ too. (see Lemma 6.1.16), and

$K = K_{\rm crit} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), U)$. Therefore, $Kx_0 = K\mathbb{A}_{\rm crit}(0)x_0 = u_{\rm crit}(x_0)(0)$. See Theorem 9.9.1(e2) and (h) for the other claims.

(c) This follows from the above and Theorem 9.9.1 (note that (1.) and (5.) of Hypothesis 9.2.2 are satisfied).

(d) This follows from Lemma 6.3.16(b)&(d) and Theorem 6.9.1(a), because $X = I$ in 1° above (note that all possible $\mathbb{X}$'s are given by $E\mathbb{X}$, $E \in \mathcal{GB}(U)$).

(e1)&(e2) The assumption that S is one-to-one was used above only for the uniqueness of $K_{\rm crit}$ and for the existence of a J-critical control in WPLS form, hence (e1)&(e2) hold.

Remark: If we use the actual eCARE (see Definition 9.8.1, we obtain all state feedback pairs; these are exactly the pairs generated by $\left[\begin{array}{c|c} XK & I-X \end{array}\right]$ $(X \in \mathcal{GB}(U))$, where K is a solution of the eCARE (9.137). In particular, the solutions of (9.137) are solutions of the eCARE. □

See Remark 10.2.18 for a different formulation for the cost function when C is bounded. However, the case with a bounded C is not at all as easy as that with a bounded B; cf. Example 9.13.8.

Often $S, X \in \mathcal{GB}(U)$, i.e., the eCARE is equivalent to the CARE (see Remark 9.8.2):

Lemma 9.9.7 ($S, X \in \mathcal{GB}(U)$) *We often require the* signature operator *S to be one-to-one, or even invertible. This is often the case with standard assumptions on $\mathbb{D}$ and J, see, e.g., Section 10.1. We make here some additional remarks on this, assuming that $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE (cf. Theorem 9.9.1(a1)).*

- *(a1) S is one-to-one [invertible] iff $\mathbb{X}^{t\,*} S \mathbb{X}^t$ is one-to-one [invertible] (for any $t > 0$).*
- *(a2) S is one-to-one iff the J-critical control for Σ is unique.*
- *(b1) Assume that $S = S^*$. Then S is invertible iff $S^* S \gg 0$.*
- *(b2) If $\dim U < \infty$, then $S \in \mathcal{GB}(U)$ iff S is one-to-one.*
- *(c1) If $\mathcal{J}(0, \cdot) > 0$, then $S > 0$.*
- *(c2) If $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_*^*$, then $S \in \mathcal{GB}(U)$ [$S \gg 0$].*
- *(c3) If $\mathcal{P} \geq 0$, $t > 0$, and $\mathbb{D}^{t\,*} J \mathbb{D}^t > 0$ [$\gg 0$] on $\mathrm{L}^2([0,t); U)$, then $S > 0$ [$\gg 0$].*
- *(c4) If $\langle \mathbb{D}u, J\mathbb{D}u \rangle \geq \varepsilon \|u\|^2_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{U}_*^*(0)$ and some $\varepsilon > 0$, $\omega \in \mathbf{R}$, then $S \gg 0$.*
- *(c5) Assume that there are $\varepsilon > 0$ and $\omega \in \mathbf{R}$ s.t. for all nonzero $u \in \mathcal{U}_*^*(0)$ there is a nonzero $v \in \mathcal{U}_*^*(0)$ s.t. $\langle \mathbb{D}v, J\mathbb{D}u \rangle \geq \varepsilon \|u\|_{\mathrm{L}^2_\omega} \|v\|_{\mathrm{L}^2_\omega}$. Then $S \in \mathcal{GB}(U)$.*

If, in addition, $\mathbb{X} := I - \mathbb{F}$ is WR, then the following hold:

- *(d) If $\mathbb{X} \in$ UR (e.g., when $\dim U, \dim Y < \infty$), or $\mathbb{X}, \mathbb{X}^{\rm d} \in$ SR, then $X \in \mathcal{GB}(U)$.*

  However, we do not know, whether X can be noninvertible for WR $\mathbb{X}$.
- *(e) If $\dim U < \infty$ and $X^* S X$ is one-to-one, then $X, S \in \mathcal{GB}(U)$.*

Proof: (Note that most of this holds with weaker assumptions too.)

(a1) Now $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ (the admissibility of $\mathcal{P}$) implies that $\mathbb{X}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U))$, by Lemma 2.2.8), hence (a1) holds.

(a2) This follows from Theorem 9.9.1(a1)&(e2).

(b1)&(b2) See Lemma A.3.1(c4)&(c3).

(c1) This follows from Theorem 9.9.1(a2).

(c2) See Lemma 9.10.3.

(c3) This follows from (9.160).

(c4)&(c5) These follow from the proof of Lemma 9.10.3 (set $M := \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|_{\mathcal{B}(\mathrm{L}^2_\omega)}$ etc.).

(d) See Lemma 6.3.2(a1)&(a2) and Proposition 6.3.1(a2)&(b1).

(e) Now $X^*SX \in \mathcal{GB}(U)$ implies that X^*, S, X must be invertible matrices, hence $X, S \in \mathcal{GB}$. □

Thus, by altering K and F, a smooth solution of the IARE can be converted into a solution of the CARE:

Corollary 9.9.8 *Let $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ be a $\mathcal{U}_*^*$-stabilizing solution of the IARE. If $\mathbb{D} \in \mathrm{WR}$ and $\mathbb{F} \in \mathrm{UR}$, then there are unique $\widetilde{S} \in \mathcal{GB}(U)$ and $\widetilde{K} \in \mathcal{B}(H_1, U)$ s.t. $(\mathcal{P}, \widetilde{S}, \widetilde{K})$ is a $\mathcal{U}_*^*$-stabilizing solution of the CARE; moreover, then $\widetilde{S} = X^*SX$ and $\widetilde{\mathbb{X}} = X^{-1}\mathbb{X}$.*

Proof: (Here $\widetilde{\mathbb{X}} = I - \widetilde{\mathbb{F}}$, where $[\ \widetilde{\mathbb{K}} \mid \widetilde{\mathbb{F}}\ ]$ is the pair generated by $\widetilde{K}$.)

Uniqueness follows from Theorem 9.8.12(b)&(s1); the existence follows from Proposition 9.8.10, Lemma 9.9.7(d) and Remark 9.8.2. □

As noted above (Corollary 9.9.4 and Theorem 8.4.5), optimization over $\mathcal{U}_{\mathrm{exp}}$ can be reduced to optimization over a preliminarily exponentially stabilized system. If Σ is q.r.c.-SOS-stabilizable, then the situation is analogous for optimization over $\mathcal{U}_{\mathrm{out}}$. This case and its special cases will be studied in Theorem 9.9.10 below (see (c1) for $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{sta}}$), but we first motivate it briefly:

Remark 9.9.9 (q.r.c.-stabilization and $\mathcal{U}_{\mathrm{out}}$) *The assumption that Σ is q.r.c.-SOS-stabilizable (cf. Corollary 6.7.16) and the use of q.r.c.-SOS-stabilizing solutions of the Riccati equation (Theorem 9.9.10) have several advantages:*

(1.) The theory for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ become roughly as easy as that for $\mathcal{U}_{\mathrm{exp}}$.

(2.) The control and output of $\Sigma_\circlearrowleft$ depend continuously on closed-loop input (the signal $u_\circlearrowleft$ in Figure 9.1, p. 408); i.e., $\mathbb{D}_\circlearrowleft$ and $\mathbb{F}_\circlearrowleft$ become stable.

(3.) If Σ is assumed to be [strongly] q.r.c.-stabilizable, then the closed-loop system becomes [strongly] stable (see (c1) below).

(4.) We can establish the standard equivalence on optimization, coprime factorizations and Riccati equations.

In general, a J-critical state feedback pair over $\mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$ or $\mathcal{U}_{\mathrm{str}}$) stabilizes the output (resp. and state) w.r.t. the initial state, but arbitrarily small disturbances in the (closed-loop) input may cause arbitrarily big perturbations in the state and output (as in Example 11.3.7(b)). □

Under certain assumptions, such solutions become the "correct" ones; see, e.g., (d3) below, Theorem 11.1.5 and Corollary 10.2.12(Crit3+).

We give below a variant of the equivalence of (I)–(III) on p. 9 (with (Crit2) and (Crit3) corresponding to (III) and (Crit4) to (II)); the equivalence will be enhanced in parts (d1)–(d4) below, in Section 9.1, and in certain later results.

Theorem 9.9.10 (eIARE⇔ $(J,*)$-inner r.c.f.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$.*
Then (Crit1)⇔(Crit2)⇔(Crit3)⇔(Crit4), where

(Crit1) **(J-critical $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$)** *There is a J-critical q.r.c.-SOS-stabilizing state feedback pair for Σ and $\mathbb{D}$ is J-coercive.*

(Crit2) **(IARE)** *The IARE has a q.r.c.-SOS-P-stabilizing solution.*

(Crit3) **(IARE/DARE)** *There are $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{GB}$ and a q.r.c.-SOS-stabilizing pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ satisfying the "DARE" (9.111) for some $t > 0$, s.t. $\langle \mathbb{A}_\circlearrowleft^{nt} x_0, \mathcal{P}\mathbb{A}_\circlearrowleft^{nt} x_0 \rangle \to 0$ as $n \to +\infty$, $n \in \mathbf{N}$.*

(Crit4) **(R.c.f.)** *The map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, and Σ is q.r.c.-SOS-stabilizable.*

Moreover, we have the following:

(a1) Let $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ solve (Crit1). Then $S := \mathbb{N}^* J \mathbb{N} = S^* \in \mathcal{GB}(U)$, $\mathcal{P} := \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = \mathcal{P}^* \in \mathcal{B}(H)$, and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ solve (Crit2) and $\mathbb{N}$, $\mathbb{M}$ solve (Crit4), where $\mathbb{M} := (I - \mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\mathbb{C}_\circlearrowleft := \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K}$.

(a2) Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ solve (Crit2) or (Crit3). Then $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ solves (Crit1) and S and $\mathcal{P}$ are as in (a1).

(a3) Let $\mathbb{N}, \mathbb{M}$ solve (Crit4). Then a solution of (Crit1) and (Crit2) can be constructed as in (g2) (also (g1) applies if $\Sigma \in \text{SOS}$).

(a4) This theorem also holds with "r.c." or "p.r.c." in place of "q.r.c.".

(b) A solution $\mathcal{P}$ of (Crit2) is unique. Given one solution $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ of (Crit1) or (Crit2) or $(\mathbb{N}, \mathbb{M})$ and S of (Crit4), all solutions are given by

$$\begin{bmatrix} E\mathbb{K} \mid E\mathbb{F} + I - E \end{bmatrix}, \quad (\mathbb{N}E^{-1}, \mathbb{M}E^{-1}), \quad E^{-*}SE^{-1}, \qquad E \in \mathcal{GB}(U). \tag{9.138}$$

The corresponding closed-loop systems are given by $\Sigma_{\circlearrowleft E} := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft E^{-1} \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft E^{-1} \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft E^{-1} \end{array}\right]$.

(c1) Assume that Σ is [strongly [exponentially]] q.r.c.-stabilizable.

Then any q.r.c.-SOS-stabilizing state feedback pair is [strongly [exponentially]] q.r.c.-stabilizing. Consequently, then this theorem holds with $\mathcal{U}_{\text{sta}}$ [or $\mathcal{U}_{\text{str}}$ [or $\mathcal{U}_{\text{exp}}$]] in place of $\mathcal{U}_{\text{out}}$.

[Moreover, if $J \geq 0$ and (Crit2) has a solution $\mathcal{P}$, then $\mathcal{P}$ is the greatest nonnegative admissible and the unique nonnegative output-stabilizing solution of the eIARE.]

(c2) Assume that Σ is exponentially q.r.c.-stabilizable. Then any I/O-stabilizing or input-stabilizing solution or the IARE (i.e., with stable $\mathbb{N}$ and $\mathbb{M}$ or $\mathbb{B}\mathbb{M}$) is exponentially q.r.c.-stabilizing.

Moreover, then any $(J,*)$-inner q.r.c.f. of $\mathbb{D}$ is exponentially q.r.c.

(c3) Assume that Σ is estimatable. Then any output-stabilizing solution of the IARE is exponentially q.r.c.-stabilizing (hence the greatest nonnegative admissible solution if $J \geq 0$).

(d1) Let $\mathbb{D}$ be WR. If we add to (Crit1)–(Crit4) the requirement that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ must correspond to some WR $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ (equivalently, that $\mathbb{X} = \mathbb{M}^{-1} = I - \mathbb{F}$ is WR *and $X = I$), then a fifth equivalent condition is:*

(Crit5) **(CARE)** *The CARE has a q.r.c.-SOS-P-stabilizing solution.*

Moreover, with these extra requirements any solutions of (Crit1)–(Crit5) are unique and equal (cf. (a1)–(a3)); the same applies to (d2).

(d2) Assume Hypothesis 9.2.1 and that $D^*JD \in \mathcal{GB}(U)$. Then (Crit1)–(Crit6) are equivalent (they are also equivalent to (Crit7) if Σ is optimizable and estimatable), where

(Crit6) **(B_{w}^*-CARE)** *The B_{w}^*-CARE has a q.r.c.-SOS-P-stabilizing solution.*

(d3) **(MTIC)** *Assume that 1. Σ is q.r.c.-SOS-stabilizable in $\widetilde{\mathcal{A}}$, or that 2. Σ has a q.r.c.-SOS-stabilizing SR state feedback operator s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$, or that 3. $D^*JD \in \mathcal{GB}(U)$ and Σ has an exponentially q.r.c.-stabilizing SR state feedback operator s.t. $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies Hypothesis 9.2.1, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7.*

Then (Crit1)–(Crit5) and (Crit7) are equivalent and imply that $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}$, where

(Crit7) The map $\mathbb{D}$ is J-coercive.

(d4) In the general case we still have (Crit6)$\Rightarrow$(Crit5)$\Rightarrow$(Crit1–4)$\Rightarrow$(Crit7).

Assume that (Crit1) has a solution, and use the notation of Definition 9.1.3. Then the following hold:

(e1) The closed loop cost function $\mathcal{J}_\circlearrowleft(x_0, u_\circlearrowleft)$ for $y = \mathbb{C}_\circlearrowleft x_0 + \mathbb{D}_\circlearrowleft u_\circlearrowleft$, $u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+; U)$ and $x_0 \in H$ is given by

$$\mathcal{J}_\circlearrowleft(x_0, u_\circlearrowleft) := \langle y, Jy \rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \langle x_0, \mathcal{P} x_0 \rangle_H + \langle u_\circlearrowleft, S u_\circlearrowleft \rangle_{\mathrm{L}^2(\mathbf{R}_+;U)}. \tag{9.139}$$

(e2) **(Minimization)** *The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is minimizing $\Leftrightarrow S \gg 0 \Leftrightarrow \langle \mathbb{D}u, J\mathbb{D}u \rangle \geq 0$ for all $u \in \mathcal{U}_{\mathrm{out}}(0) \Leftrightarrow \mathbb{D}$ is positively J-coercive.*

(f) Theorem 9.9.1(g)–(j) and Theorem 8.3.9 apply, and (PB) is satisfied.

(g1) **(Stable case)** *Assume that $\mathbb{C}$ and $\mathbb{D}$ are stable (i.e., that $\Sigma \in$ SOS). Assume (Crit4) (see (Crit1SOS)–(Crit4SOS) of Corollary 9.9.11 for further equivalent conditions). Then the corresponding solution of (Crit2) is*

$$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{c|c} -S^{-1}\pi_+ \mathbb{N}^* J \mathbb{C} & I - \mathbb{M}^{-1} \end{array}\right]. \tag{9.140}$$

Moreover, this $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ *is stable and q.r.c.-SOS-stabilizing. The corresponding* $\mathcal{P}$ *and* $\Sigma_\circlearrowleft$ *satisfy*

$$\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = \mathbb{C}^* J \mathbb{C}_\circlearrowleft = \mathbb{C}_\circlearrowleft^* J \mathbb{C} = \mathbb{C}^* \left(J - J\mathbb{N}S^{-1}\pi_+\mathbb{N}^* J \right) \mathbb{C} \tag{9.141}$$

$$= \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K} = \mathbb{C}^* (J - J\mathbb{D}\pi_+ (\pi_+\mathbb{D}^* J \mathbb{D}\pi_+)^{-1}\pi_+\mathbb{D}^* J)\mathbb{C}, \tag{9.142}$$

$$\mathbb{A}_\circlearrowleft = \mathbb{A} + \mathbb{B}\mathbb{K}_\circlearrowleft, \tag{9.143}$$

$$\mathbb{C}_\circlearrowleft = \left(I - \mathbb{N}S^{-1}\pi_+\mathbb{N}^* J \right) \mathbb{C}, \tag{9.144}$$

$$\mathbb{K}_\circlearrowleft = -\mathbb{M}S^{-1}\pi_+\mathbb{N}^* J \mathbb{C}. \tag{9.145}$$

(g2) Given $\mathbb{N}$ *and* $\mathbb{M}$ *as in (Crit4), the pair* $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$*, the Riccati operator* $\mathcal{P}$*, and the closed-loop system* $\Sigma_\circlearrowleft$ *can be constructed as follows:*

Choose any q.r.c.-SOS-stabilizing pair $\begin{bmatrix} \mathbb{K}' & | & \mathbb{F}' \end{bmatrix}$ *for* Σ*, and let* $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ *be the two top (block) rows of the corresponding closed-loop system* $\Sigma_\flat$*. Set* $\mathbb{M}' := (I - \mathbb{F}')^{-1}$*,* $\mathbb{X}_\natural := \mathbb{M}^{-1}\mathbb{M}'$*, so that* $\mathbb{X}_\natural \in \mathcal{G}\mathrm{TIC}(U)$*.*

Set $\begin{bmatrix} \mathbb{K}_\natural & | & \mathbb{F}_\natural \end{bmatrix} := \begin{bmatrix} -S^{-1}\pi_+\mathbb{N}^* J \mathbb{C}_\flat & | & I - \mathbb{X}_\natural \end{bmatrix}$*. Then a solution* $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ *of (Crit2) and corresponding* $\Sigma_\circlearrowleft$ *and* $\mathcal{P}$ *are obtained as follows:*

$$\mathbb{K} = \mathbb{M}^{-1}\mathbb{K}_\flat' + \mathbb{K}_\natural = \mathbb{X}_\natural \mathbb{K}' + \mathbb{K}_\natural, \quad \mathbb{F} = I - \mathbb{M}^{-1}, \tag{9.146}$$

$$\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}_\flat + \mathbb{B}\mathbb{M}\tau\mathbb{K}_\natural & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C}_\flat + \mathbb{N}\mathbb{K}_\natural & \mathbb{N} \\ \mathbb{K}_\flat + \mathbb{M}\mathbb{K}_\natural & \mathbb{M} - I \end{array}\right] \tag{9.147}$$

$$\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = \mathbb{C}_\flat^* J \mathbb{C}_\circlearrowleft = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\flat \tag{9.148}$$

$$= \mathbb{C}_\flat^* J \mathbb{C}_\flat - \mathbb{K}_\natural^* S \mathbb{K}_\natural^* = \mathbb{C}_\flat^* \left(J - J\mathbb{N}S^{-1}\pi_+\mathbb{N}^* J \right) \mathbb{C}_\flat. \tag{9.149}$$

(Recall that $\mathbb{A}_\flat = \mathbb{A} + \mathbb{B}\tau\mathbb{K}_\flat'$*,* $\mathbb{C}_\flat = \mathbb{C} + \mathbb{D}\mathbb{K}_\flat'$*, and* $\mathbb{K}_\flat' = (I - \mathbb{F}')^{-1}\mathbb{K}'$*.)*

Note that $\mathbb{K}_\natural$*,* $\mathbb{C}_\flat$*,* $\mathbb{D}_\flat$*,* $\mathbb{K}_\flat'$*,* $\mathbb{C}_\circlearrowleft$ *and* $\mathbb{D}_\circlearrowleft = \mathbb{N}$ *are stable, and that* $\mathcal{P}$ *and* $\begin{bmatrix} \mathbb{K}_\natural & | & \mathbb{F}_\natural \end{bmatrix}$ *correspond to this theorem (including (g1)) applied to* $\Sigma_\flat^1$ *and* J*.*

(g3) The constructions (g1), (g2) and (a1) can be used in (c1) and (c2) too.

See Section 9.1 for related results and further equivalent conditions. Further equivalent conditions for optimizable and estimatable systems are given in Corollary 9.2.15.

Quasi-coprimeness is essential in the above theorem; an arbitrary SOS-stabilizing (even exponentially stabilizing solution) need not correspond to the minimizing control over $\mathcal{U}_{\mathrm{out}}$ (though necessarily to that over $\mathcal{U}_{\mathrm{exp}}$), as illustrated in Example 9.13.2.

Indeed, the q.r.c.-property guarantees that $\mathcal{U}_{\mathrm{out}}(x_0)$ corresponds one-to-one and onto to $\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0)$; i.e., we obtain the situation of Theorem 8.4.5(e), indeed, if $\begin{bmatrix} \widetilde{\mathbb{K}} & | & \widetilde{\mathbb{F}} \end{bmatrix}$ is q.r.c.-SOS-stabilizing, then

$$\mathcal{U}_{\mathrm{out}}(x_0) = \{\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat \,|\, u_\flat \in \mathrm{L}^2(\mathbf{R}_+; U) = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0)\} \quad (x_0 \in H). \tag{9.150}$$

Indeed, obviously such u and $y := \mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}_\flat x_0 + \mathbb{D}_\flat u_\flat$ are stable. Conversely, if $u, y \in \mathrm{L}^2$, then $\mathbb{D}_\flat u_\flat = y - \mathbb{C}_\flat x_0 \in \mathrm{L}^2$ and $\widetilde{\mathbb{M}} u_\flat = -\widetilde{\mathbb{K}}_\flat x_0 + u \in \mathrm{L}^2$, where $u_\flat := -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}} u$, so that $u_\flat \in \mathrm{L}^2$, since $\mathbb{D}_\flat := \mathbb{D}\widetilde{\mathbb{M}}$ and $\widetilde{\mathbb{M}}$ are q.r.c.

For general SOS-stabilizing state feedback pairs, some elements of $\mathcal{U}_{\text{out}}(x_0)$ may correspond to unstable inputs $u_\flat$ for $\Sigma_\flat$, so that a P-SOS-stabilizing solution optimizes over a too small class of inputs; cf. the (non-q.r.c.-)exponentially (P-)stabilizing state feedback operator $K = -2$ of Example 9.13.2. If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ were merely output-stabilizing, then also some stable inputs $u_\flat$ for $\Sigma_\flat$ might correspond to unstable inputs u for Σ.

Also the construction formulae of (g1)–(g2) (in particular, (9.140)) base on quasi-coprimeness, hence we cannot give such formulae for non-q.r.c.-SOS-stabilizable systems. Indeed, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ corresponds to a WR state feedback operator K, then $\widehat{\mathbb{M}}^{-1}$ may only have the singularities of $(s-A)^{-1}$, as noted below Definition 6.6.10; thus we must somehow guarantee that $\widehat{\mathbb{M}}^{-1}$ does not have too many poles (cf. Lemma 6.5.4).

Proof of Theorem 9.9.10: (Note that the stability of $\mathbb{K}_\flat$ and $\mathbb{K}_\circlearrowleft$ might be omitted from the requirements and conclusions, whereas the invertibility of S is essential for, e.g., (Crit4)⇒(Crit2).)

1° "(Crit1)⇔(Crit1$\frac{1}{2}$)": We have one more equivalent condition:

(Crit1$\frac{1}{2}$) $\mathbb{D}$ is J-coercive, and there is a q.r.c.-SOS-stabilizing state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ s.t. $\pi_+ \mathbb{D}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = 0$, where $\Sigma_\circlearrowleft$ is the corresponding closed-loop system.

Now $\mathcal{U}_{\text{out}}(0) = \mathbb{M}\pi_+ \mathrm{L}^2$, by Proposition 9.10.2(e2), hence $\langle \mathbb{D}_\circlearrowleft \pi_+ u, \mathbb{C}_\circlearrowleft x_0 \rangle = 0$ for all $x_0 \in H$, $u \in \pi_+ \mathrm{L}^2$ iff $\langle \mathbb{D}\pi_+ u, \mathbb{C}_\circlearrowleft x_0 \rangle = 0$ for all $x_0 \in H$, $u \in \mathcal{U}_{\text{out}}(0)$ (because $\mathbb{D}_\circlearrowleft \pi_+ u = \mathbb{D}\mathbb{M}\pi_+ u = \mathbb{D}\pi_+ \mathbb{M}\pi_+ u$), i.e., iff $\mathbb{K}_\circlearrowleft$ is J-critical.

2° "(Crit1)⇔(Crit2)⇔(Crit3)"&(a1)&(a2) This follows Theorem 9.9.1(b)&(d), and Lemma 8.4.11(b2) (note that (Crit2)–(Crit4) require $S \in \mathcal{GB}(U)$).

3° "(Crit1)⇒(Crit4)"&(a1): These follow from equations $\mathbb{N} = \mathbb{D}\mathbb{M}$ and $\mathbb{N}^* J \mathbb{N} = S$ (see Theorem 9.9.1(b)&(g)) and Lemma 9.10.3.

4° "(Crit4)⇒(Crit1$\frac{1}{2}$)" for stable $\mathbb{C}$ and $\mathbb{D}$: By Lemma 8.4.14(a), $\mathbb{D}$ is J-coercive. Now $\mathbb{M} \in \mathcal{G}$TIC, by Lemma 6.5.6(b). Obviously, the pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ from (9.140) are stable. By using (9.140) and equation $S^{-1}\mathbb{N}^* J \mathbb{D} = \mathbb{M}^{-1}$, it is straightforward to verify that $\mathbb{K}\mathbb{A} = \pi_+ \tau \mathbb{K}$ and $\pi_+ \mathbb{F} \pi_- = \mathbb{K}\mathbb{B}$ (see the proof of Theorem 27 of [S97b] for details), hence $\Sigma_{\text{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right] \in \text{SOS}(U, H, Y \times U)$. By Corollary 6.6.9, we have $\Sigma_\circlearrowleft \in$ SOS. Now

$$\pi_+ \mathbb{N}^* J \mathbb{C}_\circlearrowleft = \pi_+ \mathbb{N}^* J \mathbb{N}\mathbb{K} + \pi_+ \mathbb{N}^* J \mathbb{C} = \pi_+ \mathbb{N}^* J \left(-\mathbb{N} S^{-1} \pi_+ \mathbb{N}^* J \mathbb{C} + J\mathbb{C}\right) = 0 \tag{9.151}$$

as required.

(N.B. We do not know whether $S^{-1}\mathbb{N}^* J \mathbb{D} = \mathbb{M}^{-1}$ holds for unstable $\mathbb{M}^{-1}$, therefore we have required the (J,S)-inner right factorization to be q.r.c.)

5° "(Crit4)⇒(Crit1$\frac{1}{2}$)": By Lemma 8.4.14(a), $\mathbb{D}$ is J-coercive. By Lemma 6.4.5(c), $\mathbb{X}_\natural \in \mathcal{G}\text{TIC}(U)$. Therefore, $\mathbb{D}_\flat = \mathbb{N}\mathbb{X}_\natural$ is a (J,S)-inner r.c.f.

By 4°, the state feedback pair $\begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix}$ is stable and q.r.c.-SOS-stabilizing for $\Sigma_\flat^1$, and the corresponding closed-loop system satisfies $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$. Moreover, $\mathbb{D}_\circlearrowleft = \mathbb{D}_\flat \mathbb{X}_\natural^{-1} = \mathbb{N}$.

Apply Lemma 6.7.12 (with $\mathbb{K}'$ and $\mathbb{K}$ interchanged, etc.) to obtain (9.146)–(9.147). Then $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$, and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ q.r.c.-SOS-stabilizes Σ into $\Sigma_\circlearrowleft$ (indeed, by (9.147), $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is q.r.c.-SOS-stabilizing). Thus, (Crit1$\frac{1}{2}$) holds. The remaining formulae of (g2) follow from those of (g1).

(a1)–(a3) Parts (a1)&(a2) were proved above; see (g1) and (g2) for (a3).

(a4) If any of (Crit1)–(Crit4) holds with "r.c." in place of "q.r.c.", then so do the others, by (a1)–(a3). This property is then inherited in by (b)–(g3). The proof for "p.r.c." is analogous.

(b) This follows from Theorem 9.9.1(f).

(c1) The first claims follows from Theorem 6.7.15(a1)[(a2)[(b1)]], the second claim is a consequence of the first one.

[By Theorem 9.9.1(a2), $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE. But $S \gg 0$, by (e2), and $\mathcal{P}$ is strongly stabilizing, hence $\mathcal{P}$ is the greatest nonnegative admissible solution of the eIARE, by Theorem 9.8.13 (since $S \geq 0$ for admissible nonnegative solutions, by (9.160)). Since $\mathcal{P} \geq \mathcal{P}' \geq \mathcal{P}$ for any nonnegative output-stabilizing solution $\mathcal{P}'$, $\mathcal{P}$ must be unique.]

(c2) This follows from Theorem 6.7.15(b1) and Lemma 6.4.5(e).

(c3) This follows from Theorem 6.7.15(c2) (and from the last claim of (c1)).

(d1) This follows from Proposition 9.8.10 (and (a1)–(a3)).

(d2) We obtain (Crit2)⇔(Crit5)⇔(Crit6) from Theorem 9.2.9 (the claim on (Crit7) will be proved in Corollary 9.2.15).

(d3) By (d4), we only have to establish (Crit7)⇒(Crit5). Assume (Crit7).

SOS-stabilizability implies that $\mathbb{K}_\flat x_0 \in \mathcal{U}_{\rm out}(x_0) \neq 0$ for all $x_0 \in H$, hence there is a unique J-critical control over $\mathcal{U}_{\rm out}$ for each $x_0 \in H$, by Theorem 8.4.3. Thus, (Crit5) follows from (7.) or (8.) of Remark 9.9.14 (note that "1." is a special case of "2.", by Proposition 6.3.1(c) and Lemma 6.6.12).

(d4) This follows from Proposition 9.2.7(a), Proposition 9.8.10 and Corollary 8.4.14(a).

(e1) This follows from $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$ and $\mathbb{D}_\circlearrowleft^* J\mathbb{D}_\circlearrowleft = S$.

(e2) See the proof of Corollary 10.2.12.

(f) This follows from (b) and Theorem 9.9.1(b).

(g1) This was proved in 4° except for the $\mathcal{P}$ formula, which follows from formulae $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, $\mathbb{K}\mathbb{N}^* J\mathbb{C}_\circlearrowleft = 0$ (see (Crit1)), $\mathbb{K}\mathbb{N}^* J\mathbb{N}\mathbb{K} = \mathbb{K}S\mathbb{K}$, and Lemma 6.4.7(b), in that order.

(g2) This follows from 5° and (g1) with straightforward computations.

(g3) This follows from the proof of (c) above. □

Note that a q.r.c.-I/O-P-stabilizing solution of the IARE (or CARE) determines a (J,S)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, by Lemma 9.10.1(b5)&(b6). If Σ is approximately reachable, then this defines $\mathbb{K}$ uniquely modulo (9.138) (because $\pi_+\mathbb{M}^{-1}\pi_- = \mathbb{K}\mathbb{B}$), hence then $\mathcal{P}$ is unique (a q.r.c.-SOS-P-stabilizing solution is always unique, by Theorem 9.9.10(b)).

In the stable case, "q.r.c.-SOS-stabilizing" is equivalent to "stable, SOS-stabilizing" (cf. also Corollary 8.3.11):

Corollary 9.9.11 (SOS-stable IARE) *Let* $\Sigma \in \mathrm{SOS}$ *and* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.

Then conditions (Crit1)–(Crit5) in Theorem 9.9.10 can be written in following forms:

(Crit1SOS) There is a J-critical stable, SOS-stabilizing state feedback pair for Σ*, and* $\mathbb{D}$ *is J-coercive.*

(Crit2SOS) The IARE has a stable, SOS-P-stabilizing solution.

(Crit3SOS) **(IARE/DARE)** *There are* $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{GB}$ *and a stable, SOS-stabilizing pair* $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *satisfying the "DARE" (9.111) for some* $t > 0$*, s.t.* $\langle \mathbb{A}_{\circlearrowleft}^{nt} x_0, \mathcal{P}\mathbb{A}_{\circlearrowleft}^{nt} x_0 \rangle \to 0$ *as* $n \to +\infty$, $n \in \mathbf{N}$.

(Crit4SOS) There is a spectral factorization $\mathbb{X}^* S \mathbb{X}$ *of* $\mathbb{D}^* J \mathbb{D}$.

(Crit5SOS) The CARE has a stable, SOS-P-stabilizing solution.

We may replace "stable, SOS-P-stabilizing solution" above by "P-admissible solution s.t. $\mathbb{D}, \mathbb{F}, \mathbb{M} \in \mathrm{TIC}$*", as well as by "[q.]r.c.-I/O-stabilizing" (and by "exponentially stabilizing" or by "*$\mathbb{M}$*-stabilizing" if* Σ *is exponentially stable, and by "stable, P-stabilizing" if* Σ *is stable).*

Moreover, the solutions of (Crit1SOS)–(Crit5SOS) (if any) are the ones of (Crit1)–(Crit5), with $\mathbb{X} = I - \mathbb{F}$. □

(This follows from Lemma 6.6.17(a) and Lemma 6.4.8(a); note that corresponding pairs $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ are the same for each condition. Recall from Definition 9.8.4 that the solution being stable or stabilizing means that $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is stable (which is redundant if Σ is exponentially stable) or stabilizing, respectively.)

Proposition 9.8.11 contains related results and the positive case is given in Corollary 10.2.13. We remind that (Crit5SOS) (or (Crit5)) is stronger than (Crit1SOS)–(Crit4SOS), which are equivalent.

J-coercivity is roughly equivalent to the existence of a unique J-critical control:

Proposition 9.9.12 ($\mathcal{U}_{\mathrm{exp}}$: IARE⇒unique optimum⇔ J-coercive) *We have (i)⇔(ii).*

(i) There is a unique J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ *for each* $x_0 \in H$*, and* $\mathbb{S}^t := \mathbb{D}^{t*} J \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U)$ *for some (hence all)* $t > 0$.

(ii) $\mathbb{D}$ *is J-coercive over* $\mathcal{U}_{\mathrm{exp}}$*, and* Σ *is optimizable.*

Moreover,

(a) Assume (i). Then $\mathbb{S}^t \gg 0 \Leftrightarrow$ *the J-critical control is minimizing* $\Leftrightarrow \mathbb{D}$ *is positively J-coercive over* $\mathcal{U}_{\mathrm{exp}}$.

(b) Assume that the IARE has an exponentially stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$*. Then (i) and (ii) hold. Moreover, then* $S \gg 0 \Leftrightarrow \mathbb{S}^t \gg 0 \Leftrightarrow \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *is minimizing* $\Leftrightarrow \mathbb{D}$ *is positively J-coercive over* $\mathcal{U}_{\mathrm{exp}}$.

(c) Condition "$\mathbb{S}^t \in \mathcal{GB}$" is redundant in (i) if any of (1.)–(4.) holds, where

(1.) $J \geq 0$ and $\mathbb{D}^{t*} J \mathbb{D}^t \gg 0$ for some $t > 0$;
(2.) $J \geq 0$, $D^* J D \gg 0$ and $\mathbb{D} \in \mathrm{MTIC}_\infty$;
(3.) $\mathbb{D} \in \mathrm{MTIC}_\infty$, $\mathbb{A}B \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; \mathcal{B}(U,H))$ and $D^* J D \in \mathcal{GB}(U)$.
(4.) Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, and $D^* J D \in \mathcal{GB}(U)$.

Thus, when minimizing over $\mathcal{U}_{\exp}$ with a some coercivity or regularity, the cost must be J-coercive over $\mathcal{U}_{\exp}$. See, e.g., Theorems 9.2.16 and 9.2.18 and Corollary 9.2.19 for enhanced versions of the proposition, and Section 10.2 for a positive variants.

Proof: (Naturally, $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$ in (i).)

The equivalence, (a), (b) and (c)(1.) follow from Theorems 14.2.7 and 13.4.4 and Remark 13.4.6.

(c) (2.) Now $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} \geq 0$, hence $\mathbb{S}^t \geq \mathbb{D}^{t*} J \mathbb{D}^t$. But $\mathbb{D}^{t*} J \mathbb{D}^t \geq D^* J D - \varepsilon \gg 0$ for $t, \varepsilon > 0$ small enough, by Theorem 2.6.4(i1), hence $\mathbb{S}^t \gg 0$ for such t.

(3.) By Lemma A.3.1(c4)&(c1), there is $\varepsilon > 0$ s.t. $\|D^* J D u_0\|_U \geq \varepsilon \|u_0\|_U$ for all $u_0 \in U$. It follows that

$$\|\pi_{[0,t)} D^* J D u\|_2 \geq \varepsilon \|u\|_2 \qquad (u \in \mathrm{L}^2([0,t);U)). \tag{9.152}$$

By Theorem 2.6.4(i1)&(i2), $\pi_{[0,t)}(\mathbb{D} - D)\pi_{[0,t)} \to 0$ and $\mathbb{B}^t \to 0$ on L^2, as $t \to 0+$, hence there is $t > 0$ s.t. $\|\pi_{[0,t)}(\mathbb{S}^t - D^* J D)\pi_{[0,t)}\|_{\mathcal{B}(\mathrm{L}^2)} < \varepsilon/2$ (note that $\mathbb{S}^t = \pi_{[0,t)} \mathbb{S}^t \pi_{[0,t)}$). Consequently, $\|\mathbb{S}^t u\|_2 \geq \varepsilon/2 \|u\|_2$ for all $u \in \mathrm{L}^2([0,t);U)$. By Lemma A.3.1(c4)&(c1), this means that $\mathbb{S}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U))$.

(4.) This follows from Theorem 9.2.16 and (b). □

However, uniqueness is sometimes possible under weaker conditions:

Remark 9.9.13 (Unique J-critical control vs. $S \in \mathcal{GB}(U)$ vs. J-coercivity) *Assume that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$ (this is obviously necessary for the existence of a J-critical control) and that Z^{s} is a Hilbert space (e.g., $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\exp}$).*

By Theorem 8.4.3 and Lemma 9.10.3, J-coercivity implies the existence of a unique J-critical control, and also the invertibility of S when the eIARE has a solution (these three are equivalent for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ if Hypothesis 9.2.2 holds (or if Σ is a wpls), by Theorem 9.2.16).

However, the invertibility of S is not necessary, nor is J-coercivity over $\mathcal{U}_*^*$, for the existence of a unique J-critical control (take Σ exponentially stable (so that $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{out}}$), $0 < D^* D \not\gg 0$, $B = 0 = C$, $J = I$).

On the other hand, Example 9.13.4 shows that even for very regular ($\mathbb{D} = D \in \mathcal{B}(U,Y)$) exponentially stable systems, $S = D^* D > 0$ is not sufficient for the existence of a J-critical control for all $x_0 \in H$ (over $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{out}}$).

It is easy to formulate shortly the necessary and sufficient condition for the existence of a unique J-critical control, but we have found no useful formulations (see the comments below Theorem 8.2.5).

As mentioned above, when B is bounded, any unique J-critical control corresponds to a $\mathcal{U}_*^*$-stabilizing solution of the CARE (and of the B_w^*-CARE if $D^*JD \in \mathcal{GB}$ or $\mathbb{D}$ is J-coercive). Unfortunately, this is not the case in general, by Example 11.3.7. Therefore, we summarize several sufficient conditions below:

Remark 9.9.14 (Necessity of the CARE) *We write* $(\Sigma, J) \in$ coerciveCARE *(over $\mathcal{U}_*^*$) if 1. Σ is WR, and 2. if Σ is J-coercive and there is a J-critical control for each $x_0 \in H$, then the CARE has a SR $\mathcal{U}_*^*$-stabilizing solution (equivalently, then K_{crit} corresponds to a SR state feedback operator).*

If any of the following conditions holds, then $(\Sigma, J) \in$ coerciveCARE*:*

(1.) $B \in \mathcal{B}(U, H)$;

(2.) Hypothesis 9.2.1 holds and $D^*JD \in \mathcal{GB}(U)$;

(3.) Hypothesis 9.2.1 holds, $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,H))$ and $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$;

(4.) Hypothesis 9.5.1 holds and $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$;

(5.) $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,H))$, $\pi_{[0,1)}C_w\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H,Y))$, and $\pi_{[0,1)}C_w\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,Y))$;

(6.) $\Sigma \in$ SOS, $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ and $\mathbb{D} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7.

(7.) $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, and Σ has a SR q.r.c.-SOS-stabilizing state feedback operator s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$.

(8.) $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, and Σ has a SR exponentially q.r.c.-stabilizing state feedback operator s.t. $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ some of (1.)–(6.).

Moreover, the following holds:

(a) In cases (1.)–(6.), we have $\mathbb{D} \in$ ULR (and $\mathbb{F} \in$ ULR for the $\mathcal{U}_*^*$-stabilizing solution).

(b) In cases (1.)–(5.), we necessarily have $S = D^*JD \in \mathcal{GB}(U)$ for the $\mathcal{U}_*^*$-stabilizing solution.

(c) In cases (1.)–(3.), the CARE becomes a B_w^*-CARE.

(d) In cases (1.)–(4.), "Σ is J-coercive and there is a" can be replaced by "there is a unique" if we assume that $D^*JD \in \mathcal{GB}(U)$.

By Lemma 8.4.4, the control mentioned in "2." is necessarily unique. By Theorem 14.1.6 and Lemma 9.9.7(c2), "$(\Sigma, J) \in$ coerciveCARE" is redundant in discrete time (i.e., it is true for any Σ and J).

Recall from Theorem 8.4.3 that "there is a J-critical control for each $x_0 \in H$" can usually (e.g., for $\mathcal{U}_{\text{out}}$ and $\mathcal{U}_{\text{exp}}$) be replaced by the finite cost condition "$\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$". If Σ is exponentially stable (or estimatable), then $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{out}}$, by Lemma 8.3.3.

See Theorems 9.5.13 and 9.2.18 and Corollary 9.1.11 for more explicit variants of (4.) and (5.) and (7.), respectively.

Proof: (Note that (1.) and (4.)–(7.) are independent of J, and that the $\mathcal{U}_*^*$-stabilizing solution is unique. The requirement "SR" could be "WR" for most but not all applications of the above definition.)

Note first that $\mathbb{D} \in \text{ULR}$ if any of (1.)–(6.) holds and $\mathbb{D} = \mathbb{D}_\flat \mathbb{X}^{-1} \in \text{SR}$ when (7.) or (8.) holds, so that condition "1." is satisfied by any of (1.)–(8.).

Case (1.) is treated in Theorem 9.9.6 (the eCARE becomes a CARE, because J-coercivity implies the invertibility of S $(= D^*JD)$, by Lemma 9.9.7(c2)).

Case (2.) is treated in Theorem 9.2.9. Condition (3.) implies condition (2.), by Lemma 9.2.17 (which contains also alternative assumptions).

Condition (4.) implies condition (5.), by Lemma 9.5.2.

Case (5.) follows from Theorem 9.2.18. Case (6.) follows from "(iv)⇒(iii)" of Corollary 9.1.12.

Case (7.) follows from (6.) and Theorem 8.4.5(g1)&(d)&(a)&(c2) (and Lemma 6.2.5 and Corollary 9.9.8; Proposition 9.12.4 would lead to an alternative proof and (9.226) holds). Analogously, case (8.) can be reduced (1.)–(6.) (use the fact that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$, by Lemma 8.3.3; naturally, by "$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies (n.)" we mean that "(n.)" is satisfied with $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ in place of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$).

(a)&(b) The proof sketched above applies for (a)&(b) too. (See Theorem 8.4.9(a) for extending (b) to case (6.).)

(c) See Theorem 9.2.9 (use the fact that $\text{Dom}(B_{\text{w}}^*) = H$ in case (1.)).

(d) The above proofs for cases (1.)–(3.) did not use J-coercivity. In cases (4.) and (5.), we can remove "Σ is J-coercive and there is a" completely if we assume that $\mathbb{A}B \in \text{L}^2_{\text{loc}}$ (this is redundant in case (4.), by Lemma 9.5.2) and that $D^*JD \in \mathcal{GB}(U)$, by Corollary 9.2.19.

(N.B. if we assume (1.) and replace "Σ is J-coercive and there is a" by "there is a unique", then the CARE might become an eCARE (instead of a B_{w}^*-CARE) if we would not explicitly assume D^*JD to be invertible.) □

A comparison of $\mathcal{U}_{\text{exp}}$, $\mathcal{U}_{\text{str}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{out}}$

In principle, the above theory on optimization and Riccati equations can be applied over any $\mathcal{U}_*^*$. However, it is not always clear a priori whether a control problem is coercive enough to guarantee the existence of a unique solution (particularly in the case of $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$), and when a solution of the corresponding Riccati equation is found, it is not always easy to verify that it is a correct one ($\mathcal{U}_*^*$-stabilizing).

For optimization over $\mathcal{U}_{\text{out}}$, it is often easy to find sufficient coercivity conditions. For $\mathcal{U}_{\text{exp}}$ one needs stronger assumptions, and for $\mathcal{U}_{\text{str}}$ (resp. $\mathcal{U}_{\text{sta}}$), we have to optimize over $\mathcal{U}_{\text{out}}$ and make suitable stabilizability or other assumptions to guarantee that the closed-loop system actually becomes strongly stable (resp. stable) (see the comments below Theorem 8.4.3).

On the other hand, J-critical state feedback pairs over $\mathcal{U}_{\text{exp}}$ correspond to exponentially stabilizing solutions of the Riccati equation, and the situation with $\mathcal{U}_{\text{str}}$ is analogous, whereas the situation with $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{out}}$ requires the complicated residual condition (PB) (see Theorem 9.8.5); therefore, for these two sets it is not easy to verify for a solution of the Riccati equation that it corresponds to optimal control, unless additional assumptions are made.

By the above, the closed-loop system corresponding to $\mathcal{U}_{\text{exp}}$ is exponentially stable. For $\mathcal{U}_{\rm str}$, $\mathcal{U}_{\rm sta}$ and $\mathcal{U}_{\rm out}$, we only know about the stability of the left column of the closed-loop system (see Theorem 9.8.5). If we wish to pose stability requirements also on the right column, we have to make additional detectability or q.r.c.-stabilizability assumptions. An LQR application with the former assumption is given by Theorem 10.1.4(c2), and an H^∞ application of the latter in Theorem 11.1.5.

This latter "q.r.c." approach is based on the fact that whereas optimization over $\mathcal{U}_{\rm exp}$ can always be reduced to the stable case, the analogous reduction for $\mathcal{U}_{\rm out}$, $\mathcal{U}_{\rm sta}$ and $\mathcal{U}_{\rm str}$ requires quasi-coprimeness; see Theorem 8.4.5(g1) and Remark 9.9.9. Although q.r.c.-stabilizability is trivially possessed by stable systems, it is often difficult to verify for unstable ones, although still popular in articles based on fractional representations of the I/O maps. See Theorem 9.9.1(b)–(c3) and Theorem 9.9.10 for more on this approach.

A third approach for $\mathcal{U}_{\rm out}$ is used in Theorem 10.1.4(b1), where we only have to study the minimal nonnegative solution.

Job Oostveen has developed a rather extensive optimization theory over $\mathcal{U}_{\rm str}$ for WPLSs with bounded input and output operators (B and C), and he avoids some of the problems described above by using suitable detectability assumptions (a most elegant example of his results is the one extended in Theorem 10.1.4(c2)). It seems that most of his results can be generalized to more general WPLSs in the same way; we recommend this for a reader interested in $\mathcal{U}_{\rm str}$.

We conclude that the theory on optimization and Riccati equations becomes most elegant for $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$, even more beautiful when one assumes estimatability (e.g., a cost on the state, by Lemma 6.6.25), so that $\mathcal{U}_{\rm exp} = \mathcal{U}_{\rm str} = \mathcal{U}_{\rm sta} = \mathcal{U}_{\rm out}$. However, often it is too restricting to require an exponentially stabilizing controller, and in several settings also the theory for $\mathcal{U}_{\rm out}$ (or $\mathcal{U}_{\rm sta}$ or $\mathcal{U}_{\rm str}$) can be substantially simplified.

Notes

The necessity part of Corollary 9.9.2 is contained in [S98b] and [WW] to some extent, in the generality explained in the notes on p. 520. In the same setting, the implication from (Crit4) to (Crit5) (for regular $\mathbb{D}$ and $\mathbb{F}$) and the formulae of (b), (e1), (g1) and (g2) of Theorem 9.9.10 are contained in [S98b, Sections 5–7].

The earlier history of infinite-dimensional Riccati equations is documented in the notes to Section 6 of [CZ]. For future research, a very important task is to determine further sufficient assumptions for a unique optimal control to exist in regular state feedback form, besides those presented in Remark 9.9.14 or elsewhere in this monograph.

One interesting candidate is the condition that $\mathbb{D} \in \mathrm{MTIC}_\infty$ (and it might imply that $\mathbb{F} \in \mathrm{MTIC}_\infty$). By Example 9.8.15, the approach of Section 9.2 does not work for this assumption, not even in the stable case, although that approach might be useful for some other candidate conditions.

9.10 Proofs for Section 9.9: Crit↔eIARE

When eating an elephant take one bite at a time.
— Gen. C. Abrams

In this section, we shall establish the equivalence between the existence of J-critical state feedback pairs and the existence of $\mathcal{U}_*^*$-stabilizing solutions of the eIARE, and also state some related results that are needed for further results. See Definition 9.8.4 (and Definition 6.6.10) for Σ_{ext} and $\Sigma_{\circlearrowleft}$.

First we explore in detail the connection between J-critical control and the admissible solutions of the eIARE (cf. (b4)):

Lemma 9.10.1 *Let $S \in \mathcal{B}(U)$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$. Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ, and let $\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] \in \mathrm{WPLS}(U,H,Y \times U)$ be the corresponding closed-loop system. Set $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_{\circlearrowleft}$.*

We consider, for $t \geq 0$, the equations

$$0 = \mathbb{D}^{t\,*} J \mathbb{C}^t_{\circlearrowleft} + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t_{\circlearrowleft}, \tag{9.153}$$

$$0 = \mathbb{D}^t_{\circlearrowleft}{}^* J \mathbb{C}^t_{\circlearrowleft} + \mathbb{B}^t_{\circlearrowleft}{}^* \mathcal{P} \mathbb{A}^t_{\circlearrowleft}, \tag{9.154}$$

$$\mathcal{P} = \mathbb{A}^t_{\circlearrowleft}{}^* \mathcal{P} \mathbb{A}^t_{\circlearrowleft} + \mathbb{C}^t_{\circlearrowleft}{}^* J \mathbb{C}^t_{\circlearrowleft}, \tag{9.155}$$

$$\mathcal{P} = \mathbb{A}^t_{\circlearrowleft}{}^* \mathcal{P} \mathbb{A}^t + \mathbb{C}^t_{\circlearrowleft}{}^* J \mathbb{C}^t, \tag{9.156}$$

$$\pi_{[0,t)} S = \mathbb{N}^{t\,*} J \mathbb{N}^t + \mathbb{B}^t_{\circlearrowleft}{}^* \mathcal{P} \mathbb{B}^t_{\circlearrowleft}, \tag{9.157}$$

$$S\mathbb{K}^t = -\left(\mathbb{N}^{t\,*} J \mathbb{C}^t + \mathbb{M}^{t\,*} \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t\right), \tag{9.158}$$

$$\mathbb{K}^{t\,*} S \mathbb{K}^t = \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*} J \mathbb{C}^t, \tag{9.159}$$

$$\mathbb{X}^{t\,*} S \mathbb{X}^t = \mathbb{D}^{t\,*} J \mathbb{D}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t, \tag{9.160}$$

$$\mathbb{X}^{t\,*} S \mathbb{K}^t = -\left(\mathbb{D}^{t\,*} J \mathbb{C}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t\right). \tag{9.161}$$

Claims (a1)–(b4) hold:

(a1) For any $t \geq 0$ we have (9.154)⇔(9.153), as well as (9.160)⇔(9.157), and (9.161)⇔(9.158).

(b1) Let $t \geq 0$ and let (9.158) hold. Then (9.156)⇔(9.159).

(b2) Let $t \geq 0$ and let (9.160) hold. Then (9.154)⇔(9.158).

(b3) Let $t \geq 0$ and let (9.154) hold. Then (9.155)⇔(9.156).

(b4) For each $t \geq 0$, conditions (i)–(iv) are equivalent, where

(i) Equations (9.159)–(9.161) (the eIARE) are satisfied;
(ii) Equations (9.153)–(9.161) are satisfied;
(iii) Equations (9.154), (9.155) and (9.157) hold.
(iv) Equations (9.158), (9.155) and (9.157) hold.

(b5) If (i), (ii) or (iii) holds for some $t > 0$, then (i)–(iii) hold for nt ($n \in \mathbf{N}$).

(b6) If (P4) of (d1) holds, (9.157) holds for each $t = t_n$ ($n \in \mathbf{N}$), and $\mathbb{N} \in \mathrm{TIC}$, then $\mathbb{N}^* J \mathbb{N} = S$.

If $\mathbb{C}_\circlearrowleft$ is stable, then (c1)–(d2) hold:

(c1) We have $\mathbb{N}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2)$ *for all* $t \geq 0$.

(c2) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$. *Then (9.160) is equivalent to*

$$\langle \mathbb{D}_\circlearrowleft u, J\mathbb{D}_\circlearrowleft v\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} = \langle u, Sv\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} \quad (u,v \in \mathrm{L}^2([0,t);U)). \tag{9.162}$$

Moreover, if either holds for all $t > 0$, *then (9.162) holds for all* $u,v \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) + \mathrm{L}^2_\mathrm{c}$ *and all* $\omega < 0$.

(c3) If $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, *then (9.154) is equivalent to*

$$\langle \mathbb{D}_\circlearrowleft \pi_+ u, J\mathbb{C}_\circlearrowleft x_0\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} = 0 \quad (u \in \mathrm{L}^2([0,t);U),\ x_0 \in H). \tag{9.163}$$

Moreover, if either holds for all $t > 0$, *then (9.163) holds for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) + \mathrm{L}^2_\mathrm{c}$ *and all* $\omega < 0$.

(c4) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, *and* $\langle \mathbb{D}_\circlearrowleft \pi_+ u, J\mathbb{C}_\circlearrowleft x_0\rangle = 0$ *for all* $u \in \mathrm{L}^2_\mathrm{c}$.

Then there is a unique $\widetilde{S} \in \mathcal{B}(U)$ *s.t.* $\langle \mathbb{N}u, J\mathbb{N}u\rangle = \langle u, \widetilde{S}u\rangle$ *(*$u \in \mathrm{L}^2_\mathrm{c}$*).*

Moreover, $\widetilde{S} = \widetilde{S}^* \in \mathcal{B}(U)$, *all of (P1)–(P4) hold, and (9.153)–(9.163) are satisfied for all* $t \geq 0$ *with* $\widetilde{S}$ *in place of* S.

(d1) We have

(P1) $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$

iff (9.155) holds for all $t \geq 0$ *and any (hence all) of (P2)–(P4) holds, where*

(P2) $\langle \mathbb{A}_\circlearrowleft^t x_0, \mathcal{P}\mathbb{A}_\circlearrowleft^t x_0\rangle \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$;
(P3) $\mathbb{A}_\circlearrowleft^{t\,*} \mathcal{P}\mathbb{A}_\circlearrowleft^t x_0 \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$;
(P4) There is a sequence $\{t_n\}$ *s.t.* $t_n \to +\infty$ *as* $n \to +\infty$ *and* $\langle \mathbb{A}_\circlearrowleft^{t_n} x_0, \mathcal{P}\mathbb{A}_\circlearrowleft^{t_n} x_0\rangle \to 0$ *for all* $x_0 \in H$.

All this holds even if we restrict t *to an unbounded set* $R \subset \mathbf{R}_+$. *If (P1) holds, then* $\mathcal{P}\mathbb{A}_\circlearrowleft^t x_0 \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$. *(Note that if* $\mathbb{A}_\circlearrowleft$ *is strongly stable, then (P2)–(P4) hold.)*

(d2) Let $\mathbb{C}$ *and* $\mathbb{K}$ *be stable. Then* $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ *iff (9.159) holds for all* $t \geq 0$ *and any (hence all) of (P2)–(P4) holds with* $\mathbb{A}$ *in place of* $\mathbb{A}_\circlearrowleft$. *Moreover,* $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ *implies that* $\mathcal{P}\mathbb{A}^t x_0 \to 0$ *for all* $x_0 \in H$.

If both $\mathbb{C}_\circlearrowleft$ *and* $\mathbb{N} = \mathbb{D}_\circlearrowleft$ *are stable, then (e1)–(f2) hold:*

(e1) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$. *Then (9.154) holds for all* $t \geq 0$ *iff* $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$. *In fact, it is sufficient that (9.154) holds for* $t = t_n$, $n \in \mathbf{N}$, *where* $t_n \to +\infty$ *as* $n \to +\infty$.

(e2) If $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$, *then* $\widetilde{S} := \mathbb{N}^* J\mathbb{N} = \widetilde{S}^* \in \mathcal{B}(U)$.

(f1) If $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$ *and* $\mathbb{N}^* J\mathbb{N} = S$, *then (9.160) holds for all* $t \geq 0$.

(f2) Conversely, assume that (9.160) holds for all $t \geq 0$. Then the following hold:

If $\mathbb{N}^* J \mathbb{N} = S$, then (P1)–(P4) hold on $H_{\mathbb{B}}$ (instead of H). If (P1), (P2), (P3) or (P4) holds on $H_{\mathbb{B}}$, then $\mathbb{N}^* J \mathbb{N} = S$.

For (f2), we recall from Lemma 6.3.26(b3), that the *reachability subspace* $H_{\mathbb{B}_\circlearrowleft}$ of $\Sigma_\circlearrowleft$ equals that of $H_{\mathbb{B}}$, i.e., the closure (in H) of

$$\{\mathbb{B}^t u \,\big|\, u \in \pi_{[0,t)} \mathrm{L}^2,\ t \geq 0\}. \tag{9.164}$$

Proof:

(a) Multiply by $\mathbb{M}^t$ or $\mathbb{X}^t$ to the left.

(b1) Insert (9.158) into (9.159) to obtain (9.156) (recall that $\mathbb{C}^t_\circlearrowleft = \mathbb{C}^t + \mathbb{N}^t \mathbb{K}^t$ and $\mathbb{A}^t_\circlearrowleft = \mathbb{A}^t + \mathbb{B}^t \mathbb{M}^t \mathbb{K}^t$).

(b2) Use equations $\mathbb{C}^t_\circlearrowleft = \mathbb{C}^t + \mathbb{D}^t_\circlearrowleft \mathbb{K}^t$, $\mathbb{A}^t_\circlearrowleft = \mathbb{A}^t + \mathbb{B}^t_\circlearrowleft \mathbb{K}^t$, and (9.160) to obtain that (cf. Lemma 5.5 of [S98b])

$$\mathbb{D}^{t}_{\circlearrowleft}{}^{*} J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^{t}_{\circlearrowleft}{}^{*} \mathcal{P} \mathbb{A}^t_\circlearrowleft = S \mathbb{K}^t + \mathbb{N}^{t\,*} J \mathbb{C}^t + \mathbb{M}^{t\,*} \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t. \tag{9.165}$$

(b3) By (6.132), the difference (9.155)$-$(9.156)* is equal to

$$\mathbb{K}^{t\,*} \left(\mathbb{D}^{t}_{\circlearrowleft}{}^{*} J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^{t}_{\circlearrowleft}{}^{*} \mathcal{P} \mathbb{A}^t_\circlearrowleft \right) = \mathbb{K}^* 0 = 0. \tag{9.166}$$

(b4) "(i)$\Rightarrow$(ii)"&"(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)": These follow easily from (a)–(b3); trivially (ii) implies (i), (iii) and (iv).

(b5) Apply Lemma 14.2.1 for $A := \mathbb{A}^t$, $B := \mathbb{B}^t$, ...; equivalently, use discretization (cf. Proposition 9.8.7) to see that if (i) holds for some t, then (i) holds for $t\mathbf{N}$. Combine this with (b4) to obtain (b5).

(b6) From (9.157) we obtain that $\langle \mathbb{N}u, J\mathbb{N}u \rangle = \langle u, Su \rangle$ for $u \in \pi_{[0,T)} \mathrm{L}^2$, because $\mathbb{B}^t_\circlearrowleft u = \mathbb{A}^{t-T}_\circlearrowleft \mathbb{B}^T_\circlearrowleft u$. Because T was arbitrary, $\mathbb{N}^* J \mathbb{N} = S$, by density.

(c1) This follows from Lemma 6.1.11 and the stability of $\mathbb{C}_\circlearrowleft$ (alternatively, from (9.167)).

(c2) (In discrete time, we have (9.162) for all $u, v \in \ell^1(\mathbf{Z}; U)$, by Lemma 13.3.8(b3); cf. Theorem 14.1.3.) Now

$$\pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} = \tau^{-t} \pi_+ \mathbb{N} \pi_- \tau^t \pi_{[0,t)} = \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2), \tag{9.167}$$

hence

$$\langle \pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} u, J \pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} v \rangle = \langle \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} u, J \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} v \rangle \tag{9.168}$$

$$= \langle \mathbb{B}^t_\circlearrowleft u, \mathcal{P} \mathbb{B}^t_\circlearrowleft v \rangle \quad (u, v \in \mathrm{L}^2). \tag{9.169}$$

Consequently, $\langle u, \pi_{[0,t)} S v \rangle = \langle \mathbb{N} \pi_{[0,t)} u, (\pi_{[0,t)} + \pi_{[t,\infty)}) J \mathbb{N} \pi_{[0,t)} v \rangle$ for $u \in \mathrm{L}^2$ iff (9.157) holds (equivalently, (9.160) holds, by (a1)).

The last claim follows from Lemma 2.1.13 (both sides are valid on $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U)$ and continuous functions on $\mathrm{L}^2_\omega(\mathbf{R}_+; U) \times \mathrm{L}^2_\omega(\mathbf{R}_+; U)$).

(c3) This follows from

$$\pi_{[0,t)}\tau(-t)\mathbb{B}_{\circlearrowleft}^*(\mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft})\mathbb{A}_{\circlearrowleft}(t) = \pi_{[0,t)}\tau(-t)\pi_-\mathbb{D}_{\circlearrowleft}^*\pi_+J\pi_+\tau(t)\mathbb{C}_{\circlearrowleft} \tag{9.170}$$

$$= \pi_{[0,t)}\mathbb{D}_{\circlearrowleft}^*J\tau(-t)\pi_+\tau(t)\mathbb{C}_{\circlearrowleft} = \pi_{[0,t)}\mathbb{D}_{\circlearrowleft}^*J\pi_{[t,\infty)}\mathbb{C}_{\circlearrowleft}. \tag{9.171}$$

The last claim follows from Lemma 2.1.13 as in (c2).

(c4) Now $\langle\mathbb{N}\pi_+v, J\mathbb{N}\pi_-u\rangle = \langle\mathbb{N}\pi_+v, J\mathbb{C}_{\circlearrowleft}\mathbb{B}_{\circlearrowleft}u\rangle = 0$ for all $u,v \in \mathrm{L}^2_{\mathrm{c}}$, hence there is a unique $\widetilde{S} = \widetilde{S}^* \in \mathcal{B}(U)$ s.t. $\langle\mathbb{N}u, J\mathbb{N}u\rangle = \langle u, Su\rangle$ ($u \in \mathrm{L}^2_{\mathrm{c}}$), by (c1) and Lemma 2.3.1.

By (9.167), we have $\mathbb{B}_{\circlearrowleft}^{t}{}^*\mathcal{P}\mathbb{B}_{\circlearrowleft}^t = (\pi_{[t,\infty)}\mathbb{N}\pi_{[0,t)})^*J\pi_{[t,\infty)}\mathbb{N}\pi_{[0,t)}$. It follows that (9.157) holds with $\widetilde{S}$ in place of S; this for all $t \geq 0$.

From (d1) it follows that (P1)–(P4) and (9.155) hold for all $t \geq 0$; by (c3), (9.154) holds for all $t \geq 0$.

By (a1), (b3), (b2) and (b1), (9.153)–(9.158) hold for all $t \geq 0$ with $\widetilde{S}$ in place of S.

(d1) 1° *Equivalence:* Assume (9.155). For any $x_0 \in H$, we have $\pi_{[0,t)}\mathbb{C}_{\circlearrowleft}x_0 \to \mathbb{C}_{\circlearrowleft}x_0$ in L^2, by Corollary B.3.8. Therefore, $\mathbb{C}_{\circlearrowleft}^*J\pi_{[0,t)}\mathbb{C}_{\circlearrowleft}x_0 \to \mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}x_0$.

Consequently, $\mathbb{A}_{\circlearrowleft}^{t}{}^*\mathcal{P}\mathbb{A}_{\circlearrowleft}^t \to \mathcal{P} - \mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}$ strongly (where $t \in R$ if the assumption holds on R only), hence (P1)–(P4) are equivalent.

2° *Claim* $\mathcal{P}\mathbb{A}_{\circlearrowleft}^t x_0 \to 0$*:* Assume (P1). Then (we use Definition 6.1.1(3.) and Corollary B.3.8)

$$\mathcal{P}\mathbb{A}_{\circlearrowleft}^t x_0 = \mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}\mathbb{A}_{\circlearrowleft}^t x_0 = \mathbb{C}_{\circlearrowleft}^*J\pi_+\tau^t\mathbb{C}_{\circlearrowleft}x_0 \to \mathbb{C}_{\circlearrowleft}^*0 = 0. \tag{9.172}$$

However, (9.172) is not sufficient, by Example 9.13.11, unless $\mathbb{A}_{\circlearrowleft}$ is stable.

(d2) The proof is analogous to that of (d1) ($\mathbb{C}_{\circlearrowleft}$ need not be stable here; also here t can be restricted to R).

(e1) This follows from (c3), because $\pi_{[0,t)}\mathbb{D}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft} = 0$ for all $t \geq 0$ iff $\pi_+\mathbb{D}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft} = 0$.

(e2) Now $(\pi_-\mathbb{N}^*J\mathbb{N}\pi_+)^* = \pi_+\mathbb{N}^*J\mathbb{N}\pi_- = \pi_+\mathbb{N}^*J\mathbb{C}_{\circlearrowleft}\mathbb{B}_{\circlearrowleft} = 0$, hence $S' := \mathbb{N}^*J\mathbb{N} \in \mathcal{B}(U)$, by Lemma 2.1.7, and $S' = S'^*$.

(f1) This follows from (c2) and Corollary B.3.8.

(f2) By the proof of (c2), we have $\pi_{[0,t)}\mathbb{N}^*J\mathbb{N}\pi_{[0,t)} = S$ iff $\mathbb{B}_{\circlearrowleft}^{t}{}^*\mathcal{P}\mathbb{B}_{\circlearrowleft}^t = \mathbb{B}_{\circlearrowleft}^{t}{}^*\mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}\mathbb{B}_{\circlearrowleft}^t$. But $\pi_{[0,t)}\mathbb{N}^*J\mathbb{N}\pi_{[0,t)} = S$ for all $t > 0$ iff $\mathbb{N}^*J\mathbb{N} = S$, by Corollary B.3.8, hence we must have $\mathbb{B}_{\circlearrowleft}^{t}{}^*\mathcal{P}\mathbb{B}_{\circlearrowleft}^t = \mathbb{B}_{\circlearrowleft}^{t}{}^*\mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}\mathbb{B}_{\circlearrowleft}^t$ for all $t > 0$, equivalently, $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^*J\mathbb{C}_{\circlearrowleft}$ on $H_{\mathbb{B}_{\circlearrowleft}} = H_{\mathbb{B}}$, by continuity.

But (P1) on $H_{\mathbb{B}}$ implies (P2)–(P4) on $H_{\mathbb{B}}$, as in the proof of (d1). Because each of (P2) and (P3) implies (P4) (on $H_{\mathbb{B}}$) it only remains to assume (P4) and prove that $\mathbb{N}^*J\mathbb{N} = S$.

Assume (P4). Let $T > 0$, and choose $u \in \pi_{[0,T)}\mathrm{L}^2$. Then $\tau(T)u = \pi_-\tau(T)u$, hence

$$\mathbb{B}_{\circlearrowleft}\tau(t)\pi_+u = \mathbb{B}_{\circlearrowleft}\tau(t-T)\pi_-\tau(T)u = \mathbb{A}_{\circlearrowleft}(t-T)x_u, \tag{9.173}$$

where $x_u := \mathbb{B}_{\circlearrowleft}\pi_-\tau(T)u$. Thus, $\langle\mathbb{B}_{\circlearrowleft}\tau(t_n)\pi_+u, \mathcal{P}\mathbb{B}_{\circlearrowleft}\tau(t_n)\pi_+u\rangle \to 0$, as $n \to \infty$, by (P4). By (9.157), it follows that (for $t_n > T$)

$$\langle u, \pi_{[0,t_n)}(S - \mathbb{N}^*J\pi_{[0,t_n)}\mathbb{N})\pi_{[0,t_n)}u\rangle = \langle u, (S - \mathbb{N}^*J\pi_{[0,t_n)}\mathbb{N})u\rangle \to 0, \tag{9.174}$$

as $n\to\infty$. But $(S-\mathbb{N}^*J\pi_{[0,t_n)}\mathbb{N})u\to(S-\mathbb{N}^*J\mathbb{N})u$, hence $\langle u,(S-\mathbb{N}^*J\mathbb{N})u\rangle=0$. Because $T>0$ and $u\in\pi_{[0,T)}\mathrm{L}^2$ were arbitrary, we have $S=\mathbb{N}^*J\mathbb{N}$. □

Now we are ready to establish the equivalence between J-critical control and the eIARE, in (c)–(e2) below:

Proposition 9.10.2 *Let Σ, $\Sigma_\circlearrowleft$, $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$, J and $\mathcal{P}$ be as in Lemma 9.10.1, and let $\mathbb{C}_\circlearrowleft$ be stable. Then the following hold:*

(a1) Conditions (i)–(iv) are equivalent, where

(i) $\langle\mathbb{D}_\circlearrowleft\pi_+u,J\mathbb{C}_\circlearrowleft x_0\rangle=0$ for all $u\in\mathrm{L}^2_\mathrm{c}$ and $x_0\in H$, and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$;

(i') $\mathbb{D}^{t\,*}J\mathbb{C}^t_\circlearrowleft+\mathbb{B}^{t\,*}\mathcal{P}\mathbb{A}^t_\circlearrowleft=0$ for all $t>0$ and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$;

(ii) Equations (9.159)–(9.161) (the eIARE) are satisfied for some $S\in\mathcal{B}(U)$ and some $T:=t>0$, and some of (P1)–(P4) holds for $t\in R:=T\mathbf{N}$;

(iii) Equations (9.153)–(9.163) are satisfied for a unique $S=S^*\in\mathcal{B}(U)$ and all $t\geq 0$, and all of (P1)–(P4) hold (for $t\in R:=\mathbf{R}_+$);

(iv) We have $\langle y,Jy\rangle=\langle x_0,\mathcal{P}x_0\rangle+\langle u_\circlearrowleft,Su_\circlearrowleft\rangle$ for all $x_0\in H$, $u_\circlearrowleft\in\mathrm{L}^2_\mathrm{c}$, where $y:=\mathbb{C}x_0+\mathbb{D}u\in\mathrm{L}^2$, $u:=\mathbb{K}_\circlearrowleft x_0+\mathbb{M}u_\circlearrowleft$.

(Note that $y=\mathbb{C}_\circlearrowleft x_0+\mathbb{D}_\circlearrowleft u_\circlearrowleft\in\mathrm{L}^2$ in (iv), by Lemma 6.1.11.)

(a2) Let $\Sigma_\mathrm{ext},\Sigma_\circlearrowleft\in\mathrm{SOS}$. Then we can replace (P1)–(P4) in (ii) by

(P') $\langle\mathbb{A}^tx_0,\mathcal{P}\mathbb{A}^tx_0\rangle\to 0$, as $t\to\infty$, for all $x_0\in H$.

(b1) Assume that $\mathbb{K}_\circlearrowleft x_0\in\mathcal{U}^*_*(x_0)$ for all $x_0\in H$. Then $\mathbb{M}\mathrm{L}^2_\mathrm{c}(\mathbf{R}_+;U)\subset\mathcal{U}^*_*(0)$.

(b2) Let $\mathbb{D}_\circlearrowleft$ be stable. Then (i) holds iff $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$ and $\pi_+\mathbb{D}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft=0$.

(b3) Assume (i). Let $u\in\mathrm{L}^2_\infty(\mathbf{R}_+;U)$ be s.t. $\mathbb{D}u\in\mathrm{L}^2$. Then $\langle\mathbb{B}\tau^tu,\mathcal{P}\mathbb{A}^t_\circlearrowleft x_0\rangle\to-\langle\mathbb{D}u,J\mathbb{C}_\circlearrowleft x_0\rangle$, as $t\to+\infty$, for each $x_0\in H$.

If, in addition, $\langle\mathbb{D}u,J\mathbb{C}_\circlearrowleft x_0\rangle=0$ for each $x_0\in H$, then (9.175) holds.

(b4) If $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is J-critical, then, for all $u\in\mathcal{U}^*_*(0)$, we have

$$\langle\mathbb{D}u,J\mathbb{D}\mathbb{M}\eta\rangle=\langle\mathbb{M}^{-1}u,S\eta\rangle\quad(\eta\in\mathrm{L}^2_\mathrm{c}(\mathbf{R};U)).\tag{9.175}$$

*(c) **($\mathcal{U}^*_*$)** The pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is J-critical over $\mathcal{U}^*_*$ and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$ iff (i) holds, $\mathbb{K}_\circlearrowleft x_0\in\mathcal{U}^*_*(x_0)$ for all $x_0\in H$, and*

$$\langle\mathbb{B}^tu,\mathcal{P}\mathbb{A}^t_\circlearrowleft x_0\rangle\to 0,\ \text{ as }\ t\to+\infty,\quad(x_0\in H,\ u\in\mathcal{U}^*_*(0)).\tag{9.176}$$

*(d) **($\mathcal{U}_\mathrm{exp}$)** Let $\Sigma_\circlearrowleft$ be exponentially stable. Then $\mathcal{U}_\mathrm{exp}(0)=\mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Moreover, (i) holds iff $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is J-critical over $\mathcal{U}_\mathrm{exp}$ and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$.*

*(e1) **($\mathcal{U}_\mathrm{out}$)** The pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is J-critical over $\mathcal{U}_\mathrm{out}$ and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$ iff $\mathbb{K}_\circlearrowleft$ is stable and (i) and (9.176) hold.*

(e2) Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ be q.r.c.-SOS-stabilizing. Then $\mathcal{U}_\mathrm{out}(0)=\mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Moreover, $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is J-critical over $\mathcal{U}_\mathrm{out}$ and $\mathcal{P}=\mathbb{C}_\circlearrowleft^*J\mathbb{C}_\circlearrowleft$ iff (i) holds.

(f1) **($\mathcal{U}_{\rm str}$, $\mathcal{U}_{\rm sta}$)** *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm sta}$ (resp. $\mathcal{U}_*^* = \mathcal{U}_{\rm str}$). Then $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ iff $\begin{bmatrix} \mathbb{A}_\circlearrowleft^{\rm T} \mid \mathbb{C}_\circlearrowleft^{\rm T} \quad \mathbb{K}_\circlearrowleft^{\rm T} \end{bmatrix}^{\rm T}$ is (resp. strongly) stable and (i) and (9.176) hold.*

(f2) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm sta}$ [or $\mathcal{U}_*^* = \mathcal{U}_{\rm str}$] and that $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is [strongly] q.r.c.-stabilizing. Then $\mathcal{U}_*^*(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Moreover, $\mathbb{K}_\circlearrowleft$ is J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ iff (i) holds.

(g) Let $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ be J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$. Then $\mathbb{K}_\circlearrowleft x_0 + \mathbb{M} u_\circlearrowleft$ is J-critical over $\mathcal{U}_*^*$ for x_0 whenever $x_0 \in H$, $u_\circlearrowleft \in \mathrm{L}^2_{\rm c}$ and $S u_\circlearrowleft \equiv 0$.

(h) We may restrict t to $R := T\mathbf{N}$ or any other unbounded $R \subset \mathbf{N}$ in (9.176), (P') and (P1)–(P4).

By (e2), a J-critical control over $\mathcal{U}_{\rm out}$ can be given in the form of q.r.c.-SOS-stabilizing state feedback iff the eIARE has a q.r.c.-SOS-stabilizing solution. In either case, we obtain $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ from (i). This is written out in Theorem 9.9.10.

By (a1), the solutions of (i) correspond one-to-one to the P-$\mathbb{C}$-stabilizing solutions of the eIARE. (For other $\mathbb{C}$-stabilizing solutions of the eIARE, the corresponding cost $\mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ is finite but $\mathcal{P}$ contains some "phantom cost", hence such solutions are not interesting; cf. Example 9.13.9.)

Proof: (a1) (Here "(b5)", "(c3)", "(c4)", "(d1)" and "(P1)" refer to Lemma 9.10.1.)

"(iii)⇒(ii)": This is trivial. "(i)⇒(i')": This follows from (c3) and (a1). "(i)⇒(iii)": This follows from (c4) with $S := \widetilde{S}$.

"(ii)⇒(i)": By (b5), the equations hold for nt ($n \in \mathbf{N}$); by (d1), equation $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ holds; by (c3) (applied to nt, $n \in \mathbf{N}$), (i) holds.

"(iii)⇒(iv)": This follows from (P1), (9.163) and (9.162).

"(iv)⇒(i)": This is obvious (cf. the proof of Lemma 8.3.7).

(a2) 1° Assume (iii). Then $\mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}$, by a direct computation using equations $\mathbb{C}_\circlearrowleft = \mathbb{C} + \mathbb{N}\mathbb{K}$, $\mathbb{N}^* J \mathbb{N} = S$ and $\pi_+ \mathbb{N}^* J \mathbb{C}_\circlearrowleft = 0$, hence (P') holds, by Lemma 9.9.1(d2).

2° Assume (ii) with (P') in place of (P1)–(P4). By Lemma 9.9.1(d2), we have $\mathcal{P} = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}$ and $\mathcal{P}\mathbb{A}^{nT} x_0 \to 0$ for each $x_0 \in H$. By the former,

$$\mathcal{P}\mathbb{B}\tau\mathbb{K}_\circlearrowleft = \mathbb{C}^* \tau J \tau^* \mathbb{C}\mathbb{B}\tau^t \mathbb{K}_\circlearrowleft + \mathbb{K}^* \tau S \tau^* \mathbb{K}\mathbb{B}\tau^t \mathbb{K}_\circlearrowleft. \tag{9.177}$$

Fix $x_0 \in H$. Now $\tau^{-nT} \mathbb{C}\mathbb{B}\tau^{nT} \mathbb{K}_\circlearrowleft x_0 = \pi_{[nT,+\infty)} \mathbb{D} \pi_{(-\infty,nT)} \mathbb{K}_\circlearrowleft x_0 \to 0$, because $\pi_{[nT,+\infty)} \mathbb{D}\mathbb{K}_\circlearrowleft x_0 \to 0$ and $\pi_{[nT,+\infty)} \mathbb{K}_\circlearrowleft x_0 \to 0$, as $n \to \infty$. Analogously, $\tau^{-nT} \mathbb{K}\mathbb{B}\tau^{nT} \mathbb{K}_\circlearrowleft x_0 \to 0$, hence

$$\mathcal{P}\mathbb{A}_\circlearrowleft^{nT} x_0 = \mathcal{P}\mathbb{A}^{nT} x_0 + \mathcal{P}\mathbb{B}\tau^{nT} \mathbb{K}_\circlearrowleft x_0 \to 0. \tag{9.178}$$

Therefore, (ii) holds.

(b1) Let $t > 0$ and $\eta_\circlearrowleft \in \mathrm{L}^2([0,t);U)$. Set $\widetilde{\eta} := \mathbb{M}^t \eta_\circlearrowleft$ and define $\eta \in \mathcal{U}_*^*(0)$ as in Lemma 9.7.10. Then

$$\eta = \mathbb{M}^t \eta_\circlearrowleft + \tau^{-t} \mathbb{K}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \eta_\circlearrowleft = \tau^{-t}(\pi_- \mathbb{M} \pi_- + \pi_+ \mathbb{M} \pi_-) \tau^t \eta_\circlearrowleft = \mathbb{M}\eta_\circlearrowleft. \tag{9.179}$$

(b2) This follows from the fact that now $\mathbb{D}_\circlearrowleft \pi_{[0,t)} u \to \mathbb{D}_\circlearrowleft \pi_+ u$, as $t \to +\infty$, by Corollary B.3.8.

(b3) Choose $\omega \in \mathbf{R}$ s.t. $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. Since (b4) contains no reference to $\mathcal{U}^*_*$, we can (and will) assume that $\mathcal{U}^*_* := \mathcal{U}^\omega_{[0\ 0]}$, so that $u \in \mathcal{U}^*_*(0)$.

1° The convergence claim follows from (iii) and (9.62).

2° For the latter claim, choose $t > 0$ s.t. $\eta = \pi_{(-\infty,t)}\eta$, and set $u_\circlearrowleft := \mathbb{M}^{-1}u$, so that $u_1 := \mathbb{M}\pi_{[0,t)}u_\circlearrowleft \in \mathbb{M}\pi_+\mathrm{L}^2_\mathrm{c}$, hence $\mathbb{D}u_1 \in \mathrm{L}^2$, i.e., $u_1 \in \mathcal{U}^*_*(0)$. Consequently, $u_2 := \mathbb{M}\pi_{[t,\infty)}u_\circlearrowleft = u - \mathbb{M}\pi_{[0,t)}u_\circlearrowleft \in \mathcal{U}^*_*(0)$, hence $\tau^t u_2 \in \mathcal{U}^*_*(0)$ too (because $\pi_{[0,t)}u_2 = 0$). Consequently,

$$\begin{aligned}\langle \mathbb{D}u, J\mathbb{D}\mathbb{M}\eta\rangle &= \langle \mathbb{D}\mathbb{M}\pi_{[0,t)}u_\circlearrowleft, J\mathbb{N}\eta\rangle + \langle \pi_{[t,\infty)}\mathbb{D}\mathbb{M}\pi_{[t,\infty)}u_\circlearrowleft, J\mathbb{N}\eta\rangle \\ &= \langle \pi_{[0,t)}u_\circlearrowleft, S\eta\rangle + 0 = \langle \mathbb{M}^{-1}u, S\eta\rangle,\end{aligned} \tag{9.180}$$

(note that (9.162) holds for all $u,v \in \mathrm{L}^2_\mathrm{c}$, by time-invariance) because $\langle \pi_{[0,t)}u_\circlearrowleft, S\eta\rangle = \langle u_\circlearrowleft, S\eta\rangle$ and

$$\pi_{[t,\infty)}J\mathbb{N}\eta = \tau^{-t}J\pi_+\mathbb{N}\tau^t\eta = \tau^{-t}J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft\tau^t\eta, \tag{9.181}$$

hence $\langle \mathbb{D}u_2, \pi_{[t,\infty)}J\mathbb{N}\eta\rangle = \langle \mathbb{D}\tau^t u_2, J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft\tau^t\eta\rangle = 0$, by the assumption.

(b4) This follows from (b3). (Note that in (9.175), the (inner product) integral can be taken over over a finite interval only, hence we have allowed $\mathbb{M}^{-1}u \notin \mathrm{L}^2$.)

(c) Since (i) is equivalent to (i'), we obtain from (b3) that (9.176) holds iff $\langle \mathbb{D}u, J\mathbb{C}_\circlearrowleft x_0\rangle = 0$ for all $x_0 \in H$ and all $u \in \mathcal{U}^*_*(0)$. Therefore, (c) holds.

(d) By (8.74), we have $\mathcal{U}_{\mathrm{exp}}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Therefore, the equivalence follows from (b2).

(e1)&(f1) These follows from (c).

(e2) Let (ii) hold. Then $\mathcal{U}_{\mathrm{out}}(0) = \mathbb{M}\pi_+\mathrm{L}^2$, because $u, \mathbb{D}u \in \mathrm{L}^2 \Leftrightarrow \mathbb{M}^{-1}u \in \mathrm{L}^2$, by Lemma 6.5.6(a1)&(f). Therefore, the equivalence follows from (b2).

(f2) The proof is analogous to that of (e2).

(g) Let $x_0 \in H$, $u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}$ and $Su_\circlearrowleft \equiv 0$. Set $\widetilde{u} := \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft$. Then $\langle \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}\widetilde{u})\rangle = 0 + \langle \mathbb{D}u, J\mathbb{D}\mathbb{M}u_\circlearrowleft\rangle = 0$, by (9.175), for all $u \in \mathcal{U}^*_*(0)$. Therefore, $\widetilde{u}$ is J-critical over $\mathcal{U}^*_*$ for x_0.

(h) One observes this from above proofs. □

We have already shown that J-coercivity implies the existence of a unique J-critical control (if the system is stabilizable); here we show that it also implies that the signature operator is invertible:

Lemma 9.10.3 (J-coercive $\Rightarrow S \in \mathcal{GB}(U)$) *Assume that $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}^*_*$. If $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is a $\mathcal{U}^*_*$-stabilizing solution of the eIARE, then $S \in \mathcal{GB}(U)$ [$S \gg 0$].*

See also Lemma 9.9.7(c4)&(c5).

Proof: Choose $\beta \geq \max\{0, \vartheta\}$ s.t. $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\beta$. Set $\varepsilon' := \|\mathbb{M}^{-1}\|^{-1}_{\mathrm{TIC}_\beta}\varepsilon$, $\varepsilon'' := \varepsilon' \int_0^1 e^{-2\beta s}\,ds$. Let ε be as in Definition 8.4.1. Choose $M := \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|_{\mathcal{B}(\mathrm{L}^2)}$.

Let $u_0 \in U$ be given. Set $u_\circlearrowleft := \chi_{[0,1)}u_0$, so that $u := \mathbb{M}u_\circlearrowleft \in \mathcal{U}^*_*(0)$, by Proposition 9.10.2(c)&(b1). Choose $v \in \mathcal{U}^*_*(0)$ s.t. $\|v\|_{\mathcal{U}^*_*} \leq 1$ and $\langle \mathbb{D}v, J\mathbb{D}u\rangle \geq$

$\varepsilon\|u\|_{\mathcal{U}_*^*}$. It follows that $\|v\|_2 \le 1$. The function $\|u\|^2_{L^2_\omega} = \int_{\mathbf{R}} e^{-2\omega t}\|u(t)\|^2_U\,dt$ is decreasing in ω; therefore, by (9.175), we have

$$M\|Su_0\|_U \ge \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|\|v\|_2\|Su_0\|_U \ge |\langle \mathbb{M}^{-1}v, Su_\circlearrowleft\rangle| \tag{9.182}$$

$$= |\langle \mathbb{D}v, J\mathbb{D}u\rangle| \ge \varepsilon\|u\|_{\mathcal{U}_*^*} \ge \varepsilon\|u\|_{L^2_\beta} \ge \varepsilon'\|u_\circlearrowleft\|_{L^2_\beta} = \varepsilon''\|u_0\|_U. \tag{9.183}$$

Because $u_0 \in U$ was arbitrary, S is coercive, hence $S \in \mathcal{GB}(U)$, by Lemma A.3.1(c4) [and necessarily $S \ge 0$, hence $S \gg 0$, Lemma A.3.1(b1)]. □

(See the notes on p. 520.)

9.11 Proofs for Section 9.8: eCARE↔eIARE

If I had only known, I would have been a locksmith.
— Albert Einstein (1879–1955)

Having established the connection between optimal control and a $\mathcal{U}_*^*$-stabilizing solution of the eIARE in the previous section, we now go on to show the eIARE equivalent to the eCARE in the regular case. We start with a technical result:

Lemma 9.11.1 *Let* $(\mathcal{P},S,\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *be a solution of the eIARE s.t.* $\Sigma_{\text{ext}} \in \text{WPLS}$.

Let x and y be the state and output of Σ corresponding to initial state $x_0 \in H$ and input $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. Fix $t \geq 0$, and let x^{d} and u^{d} be the state and output of $\Sigma^{\mathrm{d}}_{\text{ext}}$ corresponding to initial state $x_0^{\mathrm{d}} := \mathcal{P}x(t)$, and inputs $y^{\mathrm{d}}(s) = Jy(t-s)$, $z^{\mathrm{d}}(s) = S(\mathbb{X}u - \mathbb{K}x_0))(t-s)$ ($s \in [0,t]$). Then, for $s \in [0,t]$, we have $x^{\mathrm{d}}(t-s) = \mathcal{P}x(s)$ and

$$\pi_{[0,t)}u^{\mathrm{d}}(t-\cdot) = -S\pi_{[0,t)}\left(\mathbb{K}x_0 - \mathbb{X}\pi_{[0,t)}u\right). \tag{9.184}$$

Proof: (a) Now $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, $y := \mathbb{C}x_0 + \mathbb{D}u$, $x^{\mathrm{d}}(t-s) := \mathbb{A}(t-s)^* x_0^{\mathrm{d}} + \mathbb{C}^* \boldsymbol{Я}\tau(t-s)y^{\mathrm{d}} + \mathbb{K}^* \boldsymbol{Я}\tau(t-s)z^{\mathrm{d}}$, and $u^{\mathrm{d}}(t-\cdot) = \mathbb{B}^* x_0^{\mathrm{d}} + \boldsymbol{Я}\mathbb{D}^* \boldsymbol{Я} y^{\mathrm{d}} + \boldsymbol{Я}\mathbb{F}^* \boldsymbol{Я} z^{\mathrm{d}}$ (see Lemma 6.1.4 and Definition 6.1.5).

Note that $\mathbb{C}^* \boldsymbol{Я} = \mathbb{C}^* \boldsymbol{Я}\pi_-$, $y^{\mathrm{d}}(t-s) = J\pi_+\boldsymbol{Я}\tau(t)y$ and

$$\boldsymbol{Я}\pi_-\tau(t-s)J\pi_+\boldsymbol{Я}\tau(t)\pi_+ = \pi_+\tau(-(t-s))\pi_-\tau(t)\pi_+J = \pi_{[0,t-s)}\tau(s)J. \tag{9.185}$$

Therefore, (recall from Definition 6.1.1, that $\pi_+\tau(s)\mathbb{C} = \mathbb{C}\mathbb{A}(s)$, $\mathbb{B}\tau(t-s)\pi_- = \mathbb{A}(t-s)\mathbb{B}$, and $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$)

$$\begin{aligned}
x^{\mathrm{d}}(t-s) &= \mathbb{A}(t-s)^*\mathcal{P}\left(\mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u\right) \\
&\quad + \mathbb{C}^*\pi_{[0,t-s)}\tau(s)J\left(\mathbb{C}x_0 + \mathbb{D}u\right) + \mathbb{K}^*\pi_{[0,t-s)}\tau(s)S\left(-\mathbb{K}x_0 + \mathbb{X}u\right) \\
&= \mathbb{A}(t-s)^*\mathcal{P}\mathbb{A}(t-s)\mathbb{A}(s)x_0 + \mathbb{C}^*\pi_{[0,t-s)}J\mathbb{C}\mathbb{A}(s)x_0 - \mathbb{K}^*\pi_{[0,t-s)}S\mathbb{K}\mathbb{A}(s)x_0 \\
&\quad + \left(\mathbb{A}(t-s)^*\mathcal{P}\mathbb{B}\tau(t-s) + \mathbb{C}^*\pi_{[0,t-s)}J\mathbb{D} + \mathbb{K}^*\pi_{[0,t-s)}S\mathbb{X}\right)(\pi_{[0,t-s)} + \pi_-)\tau(s)u \\
&= \mathcal{P}\mathbb{A}(s)x_0 + 0 + \left(\mathbb{A}(t-s)^*\mathcal{P}\mathbb{B}\tau(t-s) + \mathbb{C}^*\pi_{[0,t-s)}J\mathbb{D} + \mathbb{K}^*\pi_{[0,t-s)}S\mathbb{X}\right)\pi_-\tau(s)u \\
&= \mathcal{P}\mathbb{A}(s)x_0 + \mathcal{P}\mathbb{B}\tau(s)u = \mathcal{P}x(s),
\end{aligned} \tag{9.186}$$

where the last three identities follow from (9.161) (in fact, (9.161)*) and (9.159) (with $\mathbb{B}\tau(t-s)\pi_- = \mathbb{A}(t-s)\mathbb{B}$, $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$ and $\pi_+\mathbb{X}\pi_- = -\mathbb{K}\mathbb{B}$). Similarly (we omit the details),

$$\begin{aligned}
\pi + \boldsymbol{Я}\tau(t)u^{\mathrm{d}} &= \pi_+\tau(-t)\mathbb{B}^*\mathcal{P}x(t) + \pi_+\tau(-t)\mathbb{D}^*\boldsymbol{Я}J\pi_+\boldsymbol{Я}\tau(t)y \\
&\quad + \pi_+\tau(-t)\mathbb{F}^*\boldsymbol{Я}S\pi_+\boldsymbol{Я}\tau(t)(\mathbb{X}u - \mathbb{K}x_0)) \\
&= \pi_{[0,t)}S\mathbb{X}u - S\pi_{[0,t)}\mathbb{K}x_0,
\end{aligned} \tag{9.187}$$

by (9.161) and (9.160), as above. □

In Lemma 9.11.2 and Proposition 9.11.4 we establish the implication eIARE⇒eCARE.

Since the Lyapunov equation does not contain feedthrough operators, it is very handy to move from its differential (instantaneous) form to the integrated one and conversely:

Lemma 9.11.2 (Lyapunov equation) *Let* $S \in \mathcal{B}(U)$ *and* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y\times U)$.

Then $\mathcal{P} \in \mathcal{B}(H)$ *satisfies the* Lyapunov equation $A^*\mathcal{P} + \mathcal{P}A + C^*JC = K^*SK$ *(*$\in \mathcal{B}(H_1, H_{-1}^*)$*) iff (9.159) holds for all* $t \geq 0$. □

(This follows from Lemma 9.7.8 with $\widetilde{J} \mapsto \left[\begin{smallmatrix} -J & 0 \\ 0 & S \end{smallmatrix}\right]$ and $P \mapsto \mathcal{P}$.)

Note that this equation does not require any regularity assumptions unlike the second and third equations of the CARE (which contain explicit feedthrough operators), treated in Proposition 9.11.4.

Now we have obtained the first equation (the Lyapunov equation) of the eCARE. The rest is not as simple. For S, we need different formulae in different occasions. Several such formulae, including the middle equation of the eCARE, can be derived from (9.190) or (9.188), that will be established below.

Proposition 9.11.3 ($X^*SX = D^*JD + \ldots$) *Let the eIARE have a solution* $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$. *Set* $\omega = \max\{0, \omega_A\}$. *Then, for all* $s, z \in \mathbf{C}^+_\omega$ *and* $u_0, v_0 \in U$, *we have*

$$\langle \widehat{\mathbb{X}}(s)u_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U = \langle \widehat{\mathbb{D}}(s)u_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y + (s+\bar{z})\langle (s-A)^{-1}Bu_0, \mathcal{P}(z-A)^{-1}Bv_0\rangle_Y. \tag{9.188}$$

In particular,

(a) If $\mathbb{D}, \mathbb{X} \in \mathrm{WR}$*, then* $X^*SX = \text{w-lim}_{s\to+\infty} D^*JD + B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$.

(b) If $\mathbb{D}, \mathbb{X} \in \mathrm{SR}$*, then* $X^*SX = D^*JD + \text{s-lim}_{s\to+\infty} 2sB^*(s-A^*)^{-1}\mathcal{P}(s-A)^{-1}B$.

(c) If $\mathbb{D}, \mathbb{X}, \mathbb{B}\tau \in \mathrm{SVR}$*, then* $D^*JD = X^*SX$.

Note that the conclusions include the convergence of the limits (including the fact that $\mathcal{P}(s-A)^{-1}Bu_0 \in \mathrm{Dom}(B_{\mathrm{w}}^*)$ for all $u_0 \in U$ in (a), etc.). From (b) we observe that if $\mathbb{D}$ and $\mathbb{X}$ are SR and $\|(s-A)^{-1}B\| \leq s^{-r}$ for some $r > 1/2$ and all real s big enough (this is true for $r = 1/2$, by Theorem 6.2.11(b3)), then $X^*SX = D^*JD$. Thus, the w-lim term can be nonzero only when B is "maximally unbounded". See Remark 9.9.14(b) for some further sufficient assumptions for $X^*SX = D^*JD$.

Proof: (In fact, it suffices that $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ for all $t > 0$, $\Sigma \in \mathrm{WPLS}(U,H,Y)$, $\mathbb{X} \in \mathrm{TIC}_\infty$, $S, J, \mathcal{P} \in \mathcal{B}$, and that there is $\alpha \geq 0$ s.t. $s, z \in \mathbf{C}^+_\alpha$ and $\mathbb{B}\tau, \mathbb{D}, \mathbb{X} \in \mathrm{TIC}_\alpha$.)

Let $u := e^{st}u_0$, $v := e^{zt}v_0$, so that $\pi_- u, \pi_- v \in \mathrm{L}^2_\alpha$ for some $\alpha > \omega$, and work as in the proof of Lemma 2.2.4 (note that $\langle \mathbb{B}^t\tau^t u, \mathcal{P}\mathbb{B}^t\tau^t u\rangle_H \to \langle (s-A)^{-1}Bu_0, \mathcal{P}(s-A)^{-1}Bu_0\rangle_H$, by Lemma 6.2.10). Divide the result by $\int_{-\infty}^0 e^{ts+t\bar{z}}dt = (s+\bar{z})^{-1}$ to obtain (9.188).

In the sequel we shall use the facts that $\widehat{\mathbb{B}\tau u_0}(s) = (s-A)^{-1}Bu_0$, by Theorem 6.2.11(b1), and $\mathbb{B}\tau$ is ULR, by Lemma 6.3.16(c).

(a) Let $s \to +\infty$ in (9.188) (recall that "$s \to +\infty$" means "$s \in \mathbf{R}$ and $s \to +\infty$") to obtain

$$\langle Xu_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U = \langle Du_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y + \langle u_0, B_{\mathrm{w}}^*\mathcal{P}(z-A)^{-1}Bv_0\rangle_Y + \overline{z}\cdot 0 \tag{9.189}$$

(indeed, the limit $\langle Xu_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U - \langle Du_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y$ exists and u_0 is arbitrary, we have $\mathcal{P}(z-A)^{-1}Bv_0 \in \mathrm{Dom}(B_{\mathrm{w}}^*)$). Then let $z \to +\infty$ to obtain (a).

(b) Substitute $z \mapsto s$ into (9.188), and let $s \to +\infty$ (use Lemma A.3.1(i2)).

(c) Just substitute $z, s \mapsto \beta + iy$ into (9.188), and let $y \to +\infty$. □

When $\mathbb{D}$ and $\mathbb{F}$ are WR, we can derive also the second (at least) and third equations of the eCARE:

Proposition 9.11.4 (WR eIARE⇒eCARE) *Let $\mathbb{D}$ be WR, and let the eIARE have a WR solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$. Then we have the following:*

(a) If $x_0 \in H$, $u_0 \in U$ and $Ax_0 + Bu_0 \in H$, then

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_{\mathrm{w}} + X^*SK_{\mathrm{w}})x_0 = (X^*SX - D^*JD)u_0. \tag{9.190}$$

(b1) For any $u_0 \in U$ we have

$$X^*SXu_0 = D^*JDu_0 + \operatorname*{w-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}Bu_0. \tag{9.191}$$

(b2) If $\mathbb{N} \in \mathrm{TIC} \cap \mathrm{SHPR}$ and $\mathbb{X} \in \mathrm{SR}$, then $D^*JD = X^*SX$.

(b3) If $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$, then $X^*SX = D^*JD$.

(c1) If $\mathbb{D}, \mathbb{F} \in \mathrm{SR}$, then $\mathcal{P} \geq 0 \Rightarrow X^*SX \geq D^*JD$.

(c2) If $\mathbb{F} \in \mathrm{SR}$, then $\mathcal{P}, J \geq 0 \Rightarrow X^*SX \geq D^*JD$.

(d) $X^*SKx_0 = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_0$ for all $x_0 \in \mathrm{Dom}(A)$.

If $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is admissible and $I - F$ is left-invertible (this is the case if $\mathbb{F}$ is SR, by Proposition 6.3.1(a1)), then we can apply (d1) (and (d2) if $F = 0$) of Proposition 6.6.18.

Proof: (a) In the proof of Lemma 9.8.9, we have, by Lemma 9.11.1, that

$$u^{\mathrm{d}} = \mathbb{B}^{\mathrm{d}}x_0^{\mathrm{d}} + \mathbb{D}^{\mathrm{d}}y^{\mathrm{d}} + \mathbb{F}^{\mathrm{d}}z^{\mathrm{d}} = B_{\mathrm{w}}^*x^{\mathrm{d}} + D^*y^{\mathrm{d}} + F^*z^{\mathrm{d}}, \tag{9.192}$$

hence, by (9.184), Theorem 6.2.13(a2), Lemma 6.2.9(b),

$$S\pi_{[0,t)}z = u^{\mathrm{d}}(t-\cdot) = B_{\mathrm{w}}^*\mathcal{P}x + D^*Jy + F^*Sz, \quad \text{equivalently}, \tag{9.193}$$

$$X^*Sz = B_{\mathrm{w}}^*\mathcal{P}x + D^*Jy \quad (\text{on } [0,t)). \tag{9.194}$$

But z is the output of $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$, hence $z = -K_{\mathrm{w}}x + Xu$. Thus, by setting the argument equal to zero, (9.194) becomes (9.190) when we note that $y(0) = C_{\mathrm{w}}x_0 + Du_0$.

(b1) Take $x_0 := (\alpha - A)^{-1}Bu_0 \in H_B$ and apply (a) (note that, $\begin{bmatrix} C_{\mathrm{w}} \\ K_{\mathrm{w}} \end{bmatrix}(\alpha - A)^{-1}Bu_0 \to 0$ weakly as $\alpha \to +\infty$, by Theorem 6.2.11(d1)).

(b2) We have $\mathbb{D} = \mathbb{N}\mathbb{X}$, hence $\mathbb{D} \in$ SR and $D = NX$, by Lemma 6.2.5. But $\mathbb{N}^* J\mathbb{N} = S$, by Lemma 9.10.1(c2) and continuity, hence $N^*JN = S$, by Lemma 6.3.6(b), hence $D^*JD = X^*SX$.

(b3) By Lemma A.4.4(d3) (with H_{-1} in place of H), we have $(s-A)^{-1}Bu_0 \to 0$ in H, as $s \to +\infty$, for all $u_0 \in U$, hence (b3) follows from (b1). (Note that now also $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$.)

(c1) This follows from Proposition 9.11.3(b).

(c2) This follows as in (c1), because the term $\langle C_{\mathrm{w}}(s-A)^{-1}Bu_0, JDu_0\rangle$ and its adjoint converge to zero, the term $\langle Du_0, JDu_0\rangle$ is constant, and $\langle C_{\mathrm{w}}(s-A)^{-1}Bu_0, JC_{\mathrm{w}}(s-A)^{-1}Bu_0\rangle \geq 0$.

(d) Take $u_0 = 0$ in (a). □

Having shown the implication eIARE⇒eCARE in Lemma 9.11.2 and in (b1) and (d) of Proposition 9.11.4, we now turn our attention to the converse direction eCARE⇒eIARE. We start by some technical implications of the eCARE:

Lemma 9.11.5 (eCARE⇒) *Let $\mathbb{D}$ be WR, and let $(\mathcal{P}, S, \begin{bmatrix} K \mid I - X \end{bmatrix})$ be a solution of the eCARE (in fact, equation $K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC$ need not hold). Then $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and*

(a) $(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_{\mathrm{w}} + (X^*SK)_{\mathrm{w}})x_0 = (X^*SX - D^*JD)u_0 = \text{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0$ whenever $Ax_0 + Bu_0 \in H$.

If, in addition, $X, S \in \mathcal{GB}(U)$, then we have the following:

(b) $K_{\mathrm{w}}, B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U) \subset \mathcal{B}(H_1, U)$.

(c) $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}, B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U)$, but $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P} - \text{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0$ although $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P}$ on H_1.

(d) If $\left[\begin{smallmatrix} A & B \\ C & \\ K & \end{smallmatrix}\right]$ generate a WPLS Σ_{ext}, then Σ_{ext} is WR.

(e) Σ_{ext} in (d) is SR [UR] iff Σ is SR [UR] and the weak limit in the CARE exists as strong [uniform] limit too.

As one observes from 2° of the proof, for any WR $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $J = J^* \in \mathcal{B}(U,Y)$, $\mathcal{P} \in \mathcal{B}(H)$, and $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_1,U)$, we have $\exists (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)_{\mathrm{w}}x_0$ for all $x_0 \in H_B \Leftrightarrow \exists\, \text{w-lim}_{\alpha\to\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}Bu_0$ for all $u_0 \in U$. Thus, assuming $X, S \in \mathcal{GB}(U)$, the weak regularity of K (i.e., that of $\mathbb{F}$ and $\mathbb{X}$) is contained in (equivalent to) the assumption on convergence of the second equation of eCARE!

Proof: (The first equation of the eCARE is not used in this proof.)

(a) (We prove here also a part of (b).)

1° The inclusion $\mathcal{P}[H_B] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ was noted in Remark 9.1.6. By Lemma A.3.6, it follows that $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$, hence $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U) \subset \mathcal{B}(H_1, U)$.

2° "$H_B \subset \mathrm{Dom}((X^*SK)_{\mathrm{w}})$" & (a): Let $x_0 \in H_B$ and $u_0 \in U$ be s.t. $z_0 := Ax_0 + Bu_0 \in H$ (see Definition 6.1.17). Set $x_s := s(s-A)^{-1}x_0 \in H_1$ so that $(X^*SK)_{\mathrm{w}}x_0 := \text{w-lim}_{s\to+\infty}(X^*SK)x_s$, by Proposition 6.2.8(a1), once the convergence of this limit is established; this will be done below: using the eCARE, we get

$$\begin{aligned} X^*SKx_s &= -D^*JCx_s - B_{\mathrm{w}}^*\mathcal{P}x_s \\ &\rightharpoonup -D^*JC_{\mathrm{w}}x_0 - B_{\mathrm{w}}^*\mathcal{P}x_0 + \operatorname*{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0, \end{aligned}$$

since

$$\begin{aligned} B_{\mathrm{w}}^*\mathcal{P}x_s &= B_{\mathrm{w}}^*\mathcal{P}s(s-A)^{-1}x_0 = B_{\mathrm{w}}^*\mathcal{P}(I+A(s-A)^{-1})x_0 \\ &= B_{\mathrm{w}}^*\mathcal{P}x_0 + B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}(z_0 - Bu_0) \\ &\rightharpoonup B_{\mathrm{w}}^*\mathcal{P}x_0 + 0 - \operatorname*{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0, \end{aligned}$$

because $(s-A)^{-1}z_0 \xrightarrow[H_1]{} 0$, by Lemma A.4.4(d3), and $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_1, U)$.

(b) Since $X^*S \in \mathcal{B}(U)$, we now have $H_B \subset \mathrm{Dom}(K_{\mathrm{w}})$, by 2°. By Lemma A.3.6, this implies that $K_{\mathrm{w}} \in \mathcal{B}(H_B, U)$.

(c) If $x_0 \in H_B$ and $u_0 \in U$ are s.t. $Ax_0 + Bu_0 \in H$ (see Definition 6.1.17), then the latter limit in 2° shows that $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}x_0 = B_{\mathrm{w}}^*\mathcal{P}x_0 - \text{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0$ (and that this limit exists). Thus, $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U)$ (cf. the proof of (b)). Naturally, both $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}$ and $B_{\mathrm{w}}^*\mathcal{P}$ are (continuous) extensions of $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_1, U)$.

To get an example where $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}x_0 \neq B_{\mathrm{w}}^*\mathcal{P}x_0$, apply (9.117) to any $u_0 \neq 0$ (with, e.g., $x_0 := (s-A)^{-1}Bu_0$).

(d) Σ_{ext} is WR iff $H_B \subset \mathrm{Dom}(\left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right]_{\mathrm{w}}) = \mathrm{Dom}(C_{\mathrm{w}}) \cap \mathrm{Dom}(K_{\mathrm{w}})$, by Proposition 6.2.8(a1). If $x_0 \in H_B$, then $x_0 \in \mathrm{Dom}(C_{\mathrm{w}})$ by the weak regularity of Σ and $x_0 \in \mathrm{Dom}(K_{\mathrm{w}})$ by 2°.

(e) Certainly $\mathbb{D} \in \mathrm{SR}$ is necessary, hence we assume that $\mathbb{D}$ is SR, and find out when $\mathbb{F}$ is SR too, i.e., when $K_{\mathrm{w}}(s-A)^{-1}B \to 0$ strongly, equivalently, when $X^*SK_{\mathrm{w}}(s-A)^{-1}B \to 0$ strongly (because $X^*S \in \mathcal{GB}$).

By substituting $x_0 = s(s-A)^{-1}Bu_0$ into (a), we see that

$$X^*SK_{\mathrm{w}}(s-A)^{-1}B - B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B = -D^*JC_{\mathrm{w}}(s-A)^{-1}B + X^*SX - D^*JD. \tag{9.195}$$

Because $\mathbb{D}$ is SR, the right-hand-side converges strongly, as $s \to +\infty$. Therefore, $X^*SK_{\mathrm{w}}(s-A)^{-1}B$ converges strongly iff $B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ converges strongly.

The same proof applies for uniform regularity (and for any other form of regularity), mutatis mutandis. □

Since the second and third formulae of the eCARE contain feedthrough operators and the term $B_{\mathrm{w}}^*\mathcal{P}$, we need to be careful when we "integrate" them to obtain the eIARE, unlike in the simple case of the Lyapunov equation (see Lemma 9.11.2). However, we write the proofs in detail, so that the reader should be able to follow the steps. We start with the slightly simpler one, namely the third equation:

Lemma 9.11.6 ($\mathbb{X}^{t*}S\mathbb{K}^t = -(\mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t)$) *Let Σ be WR, and let the eCARE have a WR solution $(\mathcal{P}, S, [\ K \mid I-F\ ])$. Then, for all $t \geq 0$,*

$$\mathbb{X}^{t*}S\mathbb{K}^t = -\left(\mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t\right). \tag{9.196}$$

Proof: Set $\mathbb{T}(t) := \mathbb{X}^{t*} S \mathbb{K}^t + \mathbb{D}^{t*} J \mathbb{C}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{A}^t \in \mathcal{B}(H, \mathrm{L}^2([0,t);U))$ for $t \geq 0$. By density (see Theorem B.3.11), it is enough to show that

$$f(t) := f_{u,x_0}(t) := \langle u, \mathbb{T}(t) x_0 \rangle_{\mathrm{L}^2} = 0 \tag{9.197}$$

for arbitrary $t \geq 0$, $u \in \mathcal{C}_c^\infty((0,t);U)$ and $x_0 \in H_1$. Let v, x_0 be as above, and set (see Theorem 6.2.13(b))

$$x := \mathbb{A}(\cdot)x_0 \in \mathcal{C}^1([0,\infty);H), \;\; z := \mathbb{B}^t u \in \mathcal{C}^1(\mathbf{R};H) \cap \mathcal{C}(\mathbf{R};H_B), \mathbb{X}u \in \mathrm{W}^{1,2}_{\mathrm{loc}} \subset \mathcal{C} \tag{9.198}$$

to obtain $x' = Ax$, $z' = Az + Bu$, $x', z' \in \mathcal{C}(\mathbf{R};H)$; $(\mathbb{X}u)(t) = u(t) - K_{\mathrm{w}} z(t)$, $(\mathbb{D}u)(t) = Du(t) + C_{\mathrm{w}} z(t)$, $\mathbb{C}x_0 = Cx(t)$, $\mathbb{K}x_0 = Kx(t)$ (see (6.46)). Thus, (to be brief, we drop here the (t)'s after u, x, x', z, z')

$$f(t) = \int_0^t \langle \mathbb{X}u, S\mathbb{K}x_0 \rangle_U \, dt + \int_0^t \langle \mathbb{D}u, J\mathbb{C}x_0 \rangle_Y \, dt + \langle z, \mathcal{P}x \rangle_H. \tag{9.199}$$

Because $f(0) = 0$ (since $z(0) = 0$), it is enough to show that $f'(t) = 0$ for $t \geq 0$, since, by Lemma B.5.4, $f \in \mathcal{C}^1([0,\infty))$ and

$$f'(t) = \langle Xu - K_{\mathrm{w}} z, SKx \rangle_U + \langle Du + C_{\mathrm{w}} z, JCx \rangle_Y + \langle z, \mathcal{P}x \rangle_H'. \tag{9.200}$$

Now $\langle z, \mathcal{P}x \rangle_H' = \langle z, \mathcal{P}x' \rangle_H + \langle z', \mathcal{P}x \rangle = \langle z, \mathcal{P}Ax \rangle_H + \langle Az + Bu, \mathcal{P}x \rangle_H$. Set $z_r := r(r-A)^{-1} z \in H_1$ for $r > \omega_A$, so that $\begin{bmatrix} C_{\mathrm{w}} \\ K_{\mathrm{w}} \end{bmatrix} z = \text{w-lim}_{r \to +\infty} \begin{bmatrix} C \\ K \end{bmatrix} z_r$, by Proposition 6.2.8. Then

$$f'(t) = \lim_{r \to +\infty} \big(\langle u, (X^*SK + D^*JC)x \rangle_U + \langle z_r, (C^*JC - K^*JK)x \rangle_H \tag{9.201}$$

$$+ \langle z, \mathcal{P}Ax \rangle_H + \langle Az_r, \mathcal{P}x \rangle + \langle r(r-A)^{-1}Bu, \mathcal{P}x \rangle \big). \tag{9.202}$$

But $C^*JC - K^*JK = -A^*\mathcal{P} - \mathcal{P}A$, by the eCARE, hence

$$f'(t) = \lim_{r \to +\infty} \big(\langle u, (X^*SK + D^*JC)x \rangle_U + \langle u, B^* r(r-A^*)^{-1} \mathcal{P}x \rangle \big) \tag{9.203}$$

$$= \langle u, (X^*SK + D^*JC + B_{\mathrm{w}}^* \mathcal{P})x \rangle_U = 0, \tag{9.204}$$

by the definition of K. □

Now only the hardest part of eCARE⇒eIARE remains:

Lemma 9.11.7 ($\mathbb{X}^{t*} S \mathbb{X}^t = \mathbb{D}^{t*} J \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t$) *Let Σ be WR, and let the eCARE have a WR solution $(\mathcal{P}, S, \begin{bmatrix} K & \big| & I - X \end{bmatrix})$. Then, for all $t \geq 0$, we have*

$$\mathbb{X}^{t*} S \mathbb{X}^t = \mathbb{D}^{t*} J \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t. \tag{9.205}$$

Proof: The proof below requires even more patience than the one above. One might ask whether the proof could be simplified in most special cases, e.g., when $\mathbb{D} \in \mathrm{MTIC}$. However, it seems that this is not the case; in the SR case, some details would be slightly but not essentially simpler, and even for $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, we do not know any way to avoid the tricks with the non-commuting limits or similar difficulties in the proof.

Fix $t > 0$ (for $t = 0$ (9.205) is trivial). Let $u \in \mathrm{W}^{1,2}(\mathbf{R};U)$, $\pi_- u = 0$, and set $x := \mathbb{B}\tau u$. Then, by Theorem 6.2.13(b1)(ii), we have $x \in \mathcal{C}^1(\mathbf{R};H) \cap \mathcal{C}(\mathbf{R};H_B)$,

$x' = Ax + Bu = \mathbb{B}\tau u' \in \mathcal{C}(\mathbf{R}; H)$, $\mathbb{D}u = C_w x + Du$, $\mathbb{X}u = Xu - K_w x$, $x(0) = \mathbb{B}u = \mathbb{B}\pi_- u = 0$. Because C_w and K_w are continuous on H_B, by Proposition 6.2.8(a1), functions $\mathbb{D}u$ and $\mathbb{X}u$ are continuous, hence so are functions f, g, h, h_1 and h_2, that we will define later. Let $T \in [0,t)$ be arbitrary. Set (we will write u for $u(T)$, x for $x(T)$ and x' for $x'(T)$ except in "$\mathbb{D}u$" and "$\mathbb{X}u$")

$$\begin{aligned} g(T) :&= \langle (\mathbb{D}u)(T), J(\mathbb{D}u)(T)\rangle_Y = \langle Du + C_w x, JDu + JC_w x\rangle_Y \\ &= \langle u, D^*JDu\rangle_U + \langle u, D^*JC_w x\rangle_U + \langle D^*JC_w x, u\rangle_U + \langle C_w x, JC_w x\rangle_Y . \end{aligned}$$

Set also

$$f(T) := \langle (\mathbb{X}u)(T), S(\mathbb{X}u)(T)\rangle_U \tag{9.206}$$

$$= \langle Xu, SXu\rangle_U - \langle Xu, SK_w x\rangle_U - \langle K_w x, SXu\rangle_U + \langle K_w x, SK_w x\rangle_U . \tag{9.207}$$

Now $x(T) \in H_B$ need not belong to $\mathrm{Dom}(A)$, yet the definition of K and the Riccati equation give us information for $x_0 \in \mathrm{Dom}(A)$ only. To overcome this problem, we define $x_s := s(s-A)^{-1}x(T) \in \mathrm{Dom}(A)$ for $s > \omega_A$, getting (we first use Proposition 6.2.8(a1), then the third and first equation of the eCARE with $S = S^*$)

$$\begin{aligned} f(T) &= \lim_{r\to+\infty}\lim_{s\to+\infty} [\langle u, X^*SXu\rangle_U - \langle u, X^*SKx_s\rangle_U - \langle X^*S^*Kx_r, u\rangle_U + \langle Kx_r, SKx_s\rangle_U] \\ &= \lim_{r\to+\infty}\lim_{s\to+\infty} \big[\langle u, X^*SXu\rangle_U + \langle Xu, (B_w^*\mathcal{P} + D^*JC)x_s\rangle_U \\ &\quad + \langle (B_w^*\mathcal{P} + D^*JC)x_r, Xu\rangle_U + \langle Kx_r, SKx_s\rangle_U\big] \\ &= \lim_{r\to+\infty}\lim_{s\to+\infty} [\langle u, D^*JDu\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U \\ &\quad + \langle u, (B_w^*\mathcal{P} + D^*JC)x_s\rangle_U + \langle (B_w^*\mathcal{P} + D^*JC)x_r, u\rangle_U \\ &\quad + \langle Ax_r, \mathcal{P}x_s\rangle_H + \langle \mathcal{P}x_r, Ax_s\rangle_H + \langle Cx_r, JCx_s\rangle_Y] = g(T) + h(T), \end{aligned}$$

where $g(T) := \lim_{r\to+\infty}\lim_{s\to+\infty}[\langle u, D^*JDu\rangle_U + \langle u, D^*JCx_s\rangle_U + \langle D^*JCx_r, u\rangle_U + \langle Cx_r, JCx_s\rangle_Y] = \langle (\mathbb{D}u)(T), (J\mathbb{D}u)(T)\rangle_Y$, and

$$h(T) := \lim_{r\to+\infty}\lim_{s\to+\infty} \big[\langle u, (X^*SX - D^*JD)u\rangle_U + \langle u, B_w^*\mathcal{P}x_s\rangle_U \tag{9.208}$$

$$+ \langle Ax_r, \mathcal{P}x_s\rangle_H + \langle B_w^*\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\big] . \tag{9.209}$$

On the other hand, $h(T) = h_1(T) + h_2(T)$, where[3]

$$\begin{aligned} h_2(T) &:= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B_{\mathrm w}^*\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\right] \\ &= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B^* s(s-A^*)^{-1}\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\right] \\ &= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \mathcal{P}x_r, s\overline{(s-A)^{-1}}Bu\rangle_H + \langle \mathcal{P}x_r, s\overline{(s-A)^{-1}}Ax\rangle_H\right] \\ &= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \mathcal{P}x_r, s(s-A)^{-1}x'\rangle_H\right] \\ &= \langle \mathcal{P}x, x'\rangle_H = \langle x, \mathcal{P}x'\rangle_H \end{aligned}$$

(the next to last identity is from Lemma A.4.4(d1)), and

$$h_1(T) := \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle u, (X^*SX - D^*JD)u\rangle_U + \langle u, B_{\mathrm w}^*\mathcal{P}x_s\rangle_U + \langle Ax_r, \mathcal{P}x_s\rangle_H\right].$$

Now (we use first the eCARE, then Lemma 9.11.5(a))

$$\begin{aligned} &\lim_{s\to\infty}\langle u, B_{\mathrm w}^*\mathcal{P}x_s\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U && (9.210)\\ &= \lim_{s\to\infty}\langle u, -(D^*JC + XSK)x_s\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U && (9.211)\\ &= \langle u, -(D^*JC_{\mathrm w} + XSK_{\mathrm w})x + (X^*SX - D^*JD)u\rangle_U = \langle u, B_{\mathrm w}^*\mathcal{P}x\rangle_U, && (9.212) \end{aligned}$$

because $\overline{A}x + Bu = x' \in H$, as required in Lemma 9.11.5. Therefore

$$\begin{aligned} \langle x', \mathcal{P}x\rangle_H &= \lim_{r\to\infty}\langle r(r-A)^{-1}\left[x'\right], \mathcal{P}x\rangle_H \\ &= \lim_{r\to\infty}\langle r\overline{(r-A)^{-1}}\left[Ax + Bu\right], \mathcal{P}x\rangle_H \\ &= \lim_{r\to\infty}\left[\langle r\overline{(r-A)^{-1}}Ax, \mathcal{P}x\rangle_H + \langle r\overline{(r-A)^{-1}}Bu, \mathcal{P}x\rangle_H\right] \\ &= \lim_{r\to\infty}\langle Ax_r, \mathcal{P}x\rangle_H + \langle u, B_{\mathrm w}^*\mathcal{P}x\rangle_U = h_1(T). \end{aligned}$$

(N.B. $\lim_{r\to\infty}\langle Ax_r, \mathcal{P}x\rangle_H$ exists.)[4] Thus, still for an arbitrary $T \in \mathbf{R}$, we have

$$h(T) = h_1(T) + h_2(T) = \langle x', \mathcal{P}x\rangle_H + \langle x, \mathcal{P}x'\rangle_H = \langle x, \mathcal{P}x\rangle_H'(T). \tag{9.213}$$

Because $T \in \mathbf{R}$ was arbitrary, we get (recall that $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$)

$$\begin{aligned} \langle \mathbb{X}^t u, S\mathbb{X}^t u\rangle_{\mathrm L^2} - \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle_{\mathrm L^2} &= \int_0^t f(T)\,dT - \int_0^t g(T)\,dT && (9.214)\\ = \int_0^t h(T)\,dT &= \int_0^t \langle x, \mathcal{P}x\rangle_H'(T)\,dT && (9.215)\\ = \langle x(t), \mathcal{P}x(t)\rangle - 0 &= \langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle_H. && (9.216) \end{aligned}$$

[3]To clarify this part of the proof, we use here bars for (unique continuous) extensions, e.g., by $\overline{(s-A)^{-1}} \in \mathcal{B}(H_{-1}, H)$ we denote the extension of $(s-A)^{-1} \in \mathcal{B}(H, H_1)$. One can easily verify that $\overline{(s-A)^{-1}}$ is the inverse of $s - \overline{A}$ and $\langle x, \overline{(s-A)^{-1}}z\rangle_H = \langle (s-A^*)^{-1}x, z\rangle_{\langle H_1^*, H_{-1}\rangle}$ for all $x \in H$, $z \in H_{-1}$. Consequently, $\langle B_{\mathrm w}^* x, u\rangle_U = \lim_{s\to+\infty}\langle sx, \overline{(s-A)^{-1}}Bu\rangle_H$ for any $u \in U$, $x \in \mathrm{Dom}(B_{\mathrm w}^*)$ (see Definition 6.1.17). (All this holds for any $\Sigma \in$ WPLS.)

[4]Note that the commutator $(\lim_s \lim_r - \lim_r \lim_s)\left(\langle u, B_{\mathrm w}^*\mathcal{P}x_s\rangle_U + \langle Ax_r, \mathcal{P}x_s\rangle_H\right)$ of the expression in $h_1(t)$ is equal to the term $\langle u, (X^*SX - D^*JD)u\rangle_U$ (which is frequently zero, cf. Remark 9.9.14). When this commutator is zero, we can compute $h_1(T)$ in the same way as $h_2(T)$.

Since $C_c^\infty((0,t);U) \subset \mathrm{W}^{1,2}(\mathbf{R};U)$ is continuous in $\mathrm{L}^2([0,t);U)$, we obtain (9.205). □

Any solution of the CARE with $S \gg 0$ is WR:

Proposition 9.11.8 ($S \gg 0 \Rightarrow \mathcal{P}$ is WR) *Let Σ be WR, and let $(\mathcal{P},S,K)$ be a solution of the eCARE. Then we have $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ on $\mathrm{W}_0^{1,2}(\mathbf{R}_+;U) = \{u \in \mathrm{W}^{1,2}(\mathbf{R};U) \,\big|\, \pi_- u = 0\}$, for all $t \ge 0$.*

If, in addition, $S \gg 0$ and $X = I$, then $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a WR WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right]$, and $(\mathcal{P},S,\left[\begin{smallmatrix} \mathbb{K} & \mathbb{F} \end{smallmatrix}\right])$ is a WR solution of the IARE.

(In the first claim, we have set $(\mathbb{X}u)(t) := -K_\mathrm{w}x(t) + Xu(t)$ (a.e.), where $x := \mathbb{B}\tau u$. The claim means that $\langle \mathbb{X}^t u, S\mathbb{X}^t u\rangle = \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle + \langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle$ for all $(u \in \mathrm{W}_0^{1,2})$.)

Proof: The original proof of Lemma 9.11.7 will do for the claim on $\mathrm{W}_0^{1,2}$ (see Lemma B.7.9 for $\mathrm{W}_0^{1,2}$). Assume then that $S \gg 0$ and $X = I$.

By Lemma 9.12.2(a1), $\left[\begin{smallmatrix} A \\ K \end{smallmatrix}\right]$ generate a WPLS, hence (1.)–(4.) of Lemma 6.3.13 are satisfied by $\left[\begin{smallmatrix} A & B \\ -K_\mathrm{w} & I \end{smallmatrix}\right]$ (with $C_\mathrm{c} \mapsto -K_\mathrm{w}$ and $D_\mathrm{c} \mapsto I$). From equation $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ we deduce that $\|S^{1/2}\mathbb{X}^t u\|_2 \le M\|u\|_2$ for all $u \in C_c^\infty((0,t);U)$, so that also (5.) of Lemma 6.3.13 holds. Therefore, $\left[\begin{smallmatrix} A & B \\ -K_\mathrm{w} & I \end{smallmatrix}\right]$ generate a WPLS.

By Lemma 9.11.5(b), we have $K_\mathrm{w} \in \mathcal{B}(H_B,U)$, hence this WPLS is WR. □

In the SR case, the above proofs can be modified to cover the case where the [e]CARE is replaced by the corresponding inequality:

Proposition 9.11.9 (Riccati inequality) *Assume that Σ is SR and that some $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S = S^*, X \in \mathcal{B}(U)$ and $K \in \mathcal{B}(H_1,U)$ satisfy*

$$\begin{cases} K^*SK \le A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A)^*) \\ X^*SX = D^*JD + \operatorname*{s-lim}_{s\to+\infty} B_\mathrm{w}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ X^*SK = -(B_\mathrm{w}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\mathrm{Dom}(A),U). \end{cases} \tag{9.217}$$

(a) We have $\mathbb{K}^{t*}S\mathbb{K}^t \le \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t*}J\mathbb{C}^t$ on $\mathrm{Dom}(A)$, where $(\mathbb{K}x_0)(t) := K\mathbb{A}^t x_0$ for all $t \ge 0$, $x_0 \in H_1$.

If $S \gg 0$, then $\left[\begin{smallmatrix} A \\ K \end{smallmatrix}\right]$ generate a WPLS and $\mathbb{K}^{t*}S\mathbb{K}^t \le \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t*}J\mathbb{C}^t$ (on H) for all $t \ge 0$.

(b) If $H_B \subset \mathrm{Dom}(K_\mathrm{s})$ (this holds if $X,S \in \mathcal{GB}(U)$), then we have $\mathbb{X}^{t*}S\mathbb{X}^t \le \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ on $\mathrm{W}_0^{1,2}(\mathbf{R}_+;U)$, for all $t \ge 0$.

(c) If $\left[\begin{smallmatrix} A & B \\ -K & X \end{smallmatrix}\right]$ generate a SR WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ -\mathbb{K} & \mathbb{X} \end{smallmatrix}\right]$, then we have $\mathbb{X}^{t*}S\mathbb{X}^t \le \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ for all $t \ge 0$ and $X^*SX \le D^*JD + \text{s-lim}_{s\to+\infty} 2sB^*(s-A^*)^{-1}\mathcal{P}(s-A)^{-1}B$.

(d) If we can have "$K^*SK+\varepsilon I+$" in place of "K^*SK" in (9.217), then we can have $\mathbb{X}^{t*}S\mathbb{X}^t+\varepsilon\mathbb{L}^{t*}\mathbb{L}^t$ in place of $\mathbb{X}^{t*}S\mathbb{X}^t$ in both (b) and (c).

(e) If $S \gg 0$ and $X = I$, then $\left[\begin{array}{c|c} A & B \\ \hline -K & X \end{array}\right]$ generate a SR WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$, hence then (c) applies.

(In (a), we have set $(\mathbb{X}u)(t) := -K_{\mathrm{w}}x(t)+Xu(t)$ (a.e.), where $x := \mathbb{B}\tau u$. In (b), we have set $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}_\infty(U,H)$; do not mix up $\mathbb{L}^t := \pi_{[0,t)}\mathbb{B}\tau\pi_{[0,t)} : \mathrm{L}^2_\omega \to \mathrm{L}^2_\omega$ with $\mathbb{B}^t := \mathbb{B}\tau^t\pi_+ : \mathrm{L}^2_\omega \to H$.)

Thus, for the Riccati inequality (9.217), the first and the third equation of the eIARE become inequalities. However, the third equation seems to be lost in the above case, due to its asymmetry (cf. the proof of Lemma 9.11.6).

By (d), in the case of "$\ll$", we have $\mathbb{X}^{t*}S\mathbb{X}^t+\varepsilon\mathbb{L}^{t*}\mathbb{L}^t \le \mathbb{D}^{t*}J\mathbb{D}^t+\mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ $(t \ge 0)$ for some $\varepsilon > 0$. To treat the opposite signs, multiply $S, J, \mathcal{P}$ by -1.

See Lemma 9.12.2 for analogous results.

Proof of Proposition 9.11.9: (a) This is contained in Lemma 9.12.2(a1) and (9.220).

(b) 1° We observe from 2° of the proof of Lemma 9.11.5(a) that if $X,S \in \mathcal{GB}(U)$, then $X^*SK_{\mathrm{s}} \in \mathcal{B}(H_B,U)$, hence then $K_{\mathrm{s}} \in \mathcal{B}(H_B,U)$.

2° Set $R := A^*\mathcal{P}+\mathcal{P}A+C^*JC-K^*SK \in \mathcal{B}(H_1,H_{-1}^*)$, so that $\langle x_0,Rx_0\rangle \ge 0$ for all $x_0 \in H_1$. Since $Cx_r \to C_{\mathrm{w}}x$ strongly and $Kx_r \to K_{\mathrm{w}}x$ strongly, as $r \to +\infty$, we may replace "$\lim_{r\to+\infty}\lim_{s\to+\infty}$" by "$\lim_{r=s\to+\infty}$" in the proof of the lemma (by Lemma A.3.1(i2)). (This is the reason for the explicit and implicit "strong regularity assumptions on $\mathbb{D}$ and $\mathbb{X}$").

Then "$f(T) = g(T)+h(T)$" must be replaced by "$f(T) = g(T)+h(T)-k(T)$", where $k(T) := \lim_{r\to+\infty}\langle x_r,Rx_r\rangle \ge 0$ (the limit exists since $g(T)-h(T)-f(T)$ converges, as shown in the proof). Consequently, we have obtained (9.215) with "$\le$" in place of "$=$".

(c) The first claim follows from (b), by density (see Theorem B.3.11(b1)). The second claim follows as in the proof of Proposition 9.11.3(b).

(d) Now, in the proof of (b), there is $\varepsilon > 0$ s.t. $\langle x_0,Rx_0\rangle-\varepsilon\langle x_0,x_0\rangle \ge 0$ for all $x_0 \in H_1$. Set $\widetilde{C} := \begin{bmatrix} C \\ I \end{bmatrix}$, $\widetilde{D} := \begin{bmatrix} D \\ 0 \end{bmatrix}$, $\widetilde{J} := \begin{bmatrix} J & 0 \\ 0 & -\varepsilon I \end{bmatrix}$ to get the setting of (b) with $\widetilde{\Sigma} := \left(\begin{array}{c|c} A & B \\ \hline \widetilde{C} & \widetilde{D} \end{array}\right) \in \mathrm{WPLS}(U,H,Y\times U)$, $\widetilde{J}$ and $\widetilde{R} := R-\varepsilon I \ge 0$ in place of Σ, J and R, respectively. Then $\widetilde{\mathbb{D}}^{t*}\widetilde{J}\widetilde{\mathbb{D}}^t = \mathbb{D}^{t*}J\mathbb{D}^t-\varepsilon\mathbb{L}^{t*}\mathbb{L}^t$, where $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}_\infty(U,H)$, hence this follows from (b). (Analogously, we have $\mathbb{K}^{t*}S\mathbb{K}^t+\varepsilon\mathbb{R}^{t*}\mathbb{R}^t \le \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t-\mathcal{P}+\mathbb{C}^{t*}J\mathbb{C}^t$ on $\mathrm{Dom}(A)$, where $(\mathbb{R}^t x_0)(t) := \mathbb{A}^t x_0$ $x_0 \in H$, $t \ge 0$.)

(e) By (a), $\left[\begin{array}{c} A \\ \hline K \end{array}\right]$ generate a WPLS, hence (1.)–(4.) of Lemma 6.3.13 are satisfied by $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I \end{array}\right]$ (with $C_{\mathrm{c}} \mapsto -K_{\mathrm{w}}$ and $D_{\mathrm{c}} \mapsto I$).

Also (5.) of Lemma 6.3.13 holds, by (b). Therefore $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I \end{array}\right]$ generate a WPLS. Since $K_{\mathrm{s}} \in \mathcal{B}(H_B,U)$ (by (b)), this WPLS is SR. □

Notes for Sections 9.10 and 9.11

In [S97b], Olof Staffans showed that the existence a spectral factorization leads to an optimal state feedback pair. In [S98b, Section 4], this was applied to the stabilized form of a jointly stabilizable and detectable system. In [S98b, Section 5], the state feedback pair was shown to lead to formulae (9.153)–(9.161). The proofs of ("necessity parts" of the) Lemma 9.10.1(a1)–(b4)&(c2) are essentially from there.

In the same sense, Lemma 9.11.1 equals Corollary 5.7 of [S98b], and also Lemma 9.11.2 and formulae (a), (b1), (c1) and (d) of Proposition 9.11.4 are from [S98b, Sections 6–7]. Most of Proposition 9.11.3 is from a preprint of [SW01a] in the same sense.

In the generality of [S98b], most of the stable case of Lemmas 9.11.2, 9.11.5 and 9.11.7 and Proposition 9.11.4 is contained in [Mik97b] and [Mik98].

9.12 Further eIARE and eCARE results

'My feet hath fate, O king,' he said,
'here over the mountains bleeding led,
and what I sought not I have found,
and love it is hat here me bound.
For fairer than are born to Men
A daughter hast thou, Lúthien.'
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section, we shall extend some classical results such as the correspondence between open-loop and closed-loop Riccati equations; we shall also study "pseudospectral factorizations" (something close to a spectral factorization).

However, we start by making a remark on "irregular CAREs":

Remark 9.12.1 ("Compatible CARE") *By applying Lemma 6.3.10(b) instead of Theorem 6.2.13 in the proofs, we see that Proposition 9.11.4(a)&(d) hold for any WPLS provided that we make replacements* $(C_{\mathrm{w}}, D) \mapsto (C_{\mathrm{c}}, D_{\mathrm{c}})$, $(B^*_{\mathrm{w}}, D^*) \mapsto (B^*_{\mathrm{c}}, D^*_{\mathrm{d}})$ *etc. We have to remind that compatible pairs are not unique in general (the equations hold for any such pairs).*

In particular, the third equation of the eCARE hold in the compatible case too. The first (Lyapunov) equation of the eCARE holds for any WPLS, by Lemma 9.11.2.

Unfortunately, we have no decent formulae for S in the general case, and thus any attempts to define an "eCARE" that would imply the eIARE seem doomed. Therefore, it seems more adviseable to use Section 9.7 with some admissibility condition in the case where the state feedback pair is not known to be regular.

Note that in the above case, formulae such as $\widehat{\mathbb{X}}(s) = X_{\mathrm{c}} - K_{\mathrm{c}}(s-A)^{-1}B$ hold, by Lemma 6.3.10(a), and that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is uniquely determined by K_{c} and X_{c}.

The output stability of Σ is equivalent the solvability of a CARE:

Lemma 9.12.2 ($\mathbb{A}/\mathbb{C}$ is stable $\Leftrightarrow$ CARE) *Let* $\widetilde{C} \in \mathcal{B}(H_1, Z)$, $\widetilde{J} \in \mathcal{B}(Z)$. *Define* $\widetilde{\mathbb{C}} : H_1 \to \mathcal{C}(\mathbf{R}_+; Z)$ *by* $\widetilde{\mathbb{C}}x_0 := \widetilde{C}\mathbb{A}x_0$ $(x_0 \in H_1)$. *We say that "$\widetilde{\mathbb{C}}$ is stable" if* $\left[\begin{smallmatrix} A \\ \widetilde{C} \end{smallmatrix}\right]$ *generate a WPLS* $\left[\begin{smallmatrix} \mathbb{A} \\ \widetilde{\mathbb{C}} \end{smallmatrix}\right]$ *with* $\widetilde{\mathbb{C}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+; Z))$. *The following hold:*

(a1) Assume that $\widetilde{J} \gg 0$. *If some* $\mathcal{P} \in \mathcal{B}(H)$ *satisfies*

$$\widetilde{C}^*\widetilde{J}\widetilde{C} \leq A^*\mathcal{P} + \mathcal{P}A + C^*JC \quad \text{on } \mathrm{Dom}(A), \tag{9.218}$$

then $\left[\begin{smallmatrix} A \\ \widetilde{C} \end{smallmatrix}\right]$ *generate a WPLS.*

(a2) Assume that $\widetilde{J} \gg 0$. *Then* $\widetilde{\mathbb{C}}$ *is stable iff there is* $\mathcal{P} \in \mathcal{B}(H)$ *s.t.* $\mathcal{P} \geq 0$ *and*

$$A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{J}\widetilde{C} \leq 0 \quad \text{on } \mathrm{Dom}(A). \tag{9.219}$$

(b) Assume that $\widetilde{\mathbb{C}}$ *is stable. Then* $\mathcal{P} = \widetilde{\mathbb{C}}^*\widetilde{J}\widetilde{\mathbb{C}}$ *satisfies* $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{J}\widetilde{C} = 0$, *and* $\widetilde{\mathcal{P}} \geq \mathcal{P}$ *for any* $\widetilde{\mathcal{P}} \geq 0$ *that solves (9.219).*

In particular, if $\widetilde{J} \geq 0$, then $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$ is the smallest nonnegative solution of (9.219).

(c) Assume, that $\mathbb{A}$ is strongly stable and $\widetilde{\mathbb{C}}$ stable. Then $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$ is the unique solution (in $\mathcal{B}(H)$) of $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^* \widetilde{J}\widetilde{C} = 0$.

(d) The semigroup $\mathbb{A}$ is exponentially stable iff $A^*\mathcal{P} + \mathcal{P}A \ll 0$ for some nonnegative $\mathcal{P} \in \mathcal{B}(H)$ (and any such $\mathcal{P}$ necessarily satisfies $\mathcal{P} > 0$).

Note that we can take $\widetilde{C} := I =: \widetilde{J}$ in (a2) to check the exponential stability of $\mathbb{A}$, by (d). Naturally, we obtain analogous results for $(A, \widetilde{B})$, by duality (recall that strong stability is then mapped to strong-$*$ stability).

Proof: (a1) Fix $t > 0$. By Lemma 9.7.8(b), (9.218) is equivalent to

$$\widetilde{\mathbb{C}}^{t\,*}\widetilde{J}\widetilde{\mathbb{C}}^t \leq \mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*}J\mathbb{C}^t \quad (t \geq 0), \tag{9.220}$$

on $\mathrm{Dom}(A)$. Thus, there is $M < \infty$ s.t. $\|\widetilde{J}^{1/2}\widetilde{\mathbb{C}}^t x_0\|^2_{\mathrm{L}^2([0,t);Z)} \leq M\|x_0\|^2_H$ for all $x_0 \in \mathrm{Dom}(A)$. Since $\widetilde{J}^{1/2} \gg 0$ (by Lemma A.3.1(b4)), it follows from Corollary 6.3.14 that $\begin{bmatrix} A \\ \widetilde{C} \end{bmatrix}$ generate a WPLS.

(a2) 1° Let $\mathcal{P}$ be any nonnegative solution of (9.219), and assume that $\widetilde{J} \gg 0$. By Lemma 9.7.8)(b), inequality (9.219) is equivalent to

$$\mathcal{P} \geq \mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t + \widetilde{\mathbb{C}}^{t\,*}\widetilde{J}\widetilde{\mathbb{C}}^t \quad (t \geq 0), \tag{9.221}$$

on $\mathrm{Dom}(A)$, hence $\|\widetilde{J}^{1/2}\pi_{[0,t)}\widetilde{\mathbb{C}}x_0\|^2_2 \leq \langle x_0, \mathcal{P}x_0\rangle \quad (t \geq 0,\ x_0 \in H_1)$. Consequently, $\widetilde{\mathbb{C}}$ has a unique extension $\widetilde{\mathbb{C}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+; Z))$, hence $\begin{pmatrix} A \\ \widetilde{C} \end{pmatrix}$ is an output-stable WPLS.

2° Assume that $\begin{bmatrix} \mathbb{A} \\ \widetilde{\mathbb{C}} \end{bmatrix} \in$ WPLS, $\widetilde{\mathbb{C}}$ is stable and $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}(Z)$. Let Σ' be the system generated by $\left[\begin{array}{c|c} A & 0 \\ \hline C' & D' \end{array}\right] := \left[\begin{array}{c|c} A & 0 \\ \hline \widetilde{C} & 0 \\ 0 & I \end{array}\right]$. Set $J' := \begin{bmatrix} \widetilde{J} & 0 \\ 0 & I \end{bmatrix}$, so that the corresponding cost function becomes

$$\mathcal{J}'(x_0, u) = \langle \widetilde{\mathbb{C}}x_0, \widetilde{J}\widetilde{\mathbb{C}}x_0\rangle + \|u\|_2^2, \quad \text{and} \tag{9.222}$$

$$\langle \mathbb{C}'x_0 + \mathbb{D}'u, J'\mathbb{D}'\eta\rangle = \langle u, \eta\rangle_{\mathrm{L}^2} \quad (x_0 \in H,\ u, \eta \in \mathcal{U}'_{\mathrm{out}}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U)). \tag{9.223}$$

Thus, $u_{\mathrm{crit}}(x_0) = 0$ is the unique J'-critical control for each $x_0 \in H$, so that $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$. Since $B = 0$ is bounded and $D'^* J' D' = I$, the operator $\mathcal{P}$ corresponds to the unique $\mathcal{U}'_{\mathrm{out}}$-stabilizing solution of the CARE (or B^*_{w}-CARE) $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{J}\widetilde{C} = 0$, $S = I$, $SK = 0$.

(b) Assume that $\widetilde{\mathbb{C}}$ is stable and that $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$, so that $\mathcal{P}$ solves (9.221), by 2°. By (9.221), any other solution $\widetilde{\mathcal{P}} \geq 0$ satisfies $\widetilde{\mathcal{P}} \geq \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}} + \text{s-lim}_{t\to+\infty} \mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t \geq \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$.

If $\widetilde{J} \geq 0$, then $\mathcal{P} \geq 0$, hence then $\mathcal{P}$ is the smallest nonnegative solution of (9.219).

(c) Now $\widetilde{\mathcal{P}} = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}} + \text{s-lim}_{t\to+\infty} \mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t = \widetilde{\mathbb{C}}^* \widetilde{J}\widetilde{\mathbb{C}}$ for any solution $\mathcal{P} \in \mathcal{B}(H)$, as in (b).

(d) Naturally, the inequality "$A^*\mathcal{P} + \mathcal{P}A \ll 0$" (on $\mathrm{Dom}(A)$) means that there is $\varepsilon > 0$ s.t. $\langle Ax_0, \mathcal{P}x_0\rangle_H + \langle x_0, \mathcal{P}Ax_0\rangle_H \leq -\varepsilon\langle x_0, x_0\rangle_H$ for all $x_0 \in \mathrm{Dom}(A)$ (cf.

Definition A.3.23; it obviously follows that $\mathrm{Ker}(\mathcal{P}) = \{0\}$, i.e., that $\mathcal{P} > 0$ (even $\mathcal{P} \gg 0$ if A is bounded)).

If $A^*\mathcal{P} + \mathcal{P}A \leq -\varepsilon I$ and $\mathcal{P} \geq 0$, then $\widetilde{\mathbb{C}}$ is stable, where $\widetilde{\mathbb{C}}^t x_0 := \mathbb{A}^t x_0$ ($t > 0$, $x_0 \in H$), by (a) (set $\widetilde{C} := I$, $J := \varepsilon I \gg 0$), hence then $\mathbb{A}$ is exponentially stable, by Lemma A.4.5(i)&(ii).

Conversely, if $\mathbb{A}$ is exponentially stable, then $A^*\mathcal{P} + \mathcal{P}A + I^*II \leq 0$ on $\mathrm{Dom}(A)$ for some $\mathcal{P} \geq 0$, by (a2) (and we can have $\mathcal{P} = \widetilde{\mathbb{C}}^*\widetilde{\mathbb{C}} > 0$, where $\widetilde{\mathbb{C}}$ is as above). □

We now adopt the notation $\mathcal{P} \in \mathrm{eIARE}(\Sigma, J)$ (or $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}) \in \overline{\mathrm{eIARE}}(\Sigma, J)$) for the solutions of the eIARE for Σ and J. In the lemma below, we show how the solutions of a perturbed system correspond to the solutions for the original one:

Lemma 9.12.3 *Let $\begin{bmatrix} \mathbb{K}' & \mathbb{F}' \end{bmatrix}$ be admissible for Σ with closed-loop system $\Sigma_\flat$. Set $\mathbb{M}' := (I - \mathbb{F}')^{-1}$. Then*

$$(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}) \in \overline{\mathrm{eIARE}}(\Sigma, J) \Leftrightarrow \tag{9.224}$$

$$(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} - \mathbb{X}\mathbb{K}_\flat & I - \mathbb{X}\mathbb{M}' \end{bmatrix}) \in \overline{\mathrm{eIARE}}(\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right], J). \tag{9.225}$$

Moreover,

(a) If $\mathbb{F}'$ and $\mathbb{X}\mathbb{M}'$ (resp. and $\mathbb{F}$) are as above and have any strong or uniform regularity property, then so does $\mathbb{F}$ (resp. $\mathbb{X}\mathbb{M}'$).

(b) The two top rows ($\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$) of the corresponding closed-loop systems are equal (hence (P) is satisfied for Σ iff it is satisfied for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$) and Lemma 6.7.11(a1)–(a6) apply for the three pairs.

(c) Assume that $\begin{bmatrix} \mathbb{K}' & \mathbb{F}' \end{bmatrix}$ is [q.]r.c.-SOS-stabilizing and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.

Then $\mathcal{P}$ is [q.]r.c.-SOS-stabilizing for Σ iff $\mathcal{P}$ is q.r.c.-SOS-stabilizing (equivalently, stable and [[r.c.-]]SOS-stabilizing) for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

(d1) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then the J-critical state feedback pairs (equivalently, exponentially stabilizing solutions of the eIARE) for Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to each other through (9.224).

(d2) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{sta}}$ [$\mathcal{U}_{\mathrm{str}}/\mathcal{U}_{\mathrm{out}}$], and that $\begin{bmatrix} \mathbb{K}' & \mathbb{F}' \end{bmatrix}$ is q.r.c.-SOS-stabilizing

Then the J-critical q.r.c.-[strongly/SOS-]stabilizing state feedback pairs (equivalently, P-q.r.c.-[strongly/SOS-]stabilizing solutions of the eIARE) for Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to each other through (9.224).

Proof: Assume that $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}) \in \overline{\mathrm{eIARE}}(\Sigma, J)$. Set $\mathbb{X}_\natural := \mathbb{X}\mathbb{M}'$, $\mathbb{F}_\natural := I - \mathbb{X}_\natural$, $\mathbb{K}_\natural := \mathbb{K} - \mathbb{X}_\natural\mathbb{K}'$. Then it is easy to verify that $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K}_\natural & \mathbb{F}_\natural \end{bmatrix}) \in \overline{\mathrm{eIARE}}(\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right], J)$. Exchange the roles of Σ and $\Sigma_\flat^1$ to obtain the converse.

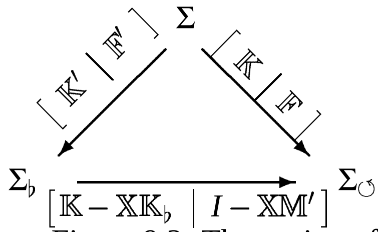

Figure 9.2: The setting of Lemma 9.12.3

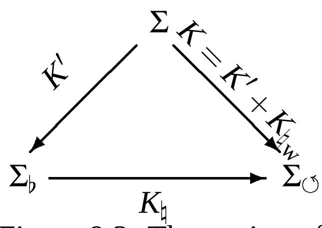

Figure 9.3: The setting of Proposition 9.12.4

(a) Set $\mathbb{X}_\natural := \mathbb{X}\mathbb{M}'$. Since $\mathbb{X}_\natural, \mathbb{X}, \mathbb{M}' \in \mathcal{G}\mathrm{TIC}_\infty(U)$, any strong or uniform regularity property shared by two of these is shared by the third too, since such properties are preserved in compositions, by Lemma 6.2.5.

(b) This is the setting of Lemma 6.7.11(a') with $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ in place of $\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]$ and $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ in place of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, hence the conclusions (a1)–(a6) apply with these replacements.

(c) This follows from Lemma 6.7.11(a2)&(a1) and Lemma 6.6.17(b).

(d1) This follows from Theorem 9.9.1(a1) and the fact that $\mathbb{A}_\circlearrowleft$ is common for both solutions, by (b) (and Lemma 6.1.10).

(d2) For $\mathcal{U}_{\mathrm{out}}$ this follows from (c) ("q.r.c.-SOS-"), (b) (for "P-") and Theorem 9.9.1(b). Since $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \end{array}\right]$ are common for both solutions, by (b), this leads to the claims on $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$. □

Thus, if $K_\natural$ is an optimal feedback for $\Sigma_\flat$, then the preliminary plus the optimal closed-loop feedback $K' + K_\natural$ produce the optimal open-loop feedback K (i.e., the same $\mathbb{A}_\circlearrowleft$ $\mathbb{C}_\circlearrowleft$, hence the same (possibly optimal) state $x_\circlearrowleft$ and output $y_\circlearrowleft$):

Proposition 9.12.4 (Σ-CARE$\cong$ $\Sigma_\flat$-CARE) *Let K' be an admissible SR state feedback operator for Σ with closed-loop system $\Sigma_\flat$.*

The WR solutions of form $(\mathcal{P}, S, \left[\begin{array}{c|c} K & 0 \end{array}\right])$ of the eCARE for Σ correspond 1-1 to the WR solutions of form $(\mathcal{P}_\natural, S_\natural, \left[\begin{array}{c|c} K_\natural & 0 \end{array}\right])$ of the eCARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ through

$$K = K' + K_{\natural_{\mathrm{w}}}, \;\; S = S_\natural, \;\; \mathcal{P} = \mathcal{P}_\natural. \tag{9.226}$$

Also (a)–(d2) of Lemma 9.12.3 apply; in particular, if K' and $K_\natural$ (resp. and K) are as above and have any strong or uniform regularity property, then so does K (resp. $K_\natural$).

(To be exact, by $K = K' + K_{\natural_{\mathrm{w}}}$ we mean that $K = K' + K_{\natural_{\mathrm{w}}|\mathrm{Dom}(A)}$.) Thus, all

WR J-critical state feedback operators K for Σ over $\mathcal{U}_{\text{exp}}$ correspond 1-1 to the WR state feedback operators $K_\natural$ for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ over $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}$ through $K = K' + K_{\natural_{\text{w}}}$. (See Theorem 8.4.5(f) for other $\mathcal{U}_*^*$'s.)

Conversely, given K, the optimal feedback for $\Sigma_\flat$ is $K_\natural = (K_s - K'_w)_{|\text{Dom}(A_\flat)}$, i.e., we must remove the preliminary feedback K' and replace it by K, the optimizing one.

Proof: The correspondence and (a) follow by combining Lemma 9.12.3 and Proposition 6.6.18(f) (interchange the roles of Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ for the other direction). The rest follows from Lemma 9.12.3. □

Two different solutions of the eIARE correspond to each other in the following way:

Lemma 9.12.5 *Let* $(\mathcal{P}_1, S_1, [\ \mathbb{K}_1 \mid \mathbb{F}_1\ ]) \in \overline{\text{eIARE}}(\Sigma, J)$. *Then*

$$(\mathcal{P}_2, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\Sigma, J) \Leftrightarrow \tag{9.227}$$

$$(\mathcal{P}_2 - \mathcal{P}_1, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\left(\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K}_1 & \mathbb{X}_1 \end{array}\right), S_1). \tag{9.228}$$

Equivalently, for any $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ]) \in \overline{\text{eIARE}}(\Sigma, J)$ we have

$$\text{eIARE}(\Sigma, J) = \mathcal{P} + \text{eIARE}(\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right], S). \tag{9.229}$$

Naturally, an analogous result holds for $(\mathcal{P}_k, S_k, K_k)$ $(k = 1, 2)$ in the regular case.

Proof: Set $\Sigma^{\mathcal{P}_1} := \left(\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K}_1 & \mathbb{X}_1 \end{array}\right) \in \text{WPLS}(U, H, U)$.

1° "$\Rightarrow$": Let $(\mathcal{P}_2, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\Sigma, J)$ and $t > 0$. Set $\mathcal{P} := \mathcal{P}_2 - \mathcal{P}_1$. Substract the two Lyapunov equations to obtain

$$(\mathbb{K}_2^t)^* S_2 \mathbb{K}_2^t = \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + (\mathbb{K}_1^t)^* S_1 \mathbb{K}_1^t. \tag{9.230}$$

This is the first (i.e., Lyapunov) equation of the eIARE for $\left(\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K}_1 & \mathbb{X}_1 \end{array}\right)$ and S_1. Now $(\mathbb{X}_2^t)^* S_2 \mathbb{X}_2^t = \mathbb{D}^{t\,*} J \mathbb{D}^t + \mathbb{B}^{t\,*} \mathcal{P}_1 \mathbb{B}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t = (\mathbb{X}_1^t)^* S_1 \mathbb{X}_1^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t$, and

$$-(\mathbb{X}_2^t)^* S_2 \mathbb{K}_2^t = \mathbb{D}^{t\,*} J \mathbb{C}^t + \mathbb{B}^{t\,*} \mathcal{P}_2 \mathbb{A}^t = -(\mathbb{X}_1^t)^* S_1 \mathbb{K}_1^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t. \tag{9.231}$$

Thus, the three equations of the eIARE are satisfied, i.e., $(\mathcal{P}, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\Sigma^{\mathcal{P}_1}, S_1)$.

2° "$\Leftarrow$": Let $(\mathcal{P}_2 - \mathcal{P}_1, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\Sigma^{\mathcal{P}_1}, S_1)$. By going 1° backwards, we see that $(\mathcal{P}_2, S_2, [\ \mathbb{K}_2 \mid \mathbb{F}_2\ ]) \in \overline{\text{eIARE}}(\Sigma, J)$. □

As Example 9.13.9 shows, the CARE may have solutions $(\mathcal{P} + \Delta, S, K)$ for infinitely many $\Delta = \Delta^* \in \mathcal{B}(H)$. If one of these, say $\mathcal{P}$, is $\mathcal{U}_*^*$-stabilizing (or at least P-stabilizing), then Δ corresponds to "fake cost", i.e., $\mathcal{P} + \Delta = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft + \Delta$, where $\mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ is the corresponding (optimal) closed-loop cost. The following corollary formulates necessary and sufficient conditions:

Corollary 9.12.6 *Let* $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ]) \in \overline{\text{eIARE}}(\Sigma, J)$ *and* $\Delta = \Delta^* \in \mathcal{B}(H)$. *Then (i)⇔(ii) (⇔(iii) provided that* $[\ \mathbb{K} \mid \mathbb{F}\ ]$ *is admissible):*

(i) $(\mathcal{P}+\Delta, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}) \in \overline{\text{eIARE}}(\Sigma, J)$*;*

(ii) $\Delta = \mathbb{A}^{t\,*}\Delta\mathbb{A}^t$ *and* $\mathbb{B}^{t\,*}\Delta\mathbb{A}^t = 0 = \mathbb{B}^{t\,*}\Delta\mathbb{B}^t$ *for all* $t \geq 0$*;*

(iii) $\Delta = \mathbb{A}^{t}_{\circlearrowleft}{}^{*}\Delta\mathbb{A}^t_{\circlearrowleft}$ *and* $\mathbb{B}^{t}_{\circlearrowleft}{}^{*}\Delta\mathbb{A}^t_{\circlearrowleft} = 0 = \mathbb{B}^{t}_{\circlearrowleft}{}^{*}\Delta\mathbb{B}^t_{\circlearrowleft}$ *for all* $t \geq 0$.

Proof: By Lemma 9.12.5, we have (i) iff $\Delta \in \text{eIARE}(\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right], S)$, i.e., iff (ii) holds.

Assume that $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is admissible for Σ. Then $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is admissible for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$ with closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline 0 & I \end{array}\right]$, so that $\Delta \in \text{eIARE}(\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right], S)$ becomes equivalent to (iii), by Lemma 9.10.1(b4)(i)&(iv). □

If, e.g., Σ is exponentially stable, then any stabilizing solution of the IARE leads to the spectral factorization $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, by Proposition 9.8.11(d1) (see also Corollary 9.9.11). Under weaker assumptions than those in Proposition 9.8.11, we can still obtain a "pseudospectral factorization", a weak form of $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$":

Proposition 9.12.7 *Let* $\Sigma_{\text{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right] \in \text{WPLS}(U, H, Y \times U)$*, and let (9.160) hold for all* $t > 0$ *and some* $J, S, \mathcal{P} \in \mathcal{B}$*, where* $\mathbb{X} := I - \mathbb{F}$*. Then*

$$\langle \mathbb{X}v, \pi_{(-\infty,t)} S\mathbb{X}u\rangle = \langle \mathbb{D}v, \pi_{(-\infty,t)} J\mathbb{D}u\rangle + \langle \mathbb{B}\tau^t v, \mathcal{P}\mathbb{B}\tau^t u\rangle \quad (u, v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}; U),\ t \in \mathbf{R}). \tag{9.232}$$

Moreover, we have the following:

(a) If $\mathbb{B}$*,* $\mathbb{D}$ *and* $\mathbb{X}$ *are stable, then*

$$\mathbb{X}^* S\mathbb{X} = \mathbb{D}^* J\mathbb{D} + \operatorname*{s-lim}_{t\to+\infty} \tau^{-t}\mathbb{B}^* \mathcal{P}\mathbb{B}\tau^t. \tag{9.233}$$

If, in addition, $\mathbb{B}$ *is strongly stable, then* $\mathbb{X}^* S\mathbb{X} = \mathbb{D}^* J\mathbb{D}$*.*

(b) If $\mathbb{B}$ *and* $\mathbb{D}$ *are strongly stable and* $u \in \mathrm{L}^2(\mathbf{R}; U)$*, then* $\mathbb{X}^*\pi_{[-T,t)} S\mathbb{X}u \to \mathbb{D}^* J\mathbb{D}u$ *in* $\mathrm{L}^2(\mathbf{R}; U)$*, as* $t, T \to +\infty$ *(independently); in particular, we have (uniformly in v)*

$$\lim_{T,t\to+\infty} \langle \pi_{[-T,t)}\mathbb{X}v, \pi_{[-T,t)} S\mathbb{X}u\rangle = \langle \mathbb{D}v, J\mathbb{D}u\rangle \qquad (u, v \in \mathrm{L}^2(\mathbf{R}; U)). \tag{9.234}$$

(Naturally, the statements include the convergence of limits presented.)

Recall that the eIARE implies (9.160). If, e.g., $\mathbb{D}$ and $\mathbb{X}$ are stable, $S \in \mathcal{GB}(U)$ and (P) holds on $\operatorname{Ran}(\mathbb{B})$, then we have the spectral factorization $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, by (9.232) and continuity.

Proof: 1° Let $v_k \in \mathrm{L}^2([-t, 0); U)$ $(k = 1, 2)$, and substitute $\tau^{-t} v_k$ $(\in \mathrm{L}^2([0, t); U))$ into (9.160) to obtain (note that $\pi_J \tau^r = \tau^r \pi_{r+J}$)

$$\langle \mathbb{X}v_2, \pi_- S\mathbb{X}v_1\rangle = \langle \mathbb{D}v_2, \pi_- J\mathbb{D}v_1\rangle + \langle \mathbb{B}v_2, \mathcal{P}\mathbb{B}v_1\rangle. \tag{9.235}$$

This holds for arbitrary $t \in \mathbf{R}$, hence for arbitrary $v_k \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}; U)$ $(k = 1, 2)$, because $\pi_+ v_k$ does not affect the equation.

Let now t, u, v be as in (9.232) and substitute $v_1 := \tau^t u$, $v_2 := \tau^t v$ into (9.235) to obtain (9.232).

(a) Let $\mathbb{B},\mathbb{D},\mathbb{X}$ be stable. By (9.232), we have $\mathbb{X}^*\pi_{(-\infty,t)}S\mathbb{X}u = \mathbb{D}^*\pi_{(-\infty,t)}J\mathbb{D}u+\tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t u$ for all $u\in \mathrm{L}^2_{\mathrm{c}}$, hence for all $u\in\mathrm{L}^2$, by continuity. Let $t\to+\infty$ and use Corollary B.3.8 to obtain (9.233). The second claim is obvious (because $\tau^{-t}\mathbb{B}^* = (\mathbb{B}\tau^t)^*$ is bounded $H\to\mathrm{L}^2$).

(b) Now $\mathbb{X}^{\mathrm{d}}\pi_{[-t,T)}\in\mathcal{B}(\mathrm{L}^2)$ for all $t,T>0$, by Lemma 6.1.11 (since $\mathbb{B}^{\mathrm{d}}$ is stable), hence $\mathbb{X}^*\pi_{[-T,t)},\pi_{[-T,t)}\mathbb{X}\in\mathcal{B}(\mathrm{L}^2)$. Thus, from (9.232), we obtain

$$\mathbb{X}^*S\pi_{[-T,t)}\mathbb{X}u = \mathbb{D}^*\pi_{[-T,t)}J\mathbb{D}u+\pi_{[-T,t)}\tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t\pi_{[-T,t)}u \tag{9.236}$$

(in $\mathrm{L}^2(\mathbf{R};U)$) for all $u\in\mathrm{L}^2_{\mathrm{c}}([-T,+\infty))$ (apply (9.232) to each $v\in\mathrm{L}^2_{\mathrm{c}}$ and recall that $\mathrm{L}^2_{\mathrm{c}}$ is dense in $\mathrm{L}^2=(\mathrm{L}^2)^*$). By continuity, this holds for all $u\in\mathrm{L}^2$. Since $\pi_{\mathbf{R}\setminus[-T,t)}u\to0$ (see Corollary B.3.8), we have $\mathbb{B}\tau^t\pi_{[-T,t)}u = \mathbb{B}\tau^t u-\mathbb{B}\tau^t\pi_{\mathbf{R}\setminus[-T,t)}u\to0$, as $T,t\to+\infty$. Consequently

$$\mathbb{D}^*\pi_{[-T,t)}J\mathbb{D}u+\pi_{[-T,t)}\tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t\pi_{[-T,t)}u\to\mathbb{D}^*J\mathbb{D}u, \tag{9.237}$$

i.e., $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u\to\mathbb{D}^*J\mathbb{D}u$ in L^2, as $t,T\to+\infty$ (independently). Thus, (b) holds. □

As indicated in Chapter 5, the factorization "$\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$" corresponding to a J-critical state feedback pair for a stable system need not be stable (that is, a spectral factorization), but we may have $\widehat{\heartsuit}\widehat{\mathbb{X}}\in\mathrm{H}^2\setminus\mathrm{H}^\infty$ in case $\dim U<\infty$. We state this and more general results below:

Lemma 9.12.8 ($\mathbb{B},\mathbb{D}$ stable $\Rightarrow \mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$ & $\widehat{\mathbb{X}}\in\mathcal{G}\mathrm{H}$) *Assume that $\mathbb{B}$ and $\mathbb{D}$ are stable, $\vartheta=0$, and $(\mathcal{P},S,\begin{bmatrix}\mathbb{K} & \mathbb{F}\end{bmatrix})$ is a $\mathcal{U}^*_*$-stabilizing solution of the eIARE. Set $\mathbb{M}^{-1}:=\mathbb{X}:=I-\mathbb{F}$, $\mathbb{N}:=\mathbb{D}\mathbb{M}$, $\widehat{\mathbb{X}}^{\mathrm{d}}:=\widehat{\mathbb{X}}(\bar{\cdot})^*$. Let $r>0$. Then*

(a1) $\mathbb{X}\in\mathcal{G}\mathrm{TIC}_\omega(U)$ *for all* $\omega>0$.

(a2) $(\cdot+1)^{-1}\widehat{\mathbb{N}},(\cdot+1)^{-1}\widehat{\mathbb{M}},(\cdot+1)^{-1}\widehat{\mathbb{X}}^{\mathrm{d}}\in\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,*))$*; in particular,* $\widehat{\mathbb{X}}\in\mathcal{G}\mathrm{H}(\mathbf{C}^+;\mathcal{B}(U))$.

(b1) $\mathbb{N},\mathbb{M},\mathbb{X}^*$ *map* $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)\to\mathrm{L}^2$*, and* $\mathbb{X}^*\pi_\pm\mathbb{M}^*,\mathbb{M}\pi_{[-T,t)},\mathbb{X}^*\pi_{[-T,t)}\in\mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$ *for all* $T,t>0$.

(b2) $\mathbb{M}\pi_+\mathbb{X},\pi_{[-T,t)}\mathbb{X}\in\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U))\cap\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ *for each* $\omega>0$*, and* $\mathbb{M}\pi_+\mathbb{X}$ *and* $\pi_{[-T,t)}\mathbb{X}$ *have a continuous extensions to* $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$.

(c1) $\langle\mathbb{N}u,J\mathbb{N}v\rangle=\langle u,Sv\rangle$ *for all* $u,v\in\mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)$.

(c2) $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u\to\mathbb{D}^*J\mathbb{D}u$ *in* $\mathrm{L}^2(\mathbf{R};U)$*, as* $t,T\to+\infty$*, if* $\mathbb{B}$ *is strongly stable and* $u\in\mathrm{L}^2(\mathbf{R};U)$.

(d) **($\dim U<\infty\Rightarrow\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}}=\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$)** *If* $\dim U<\infty$*, then* $(\cdot+1)^{-1}\widehat{\mathbb{X}}^{\pm1}\in\mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U))\cap\mathrm{L}^2(i\mathbf{R};\mathcal{B}(U))$*, and* $\widehat{\mathbb{X}}\in\mathcal{G}\mathcal{B}(U)$ *and* $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}}=\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ *a.e. on* $i\mathbf{R}$.

(e) **($(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1}=\mathbb{M}\pi_+S^{-1}\mathbb{M}^*$)** *If* $\mathbb{T}:=\pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ *is invertible on* $\mathrm{L}^2(\mathbf{R}_+;U)$. $\mathbb{B}$ *is strongly stable, then* $S\in\mathcal{G}\mathcal{B}(U)$ *and* $\mathbb{T}^{-1}=\mathbb{M}\pi_+S^{-1}\mathbb{M}^*\in\mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$.

(See Lemma 14.2.8 for additional information.) Recall that $\vartheta = 0$ for $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$.

Proof: (a1)–(c2) These follow as in the proof of Lemma 14.2.8 (alternatively, we can use discretization).

(d) By (a2) and Theorem 3.3.1(c3), we have $(\cdot + 1)^{-1}\widehat{\mathbb{X}}^{\pm 1} \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}^2(i\mathbf{R};\mathcal{B}(U))$. By continuity, $\widehat{\mathbb{X}}\widehat{\mathbb{M}} = I = \widehat{\mathbb{M}}\widehat{\mathbb{X}}$ and $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ a.e. on $i\mathbf{R}$ (since these hold on $\mathbf{C}^+$); in particular, $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ a.e. on $i\mathbf{R}$.

Since $\langle \mathbb{X}u, S\mathbb{X}v\rangle_{\mathrm{L}^2} = \langle \mathbb{D}u, J\mathbb{D}v\rangle_{\mathrm{L}^2}$, i.e., $\langle \widehat{\mathbb{X}}\widehat{u}, S\widehat{\mathbb{X}}\widehat{v}\rangle_{\mathrm{L}^2(i\mathbf{R};U)} = \langle \widehat{\mathbb{D}}\widehat{u}, J\widehat{\mathbb{D}}\widehat{v}\rangle_{\mathrm{L}^2(i\mathbf{R};U)}$ for all $\widehat{u},\widehat{v} \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$, we have $\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ a.e. on $i\mathbf{R}$.

(e) This follows from Lemma 14.2.8(e), by discretization. □

Notes

For bounded C, part of Lemma 9.12.2 (with "$=$" in place of "$\leq$") is known (see [CZ, Section 5]). For general C, Lemma 9.12.2(a2) was essentially given in [Grabowski91, Theorem 3] and (c) in [Sbook, Theorem 9.5.2] (both with "$=$" in place of "$\leq$"); the latter also contains further necessary and sufficient conditions for $\begin{bmatrix} A \\ C \end{bmatrix}$ to generate an output-stable WPLS.

Formula (9.224) was used in [S98b]. Lemma 14.2.4 is from [Mal00], and Lemma 9.12.5 is its IARE variant. Part of Proposition 9.12.7 was given in [Mik97b].

9.13 Examples of Riccati equations

Though the day of my destiny's over,
And the star of my fate hath declined,
Thy soft heart refused to discover
The faults which so many could find.

— Lord Byron (1788–1824), "Stanzas to Augusta"

In this section, we shall illustrate by examples several "pathological" cases due to which the general Riccati equation theory is rather complex. The part corresponding to different $\mathcal{U}_*^*$'s may be new even for finite-dimensional systems.

The studies on Riccati equations have mainly concentrated on exponentially stabilizing solutions of Riccati equations. The articles on optimization over $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{sta}}$ have usually either neglected uniqueness or used estimatability (e.g., put some cost on the state) to reduce $\mathcal{U}_{\text{out}}$ to $\mathcal{U}_{\text{exp}}$.

We found the condition (P) for the SOS-stable case in [Mik97b] (cf. Corollary 9.9.11); the condition (PB) is new. In Proposition 9.13.1(a)&(b)&(e) we show that these conditions are necessary. The difference between $\mathcal{U}_{\text{out}}$- and $\mathcal{U}_{\text{exp}}$-stabilizing solutions and the role of coprimeness are illustrated in Example 9.13.2. We also present plenty of examples that illustrate the different aspects of the role of the signature operator S and the insignificance of D^*JD (when the latter does not coincide with the former).

By discretization (see Proposition 9.8.7), all our examples apply also in discrete time (eDAREs). In particular, our CARE "counter-"examples are also eCARE, IARE, eIARE, DARE and eDARE "counter-"examples.

Most examples also have obvious discrete-time analogies (without artificial discretization); in particular, all our finite-dimensional examples have finite-dimensional discrete-time counterparts.

Our examples are mathematically motivated; for physically more motivated examples on WPLSs and Riccati equations, see, e.g., [Sal87].

The following proposition summarizes which conclusions can be drawn from the different examples:

Proposition 9.13.1 (Non-$\mathcal{U}_*^*$-stabilizing solutions)

(a) **((P)$\not\Rightarrow$(PB))** *[Example 9.13.2] There can be several P-SOS-stabilizing solutions of the CARE, only one of which is $\mathcal{U}_{\text{out}}$-stabilizing.*

A $\mathcal{U}_{\text{out}}$-stabilizing solution need not be internally stabilizing, even if there were exponentially stabilizing solutions.

(b) **((PB2)$\not\Rightarrow$(P))** *[Example 9.13.9] There can be several r.c.-stabilizing solutions $\mathcal{P}$ satisfying $\langle \mathbb{B}^t x_0, \mathcal{P}\mathbb{A}_{\circlearrowleft}^t x_0\rangle \to 0$, of the CARE for a stable minimal system, only one of which is P-stabilizing (hence $\mathcal{U}_{\text{out}}$-stabilizing and equal to the J-critical cost operator over $\mathcal{U}_{\text{out}}$ (and over $\mathcal{U}_{\text{sta}}$)).*

Moreover, our example is exactly reachable and of the standard LQR (minimization) form with $\mathbb{A}^*$ strongly stable.

(c1) **($\mathbb{D} \in \mathrm{ULR} \not\Rightarrow \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix} \in \mathrm{WR}$)** *[WW, Example 11.5] There is $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}) \cap \mathrm{ULR}$, for which $D = 0$ and $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \gg 0$ with an irregular (I-spectral factor) $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(\mathbf{C})$ (in fact, $\widehat{\mathbb{X}}(2^k) = 2 + (-1)^k$, hence $\mathbb{X}, \mathbb{X}^{-1} \notin \mathrm{WR}$).*

Alternatively, we can choose $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}, \mathbf{C}^2) \cap \mathrm{ULR}$ s.t. $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, $D_1 = 0$ and $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \gg 0$ for this same $\mathbb{X}$ multiplied by some positive constant.

In particular, if Σ is a strongly stable realization of (either) $\mathbb{D}$ (as in (6.11)), then the IARE has a strongly PB-r.c.-stabilizing solution, but there are no PB-output-stabilizing solutions of the eCARE.

(c2) **(Bounded $C \not\Rightarrow S = D^*JD$)** *[Example 9.13.8] There is a stable positively I-coercive WPLS (in standard LQR form) with $J = I$, bounded C and $D^*C = 0$ s.t. $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$ but $S \neq D^*JD$, K is unbounded and $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_{\mathrm{w}}^*)$.*

(d1) Condition $S = D^*JD \gg 0$ does not guarantee sufficient coercivity for the existence of a minimizing (J-critical) control over $\mathcal{U}_{\mathrm{exp}}$; see Example 9.13.5.

(d2) We may have $D^*JD \gg 0 \gg S$ for a maximizing solution, hence D^*JD does not characterize the signature properties of the problem; see Example 9.13.7.

(e) Let $\Sigma \in \mathrm{SOS}$. If there is a stable, stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & I - \mathbb{X} \end{bmatrix})$ of the IARE s.t. $\langle \mathbb{A}^t x_0, \mathcal{P}\mathbb{A}^t x_0 \rangle \to 0$, as $t \to +\infty$, for all $x_0 \in H_{\mathbb{B}}$, then $\mathbb{X}^* S \mathbb{X}$ is the unique spectral factorization of $\mathbb{D}^* J \mathbb{D}$, and the IARE has a (unique) stable PB-r.c.-stabilizing solution, namely $(\widetilde{\mathcal{P}}, S, \begin{bmatrix} \mathbb{K} & I - \mathbb{X} \end{bmatrix})$, where $\widetilde{\mathcal{P}} := \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$.

Thus, the open- and closed-loop systems for $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are identical; the only difference is that $\widetilde{\mathcal{P}}$ is the J-critical cost operator and $\mathcal{P} = \widetilde{\mathcal{P}} + \text{s-lim}_{t \to +\infty} \mathbb{A}(t)^* \mathcal{P} \mathbb{A}(t)$, as in Example 9.13.12.

(f) Even if Σ is strongly stable and the Popov Toeplitz operator $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+$ is invertible (as in Proposition 8.3.10), so that there is a unique J-critical control for each $x_0 \in H$ over $\mathcal{U}_*^* := \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$, 1. it may be that there is no J-critical state feedback pair (Example 11.3.7(a)), 2. it may be that the J-critical state feedback pair $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}$ and its closed-loop form $\begin{bmatrix} \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{bmatrix}$ are unstable (Example 11.3.7(b)) though they and Σ are UHPR; in particular there is no spectral factorization of $\mathbb{D}^* J \mathbb{D}$.

Proof: (a)&(b)&(c2)&(d1)&(d2)&(f) See corresponding examples.

(c1) (We do not know whether this can happen for exponentially stable $\mathbb{D}$.) The existence of $\mathbb{X}$ is proved in [WW, Example 11.5] (although any irregular $\mathbb{X}$ would do). We can then simply take $\widehat{\mathbb{D}}(s) := e^{-s}\widehat{\mathbb{X}}(s)$ in the former case; in the latter case we must multiply $\mathbb{X}$ by a positive constant so that $\mathbb{X}^*\mathbb{X} \gg I$. Let $\mathbb{Z}$ be an I-spectral factor of $\mathbb{X}^*\mathbb{X} - I$, and set $\mathbb{D}_1 := e^{-\cdot}\mathbb{Z}$ to guarantee that $\mathbb{D} \in \mathrm{ULR}$ (as in [WW]).

By Corollary 9.9.11(Crit4SOS), we have $\mathbb{X} = I - \mathbb{F}$, where $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ is the unique $\mathcal{U}_{\mathrm{out}}$-stabilizing solution of the eIARE. Because $\mathbb{F}$ is not WR, the eCARE does not have a $\mathcal{U}_{\mathrm{out}}$-stabilizing solution.

(e) Now $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ on $\mathrm{Ran}(\mathbb{B})$, by (9.159). Consequently, $\mathbb{X}^* S\mathbb{X}u = \mathbb{D}^* J\mathbb{D}u$ for $u \in \mathrm{L}^2$, by (9.160) (since $\mathbb{C}\mathbb{B}^t = \pi_+\mathbb{D}\pi_-\tau^t\pi_+ = \tau^t\pi_{[t,\infty)}\mathbb{D}\pi_{[0,t)} \to 0$, because $\pi_{[t,\infty)}\mathbb{D}\pi_{[0,t)} \to \pi_\emptyset\mathbb{D}\pi_+ = 0$, as $t \to +\infty$, by Corollary B.3.8). Because $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, the operator $\widetilde{\mathcal{P}}$ is q.r.c.-P-stabilizing, by Theorem 9.9.10(b)&(e2), hence it is stable and PB-r.c.-stabilizing (and unique). By (9.159), we have

$$\mathcal{P} - \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t = \mathbb{C}^{t*}J\mathbb{C}^t - \mathbb{K}^{t*}J\mathbb{K}^t \to \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K} = \widetilde{\mathcal{P}} \tag{9.238}$$

strongly, as $t \to +\infty$. □

We continue by presenting the actual examples, starting with a simple (unobservable) system illustrating that the solutions over $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ may be different:

Example 9.13.2 (Std. LQR: $\mathcal{U}_{\mathrm{out}}$-stabilizing $\neq$ $\mathcal{U}_{\mathrm{exp}}$-stabilizing, (P)$\not\Rightarrow$(PB)) Let $U = \mathbf{C} = H = Y$, $A = 1 = B = D = J$, $C = 0$. Then the CARE becomes

$$S = 1, \;\; SK = -\mathcal{P}, \;\; \mathcal{P} + \mathcal{P} + 0 = K^* SK = \mathcal{P}^2. \tag{9.239}$$

Thus, the solutions of the CARE are given by $(0,1,0)$ (PB-r.c.-SOS-stabilizing, hence $\mathcal{U}_{\mathrm{out}}$-stabilizing) and $(2,1,-2)$ ($\mathcal{U}_{\mathrm{exp}}$-stabilizing, hence exponentially P-stabilizing). The corresponding closed-loop systems are

$$\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \quad \text{and} \quad \Sigma'_{\circlearrowleft} := \left[\begin{array}{c|c} -1 & 1 \\ \hline 0 & 1 \\ -2 & 0 \end{array}\right]. \tag{9.240}$$

Now $\mathbb{D} = 1 = \mathbb{D}_{\circlearrowleft} = \mathbb{D}'_{\circlearrowleft}$, $\mathbb{C} = 0 = \mathbb{C}_{\circlearrowleft} = \mathbb{C}'_{\circlearrowleft}$, $\mathbb{K}_{\circlearrowleft} = 0 = \mathbb{F}_{\circlearrowleft}$, $\mathbb{X} = I$, hence $(0,1,0)$ is, indeed, the unique $\mathcal{U}_{\mathrm{out}}$-stabilizing solution, (in fact, it is PB-r.c.-SOS-stabilizing and cost-minimizing; the condition (PB) is trivially satisfied for $\mathcal{P} = 0$), whereas $(2,1,-2)$ is the unique $\mathcal{U}_{\mathrm{exp}}$-stabilizing solution.

Let us see why this happens: One easily verifies that $\widehat{\mathbb{N}}'(s) = (s-1)/(s+1) = \widehat{\mathbb{M}}'(s)$ (indeed, $\mathbb{N}'$ and $\mathbb{M}'$ are not q.r.c., because both have a zero at $s = 1$), and that $\langle \mathbb{N}' u_\flat, J\mathbb{C}'_{\circlearrowleft} x_0\rangle = 0$ for all $x_0 \in H$ and $u_\flat \in \pi_+\mathrm{L}^2$ (we have $(J\mathbb{C}'_{\circlearrowleft} x_0)(t) = -2\mathrm{e}^{-t}x_0$ ($t \geq 0$)), but $\langle \mathbb{D}u, J\mathbb{C}'_{\circlearrowleft} x_0\rangle \neq 0$ for $x_0 \neq 0$, $u \in \pi_+\mathrm{L}^2$ s.t. $\widehat{u}(1) \neq 0$.

Thus, although $\Sigma'_{\circlearrowleft}$ is J-critical over $\mathcal{U}_{\mathrm{out}}^{\Sigma_{\circlearrowleft}}$, i.e., J-critical w.r.t. "closed-loop stable" controls ($u_\flat \in \pi_+\mathrm{L}^2$), the system $\Sigma'_{\circlearrowleft}$ "does not see" signals with $\widehat{u}(1) \neq 0$, i.e., $u_\flat := (\mathbb{M}')^{-1}u \notin \mathrm{L}^2$ for such u, because $\mathbb{N}'$ and $\mathbb{M}'$ are not q.r.c. One can also verify that

$$\langle \mathbb{B}^t u, \mathcal{P}\mathbb{A}'_{\circlearrowleft} x_0\rangle_H = 2\mathrm{e}^{-t}\overline{x_0}\int_0^t \mathrm{e}^t\mathrm{e}^{-s}u(s)\,ds \to 2\overline{x_0}\widehat{u}(1) \neq 0 \tag{9.241}$$

when $x_0 \neq 0$ and $u \in \mathrm{L}^2$ is s.t. $\widehat{u}(1) \neq 0$, i.e., (PB) does not hold for $\mathcal{P} = 2$ (unless we replace $\mathcal{U}_{\mathrm{out}}$ by $\mathcal{U}_{\mathrm{exp}}$ in (PB), see Theorem 9.9.1(g1)). Trivially, (PB) holds for $\mathcal{P} = 0$. ◁

See also Example 6.6.16

Even for exponentially stable systems, the existence of a unique J-critical control does not guarantee that $\mathbb{D}$ is J-coercive, nor that the CARE has a solution:

Example 9.13.3 (Singular control: unique optimum without CARE and J-coercivity, $\nexists(D^*JD)^{-1}$) Assume that $\mathbb{A}$ is exponentially stable (e.g., $H = \mathbf{C}$ and $A = -1$), $C = 0 = B$, $J = I$, and $\mathbb{D} = D \in \mathcal{B}(U,Y)$ is one-to-one but not coercive (i.e., $S := D^*D \notin \mathcal{GB}(U)$). Let $\mathcal{U}_*^* \in \{\mathcal{U}_{\rm out}, \mathcal{U}_{\rm sta}, \mathcal{U}_{\rm str}, \mathcal{U}_{\rm exp}\}$.

Then $u = 0$ is the unique minimizing control for x_0, and $\mathcal{U}_*^*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$, for each $x_0 \in H$. The corresponding unique $\mathcal{U}_*^*$-stabilizing solution of the eCARE is given by $(0, D^*D, 0)$ (by Theorem 9.9.6, equation $SK = 0$ forces K to be zero).

However, $\mathbb{D}^*J\mathbb{D} = D^*D$ is not J-coercive over $\mathcal{U}_*^*$. ◁

Even when Σ is exponentially stable and $S > 0$, there need not exist any PB-stabilizing solutions nor minimizing control over $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm exp}$:

Example 9.13.4 (Stable Σ and $S > 0 \not\Rightarrow \exists\mathcal{U}_*^*$-stabilizing solution (or optimal control)) Take $U := \ell^2(\mathbf{N}) := Y$, $J := I$. Define $\mathbb{D} := D \in \mathcal{B}(U,Y)$ by $(Du_0)(n) := \mathrm{e}^{-2n}u_0(n)$ $(n \in \mathbf{N},\ u_0 \in U)$ (note that $0 < D^*D \not\gg 0$). Let Σ be the -1-stable exactly observable realization of D, so that $H = \mathrm{L}^2_{-1}(\mathbf{R}_+;Y)$, $\mathbb{C} = \pi_+$. Define $x_0 \in H$ and $u \in \mathrm{L}^2_{\rm loc}(\mathbf{R}_+;U)$

$$x_0(t) := \mathrm{e}^{-t}\sum_{n\in\mathbf{N}} \pi_{[n,n+1)}(t)e_n \in H, \tag{9.242}$$

$$u_\infty(t) := -\mathrm{e}^{-t}\sum_{n\in\mathbf{N}} \mathrm{e}^{2n}e_n \in H, \tag{9.243}$$

where $\{e_n\}$ is the natural base of U. Then $\mathbb{C}x_0 + Du = 0$, so that u_∞ minimizes $\mathcal{J}(x_0,u) := \|x_0 + Du\|_2^2$ over all $u \in \mathrm{L}^2_{\rm loc}(\mathbf{R}_+;U)$. Since D is one-to-one, u_∞ is the unique minimum.

However, $\mathcal{J}(x_0, \pi_{[0,T)}u_\infty) = \|\pi_{[0,T)}x_0\|_2^2 = (\mathrm{e}^{-2T}/2)^{1/2} \to 0$ as $T \to \infty$, and $\pi_{[0,T)}u_\infty \in \mathcal{U}_*^*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$ (we assume that $\mathcal{U}_*^* \in \{\mathcal{U}_{\rm exp}, \mathcal{U}_{\rm str}, \mathcal{U}_{\rm sta}, \mathcal{U}_{\rm out}\}$). Since $u_\infty \notin \mathrm{L}^2$, there is no minimum (hence no J-critical control) over $\mathcal{U}_*^*$, hence there is no $\mathcal{U}_*^*$-stabilizing solution of the eIARE (note that $S = D^*D > 0$).

◁

Now we show that for the existence of a minimum, 1. even for an exponentially stable system $(\Sigma_\flat^1)$, condition $S \gg 0$ is not sufficient, and 2. a system may be exponentially stabilizable and J-coercive over $\mathcal{U}_{\rm out}$ without being J-coercive over $\mathcal{U}_{\rm exp}$:

Example 9.13.5 (Exponentially stabilizable with $S = D^*JD \gg 0$ but no minimum over $\mathcal{U}_{\rm exp}$) Let $J = 1$.

(a) For $\Sigma := \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array}\right)$, we have $S = D^*JD = 1 \gg 0$, and $(0,1,0)$ is the unique solution of the CARE $\mathcal{P}^2 = 0$, and corresponding control $u = 0$ is minimizing over $\mathcal{U}_{\rm out}$ (and $\mathcal{U}_{\rm sta}$), but there is no minimum over $\mathcal{U}_{\rm exp}$ (nor $\mathcal{U}_{\rm str}$), and $\inf_{u\in\mathcal{U}_{\rm exp}(x_0)} \mathcal{J}(x_0,u) = 0$ for all $x_0 \in H$.

Here $\widehat{x}(s) = s^{-1}(x_0 + \widehat{u}(s))$, so that we must have $\widehat{u}(0) = -x_0$ in some sense (for the boundary function of $\widehat{u} \in \mathrm{H}^2(\mathbf{C}^+)$), but we wish to minimize $\mathcal{J}(x_0,u) =$

$\|\widehat{u}\|_2^2$ under this condition, and the latter expression can be taken arbitrarily small. In time domain, we have $x(t) = x_0 + \int_0^t u(r)\,dr$, so that $\mathcal{J}(x_0, u_n) \to 0+$, as $n \to \infty$, where $u_n := -x_0 n^{-1}\chi_{[0,n)} \in \mathcal{U}_{\rm exp}(x_0)$.

(b) By using the exponentially stabilizing state feedback operator $K = -1$ (and then removing the state feedback (= third) row), we obtain $\Sigma_\flat^1 := \left(\begin{array}{c|c} -1 & 1 \\ \hline -1 & 1 \end{array}\right)$. The CARE becomes again $\mathcal{P}^2 = 0$ with the unique solution $(0,1,1)$, hence there is no minimizing control over $\mathcal{U}_{\rm exp} = \mathcal{U}_{\rm out}$, although the infimum cost over $\mathcal{U}_{\rm exp}$ is again zero.

Now $y = x' = u - x$, so $\mathcal{J}(x_0, u) = \|u - x\|_2^2$, and it would be optimal to have $u \equiv x \equiv x_0$, but this is not allowed, since we require that $u \in \mathrm{L}^2$.

(c) The system Σ is J-coercive over $\mathcal{U}_{\rm out}$ (since $\mathbb{D}^* J \mathbb{D} = 1 \gg 0$), hence we knew that there had to be a unique minimizing control over $\mathcal{U}_{\rm out}$; however, Σ is not J-coercive over $\mathcal{U}_{\rm exp}$ (an alternative proof for this is that $(s-A)^{-1}B = s^{-1}$ is not majorized by $\widehat{\mathbb{D}} \equiv 1$ on $i\mathbf{R}$; see Proposition 10.3.2(iv)&(i)).

The system $\Sigma_\flat^1$ is not J-coercive over either $\mathcal{U}_{\rm out}$ or $\mathcal{U}_{\rm exp}$.

Note that the state feedback $K = -1$ for Σ used in (b) is not r.c.-stabilizing, since $\mathbb{N} = \mathbb{M} = s/(s+1)$ has a common zero at $s = 0$; thus, the minimizing control $u = 0$ over $\mathcal{U}_{\rm out}$ is lost in this preliminary stabilization. ◁

Even if $S = 0$ and the J-critical control is not unique, there might be only one J-critical control in feedback form:

Example 9.13.6 ($\mathcal{U}_{\rm out}$: Unique $\mathbb{K}_\circlearrowleft$ although $S = 0$) Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$ (i.e., $\mathbb{A} = I$ and $\mathbb{B} = 0 = \mathbb{C} = \mathbb{D}$). Then the eCARE becomes $0 = K^* S K$, $X^* S X = 0$, $X^* S K = 0$ (see Theorem 9.9.6(e2) and the remark in its proof). The admissible solutions are the ones with $X \in \mathcal{GB}(U)$; for them we have $S = 0$, so that $\mathcal{P}$, K and X can be arbitrary.

Since $B = 0$, we have $A_\circlearrowleft = A + BK = A$, so that $\mathbb{A}_\circlearrowleft = \mathbb{A} = I$ and hence $K = K_\circlearrowleft = 0$ is the only output-stabilizing state feedback operator (note that $H_1 := \mathrm{Dom}(A) = H$). Condition (P) requires that $\mathcal{P} = 0$. Thus, all P-output-stabilizing solutions are given by $(0, 0, \left[\begin{array}{c|c} 0 & I - X \end{array}\right])$ $(X \in \mathcal{GB}(U))$ (and they all are P-SOS-r.c.-stabilizing, hence $\mathcal{U}_{\rm out}$-stabilizing, by Theorem 9.9.1(b)).

Obviously, $\mathcal{U}_{\rm out}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U)$ for all $x_0 \in H$, and each control is J-critical (the cost is zero for each control). Nevertheless, 0 is the only J-critical control in state feedback form. ◁

(To obtain the corresponding discrete-time example we must set $A = I$, $B = C = D = 0$ (so that still $\mathbb{A} = I$, $\mathbb{B} = \mathbb{C} = \mathbb{D} = 0$).)

Even for $\mathbb{D}, \mathbb{X} \in$ MTIC, the operator $D^* J D$ need not contain any information on the signature properties of the problem:

Example 9.13.7 [$D^* J D \gg 0 \gg S$] Let $\Sigma \in$ SOS, $\mathbb{D} = \left[\begin{smallmatrix} 2\tau^{-1} \\ I \end{smallmatrix}\right] \in \mathrm{MTIC_d}(U, U^2)$, $J = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$. Then $D^* J D = I \gg 0$ but $\mathbb{D}^* J \mathbb{D} = -3I = I^* S I \in \mathrm{TI}(U)$, where $S = -3I \ll 0$.

It follows that the CARE has a ULR unique $\mathcal{U}_{\rm out}$-stabilizing solution, and this solution is q.r.c.-SOS-stabilizing and maximizing over $\mathcal{U}_{\rm out}$ (and $\mathcal{U}_{\rm sta}$ and $\mathcal{U}_{\rm str}$ if $\mathbb{A}$ is strongly stable), by (9.139).

We can set $\mathbb{D} := \begin{bmatrix} 2\tau^{-1} \\ D_2 \end{bmatrix}$, $J = \begin{bmatrix} -I & 0 \\ 0 & \widetilde{J} \end{bmatrix}$ for any $Y_2, D_2 \in \mathcal{B}(U, Y_2)$, $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}(Y_2)$ s.t. $\|D_2^* \widetilde{J} D_2\| < 4$ without affecting the above (except that S is altered but still $S \ll 0$). In particular, $D^* J D = D_2^* \widetilde{J} D_2$ may be uniformly/strictly/nonstrictly positive/negative, zero, or indefinite. $\triangleleft$

G. Weiss and H. Zwart [WZ] have shown that even if C is bounded and $\mathbb{D}, \mathbb{X} \in$ ULR, we may have $S \neq D^* J D$ and K unbounded:

Example 9.13.8 [C bounded, $B_w^* \mathcal{P}, K$ not] Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ be as in Example 9.8.15, but set

$$\mathcal{J}(x_0, u) := \int_0^\infty (\|\pi_{[0,1)} x(t)\|_H^2 + \|u(t)\|_U^2)\, dt, \tag{9.244}$$

i.e., $\left[\begin{array}{c|c} C & D \end{array}\right] := \left[\begin{array}{c|c} \pi_{[0,1)} & 0 \\ 0 & I \end{array}\right]$ and $J := I$. These operators are bounded, and one easily verifies $\mathbb{D}^* J \mathbb{D} = 2I$ and that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(\mathbf{C}, H, H \times \mathbf{C})$, where $H := \mathrm{L}^2(\mathbf{R}_+)$ (see (19) of [WZ]). Thus, $S := 2I$, $\mathbb{X} := I$ defines a spectral factorization of $\mathbb{D}^* J \mathbb{D}$ (by Corollary 9.9.11, this corresponds to the stabilizing solution over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$).

However, $D^* J D = I \neq S$, hence $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_w^*)$ and $K = -B_w^* \mathcal{P}$ is unbounded (by Proposition 9.11.4(b3)). One can verify from Proposition 8.3.10 that $\mathcal{P} = \pi_{[0,1)} + \frac{1}{2}\pi_{[1,\infty)}$. (See [WZ] for details.) $\triangleleft$

Even for a minimal system (and standard LQR cost function), condition (PB) is not always superfluous:

Example 9.13.9 ((P) is necessary even for minimal weakly stable Σ) We construct here an exactly reachable and approximately observable (weakly, even strong*) stable system Σ with scalar input s.t. the LQR for Σ has a unique solution over $\mathcal{U}_{\mathrm{out}}$, but there are also other (non-PB-)r.c.-stabilizing solutions.

Take $U = \mathbf{C}$, $H = \mathrm{L}^2(\mathbf{R}_-; \mathbf{C})$, $Y = \ell^2(\mathbf{N})$, $\mathbb{A} = \tau\pi_-$, $\mathbb{B} = \pi_-$, $\mathbb{C}_1 = (2^{-k/2}\tau_{-k}\pi_{[-k,0]})_{k\in\mathbf{N}}$, $\mathbb{D}_1 = (2^{-k/2}\tau_{-k})_{k\in\mathbf{N}}$, $\mathbb{C} := \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$, $\mathbb{D} := \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, $J = I$ to get a standard LQR form minimal (exactly reachable ($\mathbb{B}\mathbb{B}^* = \pi_- \gg 0 \in \mathcal{B}(H)$) and approximately observable ($\mathbb{C}^*\mathbb{C} = \sum_k 2^{-k}\pi_{[-k,0]} = \sum_{k=0}^\infty 2^{-k}\pi_{[-k-1,-k]} > 0$)) stable minimization problem (the minimization of $\mathcal{J}(u, x_0) := \|\mathbb{D}_1 u + \mathbb{C}_1 x_0\|^2 + \|u\|^2$).

The generators of $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ are $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A = \frac{d}{d\theta}$, $B = \delta_0$, $H_1 := \mathrm{Dom}(A) = \mathrm{W}_0^{1,2}(\mathbf{R}_-)$, $(s - A)^{-1} B : u_0 \mapsto e^{s\cdot} u_0 \in H_B = \mathrm{W}^{1,2}(\mathbf{R}_-)$, $A^* = -\frac{d}{d\theta}$, $H_1^* := \mathrm{Dom}(A^*) = \mathrm{W}^{1,2}(\mathbf{R}_-)$, $B_w^* = \delta_{0-}^*$; $C_1 := (2^{-k/2}\delta_{-k}^*)_k : H_1 \to \ell^2(\mathbf{N})$, $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (here θ is the argument of an element $x_0 \in H$; cf. Example 6.2.14).

Let $\mathcal{P}$ be a multiplication operator, say, $\mathcal{P} \in \mathrm{L}^\infty(\mathbf{R}_-)\cdot$. Then $B_w^* \mathcal{P} x_0 = (\mathcal{P} x_0)(0-) = 0$ for all $x_0 \in H_1$ (we use this only for $\mathcal{P}$ s.t. $\mathcal{P} x_0$ (left-)continuous at 0 for all $x_0 \in H_1$; the general case would follow as at the end of Example 6.2.14), hence then $K = 0$, $\mathbb{K} = 0$, $\mathbb{X} = X \in \mathcal{GB}(U)$ for any admissible multiplication operator solution (so it is optimal to have no feedback, because such would never catch up the τx_0 term moving towards $-\infty$). Thus, all admissible multiplication operator solutions are r.c.-stabilizing. The first equation of the eCARE becomes

$$\langle x_0', \mathcal{P}x_0\rangle_H + \langle x_0, \mathcal{P}x_0'\rangle_H = -\sum_{k=1}^{\infty} 2^{-k}|x_0(-k)|^2 \ \text{ for all } \ x_0 \in H_1. \tag{9.245}$$

Setting $\mathcal{P} := \sum_{k=1}^{\infty} r_k \pi_{[-k,-k+1]}$, the left-hand side becomes $\sum_{k=1}^{\infty} r_k[x_0(-k+1) - x_0(k)]$, hence we should have $r_2 - r_1 = -2^{-1}$, $r_3 - r_2 = -2^{-2}$, …, $r_{n+1} - r_n = -2^{-n}$. Thus, $r_{n+1} = r_1 - \sum_{k=1}^{n} 2^{-k} = 2^{-n} + r_1 - 1$.

Therefore, $\mathcal{P}_r := rI + \sum_{k=0}^{\infty} 2^{-k}\pi_{[-k-1,-k]} \in \mathcal{B}(H)$ is a stabilizing solution of the eCARE for each $r \in \mathbf{R}$. Note that $S = I + \lim_{s\to+\infty} B_{\rm w}^* \mathcal{P}(s-A)^{-1}B = 1 + ((r+1)e^{s\cdot})(0-) = 2 + r$, hence for $r \neq -2$ and $X = I$, the operator $\mathcal{P}$ becomes a stabilizing solution of the CARE.

Thus, $r = 0$ gives $\mathcal{P}_0 = \mathbb{C}^*\mathbb{C}$, the unique (by Proposition 9.8.11) P-r.c.-stabilizing solution (because it is the J-critical cost). For $r \in \mathbf{R} \setminus \{0\}$, the formula $\mathcal{J}(u_{\min}(x_0), x_0) = \langle x_0, \mathcal{P}_r x_0\rangle$ does not hold and

$$\mathbb{X}_r^* S_r \mathbb{X}_r = S_r = (2+r) \neq 2 = \mathbb{D}^* J \mathbb{D} \tag{9.246}$$

(but $\mathbb{D}^* J\mathbb{D} + \text{w-lim}_{t\to+\infty} \tau(t)^*\mathbb{B}^*\mathcal{P}_r \mathbb{B}\tau(t) = 2 + r$, as in (9.233)).

We conclude that the condition (PB) is, indeed, necessary, even for a minimal weakly stable WPLS. For other values of r, we get a stable, r.c.-stabilizing solution $(\mathcal{P}_r, S_r, K_r)$ s.t. $\mathcal{P}_r$ differs from the critical cost operator $\mathcal{P}_0$, and $\mathbb{X}_r^* S_r \mathbb{X}_r$ is not a spectral factorization of $\mathbb{D}^* J\mathbb{D}$ (since $K_r = 0$ and $\mathbb{X}_r = I$).

Finally, we note that since $\mathbb{A}_{\circlearrowleft} = \mathbb{A} + 0 = \tau\pi_-$ (for any $r \in \mathbf{R}$), we obtain that

$$\langle \mathbb{B}^t u, \mathcal{P}_r \mathbb{A}_{\circlearrowleft}^t x_0\rangle = \langle \pi_-\tau^t u, \mathcal{P}_r \tau^t \pi_- x_0\rangle = 0 \to 0, \quad \text{as } t \to +\infty \tag{9.247}$$

for all $r \in \mathbf{R}$ and $u \in \mathcal{U}_*^*(0)$ (even for all $u \in \mathrm{L}^2_{\rm loc}(\mathbf{R}_+; U)$), since $u = \pi_+ u$. Thus, (9.247) does not imply (P), since the latter is satisfied for $r = 0$ only. ◁

Thus, if $(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ is a $\mathcal{U}_{\rm out}$-stabilizing (or $\mathcal{U}_{\rm sta}$-stabilizing) solution of the IARE, CARE or DARE, it can happen that there are solutions of form $(\mathcal{P}', S, [\ \mathbb{K} \mid \mathbb{F}\ ])$ (hence necessarily output-stabilizing) s.t. $\mathcal{P}' \gg \mathcal{P} > 0$ or $\mathcal{P} > 0 \gg \mathcal{P}'$, by Example 9.13.9 (discretize it for the DARE example). Thus, even q.r.c.-stabilization does not suffice, but we do have to verify the condition (P) (equivalently, that $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$) to avoid "fake cost" (or residual cost at infinity).

Above we added the copy of u to the output to get a standard LQR cost function; we could as well remove it and substract I from S and S_r.

Example 9.13.10 (Unique J-critical control though $D^*JD = 0$) In Example 9.13.9, we may remove the second row of $\mathbb{C}$ and $\mathbb{D}$, to obtain exactly same results except that then $D^*JD = 0$ and hence $S = 1$ and $S_r := 1 + r$. Indeed, still $SK = 0$ and $K^*SK = 0$, so that $(\mathcal{P}_r, S_r, 0)$ is again a stable, r.c.-stabilizing solution of the CARE for each $r \neq -1$ ($\mathcal{U}_{\rm out}$-stabilizing for $r = 0$).

Since S is one-to-one, the J-critical control is unique for each $x_0 \in H$. Note also that $\mathbb{D}^*\mathbb{D} = \sum_k (2^{-k/2})^2 = 2 = S = I^*SI$ is the corresponding I-spectral factorization and that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\rm out}$. ◁

Also the system of Example 9.8.15 has $D = 0$ and a unique J-critical control over $\mathcal{U}_{\rm out}$; however, here we have several output-stabilizing solutions, so that the condition (PB) is needed also in this case.

The next example shows that for unstable $\mathbb{A}_\circlearrowleft$ (and non–internally stabilizing $\mathcal{P}$) the condition "$\mathcal{P}\mathbb{A}_\circlearrowleft^t x \to 0$ $(x \in H)$" of Lemma 9.10.1(d1) is not sufficient for (P):

Example 9.13.11 ($\mathcal{P}\mathbb{A}_\circlearrowleft^t x \to 0 \not\Rightarrow \mathbb{A}_\circlearrowleft^{t\,*}\mathcal{P}\mathbb{A}_\circlearrowleft^t x \to 0$) Let $H = \mathrm{L}^2(\mathbf{R}_+)$, $\mathbb{A}^t := \mathrm{e}^t\tau^{-t}$, $(\widetilde{\mathcal{P}}x)(s) := \mathrm{e}^{-2s}x(s)$, $D = I = J$, $B = 0$ (so that Σ is ULR). Let $\mathbb{C}$ be stable.

Then, for any solution of the eCARE with $X = I$, we have $\mathbb{A}_\circlearrowleft = \mathbb{A}$, $\mathbb{D}_\circlearrowleft = \mathbb{D} = D$, $S = I$, $K = -C$, $C_\circlearrowleft = C + DK = 0$, $C^*JC = K^*SK$, hence $(0, I, -C)$ is the unique r.c.-SOS-PB-stabilizing solution (hence $\mathcal{U}_{\mathrm{out}}$-stabilizing, by Theorem 9.9.1(b)).

However, $\widetilde{\mathcal{P}} = \mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t \geq 0$, as one easily verifies, hence also $(\widetilde{\mathcal{P}}, I, -C)$ is a r.c.-SOS-stabilizing solution of the eCARE and the eIARE. We have $\|\widetilde{\mathcal{P}}\mathbb{A}^t x\|_H^2 \leq \mathrm{e}^{-2t}\|x\|_H^2 \to 0$ for all $x \in H$, but $\mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t x = \widetilde{\mathcal{P}}x \not\to 0$ for $x \neq 0$. ◁

One can add an unobservable and unreachable part to the semigroup and thus alter the properties of the solution without affecting the J-critical cost, nor the J-critical feedback or signature operator:

Example 9.13.12 (Wrong $\mathcal{P}$, fake cost) Let $\mathcal{P}$ be a stable, P-stabilizing solution for Σ and J, and let Σ be stable. Extend Σ to $\Sigma' := \left[\begin{array}{cc|c} \mathbb{A} & 0 & \mathbb{B} \\ 0 & \widetilde{\mathbb{A}} & 0 \\ \hline \mathbb{C} & 0 & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(U, H \times \widetilde{H}, Y)$ with a non–strongly stable $\widetilde{\mathbb{A}}$ (so we have Σ and $\widetilde{\Sigma}$ put together into (stable) Σ').

$\mathcal{P}' := \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \widetilde{\mathcal{P}} \end{bmatrix} = \mathcal{P}'^* \in \mathcal{B}(H \times \widetilde{H})$ is a stable, r.c.-stabilizing solution for Σ' and $J' = J$ iff $\widetilde{A}^*\widetilde{\mathcal{P}} + \widetilde{\mathcal{P}}\widetilde{A} = 0$ (because $(B'^*)_{\mathrm{w}}\mathcal{P}' = \begin{bmatrix} B_{\mathrm{w}}^*\mathcal{P} & 0 \end{bmatrix}$ and $\mathbb{B}'^*\mathcal{P}' = \begin{bmatrix} \mathbb{B}^*\mathcal{P} & 0 \end{bmatrix}$, thus $S' = S$, $\mathbb{K}' = \begin{bmatrix} \mathbb{K} & 0 \end{bmatrix}$, $\mathbb{F}' = \mathbb{F}$, $\mathbb{X}' = \mathbb{X}$). By Lemma 9.8.11(a), such a solution is P-stabilizing iff $\langle \widetilde{\mathbb{A}}\widetilde{x}_0, \widetilde{\mathcal{P}}\widetilde{\mathbb{A}}\widetilde{x}_0 \rangle \to 0$ for all $\widetilde{x}_0 \in \widetilde{H}$, because $\langle \mathbb{A}' \begin{bmatrix} x_0 \\ \widetilde{x}_0 \end{bmatrix}, \mathcal{P}'\mathbb{A}' \begin{bmatrix} x_0 \\ \widetilde{x}_0 \end{bmatrix} \rangle = \langle \mathbb{A}x_0, \mathcal{P}\mathbb{A}x_0 \rangle + \langle \widetilde{\mathbb{A}}\widetilde{x}_0, \widetilde{\mathcal{P}}\widetilde{\mathbb{A}}\widetilde{x}_0 \rangle$, but, by the uniqueness of $\mathcal{P}'$, only $\widetilde{\mathcal{P}} = 0$ can satisfy this condition.

(a) Taking $\widetilde{\mathbb{A}} = \mathrm{e}^{it}$ $(\widetilde{A} = i = -\widetilde{A}^* \in \mathbf{C}^{1\times 1})$ we see that any $\widetilde{\mathcal{P}} \in \mathbf{R}$ defines a stable, r.c.-stabilizing solution $\mathcal{P}'$, but only $\widetilde{\mathcal{P}} = 0$ defines a P-stabilizing solution.

(b) Require, in addition, that Σ is weakly stable. Take $\widetilde{\mathbb{A}} = \tau$ on $\widetilde{H} := \mathrm{L}^2$. Then $\widetilde{\mathbb{A}}$ and hence also $\mathbb{A}'$ is weakly but not strongly stable. Moreover, $\widetilde{A}^* = -\widetilde{A}$, hence $\widetilde{\mathcal{P}} := rI$ defines a stable, r.c.-stabilizing solution for any $r \in \mathbf{R}$. ◁

Note that if $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ were strongly stable and $S \gg 0$, then the $\mathcal{U}_{\mathrm{out}}$-stabilizing solution $\begin{bmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}$ would be the greatest solution of the eIARE having $S \geq 0$, by Corollary 9.2.11. By (a) (or (b)), the is not the case in the weakly stable case.

Naturally, the minimal cost can be negative:

Example 9.13.13 (Negative minimum, $\mathcal{P} \ll 0$) The system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \tag{9.248}$$

is minimal and exponentially stable. For $J = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, we have $\mathbb{D}^* J \mathbb{D} \gg 0$ (since $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} = 2 - |s+1|^{-2} \geq 1$ for $s \in i\mathbf{R}$), so that the system is positively J-coercive over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$.

The [e]CARE has a two solutions, $\mathcal{P} = -2 \pm \sqrt{2}$ (with $S = 2, K = -\mathcal{P}/2$). The smaller one is unstable, and the bigger one, $(-2+\sqrt{2}, 2, -1+2^{-1/2})$, minimizing over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$. Thus, the minimal cost is given by $\langle x_0, \mathcal{P} x_0 \rangle = -(2-\sqrt{2})\|x_0\|^2$. $\triangleleft$

Finally, we give an example of a minimal exponentially stabilizable system, which has a $\mathcal{U}_{\text{str}}$-stabilizing (and $\mathcal{U}_{\text{str}}$-minimizing) solution that is not strongly stabilizing (hence not minimizing over $\mathcal{U}_{\text{exp}}$):

Example 9.13.14 **($\begin{bmatrix} \mathbb{A} \\ \mathbb{C} \\ \mathbb{K} \end{bmatrix}$ and $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \\ \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ stable but $\mathbb{B}, \mathbb{D}, \mathbb{F}, \mathbb{B}_\circlearrowleft$ unstable)**

Let $A = (-k^{-1})_{k \in \mathbf{N}+1}$, $B = I$, $C = (\begin{bmatrix} k^{-1/2} \\ 0 \end{bmatrix})_{k \in \mathbf{N}+1}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $U := H := Y := \ell^2(\mathbf{N}+1)$, as in Example 6.1.14 (we have added a copy of u to the output), and set $J := I$.

By Example 6.1.14(b), $\begin{bmatrix} \mathbb{A} \\ \mathbb{C} \end{bmatrix}$ is strongly stable and $\mathbb{B}$ and $\mathbb{D}$ are unstable. Since $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$, there is a unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the CARE (=B_{w}^*-CARE)

$$S = I,\ K = -\mathcal{P},\ C^*C + A^*\mathcal{P} + \mathcal{P}A = \mathcal{P}^2. \tag{9.249}$$

since B (and A and C and D) is bounded. One can verify that this solution is given by the unique nonnegative diagonal solution, namely

$$\mathcal{P}_k = -k^{-1} + k^{-1}\sqrt{1+k} = -K_k \quad (k \in \mathbf{N}+1). \tag{9.250}$$

Since $\mathcal{P}$ is also $\mathcal{U}_{\text{str}}$-stabilizing (because $A_\circlearrowleft = A + BK = A - \mathcal{P} = (-k^{-1}\sqrt{1+k})$ is strongly stable), it must be $\mathcal{U}_{\text{sta}}$-stabilizing, by Lemma 8.3.3. By Proposition 10.7.3(d), $\mathcal{P}$ is also SOS-stabilizing. One can verify that $\mathbb{K}$ is stable but $\mathbb{F}$ is unstable. The unstability of $\mathbb{B}_\circlearrowleft$ and $\mathbb{B}_\circlearrowleft\tau$ follow as in Example 6.1.14(b).

Since there is no exponentially stabilizing solution (such a solution would be diagonal and nonnegative; an alternative proof follows from Corollary 9.7.2 and the uniqueness of $\mathcal{P}$), we can deduce that there is no J-critical control over $\mathcal{U}_{\text{exp}}$ (for some $x_0 \in H$), equivalently, $\mathbb{D}$ is not J-coercive over $\mathcal{U}_{\text{exp}}$, by Theorem 9.2.16.

(Recall from Example 6.1.14(b) that Σ is minimal and exponentially stabilizable but not detectable.) $\triangleleft$

In the above example, the cost function is coercive enough to provide a minimizing control over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$, but the existence of a minimizing control over $\mathcal{U}_{\text{exp}}$ would require even further coercivity.

Notes

Most of Proposition 9.13.1(c1) is from [WW, Example 11.5] and Example 9.13.8 is from [WZ].

9.14 $(J,*)$-critical factorization ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$)

There comes a critical moment where everything is reversed, after which the point becomes to understand more and more that there is something which cannot be understood.

— Søren Kierkegaard (1813-1855)

In this section, we shall develop an extension of canonical (H^2) factorization theory of $L^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$ maps [LS] [GlaGoh] for the case where $\mathbf{C}^{n\times n}$ is replaced by $\mathcal{B}(U)$. Most of our results are given in continuous time.

We start by formulating an additional equivalent condition for a unique J-critical control to be of state feedback form:

Definition 9.14.1 (($J,*$)-critical factorization) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Let $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty(U)$ be s.t. $\pi_+ \begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix} \pi_-\mathbb{M}^{-1}[L^2_\omega] \subset L^2$ (and $\pi_+\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1}[L^2_\omega] \subset L^2$ if $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$) for some $\omega \in \mathbf{R}$, where $\mathbb{N} := \mathbb{D}\mathbb{M}$, and*

$$\langle \mathbb{N}\pi_-\mathbb{M}^{-1}v, J\mathbb{D}\eta\rangle_{L^2(\mathbf{R}_+;Y)} = 0 \quad (\eta \in \mathcal{U}_*^*(0),\ v \in L^2_\omega(\mathbf{R}_-;U)). \tag{9.251}$$

Then $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is called a $(J,*)$-critical factorization (of $\mathbb{D}$) over $\mathcal{U}_*^*$.

It follows that $\pi_+ \begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix} \pi_-\mathbb{M}^{-1} \in \mathcal{B}(L^2_\omega, L^2)$ (and $\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1} \in \mathcal{B}(L^2_\omega, \subset L^2)$ for $\mathcal{U}_{\text{exp}}$), by Lemma A.3.6. Note that we may increase ω, since $L^2_{\omega'}(\mathbf{R}_-;U) \subset L^2_\omega$ for $\omega' > \omega$.

Lemma 9.14.2 *Make the assumptions and use the notation of Definition 9.14.1 [and assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$]. Then*

(a) $\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix}[L^2_c(\mathbf{R};U)] \subset L^2$ *[and $\mathbb{B}\mathbb{M}\tau[L^2_c] \subset L^2$]; in particular, $[\mathbb{B}\mathbb{M}\tau,]\mathbb{N},\mathbb{M} \in \text{TIC}_\gamma$ for all $\gamma > 0$.*

(b) Set $\mathbb{T} := \pi_+\mathbb{M}\pi_-\mathbb{M}^{-1}$, $\mathbb{S} := \pi_- + \mathbb{T} = \mathbb{M}\pi_-\mathbb{M}^{-1}$. Then $\mathbb{T} \in \mathcal{B}(L^2_\omega, L^2)$ and $[\pi_+\mathbb{B}\tau\mathbb{S},]\mathbb{T}, \pi_+\mathbb{D}\mathbb{S} \in \mathcal{B}(L^2_\omega, L^2)$.

(c) If Σ is stable, $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, and S is one-to-one, then all choices of $(\mathbb{N},\mathbb{M})$ are given by $(\mathbb{N}E, \mathbb{M}E)$ ($E \in \mathcal{G}\mathcal{B}(U)$).

Proof: (a) Let $v_\circlearrowleft \in L^2_c(\mathbf{R}_-;U)$. Then $v := \pi_-\mathbb{M}v_\circlearrowleft \in L^2_c \subset L^2_\omega$, hence $\mathbb{N}v_\circlearrowleft = \pi_-\mathbb{N}v_\circlearrowleft + \pi_+\mathbb{N}\pi_-\mathbb{M}^{-1}v \in L^2_c + L^2 \subset L^2$. Thus, $\mathbb{N}L^2_c \subset L^2$. By Lemma 2.1.13, we have $\mathbb{N} \in \text{TIC}_\gamma$ for all $\gamma > 0$. Replace $\mathbb{N}$ by $\mathbb{M}$ [and then by $\mathbb{B}\mathbb{M}\tau$] to obtain the other claims.

(b) Now $\mathbb{S} = \pi_-\mathbb{M}\pi_-\mathbb{M}^{-1} + \pi_+\mathbb{M}\pi_-\mathbb{M}^{-1} = \mathbb{M}\pi_-\mathbb{M}^{-1}$. hence $\mathbb{D}\mathbb{S} = \mathbb{N}\pi_-\mathbb{M}^{-1}$. By assumption, $[\pi_+\mathbb{B}\tau\mathbb{S},]$ $\mathbb{T}$ and $\pi_+\mathbb{D}\mathbb{S}$ map $L^2_\omega \to L^2$, hence $[\pi_+\mathbb{B}\tau\mathbb{S},]\mathbb{T}, \pi_+\mathbb{D}\mathbb{S} \in \mathcal{B}(L^2_\omega, L^2)$, by Lemma A.3.6.

(c) This follows from Theorem 9.14.3 and Theorem 8.3.13(f) (since $\mathbb{N}\mathbb{M}^{-1}$ is independent on $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$). □

Now we are ready to show that a strictly optimal control is of state feedback form iff there is a J-critical factorization:

Theorem 9.14.3 (JCF$\Leftrightarrow \exists \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}$) *Assume that $\mathbb{D}$ has a unique J-critical control over $\mathcal{U}_*^*$ for each $x_0 \in H$, where $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) $\mathbb{D}$ has a $(J,*)$-critical factorization over $\mathcal{U}_*^*$;

(ii) there is a J-critical state feedback pair for Σ over $\mathcal{U}_*^*$;

(iii) there is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE.

Moreover, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ solves (i), then $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix} := \begin{bmatrix} \mathbb{M}^{-1}\mathbb{K}_{\text{crit}} & I - \mathbb{M}^{-1} \end{bmatrix}$ is a J-critical state feedback pair for Σ over $\mathcal{U}_*^*$. Conversely, if $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}$ solves (ii), then $\mathbb{M} := (I - \mathbb{F})^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ form a $(J,*)$-critical factorization over $\mathcal{U}_*^*$.

A sufficient condition for (i)–(iii) (and for the eIARE becoming a CARE) is that any of (1.)–(4.) of Remark 9.9.14 holds and $D^*JD \in \mathcal{GB}(U)$.

In either case, Theorem 8.3.13 applies. See Theorem 9.9.1(a1)&(e) for the correspondence of (ii) to (iii).

Proof: (In fact, it would suffice to assume that there is a J-critical control in WPLS form for Σ, as one observes from the proof of Theorem 8.3.13(c2) and from Proposition 9.3.1.)

1° *(ii)⇒(i):* Let $\begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix}$ be J-critical (as in (ii)), and set $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$. Then $(\pi_+\mathbb{N}\pi_-)\mathbb{M}^{-1} = (\mathbb{C}_{\text{crit}}\mathbb{B}\mathbb{M})\mathbb{M}^{-1} = \mathbb{C}_{\text{crit}}\mathbb{B}$, hence (9.251) holds for any $\omega > \omega_A$. Because $\pi_+\mathbb{M}\pi_-\mathbb{M}^{-1} = \mathbb{K}_{\text{crit}}\mathbb{B} =: \mathbb{T}$, the requirements of Definition 9.14.1 are satisfied in case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. For $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, we have on $\mathbf{R}_+$ that

$$\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1} = \mathbb{B}\tau(\pi_- + (\pi_+\mathbb{M}\pi_-)\mathbb{M}^{-1}) \tag{9.252}$$

$$= \mathbb{B}\tau\pi_- + \mathbb{B}\tau(\mathbb{K}_{\text{crit}}\mathbb{B})\mathbb{M}\pi_-\mathbb{M}^{-1} = \mathbb{A}^{\cdot}\mathbb{B} + \mathbb{B}\tau\mathbb{K}_{\text{crit}}\mathbb{B} = \mathbb{A}^{\cdot}_{\text{crit}}\mathbb{B}, \tag{9.253}$$

and $\pi_+\mathbb{A}^{\cdot}_{\text{crit}}\mathbb{B}$ maps $\mathrm{L}^2_\omega \to \pi_+\mathrm{L}^2$, because $\mathbb{A}_{\text{crit}}$ is exponentially stable.

2° *(i)⇒(ii):* With the notation of Lemma 9.14.2(b) (assume, in addition, that $\omega > \omega_A$), $\mathbb{T}$ is the operator of Theorem 8.3.13(a), and $\mathbb{T}\mathbb{M} = \pi_+\mathbb{M}\pi_-$, hence (ii) holds and $\begin{bmatrix} \mathbb{M}^{-1}\mathbb{K}_{\text{crit}} & I - \mathbb{M}^{-1} \end{bmatrix}$ is J-critical, by Theorem 8.3.13(c2).

3° *(ii)⇔(iii):* See Theorem 9.9.1(a1)&(e).

4° The “moreover” claims were established in 1°–2°. The sufficient condition is obtained from Remark 9.9.14(d). □

Next we establish a canonical (H^2) factorization as a special case of J-critical factorization. The main result of this section, Theorem 9.14.6, is given in discrete time, as are the classical results, but we start with two continuous time results.

If a function is H^∞ over the unit circle, then it is also H^2 over the unit circle. The same does not hold in continuous time, hence our continuous time results and assumptions differ somewhat from those of Theorem 9.14.6. We start with the exponentially stable case:

Theorem 9.14.4 (Exponential H^2-SpF) *Assume that* $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon};\mathcal{B}(U,Y))$ *for some* $\varepsilon > 0$, $J = J^* \in \mathcal{B}(Y)$, *and* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$.

Then there are $\varepsilon' > 0$ *and* $\widehat{\mathbb{X}} \in \mathcal{G}(\mathcal{B}(U) + \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon'};\mathcal{B}(U)))$, *s.t.* $\mathbb{X}^*(D^*JD)\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, $X = I$ *and* $D^*JD \in \mathcal{GB}(U)$. *In particular,* $\widehat{\mathbb{X}}^*(D^*JD)\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ *on* $i\mathbf{R}\cup\{\infty\}$.

Moreover, $\mathbb{D} = (\mathbb{D}\mathbb{X}^{-1})\mathbb{X}$ *is the unique* $(J,*)$*-spectral factorization of* $\mathbb{D}$ *over* $\mathcal{U}_{\text{out}}$ *having* $X = I$.

Note that $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon};\mathcal{B}(*)) \subset \mathrm{H}^\infty(\mathbf{C}^+_{-\varepsilon/2};\mathcal{B}(*))$, by Lemma F.3.2(a1).

By Lemma 2.2.2(d), we have $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$ iff $\mathbb{D}^*J\mathbb{D} \gg 0$, equivalently, iff $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} \gg 0$ in $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};\mathcal{B}(U))$, by Theorem 3.1.3(a1)&(e1). However, in the indefinite case, the invertibility of the Toeplitz operator is a strictly stronger condition than the invertibility of $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ (i.e., of $\mathbb{D}^*J\mathbb{D}$). This also applies to Proposition 9.14.5 and Theorem 9.14.6.

Proof of Theorem 9.14.4: See the proof of Theorem 9.2.14(c2) (and (a)2°). The uniqueness follows from Lemma 9.14.2(c). □

In the positive case, we may give up exponential stability:

Proposition 9.14.5 (Positive H^2-SpF) *Assume that* $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$, $J = J^* \in \mathcal{B}(Y)$, *and* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$.

Then there is $\widehat{\mathbb{X}} \in \mathcal{G}(\mathcal{B}(U) + \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U)))$ *s.t.* $\mathbb{X}^*(D^*JD)\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, $X = I$ *and* $D^*JD \in \mathcal{GB}(U)$. *Moreover* $\mathbb{D} = (\mathbb{D}\mathbb{X}^{-1})\mathbb{X}$ *is the unique* $(J,*)$*-critical factorization of* $\mathbb{D}$ *over* $\mathcal{U}_{\text{out}}$ *having* $X = I$.

It follows that, $\langle\widehat{\mathbb{X}}u_0, D^*JD\widehat{\mathbb{X}}u_0\rangle_U = \langle\widehat{\mathbb{D}}u_0, J\widehat{\mathbb{D}}u_0\rangle_Y$ a.e. on $i\mathbf{R}$, for any $u_0 \in U$.

An analogous indefinite result also holds, except that $\widehat{\mathbb{X}}^{\pm 1}$ need not be stable; indeed, the continuous-time equivalent of Theorem 9.14.6 also holds under the assumption that $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}$ (use Theorem 9.9.6 instead of Theorem 14.1.6 in the proof, etc.). Note that $\mathbb{D},\mathbb{X} \in$ ULR in both cases.

Proof of Proposition 9.14.5: Take a strongly stable realization Σ of $\mathbb{D}$ with B bounded (see Theorem 6.9.1(a)&(d2)). Now we obtain our claims from Theorem 10.6.3(a) (see also (d)) (which also contains the converse) combined with Lemma 10.6.2(d)(7.) except for the regularity of $\mathbb{X}$; but, by Theorem 6.9.1(a), $\mathbb{X} - I$ and $\mathbb{X}^{-1} - I$ are in $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U))$. By Theorem 9.14.3, $\mathbb{N}\mathbb{M}^{-1}$ is a $(J,*)$-critical factorization. The uniqueness follows from Lemma 9.14.2(c).

(Note that $\mathbb{X}$ corresponds to a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{out}}$ and to a strongly r.c.-stabilizing solution of the B_{w}^*-CARE.) □

Now we establish the canonical factorization for $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U))$ functions of form $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$, $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$, with an invertible Toeplitz operator:

Theorem 9.14.6 (Discrete H^2-factorization) *Assume that $\mathbb{D} \in \mathrm{tic}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$ are s.t. $\pi^+\mathbb{D}^* J\mathbb{D}\pi^+ \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.*

Then $\mathbb{D}$ has a unique $(J,*)$-critical factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ over $\mathcal{U}_{\mathrm{out}}$ s.t. $M = I$. Moreover, $\widehat{\mathbb{X}} := \widehat{\mathbb{M}}^{-1} \in \mathcal{G}\mathrm{H}(\mathbf{D};\mathcal{B}(U))$, $\mathbb{N} = \mathbb{D}\mathbb{X}^{-1} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$ and

$$\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle \qquad (u,v \in \ell^1(\mathbf{Z};U)) \tag{9.254}$$

for some $S = S^* \in \mathcal{GB}(U)$. Furthermore, $\widehat{\mathbb{M}}, \widehat{\mathbb{X}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U))$, $(\pi^+\mathbb{D}^* J\mathbb{D}\pi^+)^{-1} = \mathbb{M}\pi^+ S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$, and all claims in (a1)–(c2) of Lemma 14.2.8 hold.

If $\dim U < \infty$, then $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$, and $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ and $\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.

If $\mathbb{D}$ is exponentially stable, then so are $\mathbb{X}$ and $\mathbb{X}^{-1}$, and then we have $\mathbb{N}^* J\mathbb{N} = S$ and $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, i.e.,

$$\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}} = \widehat{\mathbb{X}}^* S\widehat{\mathbb{X}} \quad \textit{on } \partial\mathbf{D}. \tag{9.255}$$

(In the above theorem, we have given up Standing Hypotheses on Σ.) By using the Cayley transform, we obtain the corresponding continuous-time "factorization". However, this factorization need not be well-posed, since the "factor" operator $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}(\mathbf{C}^+;\mathcal{B}(U))$ (for $\widehat{\mathbb{D}} \in \mathrm{TIC}(U,Y)$) may satisfy $\mathbb{X}, \mathbb{X}^{-1} \notin \mathrm{TIC}_\infty$ (i.e., $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \notin \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for all $\omega \in \mathbf{R}$), even when $\dim U < \infty$, as shown in Example 8.4.13 (see also Example 11.3.7). Thus, there is no equivalent result in continuous time (unless we have $\mathbb{D}^* J\mathbb{D} \gg 0$ or $\mathbb{D}$ is assumed to be sufficiently regular; cf. the remark below Proposition 9.14.5).

By Lemma 14.2.8(d), we have

$$\lim_{t\to+\infty} \langle \mathbb{X}u, S\pi_{[0,t)}\mathbb{X}v\rangle = \langle \mathbb{D}u, J\mathbb{D}v\rangle \quad (u,v \in \ell^2(\mathbf{N};U)). \tag{9.256}$$

However, we cannot write "$\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}}$", because it is not even known whether $\widehat{\mathbb{X}}$ has a boundary function. Thus, to obtain $\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$, we must, e.g., assume that $\mathbb{D}$ is exponentially stable ($\widehat{\mathbb{D}} \in \mathrm{H}^\infty(r\mathbf{D};\mathcal{B}(U))$ for some $r > 1$), in which case the above factorization becomes an exponential spectral factorization).

If $\mathbb{D}^* J\mathbb{D} \gg 0$, then $S \gg 0$, as one observes from the proof and Lemma 9.10.3, hence then it follows from (9.256) that $\mathbb{X}$ is in tic, and from (9.254) that $\mathbb{M}$ is in tic; in particular, then $\mathbb{X} = \mathbb{M}^{-1} \in \mathcal{G}\mathrm{tic}$ (i.e., $\widehat{\mathbb{X}} = \widehat{\mathbb{M}}^{-1} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$), so that this would be an alternative (system-theoretic) proof for the positive spectral factorization result Lemma 5.2.1(a) (for operators of form $\mathbb{D}^* J\mathbb{D}$, as is the case with the applications of the lemma in this monograph). (Note that we obtain the corresponding continuous-time result through the Cayley transform in this positive case.)

Proof of Theorem 9.14.6: Choose a strongly stable realization $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ of $\mathbb{D}$. By Proposition 8.3.10, there is a unique J-critical control for each $x_0 \in H$. By Theorem 9.14.3, this corresponds to a $(J,*)$-critical factorization (which is unique, by Lemma 9.14.2(c)) and to a $\mathcal{U}_{\text{out}}$-stabilizing solution $(\mathcal{P}, S, K)$ of the DARE (we have $S \in \mathcal{GB}(U)$, by Lemma 9.10.3).

Let $\Sigma_{\circlearrowleft}$, $\mathbb{X}$, $\mathbb{M}$ and $\mathbb{N}$ be as in Definition 14.1.1, so that $\mathbb{A}_{\circlearrowleft}$, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$ are strongly stable by Theorem 8.3.9(a3). Now we obtain the other claims from Lemma 14.2.8. □

We finish this section by a discussion on spectral and other J-critical factorizations in continuous time.

By Example 8.4.13, J-coercivity over $\mathcal{U}_{\text{out}}$ does not imply the existence of a spectral factorization in the indefinite case (see also Example 11.3.7). Instead, we need some additional assumptions, as in Theorems 8.4.9 and 9.2.14.

In discrete time, we always have the "H^2-factor" (the $(J,*)$-critical factor) of Theorem 9.14.6, and it suffices for the purposes of optimal control (by Theorems 9.14.3 and 8.4.3). If the I/O map is exponentially stable, then this J-critical factor is a spectral factor (cf. Theorem 14.3.2).

For a continuous-time I/O map $\mathbb{D}$, the corresponding factor and its inverse may be non-well-posed (i.e., $\mathbb{X}^{\pm 1} \notin \mathrm{TIC}_\infty$, as in Example 11.3.7), and the corresponding "state feedback pair" may thus be non-well-posed.

If $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$ is $\mathrm{H}^\infty(\mathbf{K}^c; \mathcal{B}(U,Y))$ for some compact $K \subset \mathbf{C}^-$, then an analogous condition is satisfied by $\widehat{\mathbb{X}}^{\pm 1}$ (since this condition holds iff the map is a Cayley transform of an exponentially stable (discrete time) map), and in this case we do have an exponentially stable spectral factorization (whenever the Popov Toeplitz operator is invertible), but this condition is too strong for our purposes. Unfortunately, it does not seem that continuous time exponential stability would be a sufficient condition (it seems likely that both $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ might still be unbounded near infinity).

However, if $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}$ over $\mathbf{C}^+$ (for some $D \in \mathcal{B}(U,Y)$), then the "H^2-spectral factorization" mentioned above is, indeed, well-posed (and ULR) in continuous time too, as noted below Proposition 9.14.5; Then also the continuous time form of Theorem 9.14.6 and its proof are valid.

(In the above setting, the discretized $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_{\text{out}}$-stabilizing solution of the discretized eIARE, hence the original proof of Theorem 9.14.6 leads to corresponding results on $\Delta^{\mathrm{S}}\mathbb{D}$ and $\Delta^{\mathrm{S}}\mathbb{X}$, but these are not as strong as the ones obtained from the continuous time version.)

Notes for Sections 9.14 and 9.15

Theorem 9.15.3 is a basic result of classical (generalized) canonical factorization theory, and also variants of Lemmas 9.15.2 and 9.15.5 are well known.

Actually, in the classical theory, one can replace $\mathbb{D}^* J \mathbb{D}$ by any ti_0 operator (i.e., $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ by any $\mathrm{L}^\infty(\partial \mathbf{D}; \mathbf{C}^{n \times n})$ function), and $\partial \mathbf{D}$ can be replaced by any "standard contour" (and L^2 by L^p). Thus, Theorem 9.14.6 only extends the part of the factorization theory that is needed for standard control theory.

Classical references on the subject in English include [CG81] and [LS], and an up-to-date book [BKS] appeared during the referee process for this monograph. See also the notes on p. 148.

9.15 H^2-factorization when $\dim U < \infty$

Divide et impera!

— Louis XI

We present here some basic facts on a weaker form of finite-dimensional canonical (generalized) factorizations for later use. We mainly work in discrete time (i.e., on $\mathbf{D}$, not $\mathbf{C}^+$); in particular, we relax Standing Hypothesis 9.0.1 (except in Lemma 9.15.4); we still assume that H and Y are Hilbert spaces.

Let $n \in \mathbf{N}+1$, $U := \mathbf{C}^n$. Set $\mathrm{H}^2 := \mathrm{H}^2(\mathbf{D};\mathbf{C}^n)$. Recall that $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$, hence $\partial\mathbf{D} = \{z \in \mathbf{C} \mid |z| = 1\}$. Note that $\widehat{\pi^+} \sum_{n=-\infty}^{\infty} u_n z^n = \sum_{n=0}^{\infty} L_n u^n$ for each $(u_n)_{n\in\mathbf{Z}} \in \cap_{r>0} \ell_r^2(\mathbf{Z};\mathbf{C}^n)$.

Definition 9.15.1 ($\mathcal{G}\mathrm{H}^2$-factorization) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n, Y)$ and $J = J^* \in \mathcal{B}(Y)$. We say that $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ is a $\mathcal{G}\mathrm{H}^2$-factorization of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ if $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^2(\mathbf{D};\mathbf{C}^{n\times n})$, $S = S^* \in \mathcal{G}\mathcal{B}(\mathbf{C}^{n\times n})$, and $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.*

In standard factorization theory, a $\mathcal{G}\mathrm{H}^2$-factorization with the additional condition "$\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ is bounded on $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^n)$" (cf. Theorem 9.15.3) is called a "generalized canonical (right-)factorization relative to L^2" (see, e.g., pp. 142–143 of [CG81]).

A $\mathcal{G}\mathrm{H}^2$-factorization is unique up to an invertible constant matrix, and one can always redefine $\widehat{\mathbb{X}}$ so that $S = J_1$ (i.e., make S a diagonal matrix with diagonal elements ± 1):

Lemma 9.15.2 (Uniqueness) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n, Y)$ and $J = J^* \in \mathcal{B}(Y)$. Let $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ have a $\mathcal{G}\mathrm{H}^2$-factorization $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$.*

Then all $\mathcal{G}\mathrm{H}^2$-factorizations of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ are given by $(E\widehat{\mathbb{X}})^*(E^{-*}SE^{-1})(E\widehat{\mathbb{X}})$ ($E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$), and there is $E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$ s.t. $(E^{-*}SE^{-1}) = J_1 := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{B}(\mathbf{C}^k \times \mathbf{C}^{n-k})$ for some $k \in \{0, 1, \ldots, n\}$.

Moreover, then $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} = (\widehat{\mathbb{X}}^{-1} S^{-1} \widehat{\mathbb{X}}^{-*})^{-1} \in \mathcal{G}\mathrm{L}^1(\partial\mathbf{D};\mathbf{C}^{n\times n})$ and $\widehat{\mathbb{X}}^{\pm 1} \in \mathcal{G}\mathbf{C}^{n\times n}$ a.e. on $\partial\mathbf{D}$.

Proof: 1° Obviously, any $E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$ defines a $\mathcal{G}\mathrm{H}^2$-factorization. By Lemmas 2.4.4 and 2.4.1, we have $S = (VE')^* J_1 (VE')$ for some $V, E' \in \mathcal{B}(U)$ (set $k := \dim H_+$).

2° By Theorem 3.3.1(e)&(a4), the functions $\widehat{\mathbb{D}} \in \mathrm{H}^\infty \subset \mathrm{H}^2$, $\widehat{\mathbb{M}} := \widehat{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{H}^2$, $\widehat{\mathbb{N}} := \widehat{\mathbb{D}}\widehat{\mathbb{X}} \in \mathrm{H}^2$ have L^2 boundary functions on $\partial\mathbf{D}$.

By continuity, $\widehat{\mathbb{M}}\widehat{\mathbb{X}} = I = \widehat{\mathbb{X}}\widehat{\mathbb{M}}$, $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ and $\widehat{\mathbb{N}}^* J \widehat{\mathbb{N}} = S$ a.e. on $\partial\mathbf{D}$, hence $\widehat{\mathbb{X}}, \widehat{\mathbb{M}}, \widehat{\mathbb{N}}^* J \widehat{\mathbb{N}}, \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} \in \mathcal{G}\mathcal{B}(U)$ a.e. on $\partial\mathbf{D}$ and $(\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}})^{-1} = \widehat{\mathbb{X}}^{-1} S^{-1} \widehat{\mathbb{X}}^{-*} \in \mathrm{L}^1(\partial\mathbf{D};\mathcal{B}(U))$, by The Hölder Inequality.

3° *All $\mathcal{G}\mathrm{H}^2$-factorizations:* Conversely, let $\widehat{\mathbb{X}}_2^* J_1 \widehat{\mathbb{X}}_2$ be a $\mathcal{G}\mathrm{H}^2$-factorization of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$. Set $\widehat{\mathbb{X}} := \widehat{\mathbb{X}}_1$, $\widehat{\mathbb{N}}_1 := \widehat{\mathbb{N}}$ (redefined so that $S = J_1$).

By The Hölder Inequality, $E := \widehat{\mathbb{X}}_1 \widehat{\mathbb{X}}_2^{-1} \in \mathcal{G}\mathrm{H}^1(\mathbf{D};\mathcal{B}(U))$. From equations $J_1 = \widehat{\mathbb{N}}_1^* J \widehat{\mathbb{N}}_1$ and $\widehat{\mathbb{N}}_2 = \widehat{\mathbb{N}}_1 E$, we obtain that $E^* J_1 E = J_1$, hence $E = J_1^{-1} E^{-*} J_1$

a.e. on $\partial\mathbf{D}$. By Lemma B.3.6, $(E^{-1})^{\mathrm{d}} := E(\bar{z})^{-*} \in \mathcal{G}\mathrm{H}^1(\mathbf{D};\mathcal{B}(U))$. Set $E(z) := J_1^{-1}E(1/\bar{z})^{-*}J_1$ for $z \in \overline{\mathbf{D}}^c$, so that $E \in \mathcal{B}(U)$, by Proposition D.1.20. But $\widehat{\mathbb{X}}_1 = E\widehat{\mathbb{X}}_2$, as required. □

If the Popov Toeplitz operator is invertible (i.e., $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}}$), then $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization:

Theorem 9.15.3 ($\exists(\pi^+\mathbb{D}^* J\mathbb{D}\pi^+)^{-1} \Rightarrow \exists\mathcal{G}\mathrm{H}^2$-factorization) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n, Y)$ and $J = J^* \in \mathcal{B}(Y)$. If $\pi^+\mathbb{D}^* J\mathbb{D}\pi^+ \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{N};\mathbf{C}^n))$, then $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization $\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}}$.*

Moreover, $\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ is bounded on $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^n)$ and $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$.

Naturally, the Toeplitz invertibility condition can be written as $\widehat{\pi^+}\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}\widehat{\pi^+} \in \mathcal{G}\mathcal{B}(H^2(\mathbf{D};\mathbf{C}^n))$,

Proof: The existence of $\widehat{\mathbb{X}}$ and S with $\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ bounded follows from Theorem 9.14.6. By Lemma 2.2.2(d), $\mathbb{D}^* J\mathbb{D} \in \mathcal{G}\mathrm{ti}(U)$, hence $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty$, by Lemma 13.1.5. □

In fact, the coercivity assumption can be replaced by a weaker condition:

Lemma 9.15.4 (CT: $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{crit}} & \mathbb{F}_{\mathrm{crit}} \end{array}\right] \Rightarrow \mathcal{G}\mathrm{H}^2$-factorization) *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(\mathbf{C}^n, H, Y)$ be s.t. $\mathbb{B}$ and $\mathbb{D}$ are stable, and let $J = J^* \in \mathcal{B}(Y)$. Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be a J-critical state feedback pair for Σ over $\mathcal{U}_{\mathrm{out}}$.*

Then $(\widehat{\heartsuit}\widehat{\mathbb{X}})^* S(\widehat{\heartsuit}\widehat{\mathbb{X}}) = (\widehat{\heartsuit}\widehat{\mathbb{D}})^* J(\widehat{\heartsuit}\widehat{\mathbb{D}})$ is a $\mathcal{G}\mathrm{H}^2$-critical factorization, where $\mathbb{X} := I - \mathbb{F}$ and S is the corresponding signature operator. □

(This follows from Lemma 9.12.8(d) and Lemma 13.2.1(e2). In fact, any $\mathcal{U}_*^*$ with $\vartheta = 0$ would do, with the same proof.)

We shall also need the following local variant of Proposition 5.2.2, which states that if $\mathbb{D}$ is holomorphic around some subarc of $\partial\mathbf{D}$, then so are $\mathbb{X}^{\pm 1}$:

Lemma 9.15.5 (Local holomorphic extension) *Assume that $\widehat{\mathbb{E}} \in \mathrm{L}^\infty(i\mathbf{R};\mathbf{C}^{n\times n})$ and that $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathrm{H}^2(\mathbf{D};\mathbf{C}^{n\times n})$. Let $\Omega \subset \mathbf{C}$ be open and $\Gamma := \Omega \cap \partial\mathbf{D} \neq \emptyset$. If $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $\partial\mathbf{D}$ and $\widehat{\mathbb{E}}_{|\Gamma}$ has an holomorphic extension to Ω, then $\widehat{\mathbb{X}}^{\pm 1}$ and $\widehat{\mathbb{Y}}^{\pm 1}$ have holomorphic extensions to $\mathbf{D} \cup \Omega$ and $\mathbf{D} \cup \{1/\bar{s} \mid s \in \Omega\}$, respectively.*

(Note that $\Omega' := \Omega \cap \{1/\bar{s} \mid s \in \Omega\}$ contains Γ. Note also that if $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(\mathbf{C}^n, Y))$ and $\widehat{\mathbb{D}}$ has a holomorphic extension to $\mathbf{D} \cup \Omega$, then $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}_{|\Gamma}$ has a holomorphic extension to Ω', so that the lemma applies.)

Proof: 1° *Case $-1 \notin \Omega$:* Set $\widehat{\mathbb{Z}}(s) := \widehat{\mathbb{Y}}(1/\bar{s})^*$ for $s \in \mathbf{C}$ s.t. $1/\bar{s} \in \overline{\mathbf{D}}$. We have $s = 1/\bar{s}$, hence $\widehat{\mathbb{Y}}(s)^* = \widehat{\mathbb{Z}}(s)$ for all $s \in \partial\mathbf{D}$. Since $\widehat{\mathbb{Y}} \in \mathrm{H}(\mathbf{D};\mathbf{C}^{n\times n})$, we have $\widehat{\mathbb{Z}} \in \mathrm{H}(\overline{\mathbf{D}}^c;\mathbf{C}^{n\times n})$.

Consequently, $\widehat{\mathbb{Z}}\widehat{\mathbb{X}} = \widehat{\mathbb{E}}$ a.e. on $\partial\mathbf{D}$, so that $f := \widehat{\mathbb{X}} = \widehat{\mathbb{Z}}^{-1}\widehat{\mathbb{E}}$ a.e. on $\partial\mathbf{D}$ (note from Lemma 9.15.2 that $\widehat{\mathbb{X}}, \widehat{\mathbb{Z}} \in \mathcal{G}\mathbf{C}^{n\times n}$ a.e. on $\partial\mathbf{D}$). By Theorem 3.3.1(e)&(a2), $\widehat{\mathbb{X}}(r\cdot) \to \widehat{\mathbb{X}}(\cdot)$ in $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^{n\times n})$, as $r \to 1-$. It follows that the (inverse)

Cayley transform $\widehat{\heartsuit}^{-1}\widehat{\mathbb{X}}(r+i\cdot)$ converges locally in L^1 as $r \to 0+$; analogously, $\widehat{\heartsuit}^{-1}\widehat{\mathbb{Z}}(r+i\cdot)$ converges locally in L^1 as $r \to 0-$ (see Lemma 13.2.1(e1)).

We conclude from Proposition D.1.18 that $f : \partial\mathbf{D} \cap \Omega \to \mathbf{C}^{n\times n}$ has an holomorphic extension $f : \Omega \to \mathbf{C}^{n\times n}$ s.t. $f = \widehat{\mathbb{X}}$ on $\mathbf{D} \cap \Omega$ and $f = \widehat{\mathbb{Z}}^{-1}\widehat{\mathbb{E}}$ on $\overline{\mathbf{D}}^c \cap \Omega$. Thus, $\widehat{\mathbb{X}}$ has an holomorphic extension to $\mathbf{D} \cup \Omega$.

Apply the above to $\widehat{\mathbb{E}}^* = \widehat{\mathbb{X}}^*\widehat{\mathbb{Y}}$ in place of $\widehat{\mathbb{E}}$ to obtain the claim on $\widehat{\mathbb{Y}}$ (equivalently, an holomorphic extension of $\widehat{\mathbb{Z}}_{|\overline{\mathbf{D}}^c\cap\Omega}$ to Ω). But $f^{-1} = \widehat{\mathbb{E}}^{-1}\widehat{\mathbb{Z}} \in \mathrm{H}(\Omega; \mathbf{C}^{n\times n})$, and $f^{-1} = \widehat{\mathbb{X}}^{-1}$ on $\mathbf{D} \cap \Omega$, hence $\widehat{\mathbb{X}}^{-1}$ has an holomorphic extension to $\mathbf{D} \cup \Omega$.

2° *Case* $\partial\mathbf{D} \not\subset \Omega$*:* Rotate the functions so that 1° applies.

3° *Case* $\partial\mathbf{D} \subset \Omega$*:* Apply 2° to two subsets of Ω. □

(The notes for this section are given on p. 543.)