# Chapter 7

# Dynamic Stabilization

*And when winds are at war with the ocean,*
*As the breasts I believed in with me,*
*If their billows excite an emotion,*
*It is that they bear me from thee.*

— Lord Byron (1788–1824), "Stanzas to Augusta"

In this chapter we shall study different forms of dynamic stabilization, extend standard classical results (see, e.g., pp. 15–17 and 26–47 of [Francis]) for WPLSs and supplement them with new ones.

We assume that we are given a fixed *plant*, e.g., an I/O map $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ (alternatively, a WPLS) that we wish to control. In the case of *dynamic output feedback* (cf. pp. 36–42 of [Francis]) the output ($y$) of the plant is fed back to the input ($u$) through a *Dynamic Output Feedback Controller (DF-controller)* in order to stabilize and control the plant, as in Figure 7.1. Here $u_L$ is the actual input and $y$ as the final output; $y_L$ can be considered as the disturbance in the feedback loop and $u$ as the controller output. (In the literature, one sometimes uses the word "compensator" or "regulator" in place of "controller".)

By *DF-stabilization* of $\mathbb{D} \in \mathrm{TIC}_\infty(Y,U)$ we mean that we choose $\mathbb{Q} \in \mathrm{TIC}_\infty(U,Y)$ so that the map $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ (equivalently, $(I_{U\times W} - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right])^{-1} : \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$; cf. Figure 7.1) becomes stable; one often also wishes to minimize the norm $\|u_L \mapsto y\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U),\mathrm{L}^2(\mathbf{R}_+;Y))}$.

In Section 7.1 we shall extend several classical finite-dimensional results on DF-stabilization to general WPLSs; these results include the Youla parametrization of all stabilizing controllers (Corollary 7.1.8) based on a doubly coprime factorization (d.c.f.) of $\mathbb{D}$. However, it is not known whether each DF-stabilizable map has a d.c.f. (unless $\dim U, \dim Y < \infty$, see Lemma 7.1.4), hence we also present a theory for general $\mathrm{TIC}_\infty$ maps. (This applies to Sections 7.2 and 7.3 too.)

In DF-stabilization, we require that the controller ($\mathbb{Q}$) is well-posed (or proper, i.e., $\mathbb{Q} \in \mathrm{TIC}_\infty$). In finite-dimensional theory, one sometimes allows for improper controllers ("$\widehat{\mathbb{Q}} \in \mathrm{H}^\infty/\mathrm{H}^\infty$", i.e., $\mathbb{Q}$ is allowed to have a pole at infinity) while the closed-loop map ($(I_{U\times W} - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right])^{-1}$) is always required to be well-posed. The infinite-dimensional counterpart of this concept, *a DF-controller with* *internal loop*, was introduced in [WC], by George Weiss and Ruth Curtain. This generalization of the concept of DF-controllers will be treated in Section 7.2.1.

In the $\mathrm{H}^\infty$ *Four-Block Problem (*$\mathrm{H}^\infty$ *4BP)* of Chapter 12, the controller may use only a part ($y$) of the of the output ($\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$) as its input and it can control only a part ($u$) of the input ($\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) of the plant, as in Figure 7.8. Such a controller is called a *Dynamic Partial Output Feedback Controller (DPF-controller)* (cf. pp. 26–36 and 42–47 of [Francis]). We develop the theory for DPF-controllers (with or without internal loop) in Section 7.3. However, if $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, then any stabilizing DPF-controller for $\mathbb{D}$ is a stabilizing DF-controller for $\mathbb{D}_{21}$, and under reasonable assumptions also the converse holds, by Lemmas 7.3.5 and 7.3.6. Therefore, much of this theory is obtained as a corollary of Section 7.2.

We have above treated only the I/O theory, while one is often more interested in a system stabilizing another system (also internally); cf. Figures 7.2 and 7.9. However, if $\Sigma$ is a realization of the plant ($\mathbb{D}$) and $\widetilde{\Sigma}$ is a realization of the controller ($\mathbb{Q}$), then $\widetilde{\Sigma}$ stabilizes $\Sigma$ exponentially iff $\mathbb{Q}$ stabilizes $\mathbb{D}$ and $\Sigma$ and $\widetilde{\Sigma}$ are optimizable and estimatable (recall from Definition 6.7.3 and Corollary 9.2.13 that at least if $\Sigma$ is sufficiently regular, then this is equivalent to exponential stabilizability and detectability), by Theorems 7.2.3 and 7.3.11. We also present some further results on "$\widetilde{\Sigma}$ stabilizing $\Sigma$".

We give most of our results for (non-exponential) stabilization, because the exponential analogies of such results can be obtained through shifting, as in Remarks 7.2.19 and 7.3.24 (but the converse is not true). However, there are some results that seem to hold for exponential stabilization only; such results are given explicitly.

**Remark 7.0.1** *Almost all I/O results in this chapter are purely algebraic (and do not assume commutativity, neither a matrix structure over some commutative ring), hence they are valid when we replace* $\mathrm{TIC}_\infty$ *by* $\mathcal{A}'$ *and* TIC *by* $\mathcal{A}$*, where* $\mathcal{A}$ *and* $\mathcal{A}'$ *(and* $\mathcal{X}$*) are as in Remark 6.5.11.*

*Thus, one can have a given plant* $\mathbb{D} \in \mathcal{A}'(U,Y)$ *(*$U,Y \in \mathcal{X}$*) and seek for a* $\mathbb{Q} \in \mathcal{A}'(Y,U)$ *that makes* $\mathbb{D}_I^o \in \mathcal{A}$*, i.e., "stable"; see Definitions 7.1.1, 7.2.1, and 7.3.1 for details.*

*This holds for the results concerning the I/O maps only, i.e., the frequency-domain results; the generalization of state-space results requires, of course, further assumptions (which are often simple, cf. Chapter 13 for a discrete-time application).*

## 7.1 Dynamic feedback (DF) stabilization

> *A fail-safe circuit will destroy others.*
>
> — Klipstein

As explained above, in this section we generalize several classical dynamic output feedback (DF) results (cf. [Francis, Section 4]) to the infinite-dimensional case (see, e.g., Theorem 7.1.7); most others are generalized in Section 7.2.

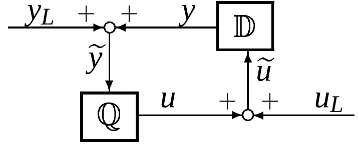


Figure 7.1: DF-controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

Every DF-stabilizable rational transfer function has a d.c.f., and classical DF-stabilization theory is based on d.c.f.'s. We believe that not every DF-stabilizable $\mathbb{D} \in \mathrm{TIC}_\infty$ has a d.c.f. (cf. Lemma 7.1.4), therefore we also develop a DF-stabilization theory for general $\mathrm{TIC}_\infty$ maps (and for general WPLSs).

In Figure 7.1, we have

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} u+u_L \\ y+y_L \end{bmatrix}, \tag{7.1}$$

or, by setting $\widetilde{u} := u+u_L$, $\widetilde{y} := y+y_L$,

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} + \begin{bmatrix} u_L \\ y_L \end{bmatrix}; \text{ or equivalently,} \tag{7.2}$$

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} = \left( I - \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} u_L \\ y_L \end{bmatrix} \tag{7.3}$$

provided that $\mathbb{Q}$ is an admissible DF-controller for $\mathbb{D}$, i.e., that $I - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty$, which is equivalent to $I - \mathbb{D}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty$ (by Lemma A.1.1(d1)).

Note that this corresponds to $L = I$ in the setting of Definition 6.6.1 (applied with substitutions $\mathbb{D} \mapsto \mathbb{D}^o := \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$, $y \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, $u_L \mapsto \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right]$; compare (7.1) and (6.123)–(6.124) with $x_0 = 0$), hence the solvability (in $\mathrm{TIC}_\infty$) of the above equations is, indeed, equivalent to the admissibility of $L$, i.e., to condition $I - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty$. We conclude that the corresponding closed-loop map is given by $\mathbb{D}^o_I$. Analogously, for the setting of Figure 7.4, the corresponding closed-loop system is given by the system $\Sigma^o_I : \begin{bmatrix} x_0 & \widetilde{x}_0 & u_L & y_L \end{bmatrix}^{\mathrm{T}} \mapsto \begin{bmatrix} x & \widetilde{x} & u & y \end{bmatrix}^{\mathrm{T}}$ defined below.

Therefore, we define:

**Definition 7.1.1 (DF-stabilization)** *We call* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ *an* admissible [stabilizing] (DF-)controller *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *if* $L = I$ *is admissible [stabilizing] for* $\mathbb{D}^o := \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$.

*We call* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y,\widetilde{H},U)$ *an* admissible [stabilizing] (DF-)controller for $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *[and we say that* $\widetilde{\Sigma}$ (DF-)stabilizes $\Sigma$*] if* $L = I$ *is admissible [stabilizing] for the (permutated) parallel connection*

$$\Sigma^o := \left[\begin{array}{cc|cc} \mathbb{A} & 0 & \mathbb{B} & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline 0 & \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ \mathbb{C} & 0 & \mathbb{D} & 0 \end{array}\right] \in \mathrm{WPLS}(U \times Y, H \times \widetilde{H}, U \times Y) \tag{7.4}$$

*(we use prefices as in Definition 6.6.4).*

*We call* $\mathbb{D}$ *(resp.* $\Sigma$*)* DF-stabilizable *if it has a stabilizing controller* $\mathbb{Q}$ *(resp.*

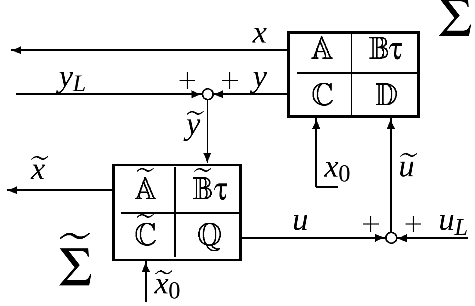


Figure 7.2: DF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U,H,Y)$

*$\widetilde{\Sigma}$), and we use prefices as in Definition 6.6.4.*

We usually say "stabilizing" instead of "admissible stabilizing" (for any meaning of these two words used in this monograph). Note that by "controller for $\mathbb{D}$" we refer to a I/O map ($\mathbb{Q}$) and by "controller for $\Sigma$" we refer to system ($\widetilde{\Sigma}$). In classical theory one often does not make any difference for these two concepts (but we do).

Thus, $\mathbb{Q}$ is admissible [stabilizing] for $\mathbb{D}$ iff the closed-loop system in Figure 7.1 is well-posed [and stable, i.e., $u, y \in \mathrm{L}^2$ for all $u_L, y_L \in \mathrm{L}^2$]. Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for $\Sigma$ iff the closed-loop system in Figure 7.2 is well-posed [and stable, i.e., $u, y \in \mathrm{L}^2$ and $x$ and $\tilde{x}$ are bounded for all $u_L, y_L \in \mathrm{L}^2(\mathbf{R}_+;*)$, $x_0 \in H$ and $\tilde{x}_0 \in \widetilde{H}$]. By Lemma A.4.5 and Lemma 6.1.10(a1), $\widetilde{\Sigma}$ is exponentially stabilizing for $\Sigma$ iff $x, \tilde{x} \in \mathrm{L}^2$ (and hence $u, y \in \mathrm{L}^2$) for all $u_L, y_L \in \mathrm{L}^2$, $x_0 \in H$ and $\tilde{x}_0 \in \widetilde{H}$.

Obviously, $\mathbb{D}^o$ and hence $\mathbb{D}^o_I$ are the same in both settings (i.e., in the setting of Figure 7.1 and in that of Figure 7.2). Thus, $\widetilde{\Sigma}$ is I/O-stabilizing for $\Sigma$ iff $\mathbb{Q}$ is stabilizing for $\mathbb{D}$. An analogous comment applies to Definitions 7.2.1 and 7.3.1 too.

Recall from Definition 6.6.10 that we follow the standard convention to use the word "stabilization" for state-feedback stabilization. Therefore, we have chosen the term "DF-stabilization" for dynamic output feedback, but we drop "DF-" when there is no danger of misinterpretation.

In some classical texts, one loosely speaks of "$\mathbb{Q}$ stabilizing $\Sigma$", but one then usually means the concept "$\widetilde{\Sigma}$ stabilizing $\Sigma$" for a suitable realization of $\mathbb{Q}$. However, we pay some attention to this "concept" in Remark 12.5.8.

From the above definition and Definition 6.6.1 we observe that $\mathbb{Q}$ is admissible for $\mathbb{D}$ iff $\widetilde{\Sigma}$ is admissible for $\Sigma$. We list here several additional equivalent conditions:

**Lemma 7.1.2 (DF-admissibility)** *A map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ is admissible [stabilizing] for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times Y)$ [and $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} \in \mathrm{TIC}(U \times Y)$]; or equivalently, if the closed-loop I/O map $\mathbb{D}^o_I : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \to \begin{bmatrix} u \\ y \end{bmatrix}$, given by*

*(cf. Figure 7.1)*

$$\mathbb{D}_I^o = \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \left( I - \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \right)^{-1} = \left( I - \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \right)^{-1} - I$$
$$= \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} - I = \begin{bmatrix} (I-\mathbb{Q}\mathbb{D})^{-1} - I & (I-\mathbb{Q}\mathbb{D})^{-1}\mathbb{Q} \\ \mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1} & (I-\mathbb{D}\mathbb{Q})^{-1} - I \end{bmatrix} \quad (7.5)$$

*is well-posed [and stable (i.e.,* $\mathbb{D}_I^o \in \mathrm{TIC}$*)].*

*Moreover,* $\mathbb{Q}$ *is admissible for* $\mathbb{D}$ *(equivalently,* $\widetilde{\Sigma}$ *is admissible for* $\Sigma$*) iff* $I - \mathbb{Q}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *(equivalently,* $I - \mathbb{D}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$*). For* $\mathbb{D}, \mathbb{Q} \in \mathrm{ULR}$ *this is equivalent to* $I - DQ \in \mathcal{G}\mathcal{B}(Y)$*.*

**Proof:** We have $I - L\mathbb{D}^o = \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}$, hence the first paragraph follows from Definition 6.6.1 and Proposition 6.6.2. Use Lemma A.1.1(d1) (note also (A.12)) and Proposition 6.3.1(c) for the second paragraph. □

The roles of $\mathbb{D}$ and $\mathbb{Q}$ (resp. $\Sigma$ and $\widetilde{\Sigma}$), are identical; e.g., $\mathbb{Q}$ stabilizes $\mathbb{D}$ iff $\mathbb{D}$ stabilizes $\mathbb{Q}$. This will not be the case in the dynamic partial (output) feedback, in Section 7.3, where the input of $\mathbb{Q}$ is only a part of the output of $\mathbb{D}$, and the input of $\mathbb{D}$ consists only partially of the output of $\mathbb{Q}$.

Unlike for admissibility, $\widetilde{\Sigma}$ being stabilizing for $\Sigma$ is a stronger condition than $\mathbb{Q}$ being stabilizing for $\mathbb{D}$, since in Figure 7.2 there are more signals (or maps between signals) to be stabilized (by the choice of $\widetilde{\Sigma}$) than in Figure 7.1.

Indeed, $\widetilde{\Sigma}$ I/O-stabilizes $\Sigma$ iff $\mathbb{Q}$ stabilizes $\mathbb{D}$. In this chapter, we will concentrate on I/O-stabilization, because for optimizable and estimatable $\Sigma$ and $\widetilde{\Sigma}$, I/O-stabilization is equivalent to exponential stabilization, by Theorem 7.2.3(d)&(c1). (See the theorem for several analogous results.)

If $\mathbb{D}$ has a d.c.f. and $\mathbb{Q}$ stabilizes $\mathbb{D}$, then $\mathbb{D}$ and $\mathbb{Q}$ have jointly [strongly] stabilizable and detectable realizations, by Theorem 6.6.28 and Proposition 7.1.6.

If $\Sigma$ and $\widetilde{\Sigma}$ are such realizations and we connect their inputs and outputs (as in Figure 7.2 and Definition 7.1.1), then the resulting combined closed-loop system becomes [strongly] stable, by Theorem 7.2.3. (If $\mathbb{D}$ has an exponential d.c.f. and $\mathbb{Q}$ stabilizes $\mathbb{D}$ exponentially, then we can choose $\Sigma$ and $\widetilde{\Sigma}$ so that the closed-loop system becomes exponentially stable.)

Note that we have assumed $\mathbb{Q}$ to be well-posed, that is, in $\mathrm{TIC}_\infty$ (i.e., $\widehat{\mathbb{Q}} \in \mathrm{H}^\infty_\infty$). See Section 7.2 for non-well-posed controllers.

A stable map (or system) is stabilized by any sufficiently small stable perturbation:

**Lemma 7.1.3 (Small Gain Theorem)** *Let* $\|\mathbb{D}\|_{\mathrm{TIC}(U,Y)} \|\mathbb{Q}\|_{\mathrm{TIC}(Y,U)} < 1$*. Then* $\mathbb{Q}$ *stabilizes* $\mathbb{D}$*.*

*If* $\Sigma$ *and* $\widetilde{\Sigma}$ *are [SOS-/strongly/exponentially] stable realizations of* $\mathbb{D}$ *and* $\mathbb{Q}$*, respectively, then* $\widetilde{\Sigma}$ *[SOS-/strongly/exponentially] stabilizes* $\Sigma$*.*

**Proof:** 1° $\mathbb{Q}$ *stabilizes* $\mathbb{D}$*:* Now $I - \mathbb{D}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.3.3(A0), hence (7.5) is stable.

2° $\Sigma$ *stabilizes* $\widetilde{\Sigma}$*:* This follows from Theorem 7.2.3(d)&(a)&(b). □

We will often assume that $\mathbb{D}$ has a d.c.f. If $U$ and $Y$ are finite-dimensional, then this does not reduce generality (we suspect that this is not the case in general):

**Lemma 7.1.4 (D.c.f.)** *If $\mathbb{D} \in \mathrm{TIC}_\infty(\mathbf{C}^n, \mathbf{C}^m)$ is DF-stabilizable, then $\mathbb{D}$ has a d.c.f.*

However, not all distributed scalar systems (functions $f/g$, where $f, g \in \mathrm{H}^\infty(\mathbf{C}^+)$ and $g \not\equiv 0$) have coprime factorizations, because $\mathrm{H}^\infty$ is not a Bezout domain (see [Vid]), although we do not know if this applies also to well-posed scalar transfer functions (those with $f/g$ bounded on some right half-plane).

If $\mathbb{D}$ has a r.c.f., then it has at least a DF-stabilizing controller with internal loop, by Corollary 7.2.13; see also Proposition 7.1.6(b1).

If $\mathbb{D} \in \mathrm{TIC}_\infty(\mathbf{C})$ has a r.c.f., then there is a *stable* stabilizing DF-controller for $\mathbb{D}$. Indeed, if $\widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+)$ are coprime, then $\widehat{\mathbb{M}} - \widehat{\mathbb{Q}}\widehat{\mathbb{N}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+)$, for some $\widehat{\mathbb{Q}} \in \mathrm{H}^\infty(\mathbf{C}^+)$, by [Treil92], hence then $\mathbb{Q} \in \mathrm{TIC}(\mathbf{C})$ is stabilizing for $\mathbb{D}$, by Proposition 7.1.6(b1). (This would not be the case if the scalar field was real, see [S92].)

Naturally, possible extensions of this "stable (Bass) rank" result by Serge Treil for multi- or infinite-dimensional Hilbert spaces would extend the above conclusion correspondingly.

**Proof of Lemma 7.1.4:** Now $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\mathbb{N} := \mathbb{D}(I - \mathbb{Q}\mathbb{D})^{-1}$ and $\mathbb{M} := (I - \mathbb{Q}\mathbb{D})^{-1}$ are stable, by (7.5). Thus, $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1} \in \mathrm{H}^\infty/\mathrm{H}^\infty$ (and $\widehat{\mathbb{D}}$ is DF-stabilizable), so by [Smith, Theorem 1], $\widehat{\mathbb{D}}$ has a generalized r.c.f. and a generalized l.c.f. in the sense that $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}_1\widehat{\mathbb{M}}_1^{-1}$ and $\widehat{\mathbb{D}} = \widehat{\widetilde{\mathbb{M}}}_1^{-1}\widehat{\widetilde{\mathbb{N}}}_1$ for some $\mathbb{N}_1, \mathbb{M}_1, \widetilde{\mathbb{N}}_1, \widetilde{\mathbb{M}}_1 \in \mathrm{TIC}$ with $\mathbb{N}_1, \mathbb{M}_1$ r.c. and $\widetilde{\mathbb{N}}_1, \widetilde{\mathbb{M}}_1$ l.c.

By Lemma 6.5.4(d2), $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{D} = \mathbb{N}_1\mathbb{M}_1^{-1}$ — this is a r.c.f. Similarly, $\mathbb{D}^{\mathrm{d}} = \widetilde{\mathbb{N}}_1^{\mathrm{d}}(\widetilde{\mathbb{M}}_1^{\mathrm{d}})^{-1}$ is a r.c.f., i.e., $\mathbb{D} = \widetilde{\mathbb{M}}_1^{-1}\widetilde{\mathbb{N}}_1$ is a l.c.f.. Thus, they can be completed to a d.c.f., by Lemma 6.5.8. □

**Lemma 7.1.5** *Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be r.c.f.'s. Then $\begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Y} \\ \mathbb{N} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M} & 0 \\ 0 & \mathbb{X} \end{bmatrix}^{-1}$ is a r.c.f. (of $\begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix}$). Moreover, we have the following:*

*(a) The DF-controller $\mathbb{Q}$ is admissible for $\mathbb{D}$ iff $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty$; if this is the case, then*

$$\mathbb{D}_I^o := \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} - I = \begin{bmatrix} 0 & \mathbb{Y} \\ \mathbb{N} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M} & -\mathbb{Y} \\ -\mathbb{N} & \mathbb{X} \end{bmatrix}^{-1} \tag{7.6}$$

*(b) The DF-controller $\mathbb{Q}$ stabilizes $\mathbb{D}$ iff $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$; if this is the case, and we set*

$$\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} := \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}^{-1}, \tag{7.7}$$

*then $\widetilde{\mathbb{M}}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, $\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is a l.c.f. of $\mathbb{D}$, and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a l.c.f. of $\mathbb{Q}$.*

*The obvious dual results for l.c.f.'s are true as well.*

So if $\mathbb{Q}$ stabilizes $\mathbb{D}$ and these maps have coprime factorizations from same side, then we actually have the d.c.f. (7.7); cf. Proposition 7.1.6(a).

**Proof:** Clearly $\begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Y} \\ \mathbb{N} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{M} & 0 \\ 0 & \mathbb{X} \end{bmatrix}^{-1}$ is a r.c.f., so (a) and the equivalence in (b) hold by Lemma 6.6.6 (and Lemma A.1.1(c3)) (recall that $L = I$ in Definition 7.1.1).

Assume now that $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ Then $\widetilde{\mathbb{M}}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma A.1.1(c1) (because $\mathbb{M}, \mathbb{X}$ do), and (7.7) shows that $\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}} = \mathbb{N}\mathbb{M}^{-1}$, $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} = \mathbb{Y}\mathbb{X}^{-1}$, and that these factorizations are coprime.

By taking (causal) adjoints, one gets the dual results. □

**Proposition 7.1.6** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U, Y)$.*

*(a) Any stabilizing DF-controller of $\mathbb{D}$ has a l.c.f. (resp. r.c.f.) iff $\mathbb{D}$ has a r.c.f. (resp. l.c.f.).*

*(b) If $\mathbb{D}$ has a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, then (b1)–(b3) hold.*

*(b1) A map $\mathbb{Q} \in \mathrm{TIC}_\infty$ DF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ has a l.c.f. $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ s.t. $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$. If $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are such, then*

$$\widetilde{\mathbb{X}} = (\mathbb{M} - \mathbb{Q}\mathbb{N})^{-1}, \quad \widetilde{\mathbb{Y}} = (\mathbb{M} - \mathbb{Q}\mathbb{N})^{-1}\mathbb{Q}, \quad \textit{and} \tag{7.8}$$

$$\mathbb{D}_I^o = \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} - I & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \to \begin{bmatrix} u \\ y \end{bmatrix}. \tag{7.9}$$

*(b2) Let $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be a l.c.f. Then $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ iff $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} \in \mathcal{G}\mathrm{TIC}$.*

*(b3) Let $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be a r.c.f. Then $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ iff $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.*

*(c) If $\mathbb{D}$ has a l.c.f. $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$, then (c1)–(c3) hold.*

*(c1) A map $\mathbb{Q} \in \mathrm{TIC}_\infty$ DF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ has a r.c.f. $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ s.t. $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I$. If $\mathbb{X}$ and $\mathbb{Y}$ are such, then*

$$\mathbb{X} = (\widetilde{\mathbb{M}} - \widetilde{\mathbb{N}}\mathbb{Q})^{-1}, \quad \mathbb{Y} = (\widetilde{\mathbb{M}} - \widetilde{\mathbb{N}}\mathbb{Q})^{-1}\mathbb{Q}. \tag{7.10}$$

*(c2) Let $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be a r.c.f. Then $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ iff $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$.*

*(c3) Let $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be a l.c.f. Then $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ iff $\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.*

*(d) Let $\mathbb{Q}$ DF-stabilize $\mathbb{D}$. Then $\mathbb{D}$ has a d.c.f. iff $\mathbb{Q}$ has a d.c.f.*

Note that $\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \begin{bmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c3).

In (b1), clearly $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix} = \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{X}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \to \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix}$.

**Proof:** (a) This follows from (b1) and (c1), because we can interchange the roles of $\mathbb{D}$ and $\mathbb{Q}$.

(b1) 1° Let $\mathbb{D}$ have a r.c.f. $(\mathbb{N},\mathbb{M})$ and $\widetilde{\mathbb{S}}\mathbb{M}-\widetilde{\mathbb{T}}\mathbb{N}=I$, $\widetilde{\mathbb{T}},\widetilde{\mathbb{S}}\in$ TIC. Let $\mathbb{Q}$ stabilize $\mathbb{D}$, so that $I-\mathbb{Q}\mathbb{D}=I-\mathbb{Q}\mathbb{N}\mathbb{M}^{-1}\in\mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{D}_I^o\in$ TIC, in particular, $\mathbb{M}-\mathbb{Q}\mathbb{N}\in\mathcal{G}\mathrm{TIC}_\infty$.

The stability of $\mathbb{D}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}]=\mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1}$ and $\mathbb{D}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}\mathbb{Q}]=\mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1}\mathbb{Q}=(I-\mathbb{D}\mathbb{Q})^{-1}-I$, from (7.5) (and Lemma A.1.1(f6)), implies that of $\widetilde{\mathbb{X}}:=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}=\mathbb{M}^{-1}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}]$ and $\widetilde{\mathbb{Y}}:=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}\mathbb{Q}$, by Lemma 6.5.6(b). Clearly $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}(\mathbb{M}-\mathbb{Q}\mathbb{N})=I$, so $\widetilde{\mathbb{X}}\in\mathcal{G}\mathrm{TIC}_\infty$ and $\widetilde{\mathbb{Y}}$ are l.c.

2° Conversely, if $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a l.c.f. and $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=I$, then $(I-\mathbb{Q}\mathbb{D})^{-1}=[\widetilde{\mathbb{X}}^{-1}(\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N})\mathbb{M}^{-1}]^{-1}=\mathbb{M}\widetilde{\mathbb{X}}$ etc., hence (7.9) holds, so $\mathbb{D}_I^o\in$ TIC, i.e., $\mathbb{Q}$ is stabilizing.

3° By Lemma 6.4.5(d), the $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ constructed in 1° are uniquely determined by $\mathbb{Q}$.

(b2) By Lemma 6.4.5, all l.c.f.'s of $\mathbb{Q}$ are given by $(\mathbb{U}\widetilde{\mathbb{Y}},\mathbb{U}\widetilde{\mathbb{X}})$ with $\mathbb{U}\in\mathcal{G}\mathrm{TIC}$, Therefore, $\mathbb{Q}$ has a l.c.f. of the form described in (a) iff $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}\in\mathcal{G}\mathrm{TIC}$.

(b3) This follows from Lemma 7.1.5.

(c) This is proved analogously (or by taking (causal) adjoints in (b)). Of course, we could write a dual formula for $\mathbb{D}_I^o$ too.

(d) This follows from (a) and from the fact that a well-posed map has a d.c.f. iff it has a r.c.f. and a l.c.f. [Lemma 6.5.8].

□

In most control theory one studies proper rational transfer functions (i.e., those with a (WPLS) realization with $\dim U,\dim H,\dim Y<\infty$); they always have a d.c.f. If $\dim U,\dim Y<\infty$, then $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ must have a d.c.f. in order to be DF-stabilizable, by Lemma 7.1.4. See Lemma 6.5.10 for further sufficient conditions for the existence of a d.c.f.

For these reasons, we shall often assume the existence of a d.c.f. This assumption enables us to generalize the Youla parameterization of all stabilizing controllers:

**Theorem 7.1.7 (Stabilizing DF-controllers)** *Let* $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ *have a r.c.f. and a l.c.f.* $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$*. Then* $\mathbb{D}$ *has the d.c.f.*

$$\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}=I=\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix} \tag{7.11}$$

*for some* $\mathbb{T},\mathbb{S},\widetilde{\mathbb{T}},\widetilde{\mathbb{S}}\in$ TIC, *and the following are equivalent:*

*(i)* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}$.

*(ii)* $\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\in\mathcal{G}\mathrm{TIC}$, $\mathbb{X}\in\mathcal{G}\mathrm{TIC}_\infty$ *and* $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$.

*(iii)* $\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\in\mathcal{G}\mathrm{TIC}$, $\widetilde{\mathbb{X}}\in\mathcal{G}\mathrm{TIC}_\infty$ *and* $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$.

*(iv)* $\mathbb{Q}$ *has a d.c.f.* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *s.t.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix}. \tag{7.12}$$

*(v) [Youla]* $\mathbb{Q} = (\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1}$ *for some* $\mathbb{U} \in \text{TIC}$ *s.t.* $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\text{TIC}_\infty$.

*(vi) [Youla]* $\mathbb{Q} = (\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}})$ *for some* $\mathbb{U} \in \text{TIC}$ *s.t.* $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\text{TIC}_\infty$.

*(vii)* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$*, where* $\left[\begin{smallmatrix} \mathbb{Y} \\ \mathbb{X} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$ *and* $\mathbb{U} \in \text{TIC}$ *is s.t.* $\mathbb{X} = \mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\text{TIC}_\infty$.

*(viii)* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*, where* $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & \mathbb{U} \end{smallmatrix}\right]\left[\begin{smallmatrix} \widetilde{\mathbb{S}} & \widetilde{\mathbb{T}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right]$ *and* $\mathbb{U} \in \text{TIC}$ *is s.t.* $\widetilde{\mathbb{X}} = \widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\text{TIC}_\infty$.

*Moreover, for* $\mathbb{U} \in \text{TIC}$ *we have* $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\text{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\text{TIC}_\infty$*, and if either is true, then*

$$(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}}). \tag{7.13}$$

Thus, any $\mathbb{D} \in \text{TIC}_\infty$ having a d.c.f. (denoted by (7.11)) is DF-stabilizable iff $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\text{TIC}_\infty$ for some $\mathbb{U} \in \text{TIC}$, or equivalently, iff the $\mathbb{S}$ in (7.11) can be chosen so that $\mathbb{S} \in \mathcal{G}\text{TIC}_\infty \cap \text{TIC}$. Those factorizations (7.11), in which $\mathbb{S} \notin \mathcal{G}\text{TIC}_\infty$, can be thought as defining non-well-posed (improper) DF-controllers; see Theorem 7.2.14 for a generalization containing also such controllers.

One faces the same problem in the finite-dimensional theory (i.e., the theory for rational transfer functions with $\dim U, \dim Y < \infty$): unless $\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}} \in \mathcal{G}\text{H}^\infty_\infty$, the controller $\mathbb{Q}$ is ill-posed (i.e., not proper, that is, unbounded in any right half-plane). If $\det(\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}}) \equiv 0$, then $\widehat{\mathbb{Q}}$ is not defined anywhere. However, regardless of $\det(\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}})$, the combined closed-loop condition (in Figure 7.1) is well-posed. This kind of non-well-posed controllers (“controllers with internal loop”) are treated in Section 7.2.

Note that all factorizations of $\mathbb{Q}$ (and $\mathbb{D}$) in the theorem are obviously coprime. We recall from Lemma A.1.1(c3) that $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\text{TIC}$ iff $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\text{TIC}$.

**Proof:** The d.c.f. (7.11) exists, by Lemma 6.5.8. Thus, any stabilizing controller of $\mathbb{D}$ has a d.c.f., by Proposition 7.1.6(d).

“(i)⇔(ii)”: This follows from Lemma 7.1.5(b). Note that $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\text{TIC}$ implies that $\mathbb{Y}$ and $\mathbb{X}$ are r.c.

“(ii)⇔(iii)”: These are adjoints of each other.

“(ii)⇔(vii)⇔(iv)”: By Lemma 6.5.9(b), all completions $\left[\begin{smallmatrix} \mathbb{Y} \\ \mathbb{X} \end{smallmatrix}\right]$ such as in (ii) are given by $\left[\begin{smallmatrix} \mathbb{T}\mathbb{V}+\mathbb{M}\widetilde{\mathbb{U}} \\ \mathbb{S}\mathbb{V}+\mathbb{N}\widetilde{\mathbb{U}} \end{smallmatrix}\right]$ with $\widetilde{\mathbb{U}} \in \text{TIC}$ and $\mathbb{V} \in \mathcal{G}\text{TIC}$. The stabilizing controllers are, by (ii), the ones corresponding to $\mathbb{S}\mathbb{V}+\mathbb{N}\widetilde{\mathbb{U}} \in \mathcal{G}\text{TIC}_\infty$, so, by Lemma 6.4.5, we may take $\mathbb{V} = I$ (and $\mathbb{U} := \widetilde{\mathbb{U}}\mathbb{V}^{-1} \in \text{TIC}$ arbitrary) without altering $\mathbb{Q}$, and thus we obtain the equivalent parametrization (vii). Moreover, in this case

$$\begin{bmatrix} \mathbb{M} & \mathbb{T}+\mathbb{M}\mathbb{U} \\ \mathbb{N} & \mathbb{S}+\mathbb{N}\mathbb{U} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} & -(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}}) \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix}, \tag{7.14}$$

by Lemma 6.5.9(c).

Claim (v) is a reformulation of (vii); claims (viii) and (vi) are the duals of (vii) and (v), respectively.

To prove the final claim about Youla parametrization, we note that, by (7.14) and Lemma A.1.1(c1), $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$. Moreover, (7.14) implies (7.13) if $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$. □

Directly from the theorem we get:

**Corollary 7.1.8 (Youla-parametrization)** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have a r.c.f. and a l.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$. Then $\mathbb{D}$ has the d.c.f.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix}. \tag{7.15}$$

*for some $\mathbb{T},\mathbb{S},\widetilde{\mathbb{T}},\widetilde{\mathbb{S}} \in \mathrm{TIC}$.*

*Moreover, the following are equivalent:*

*(i) $\mathbb{D}$ is DF-stabilizable.*

*(ii) $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ for some $\mathbb{X},\mathbb{Y}$ s.t. $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.*

*(iii) $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ for some $\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}$ s.t. $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$.*

*(vi) [Youla] $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$ for some $\mathbb{U} \in \mathrm{TIC}$.*

*(vii) [Youla] $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$ for some $\mathbb{U} \in \mathrm{TIC}$.*

*Finally, if these conditions are satisfied, then all DF-stabilizing controllers of $\mathbb{D}$ are parametrized by*

$$\mathbb{Q} = (\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}}). \tag{7.16}$$

*where $\mathbb{U}$ ranges over those $\mathbb{U} \in \mathrm{TIC}$ for which $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$ (equivalently, $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$).*

*An alternative parametrization is $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ s.t. $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ and $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$; a third one is $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ s.t. $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ and $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$.* □

Given certain regularity, we can make the controller corresponding to a r.c.f. well-posed:

**Corollary 7.1.9** *Let $\mathbb{D}$ have a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ s.t. $\widetilde{\mathbb{X}},\mathbb{M} \in \mathrm{ULR}$. Then $\mathbb{D}$ is DF-stabilizable.*

**Proof:** Define

$$\widetilde{\mathbb{S}} := M^{-1} + \widetilde{\mathbb{X}} - M^{-1}\mathbb{M}\widetilde{\mathbb{X}}, \quad \widetilde{\mathbb{T}} := \widetilde{\mathbb{Y}} - M^{-1}\mathbb{M}\widetilde{\mathbb{Y}} \tag{7.17}$$

to obtain that $\widetilde{\mathbb{S}}\mathbb{M} - \widetilde{\mathbb{T}}\mathbb{Y} = I$, $\widetilde{\mathbb{S}} \in \mathrm{ULR}$ and $\widetilde{S} = M^{-1} + \widetilde{X} - \widetilde{X} = M^{-1} \in \mathcal{G}\mathcal{B}$ (by Proposition 6.3.1(c)), hence $\widetilde{\mathbb{S}} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, $\widetilde{\mathbb{S}}^{-1}\widetilde{\mathbb{T}}$ DF-stabilizes $\mathbb{D}$, by Proposition 7.1.6(b1). □

Naturally, Youla parametrization can be applied also when one one wishes to work in a subclass of TIC:

**Proposition 7.1.10** *Assume that* $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR$\cap$TIC *(e.g.,* $\mathcal{A} =$ MTIC *or* $\mathcal{A} =$ ULR$\cap$TIC*, see Definition 6.2.4). Assume that* $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$*, i.e., (7.11) holds with* $\mathbb{M}, \mathbb{N}, \mathbb{S}, \mathbb{T}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{S}}, \widetilde{\mathbb{T}} \in \mathcal{A}$*.*

*Then all stabilizing DF-controllers of* $\mathbb{D}$ *are parametrized in Theorem 7.1.7, and the ones that have a d.c.f. over* $\mathcal{A}$ *are exactly those whose parameter(s) are in* $\mathcal{A}$*, i.e., which satisfy any (hence all) of the following equivalent conditions:*

*(ii)* $\mathbb{Q}$ *has a r.c.f.* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *s.t.* $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$*;*

*(iii)* $\mathbb{Q}$ *has a l.c.f.* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *s.t.* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$*;*

*(v)* $\mathbb{U} \in \mathcal{A}$ *in (v), (vi), (vii) or (viii) of Theorem 7.1.7.*

*If* $\mathcal{A} \underset{a}{\subset}$ ULR*, then* $\mathbb{D}$ *has stabilizing DF-controllers.*

Note that $\mathcal{A} =$ MTIC and $\mathcal{A} =$ ULR$\cap$TIC satisfy all above assumptions (cf. Definition 6.2.4). See also (the Corona) Theorem 4.1.6(d) for such d.c.f.'s.

**Proof:** Theorem 7.1.7 parametrizes all DF-stabilizing controllers of $\mathbb{D}$, in particular, by (vi'), (vi") and (7.13) of Theorem 7.1.7, they satisfy

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} := \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} I & \mathbb{U} \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} := \begin{bmatrix} I & -\mathbb{U} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \tag{7.18}$$

for some $\mathbb{U} \in$ TIC. If $\mathbb{U} \in \mathcal{A}$, then clearly $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$. Conversely, if $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$, then $\left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{A}$ (analogously, $\mathbb{U} \in \mathcal{A}$ iff $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$).

If $\mathcal{A} \underset{a}{\subset}$ ULR, then the existence of a stabilizing controller is guaranteed, by Corollary 7.1.9. (Alternatively, we can take $\mathbb{U} := -M^{-1}T$, because then $(\widetilde{\mathbb{S}} + \mathbb{U}\widetilde{\mathbb{N}})(+\infty) = M^{-1}(M\widetilde{S} - T\widetilde{N}) = M^{-1} \in \mathcal{GB}(U)$, by $(7.11)(+\infty)_{1,1}$ and Proposition 6.3.1(c).) □

## Notes

The connection between coprime factorization and dynamic stabilization (in Theorem 7.1.7 and Corollary 7.1.8) is well-known; see, e.g., pp. 36–42 of [Francis] or Chapter 12 of [ZDG] for classical presentations and [CWW96] and [CWW01] for results for WPLSs. O. Staffans has recently included some further results in [Sbook].

The class of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions" is not contained in, nor does it contain the class matrix-valued well-posed transfer functions. (Remark 7.2.20 sketches an infinite-dimensional theory that covers both classes.)

The dynamic I/O-stabilization theory based on fractional representations was first introduced in [DLMS] for rational functions. Also more general cases have been studied extensively; see, e.g., [GS] for the general case of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions", [CZ] for the special case of a Callier–Desoer class (from [CD78]), and [Logemann93] for certain other special cases (with applications to PS-systems). An excellent classical reference is [Vid], which covers all these classes to some extent. See the notes to Chapters 7 and 9 of [CZ] for further historical notes (these also cover the results postponed to Section 7.2).

Above we have presented here only the core results of the theory and those results that require the controller to be well-posed. In the rest of this chapter we shall present further results on DF-stabilization under more general assumptions.

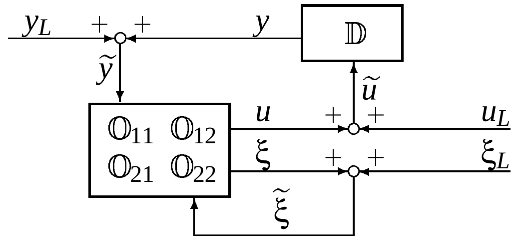


Figure 7.3: DF-controller $\mathbb{O}$ with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

## 7.2 DF-stabilization with internal loop ( $\begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix}$ )

*It's not an optical illusion, it just looks like one.*

— Phil White

The restriction $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ (or $\mathbb{S} + \mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$) in the Youla parametrization of Theorem 7.1.7 might feel somewhat artificial: it is only needed in order to have the open-loop map $\mathbb{Q} : \widetilde{y} \mapsto u$ of the controller well-posed (or proper, i.e., $\mathbb{Q} \in \mathrm{TIC}_\infty$), but even without that condition, all closed-loop maps are well-posed (once we connect the controller to the plant).

Therefore, in finite-dimensional theory, one sometimes allows for improper (or non-well-posed) controllers. To cover such controllers in addition to the proper ones, G. Weiss and R. Curtain introduced *DF-controllers with internal loop* in [WC].

This concept allows us to have mathematically more beautiful formulae and offers a solution to certain problems that cannot be solved by well-posed controllers (see the example at the beginning of [CWW01]). Nevertheless, in our most important results, we also point out when such a controller can be replaced by a well-posed controller.

Well-posed controllers, i.e., those of Section 7.1, are a subset of controllers with internal loop (and so are all $\mathrm{H}^\infty/\mathrm{H}^\infty$ fractional controllers, see Remark 7.2.8), hence many results concerning them were omitted in the previous section and are presented here under wider generality.

On the other hand, the proofs of most results for controllers with internal loop could be reduced to the well-posed case, by Lemma 7.2.6.

A map $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$, where $\Xi$ is an arbitrary Hilbert space, becomes a DF-controller with internal loop when we connect its second output to its second input, as in Figure 7.3. This resulting controller need not be well-posed, i.e., closing the internal ($\xi$) loop only need not be an admissible operation (when $\mathbb{O}$ is uncoupled from $\mathbb{D}$); it is enough that the combined closed-loop system of Figure 7.3 becomes well-posed.

As above, a DF-controller with internal loop has an internal signal $\xi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};\Xi)$, where $\Xi$ is some Hilbert space. Note that whereas a given plant fixes the signal spaces $U$ and $Y$ of any of its controllers, the space $\Xi$ may be different for different controllers.

In Figure 7.3, we obviously have

$$\begin{bmatrix} u \\ y \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix} =: \mathbb{D}^o \begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix}. \tag{7.19}$$

As under (7.1), we observe that the corresponding closed-loop map is given by $\mathbb{D}^o_I = (I-\mathbb{D}^o)^{-1}\mathbb{D}^o$, and that the corresponding closed-loop system is given by $\Sigma^o_I$ given below. Therefore, we make the following definitions:

**Definition 7.2.1 (DF-stabilization with internal loop)** *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. *A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y\times\Xi, U\times\Xi)$ *(where also* $\Xi$ *is a Hilbert space) is an* admissible [stabilizing] (DF-)controller with internal loop for $\mathbb{D}$ *if the output feedback operator* $L=I$ *is admissible [stabilizing] for*

$$\mathbb{D}^o := \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U\times Y\times\Xi). \tag{7.20}$$

*We call* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y\times\Xi, \widetilde{H}, U\times\Xi)$ *an* admissible [stabilizing] (DF-)controller with internal loop for $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *if* $L=I$ *is admissible [stabilizing] for the (permutated) parallel connection*

$$\Sigma^o := \left[\begin{array}{cc|ccc} \mathbb{A} & 0 & \mathbb{B} & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline 0 & \widetilde{\mathbb{C}}_1 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{C} & 0 & \mathbb{D} & 0 & 0 \\ 0 & \widetilde{\mathbb{C}}_2 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \in \mathrm{WPLS}(U\times Y\times\Xi, H\times\widetilde{H}, U\times Y\times\Xi). \tag{7.21}$$

*We use prefices as in Definition 6.6.4 with* $\Sigma^o_I$ *in place of* $\Sigma_L$.

*We call* $\mathbb{D}$ *(resp.* $\Sigma$*)* DF-stabilizable with internal loop *if there is a stabilizing controller with internal loop for* $\mathbb{D}$ *(resp. for* $\Sigma$*), and we use prefices as above.*

*We call two admissible DF-controllers for* $\mathbb{D}$ *(resp. for* $\Sigma$*) with internal loop* equivalent for $\mathbb{D}$ *(resp. for* $\Sigma$*) if the corresponding* $(1\text{–}2, 1\text{–}2)$*-blocks of* $\mathbb{D}^o_I := (I-\mathbb{D}^o)^{-1} - I$ *are equal, i.e., if they determine same maps from* $u_L, y_L$ *to* $u, y$.

*If* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Y\times\Xi, U\times\Xi)$*, then we may remove the words "with internal loop" everywhere in this definition and identify* $\mathbb{O}$ *with* $\mathbb{O}_{11} \in \mathrm{TIC}_\infty(Y,U)$ *(cf. Lemma 7.2.7).*

Naturally, "*[DF-]stabilizes*" means "is stabilizing for", in any of the above settings.

Note that $\mathbb{D}^o_I$ maps $(u_L, y_L, \xi_L) \mapsto (u, y, \xi)$. See also Figures 7.3 and 7.4 and the comments below Definition 7.1.1 and Summary 6.7.1.

**Lemma 7.2.2 (DF-Admissibility and equivalence)** *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. *A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y\times\Xi, U\times\Xi)$ *is an admissible [stabilizing] controller with internal loop for* $\mathbb{D}$ *iff the connection if Figure 7.3 is well-posed [and stable, i.e.,*

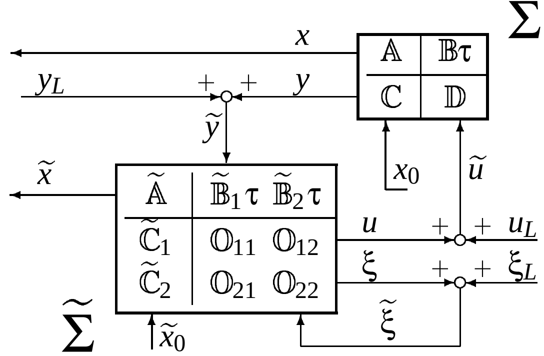


Figure 7.4: DF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma \in \mathrm{WPLS}(U,H,Y)$

*$u,y,\xi \in \mathrm{L}^2$ for all $u_L,y_L,\xi_L \in \mathrm{L}^2$], equivalently, iff $I-\mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty(U\times Y\times \Xi)$ [and $(I-\mathbb{D}^o)^{-1} \in \mathrm{TIC}$].*

*Moreover, $\widetilde{\Sigma} = \left[\begin{smallmatrix} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \widetilde{\mathbb{C}} & \mathbb{O}\end{smallmatrix}\right]$ is admissible with internal loop for $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ iff $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$.*

*Finally, if two admissible controllers for $\Sigma$ are equivalent for $\Sigma$ (i.e., their I/O maps are equivalent for $\mathbb{D}$), then the maps from $x_0,u_L,y_L$ to $x,u,y$ are equal for the two closed-loop systems.*

Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for $\Sigma$ iff the closed-loop system in Figure 7.4 is well-posed [and stable, i.e., $u,y,\xi \in \mathrm{L}^2$ and $x$ and $\widetilde{x}$ are bounded for all $u_L,y_L,\xi_L \in \mathrm{L}^2(\mathbf{R}_+;*)$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$]. (We note that exponential stability is equivalent to $x,\widetilde{x} \in \mathrm{L}^2$ (and hence $u,y,\xi \in \mathrm{L}^2$) for all $u_L,y_L,\xi_L \in \mathrm{L}^2$, $x_0 \in H$ and $\widetilde{x} \in \widetilde{H}$, by Lemma A.4.5 and Lemma 6.1.10(a1).)

We observe that only the maps concerning $\widetilde{x}$, $\widetilde{x}_0$, $\xi$ and $\xi_L$ may differ for equivalent controllers for $\Sigma$; thus, there is no difference from the part of $\widetilde{\Sigma}$ visible for $\Sigma$.

**Proof:** The claim on $I-\mathbb{D}^o$ and the "moreover claim" follow from Definitions 7.2.1 and 6.6.1. The reference to Figure 7.3 is obvious (cf. (6.122)–(6.124) and (6.127)).

The final claim (which could be observed from Figure 7.4) follows by computing $\Sigma_I^o$ from (6.125) and observing that its first, third and fourth rows and columns depend only on $\Sigma$ and $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0\end{smallmatrix}\right] \mathbb{D}_I^o \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0\end{smallmatrix}\right]^{\mathrm{T}}$ (use the fact that $\mathbb{D}_I^o := \mathbb{D}^o(I-\mathbb{D}^o)^{-1} = (I-\mathbb{D}^o)^{-1} - I$). □

The identification of $\left[\begin{smallmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0\end{smallmatrix}\right]$ and $\mathbb{O}_{11}$ is natural: all open-loop and closed-loop signals in Figures 7.3 and 7.1 become equal (except that in Figure 7.3, we have the additional, negligible signals $\xi = 0$ and $\widetilde{\xi} = \xi_L$). Thus, a (well-posed) stabilizing controller (in the sense of Definition 7.1.1) is a special case of a stabilizing controller with internal loop (see also Lemma 7.2.7). The situation with systems is the same (cf. Figures 7.4 and 7.2).

We stress that we mention the words "internal loop" explicitly whenever we speak of such controllers; all other maps are assumed to be well-posed, i.e., $\in \mathrm{TIC}_\infty$ (which is also stated explicitly in theorems and definitions), so no confusion should arise. The same applies to maps with coprime internal loop

(Definition 7.2.11) and also elsewhere is this chapter.

In connection with the $H^\infty$ Four-Block Problem, however, the theory for controllers with internal loop becomes more natural and beautiful than the part restricted to well-posed controllers. Therefore, in Chapter 12, contrary to the practice of this chapter, a "[stabilizing] controller" is allowed to possess an internal loop, and "well-posed" is always written explicitly, never tacitly.

Trivially, $\widetilde{\widetilde{\Sigma}}$ I/O-DF-stabilizes $\Sigma$ iff $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ (i.e., iff $\mathbb{D}_I^o$ becomes stable). Under standard assumptions, this is also equivalent to the stronger condition that $\widetilde{\widetilde{\Sigma}}$ DF-stabilizes $\Sigma$ (i.e., that the whole $\Sigma_I^o$ becomes stable):

**Theorem 7.2.3 ($\widetilde{\widetilde{\Sigma}}$ stabilizes $\Sigma \Leftrightarrow \mathbb{O}$ stabilizes $\mathbb{D}$)** *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\widetilde{\widetilde{\Sigma}} = \left[\begin{array}{c|c} \widetilde{\widetilde{\mathbb{A}}} & \widetilde{\widetilde{\mathbb{B}}} \\ \hline \widetilde{\widetilde{\mathbb{C}}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y\times\Xi,\widetilde{\widetilde{H}},U\times\Xi)$.

*(a) Suspect that $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are SOS-stabilizable. Then $\widetilde{\widetilde{\Sigma}}$ SOS-stabilizes $\Sigma$ with internal loop iff $\widetilde{\widetilde{\Sigma}}$ I/O-stabilizes $\Sigma$ with internal loop.*

*(b)* **([Strong] stability)** *Suppose that any of the following conditions holds:*

*(1.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are [[exponentially] strongly] q.r.c.-stabilizable;*
*(2.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are [[exponentially] strongly] q.l.c.-detectable;*
*(3.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are SOS-stabilizable and [[exponentially] strongly] detectable;*
*(4.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are detectable and [exponentially] stabilizable.*

*Then $\widetilde{\widetilde{\Sigma}}$ [[exponentially] strongly] stabilizes $\Sigma$ with internal loop iff $\widetilde{\widetilde{\Sigma}}$ I/O-stabilizes $\Sigma$ with internal loop.*

*(c1)* **(Exponential stability)** *The system $\widetilde{\widetilde{\Sigma}}$ stabilizes $\Sigma$ exponentially with internal loop iff $\widetilde{\widetilde{\Sigma}}$ I/O-stabilizes $\Sigma$ with internal loop and $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are optimizable and estimatable.*

*(c2) Suppose that any of the following conditions holds:*

*(1.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are optimizable and estimatable;*
*(2.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are optimizable and input-detectable;*
*(3.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are estimatable and output-stabilizable;*
*(4.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are optimizable and q.r.c.-stabilizable;*
*(5.) both $\Sigma$ and $\widetilde{\widetilde{\Sigma}}$ are estimatable and q.l.c.-detectable.*

*Then $\widetilde{\widetilde{\Sigma}}$ stabilizes $\Sigma$ exponentially with internal loop iff $\widetilde{\widetilde{\Sigma}}$ I/O-stabilizes $\Sigma$ with internal loop.*

*(d)* **(Well-posed controllers)** *Suppose that, instead,* $\widetilde{\widetilde{\Sigma}} = \left[\begin{array}{c|c} \widetilde{\widetilde{\mathbb{A}}} & \widetilde{\widetilde{\mathbb{B}}} \\ \hline \widetilde{\widetilde{\mathbb{C}}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y,\widetilde{\widetilde{H}},U)$. *Then (a)–(c2) hold if we delete the words "with internal loop" everywhere in this theorem.*

Thus, all maps between signals (i.e., $\Sigma_I^o : x_0,\widetilde{\widetilde{x}}_0,u_L,y_L,\xi_L \mapsto x,\widetilde{\widetilde{x}},u,y,\xi$ (and $\mapsto \widetilde{\widetilde{u}},\widetilde{\widetilde{y}},\widetilde{\widetilde{\xi}}$)) in Figure 7.4 are (SOS-/strongly/exponentially/...) stable iff the maps from $u_L$, $y_L$ and $\xi_L$ to $u$, $y$ and $\xi$ are stable and $\Sigma$ and $\widetilde{\Sigma}$ have the corresponding stabilizability listed above.

Therefore, we can often concentrate on the I/O theory. E.g., if $\Sigma$ is jointly stabilizable and detectable, and we find a stabilizing controller for $\mathbb{D}$, then any of its q.r.c.-stabilizable realizations (cf. Theorems 7.2.14 and 6.6.28) stabilizes $\Sigma$. Analogous claims hold under other assumptions for $\Sigma$.

**Proof of Theorem 7.2.3:** (a)&(b)&(c2) By Lemma 6.7.18 (and Lemma 6.7.17 with $G = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$), $\Sigma^o$ inherits the stabilizability and detectability properties of $\Sigma$ and $\widetilde{\Sigma}$. Therefore the above assumptions imply, by Proposition 6.7.14, that $L = I$ is (SOS-/strongly/exponentially, depending on the assumptions) stabilizing for $\Sigma^o$ iff it is I/O-stabilizing.

(c1) This is Theorem 7.4 of [WR00] (alternatively, "if" follows from Theorem 6.7.10(d)(viii), and the converse from (6.126) for optimizability (note that $\Sigma^o$ is obviously optimizable iff $\Sigma$ and $\widetilde{\Sigma}$ are optimizable) and by duality for estimatability (see Lemma 6.7.2(e'))).

(d) The above proofs still apply (use $\mathbb{O} = \begin{bmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{bmatrix}$ etc.). □

We can now extend the standard result (cf. p. 303 of [ZDG] and Theorem 7.3.12) to infinite-dimensional systems (although the converse in (b) is incomplete and those in (c) and (d) do not cover all WPLSs):

**Theorem 7.2.4 (Exp. DF-stabilizable ⇔ opt. & est.)** *Let* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$.

*(a) If $\Sigma$ is exponentially DF-stabilizable [with internal loop], then $\Sigma$ is optimizable and estimatable.*

*(b) Conversely, if $\Sigma$ is jointly [[exponentially] strongly] stabilizable and detectable, then $\Sigma$ is [[exponentially] strongly] DF-stabilizable with internal loop.*

*(c) Assume that $\mathbb{A}Bu_0, \mathbb{A}^*C^*y_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;H)$ for all $u_0 \in U$ and $y_0 \in Y$, and that $\mathbb{D}$ is ULR. Then the following are equivalent:*

*(i) $\Sigma$ is exponentially DF-stabilizable;*
*(ii) $\Sigma$ is exponentially DF-stabilizable with internal loop;*
*(iii) $\Sigma$ is optimizable and estimatable;*
*(iv) $\Sigma$ is exponentially jointly stabilizable and detectable;*
*(v) $\Sigma$ is exponentially jointly stabilizable and detectable by some bounded $K$ and $H$.*

*Moreover, if (v) holds, then (d) applies with the same $K$ and $H$ (hence (6.169) and (7.22) become ULR).*

*(d) If $K$ and $H$ are [[exponentially] strongly] jointly stabilizing with (6.169) being SR, and $I - \mathbb{G}_L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if (6.169) is ULR), then*

$$\left(\begin{array}{c|c} A + BK_s + HC_s + HDK_s & -H \\ \hline K & 0 \end{array}\right) \in \mathrm{WPLS}(Y,H,U) \tag{7.22}$$

*is a [[exponentially] strongly] DF-stabilizing controller for $\Sigma$. Moreover, (7.22) is SR and [[exponentially] strongly] jointly stabilizable and detectable.*

Note that the assumptions of (c) hold if $B$ and $C$ are bounded (or if $\mathbb{A}$ is somewhat smoothing, e.g., if Hypothesis 9.5.1 holds), hence always in discrete time.

A weaker form of the exponential part of the theorem is well-known for Pritchard–Salamon systems; e.g., Theorem 2.30 of [Keu] is a special case of (d) (since PS-systems are ULR and stabilizability (and detectability) are defined in a very strong sense for PS-systems; see Remark 6.6.15). However, the result (c) seems to be new in this generality

If we drop the requirement "$I - \mathbb{G}_L \notin \mathcal{G}\mathrm{TIC}_\infty$" from (d), then "(7.22)" can still be formulated as a controller with internal loop; see Proposition 5.3 of [WC] for an exponential version of this claim (or modify our proof slightly).

**Proof:** (a) This follows from Theorem 7.2.3(c1).

(b) By the proof of Theorem 6.6.28, we have the d.c.f. (6.172), and $\Sigma$ is [strongly] r.c.-stabilizable. Consequently,

$$\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_L & 0 & \mathbb{H}_L \\ \hline -\mathbb{K}_L & 0 & -\mathbb{E}_L \\ \mathbb{C}_L & I & \mathbb{G}_L \end{array}\right] \tag{7.23}$$

is an I/O-stabilizing controller with internal loop for $\Sigma$ (i.e., $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ with internal loop), by, e.g., Theorem 7.2.14(i) (or the dual of Lemma 7.2.10(a); we could use (7.13) instead if one would assume that $I - \mathbb{G}_L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$, i.e., that $\mathbb{Q}$ were well-posed).

But (7.23) is [[exponentially] strongly] stable, hence [strongly] r.c.-stabilizable [[and optimizable and estimatable]], hence it DF-stabilizes $\Sigma$ [[exponentially] strongly] with internal loop, by Theorem 7.2.3(b)[(c1)].

(c) By Corollary 9.2.13, (iii)–(v) are equivalent and the "moreover"-claim holds (with the ULR-property from Lemma 6.3.16(d)); in particular, (v) implies (i), by (d). Implication "(i)$\Rightarrow$(ii)" is trivial, and "(ii)$\Rightarrow$(iii)" was given in (a).

(d) (Note that we have adopted the notation of Definition 6.6.21. Naturally, the signs of $K$ and $H$ can be interchanged.)

If (6.169) is ULR, UR, SLR, UVR or SVR, then so are all systems and maps appearing below (including (7.22)), by Proposition 6.3.1(b2); for the same reason, all of them are always SR. (See also Lemma 6.6.27.)

1° *When (6.169) is ULR:* If (6.169) is ULR, then $I - \mathbb{G}_L$ is invertible, by Proposition 6.3.1(c), since the I/O map of (6.169) (and hence of (6.170) and (6.171)) corresponding to $K$ and $H$ is given by $\left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$; in particular, $G_L = 0 (= G)$.

2° *DF-stabilizing $\mathbb{Q}$:* By the proof of Theorem 6.6.28, we have the d.c.f. (6.172). By (6.172) and Lemma 6.5.9(a1), the invertibility of $I - \mathbb{G}_L$ implies that of $I - \mathbb{F}_{\widetilde{L}}$. By (7.13) (with $\mathbb{U} = 0$), the map

$$\mathbb{Q} := -\mathbb{E}_L (I - \mathbb{G}_L)^{-1} = -(I - \mathbb{F}_{\widetilde{L}})^{-1} \mathbb{E}_{\widetilde{L}} \in \mathrm{TIC}_\infty(Y, U) \tag{7.24}$$

is a DF-stabilizing controller for $\mathbb{D}$ and (7.24) is a [[exponential]] d.c.f.

3° *(7.22) is a SR WPLS with I/O map $\mathbb{Q}$:* By Definition 6.6.21, $L := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is admissible for $\Sigma_{\text{Total}}$. Assumption $I - \mathbb{G}_L \in \mathcal{G}\text{TIC}_\infty$ says that $\widetilde{L} := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is admissible for $(\Sigma_{\text{Total}})_L$; from (6.125) we observe that the corresponding system $(\Sigma_{\text{Total}})_{L+\widetilde{L}} = (\Sigma_{\text{Total}})_I$ has $\begin{bmatrix} * & * \\ -\mathbb{Q} & * \end{bmatrix}$ as its I/O map.

Apply (6.142) (and Proposition 6.6.18(a1) and Proposition 6.3.1(a3)) twice to observe that the generators of $(\Sigma_{\text{Total}})_I$ are given by

$$\left[\begin{array}{c|cc} A+BK_{\text{s}}+HC_{\text{s}}+HDK_{\text{s}} & H & B+HD \\ \hline C_{\text{s}}+DK_{\text{s}} & 0 & D \\ K_{\text{s}} & 0 & 0 \end{array}\right]. \tag{7.25}$$

We conclude that (7.22) is a SR WPLS with I/O map $\mathbb{Q}$.

4° *The rest:*

Let $\Sigma'$ be the system generated by

$$\left[\begin{array}{c|cc} A+BK_{\text{s}}+HC_{\text{s}}+HDK_{\text{s}} & -H & -(B+HD) \\ \hline C_{\text{s}}+DK_{\text{s}} & 0 & D \\ K_{\text{s}} & 0 & 0 \end{array}\right]. \tag{7.26}$$

Then $\Sigma'$ with its second and third columns multiplied by $-1$ equals $(\Sigma_{\text{Total}})_I$. From (6.126 we observe that $(\Sigma')_L$ with its second and third columns multiplied by $-1$ equals $((\Sigma_{\text{Total}})_I)_{-L}$ $(= (\Sigma_{\text{Total}})_{\widetilde{L}}$, by Lemma 6.6.3), and that $(\Sigma')_{\widetilde{L}}$ with its second and third columns multiplied by $-1$ equals $((\Sigma_{\text{Total}})_I)_{-\widetilde{L}} = (\Sigma_{\text{Total}})_L$.

From this and (7.26) we observe that $C_{\text{s}} + DK_{\text{s}}$ and $-(B+HD)$ are [[exponentially] strongly] jointly stabilizing for (7.22) (with "$E = 0$"). In particular, (7.22) is [[exponentially] strongly] r.c.-stabilizable, by Theorem 6.6.28 (and Lemma 6.6.22). By this, 2° and Theorem 7.2.3(b)(1.), (7.22) [[exponentially] strongly] DF-stabilizes $\Sigma$. □

Formally, a controller $\mathbb{O}$ with internal loop maps $y \mapsto u = (\mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21})y$. (If $I - \mathbb{O}_{22} \in \mathcal{G}\text{TIC}_\infty(\Xi)$, then this formula is not merely formal, by Lemma 7.2.7.)

Thus, also the controllers of form "right coprime $\text{H}^\infty/\text{H}^\infty$" (of form $\mathbb{Y}\mathbb{X}^{-1}$ with $\mathbb{X}, \mathbb{Y} \in \text{TIC}$ being r.c.) can be written as controllers with internal loop, by taking $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$. We will call such controllers maps with r.c. internal loop (they are the canonical controllers of [CWW01]); see Definition 7.2.11 for details. Here we require neither $\mathbb{X}$ nor $\widetilde{\mathbb{X}}$ to be invertible, it is enough that the system $\mathbb{D}_I^{\mathcal{O}}$ produced by closing the two loops simultaneously is well-posed.

The surprising fact is that all stabilizing controllers are of this form (modulo equivalence), whenever $\mathbb{D}$ has a l.c.f. This fact is the main theorem of [CWW01], but we give here (part (b) below) a shorter proof instead of the original seven pages long one. We also give a necessary and sufficient condition ((a) or its dual (a')) in the general case:

**Proposition 7.2.5 (I/O-DF-stabilizing controller with IL)** *Let* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$ *and* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \text{TIC}_\infty(Y \times \Xi, U \times \Xi)$.

*(a)* $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{H} := \begin{bmatrix} I-\mathbb{D}\mathbb{O}_{11} & -\mathbb{D}\mathbb{O}_{12} \\ -\mathbb{O}_{21} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$. *Moreover,* $\mathbb{O}$ *is [exponentially] stabilizing with internal loop iff the corresponding closed-loop map*

$$(I-\mathbb{D}^o)^{-1} = \begin{bmatrix} I_U + \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1}\begin{bmatrix}\mathbb{D}\\0\end{bmatrix} & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1} \\ \mathbb{H}^{-1}\begin{bmatrix}\mathbb{D}\\0\end{bmatrix} & \mathbb{H}^{-1} \end{bmatrix} : \begin{bmatrix}u_L\\y_L\\\xi_L\end{bmatrix} \mapsto \begin{bmatrix}u+u_L\\y+y_L\\\xi+\xi_L\end{bmatrix} \tag{7.27}$$

*is [exponentially] stable.*

*(a')* $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{R} := \begin{bmatrix} I-\mathbb{O}_{11}\mathbb{D} & -\mathbb{O}_{12} \\ -\mathbb{O}_{21}\mathbb{D} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times \Xi)$. *Moreover,* $\mathbb{O}$ *is [exponentially] stabilizing with internal loop iff the corresponding closed-loop map*

$$\begin{bmatrix} I_Y + \begin{bmatrix}\mathbb{D} & 0\end{bmatrix}\mathbb{R}^{-1}\begin{bmatrix}\mathbb{O}_{11}\\\mathbb{O}_{21}\end{bmatrix} & \begin{bmatrix}\mathbb{D} & 0\end{bmatrix}\mathbb{R}^{-1} \\ \mathbb{R}^{-1}\begin{bmatrix}\mathbb{O}_{11}\\\mathbb{O}_{21}\end{bmatrix} & \mathbb{R}^{-1} \end{bmatrix} : \begin{bmatrix}y_L\\u_L\\\xi_L\end{bmatrix} \mapsto \begin{bmatrix}y+y_L\\u+u_L\\\xi+\xi_L\end{bmatrix} \tag{7.28}$$

*is [exponentially] stable.*

*(b) Let* $\mathbb{D}$ *have a l.c.f.* $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$. *Then* $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{F} := \begin{bmatrix} \widetilde{\mathbb{M}}-\widetilde{\mathbb{N}}\mathbb{O}_{11} & -\widetilde{\mathbb{N}}\mathbb{O}_{12} \\ -\mathbb{O}_{21} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$*, and* $\mathbb{O}$ *is stabilizing with internal loop iff* $\mathbb{F}^{-1}, \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1} \in \mathrm{TIC}$.

*Moreover, if* $\mathbb{O}$ *is stabilizing with internal loop for* $\mathbb{D}$ *and we set*

$$\begin{bmatrix}\mathbb{Y}\\\mathbb{X}\end{bmatrix} := \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\\ I_Y & 0\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}I_Y\\0\end{bmatrix}, \tag{7.29}$$

*then* $\mathbb{O}' := \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X}\end{bmatrix} \in \mathrm{TIC}(Y \times Y, U \times Y)$, $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I_Y$, $\mathbb{O}'$ *is an equivalent (to* $\mathbb{O}$*) stabilizing controller with internal loop for* $\mathbb{D}$*, and* $\mathbb{O} = \mathbb{Y}_o\mathbb{X}_o^{-1}$ *is a r.c.f., where* $\mathbb{X}_o := \mathbb{F}^{-1}$ *and* $\mathbb{Y}_o := \mathbb{O}\mathbb{F}^{-1}$ *(in particular,* $\mathbb{O}\mathbb{F}^{-1} \in \mathrm{TIC}$*).*

*Of course, the corresponding dual result holds for* $\mathbb{D}$ *having a r.c.f.*

*(c)* $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* $\mathbb{D}$ *iff* $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) for* $\begin{bmatrix}\mathbb{D} & 0\\ 0 & I_\Xi\end{bmatrix}$.

*(d)* $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* $\mathbb{D}$ *iff* $\mathbb{O}^{\mathrm{d}}$ *is admissible (resp. [exponentially] stabilizing) for* $\mathbb{D}^{\mathrm{d}}$ *with internal loop.*

*(e)* $\mathbb{O}$ *and* $\widetilde{\mathbb{O}} \in \mathrm{TIC}_\infty(Y \times \Xi', U \times \Xi')$ *are equivalent for* $\mathbb{D}$ *iff* $(\mathbb{H}^{-1})_{11} = (\widetilde{\mathbb{H}}^{-1})_{11}$ *and* $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1})_1 = (\begin{bmatrix}\widetilde{\mathbb{O}}_{11} & \widetilde{\mathbb{O}}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1})_1$*, where* $\widetilde{\mathbb{H}}$ *corresponds to* $\widetilde{\mathbb{O}}$ *as in (a).*

If $\mathbb{O}$ is merely admissible in (b), then one observes from the proof that the conclusions of (b) still hold except that $\mathbb{O}' \in \mathrm{TIC}_\infty$ (instead of $\mathrm{TIC}_0$) and that $\mathbb{O}'$ need not be stabilizing (but it is admissible, because it is equivalent to $\mathbb{O}$).

**Proof:** (a) This follows by applying Lemma A.1.1(d1) to $A := I - \mathbb{D}^o$ (so that $\mathbb{H} = A_{22} - A_{21}A_{11}^{-1}A_{12}$).

(N.B. two admissible controllers with internal loop are equivalent for $\mathbb{D}$ iff they produce same maps $(\mathbb{H}^{-1})_{11}$ and $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1})_1$, since then the $(1–2, 1–2)$-blocks of (7.27) are the same.)

(a’) Now we set $A := T(I - \mathbb{D}^o)T$, where $T := \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$, and apply Lemma A.1.1(d1) as in (a) (note that $\mathbb{R} = A_{22} - A_{21}A_{11}^{-1}A_{12}$).

(b) 1° Clearly $\mathbb{H} = \begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix}^{-1}\mathbb{F}$, so $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$. The corresponding closed-loop map is

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I + \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix} & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix} \\ \mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix} & \mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix} \end{bmatrix}, \tag{7.30}$$

so $\mathbb{F}^{-1}, \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1} \in \mathrm{TIC}$ is clearly sufficient for $(I - \mathbb{D}^o)^{-1} \in \mathrm{TIC}$.

2° For the converse, note that (here $\widetilde{\mathbb{M}}\mathbb{S} - \widetilde{\mathbb{N}}\mathbb{T} = I$, $\mathbb{S}, \mathbb{T} \in \mathrm{TIC}$)

$$\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbb{S} & 0 \\ 0 & I\end{bmatrix} - \begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix}\begin{bmatrix}\mathbb{T} & 0\end{bmatrix} = \begin{bmatrix}I & 0 \\ 0 & I\end{bmatrix}, \tag{7.31}$$

so the stability of (7.30) implies that of $\mathbb{F}^{-1}$ and $\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}$. Therefore, also $\mathbb{O}'$ is stable in this case.

3° For the rest of the proof, we will assume that $\mathbb{O}$ stabilizes $\mathbb{D}$. Now the $(2,2)$-block of $I = (I - \mathbb{D}^o)(I - \mathbb{D}^o)^{-1}$ gives $I = -\mathbb{D}\mathbb{Y}\widetilde{\mathbb{M}} + I\mathbb{X}\widetilde{\mathbb{M}} + 0$, i.e., $I = \widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y}$. Using Lemma A.1.1(d1), one obtains that $\mathbb{O}'$ is admissible and (we set $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} = \widetilde{\mathbb{M}}^{-1}$)

$$(I - \mathbb{D}^o_{\mathbb{O}'})^{-1} = \begin{bmatrix} I + \mathbb{Y}\Delta^{-1}\mathbb{D} & \mathbb{Y}\Delta^{-1} & \mathbb{Y}\Delta^{-1} \\ (I + \mathbb{D}\mathbb{Y}\Delta^{-1})\mathbb{D} & I + \mathbb{D}\mathbb{Y}\Delta^{-1} & \mathbb{D}\mathbb{Y}\Delta^{-1} \\ \Delta^{-1}\mathbb{D} & \Delta^{-1} & \Delta^{-1} \end{bmatrix}, \tag{7.32}$$

where $\mathbb{D}^o_{\mathbb{O}'}$ is as $\mathbb{D}^o$, except that $\mathbb{O}$ is replaced by $\mathbb{O}'$. This shows that $\mathbb{O}'$ is also stabilizing.

4° $\mathbb{O}'$ is equivalent to $\mathbb{O}$, because the $(1–2, 1–2)$-block of (7.32) equals that of (7.30):

$$\begin{aligned} &I + \mathbb{Y}\Delta^{-1}\mathbb{D} = I + \mathbb{Y}\widetilde{\mathbb{N}}, && \mathbb{Y}\Delta^{-1} = \mathbb{Y}\widetilde{\mathbb{M}}; \\ &(I + \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}\mathbb{Y}\widetilde{\mathbb{M}})\mathbb{D} = \widetilde{\mathbb{M}}^{-1}(I + \widetilde{\mathbb{N}}\mathbb{Y})\widetilde{\mathbb{N}} = \mathbb{X}\widetilde{\mathbb{N}}, && I + \mathbb{D}\mathbb{Y}\mathbb{M} = \mathbb{X}\mathbb{M}. \end{aligned} \tag{7.33}$$

5° Set $\widetilde{\mathbb{M}}_o := \begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix}$, $\widetilde{\mathbb{N}}_o := \begin{bmatrix}\widetilde{\mathbb{N}} & 0 \\ 0 & I\end{bmatrix}$ to obtain $\mathbb{F} = \widetilde{\mathbb{M}}_o - \widetilde{\mathbb{N}}_o\mathbb{O}$, so that

$$\widetilde{\mathbb{M}}_o\mathbb{F}^{-1} - \widetilde{\mathbb{N}}_o\mathbb{O}\mathbb{F}^{-1} = I, \tag{7.34}$$

i.e., $\mathbb{X}_o$ and $\mathbb{Y}_o$ are r.c. (because (7.34) implies that the lower row of $\mathbb{Y}_o$ is stable, and the upper row was proved stable in 2°).

The dual result can be proved analogously from (a') (alternatively, use (d)).

(c) 0° First proof: One way to prove the rest is to interchange the second and fourth columns and the second and third rows of "$I - \mathbb{D}^o$" corresponding to $\left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$, and then apply (A.11) (with the rows and columns of $A$ and $A^{-1}$ interchanged) to the resulting matrix

$$\begin{bmatrix} \begin{bmatrix} I & -\mathbb{O}_{12} & -\mathbb{O}_{11} \\ -\mathbb{D} & 0 & I \\ 0 & -\mathbb{O}_{22} & -\mathbb{O}_{21} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \\ \begin{bmatrix} 0 & I & 0 \end{bmatrix} & -I \end{bmatrix} \tag{7.35}$$

to obtain that the invertibility of "$I - \mathbb{D}^o$" (i.e., the admissibility of $\mathbb{O}$ for $\left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$) is equivalent to the invertibility of $I - \mathbb{D}^o$, and that both inverses are stable iff either is (since "$(I - \mathbb{D}^o)$" consists of $I - \mathbb{D}^o$ and some copies of its elements).

However, for Lemma 7.2.6 we need the alternative proof given in 1°–3°:

1° The admissibility claim follows from (a), because $\mathbb{H} = I - \underline{\mathbb{D}}\mathbb{O}$, where $\underline{\mathbb{D}} := \left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$.

2° Assume that (7.27) is [exponentially] stable, so that also (7.28) is [exponentially] stable. Then, by Lemma A.1.1(f6),

$$\begin{bmatrix} \mathbb{D} & 0 \\ 0 & I \end{bmatrix} (I - \mathbb{O}\underline{\mathbb{D}})^{-1}\mathbb{O} = \mathbb{H}^{-1} - I, \tag{7.36}$$

which is [exponentially] stable, hence $\begin{bmatrix} 0 & I \end{bmatrix} (I - \mathbb{O}\underline{\mathbb{D}})^{-1}\mathbb{O} = \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{O}(I - \underline{\mathbb{D}}\mathbb{O})^{-1} = \begin{bmatrix} \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \mathbb{H}^{-1}$ is [exponentially] stable. Combine this with the right top corner of (7.27) to observe that $\mathbb{O}\mathbb{H}^{-1}$ is [exponentially] stable. By (7.27), so is also $\mathbb{H}^{-1}\underline{\mathbb{D}}$ (because so are $\mathbb{H}^{-1}\left[\begin{smallmatrix}\mathbb{D}\\ 0\end{smallmatrix}\right]$ and $\mathbb{H}^{-1}$).

Since $\mathbb{R} = I - \mathbb{O}\underline{\mathbb{D}}$, we analogously observe from (7.28) that $(I - \mathbb{O}\underline{\mathbb{D}})^{-1}$ is [exponentially] stable, hence so is (7.5) (with substitutions $\mathbb{Q} \mapsto \mathbb{O}$, $\mathbb{D} \mapsto \underline{\mathbb{D}}$), equivalently, the map

$$\begin{bmatrix} I & -\mathbb{O} \\ -\underline{\mathbb{D}} & I \end{bmatrix}^{-1} = \begin{bmatrix} \mathbb{R}^{-1} & \mathbb{O}\mathbb{H}^{-1} \\ \mathbb{H}^{-1}\underline{\mathbb{D}} & \mathbb{H}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbb{T}_{11} & \mathbb{T}_{13} & \mathbb{T}_{12} & \mathbb{T}_{13} \\ \mathbb{T}_{31} & \mathbb{T}_{33} & \mathbb{T}_{32} & \mathbb{T}_{33} - I \\ \mathbb{T}_{21} & \mathbb{T}_{23} & \mathbb{T}_{22} & \mathbb{T}_{23} \\ \mathbb{T}_{31} & \mathbb{T}_{33} & \mathbb{T}_{32} & \mathbb{T}_{33} \end{bmatrix} \tag{7.37}$$

(here $\mathbb{T} := (I - \mathbb{D}^o)^{-1}$; we have used (7.27) and (7.28) above).

3° Conversely, if (7.37) is [exponentially] stable, then so is $\mathbb{T} = (I - \mathbb{D}^o)^{-1}$, hence so is (7.27).

(d) This follows from (a) and (a'). (Note that (d) is contained in Lemma 6.7.2(e'), but this second proof will be useful later.)

(e) We observe from (7.27) that $\left[\begin{smallmatrix} I & 0 & 0\\ 0 & I & 0\end{smallmatrix}\right] \mathbb{D}^o_I \left[\begin{smallmatrix} I & 0 & 0\\ 0 & I & 0\end{smallmatrix}\right]^{\mathrm{T}}$ (equivalently, the first and second rows and columns of (7.27)$= \mathbb{D}^o_I + I$, because $\mathbb{D}^o_I = (I - \mathbb{D}^o)^{-1} - I$) depends on $(\mathbb{H}^{-1})_{11}$ and $(\begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \end{bmatrix} \mathbb{H}^{-1})_1$. □

Part (a) also shows that $\left[\begin{smallmatrix}\mathbb{D}\\ 0\end{smallmatrix}\right] = \widetilde{\mathbb{S}}^{-1}\widetilde{\mathbb{T}}$ (i.e., "$\mathrm{H}^\infty/\mathrm{H}^\infty$") with $\widetilde{\mathbb{S}}, \widetilde{\mathbb{T}} \in \mathrm{TIC}$. If $U$ and $Y$ are finite-dimensional, we can write also $\mathbb{D}$ in "$\mathrm{H}^\infty/\mathrm{H}^\infty$" form (i.e., as the inverse of a stable, ($\mathrm{TIC}_\infty$-)invertible determinant times a stable matrix), but we do not know whether these factors can be chosen to be coprime, as they are in the case of well-posed controllers, by Lemma 7.1.4.

We give here the equivalents of (c) and (d) for systems:

**Lemma 7.2.6 ($\widetilde{\Sigma}$: DF-IL vs. DF)** *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\widetilde{\Sigma} = \left[\begin{smallmatrix}\widetilde{\mathbb{A}} & \widetilde{\mathbb{B}}\\ \widetilde{\mathbb{C}} & \mathbb{Q}\end{smallmatrix}\right] \in \mathrm{WPLS}(Y\times\Xi,\widetilde{H},U\times\Xi)$.

*Then $\widetilde{\Sigma}$ is admissible (resp. [exponentially] stabilizing) with internal loop for $\Sigma$ iff $\widetilde{\Sigma}$ is admissible (resp. [exponentially] stabilizing) for $\underline{\Sigma} := \left[\begin{smallmatrix}\Sigma & 0\\ 0 & I_\Xi\end{smallmatrix}\right] \in$* $\mathrm{WPLS}(U\times\Xi,H,Y\times\Xi)$. *All prefices apply.*

*Moreover, $\widetilde{\Sigma}$ is admissible (resp. [exponentially] stabilizing) with internal loop for $\Sigma$*
*iff $\widetilde{\Sigma}^{\mathrm{d}}$ is admissible (resp. [exponentially] stabilizing) for $\Sigma^{\mathrm{d}}$ with internal loop.*

Thus, dynamic feedback with internal loop can be reduced to (proper) dynamic feedback (this could also be observed directly from Figures 7.4 and 7.3 or from equations 7.19: as $u$ goes through $\mathbb{D}$ back to $\mathbb{Q}$, we let $\xi$ go through $I$ back to $\mathbb{Q}$).

As noted below Lemma 6.7.2, the prefix "strongly" does not apply to the duality claim but several others do.

**Proof:** 1° $\underline{\Sigma}$*:* The admissibility claim is contained in Proposition 7.2.5(c), whose proof shows that $\underline{\mathbb{D}}_I^o + I = \begin{bmatrix} I & -\mathbb{Q}\\ -\underline{\mathbb{D}} & I\end{bmatrix}^{-1} - I$ contains $\mathbb{D}_I^o$ plus some copies of parts of it (plus one identity operator). We shall show below that the same holds for $\underline{\mathbb{A}}_I^o$, $\underline{\mathbb{B}}_I^o$ and $\underline{\mathbb{C}}_I^o$; this proves the claim.

From (7.37) we observe that

$$\underline{\mathbb{C}}_I^o = \begin{bmatrix} I & 0 & 0\\ 0 & 0 & I\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\mathbb{C}_I^o \quad\text{and}\quad \underline{\mathbb{B}}_I^o = \mathbb{B}_I^o\begin{bmatrix} I & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & I & 0 & 0\end{bmatrix}. \tag{7.38}$$

It follows that $\underline{\mathbb{A}}_I^o = \underline{\mathbb{A}} + \underline{\mathbb{B}}^o\tau\underline{\mathbb{C}}_I^o = \underline{\mathbb{A}} + \mathbb{B}^o I\tau\mathbb{C}_I^o = \mathbb{A}_I^o$.

2° *Duality:* This is contained in Lemma 6.7.2(e') (see the last claim of the lemma — or its proof). □

The closed-loop map $u_L, y_L \mapsto u, y$ corresponds to that of a well-posed controller iff $I - \mathbb{Q}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$:

**Lemma 7.2.7 (Well-posed $\mathbb{Q} = \mathbb{Q}_{11} + \mathbb{Q}_{12}(I-\mathbb{Q}_{22})^{-1}\mathbb{Q}_{21}$)** *Let* $\mathbb{Q} = \begin{bmatrix}\mathbb{Q}_{11} & \mathbb{Q}_{12}\\ \mathbb{Q}_{21} & \mathbb{Q}_{22}\end{bmatrix} \in$ $\mathrm{TIC}_\infty(Y\times\Xi,U\times\Xi)$ *be admissible with internal loop for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$.

*Then $\mathbb{Q}$ is equivalent to a well-posed controller iff $I - \mathbb{Q}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$; if this is the case, then that well-posed controller is given by $\mathbb{Q}_{11} + \mathbb{Q}_{12}(I-\mathbb{Q}_{22})^{-1}\mathbb{Q}_{21}$ (in particular, it is unique).*

This is exactly what one would have expected: the internal loop can be opened iff $L := \left[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}\right]$ is admissible for $\mathbb{Q}$, and in that case, $\mathbb{Q}$ is equivalent to $(\mathbb{Q}_L)_{11} = \mathbb{Q}_{11} + \mathbb{Q}_{12}(I-\mathbb{Q}_{22})^{-1}\mathbb{Q}_{21}$ (see (6.125)).

Two different well-posed controllers induce different closed loop maps $\mathbb{D}_L := \begin{bmatrix} I & -\mathbb{Q}\\ -\mathbb{D} & I\end{bmatrix}^{-1} - I \in \mathrm{TIC}_\infty$, because inverses are unique.

**Proof:** 1° For any $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$, the maps $\mathbb{D}_L$, $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right](I - \mathbb{D}^o)^{-1}\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right]^{\mathrm{T}} - I : \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ exist and are equal iff the maps $\mathbb{D}_L + I = \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1}$ and $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right](I - \mathbb{D}^o)^{-1}\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right]^{\mathrm{T}}$ exist and are equal.

If this is the case, then $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1}$ equals the $(1\text{–}2, 1\text{–}2)$-block of

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I & -\mathbb{O}_{11} & -\mathbb{O}_{12} \\ -\mathbb{D} & I & 0 \\ 0 & -\mathbb{O}_{21} & I - \mathbb{O}_{22} \end{bmatrix}^{-1} : \begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u + u_L \\ y + y_L \\ \xi_L \end{bmatrix}. \tag{7.39}$$

Therefore, by Lemma A.1.1(c1) (with $A := I - \mathbb{D}^o$ so that $B_{11} = \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1}$), from $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$ we obtain that $(A_{22} =) I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, and $(B_{11} =) \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1}$ is equal to the inverse of

$$\begin{bmatrix} I & -\mathbb{O}_{11} \\ -\mathbb{D} & I \end{bmatrix} - \begin{bmatrix} -\mathbb{O}_{12} \\ 0 \end{bmatrix}(I - \mathbb{O}_{22})^{-1}\begin{bmatrix} 0 & -\mathbb{O}_{21} \end{bmatrix} = \begin{bmatrix} I & -\mathbb{O}_{11} - \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21} \\ -\mathbb{D} & I \end{bmatrix}, \tag{7.40}$$

hence $-\mathbb{Q} = -\mathbb{O}_{11} - \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ (this also shows that $\mathbb{Q}$ is unique).

2° Conversely, if $I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$ and one defines $\mathbb{Q} := \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$, then $\mathbb{Q}$ and $\mathbb{O}$ determine same closed-loop maps $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u + u_L \\ y + y_L \end{smallmatrix}\right]$ (i.e., they are equivalent), as one can see by reversing the above calculations. □

**Remark 7.2.8 (“$\mathbb{Q} \in \mathrm{H}^\infty/\mathrm{H}^\infty$”)** *As one easily observes from the proof, Lemma 7.2.7 actually covers a more general class of systems and controllers: If there are* $\mathbb{O} \in \mathrm{TIC}_\infty$ *and a holomorphic function* $\widehat{\mathbb{Q}} \in \mathrm{H}(\Omega; \mathcal{B}(Y,U))$ *with* $\Omega \subset \mathbf{C}$ *open, s.t.* $\widehat{\mathbb{Q}}$, $\begin{bmatrix} I & -\widehat{\mathbb{Q}} \\ -\widehat{\mathbb{D}} & I \end{bmatrix}$ *and* $(I - \widehat{\mathbb{D}^o})$ *are invertible at some* $s_0 \in \mathbf{C}$ *and* $\begin{bmatrix} I & -\widehat{\mathbb{Q}} \\ -\widehat{\mathbb{D}} & I \end{bmatrix}^{-1}$ *equals the* $(1\text{–}2, 1\text{–}2)$*-block of* $(I - \widehat{\mathbb{D}^o})^{-1}$ *on a neighborhood of* $s_0$*, then* $\mathbb{Q} = \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$*. Obviously the converse holds too.*

*Thus, if* $I - \widehat{\mathbb{O}}_{22}$ *and* $(I - \widehat{\mathbb{D}^o})^{-1}$ *are invertible at any* $s_0 \in \mathbf{C}$*, then the transfer function of the controller defined by* $\mathbb{O}$ *is* $\widehat{\mathbb{O}}_{11} + \widehat{\mathbb{O}}_{12}(I - \widehat{\mathbb{O}}_{22})^{-1}\widehat{\mathbb{O}}_{21}$ *(on the open subset of* $\mathbf{C}$ *where these inverses exist).*

*Therefore, controllers with internal loop cover (but are not covered by) all controllers whose transfer functions are of the form* $\widehat{\mathbb{O}}_{11} + \widehat{\mathbb{O}}_{12}(I - \widehat{\mathbb{O}}_{22})^{-1}\widehat{\mathbb{O}}_{21}$ *(and well-defined at least at one point* $s_0 \in \mathbf{C}$*), where* $\mathbb{O} \in \mathrm{TIC}_\infty$*; in particular, all “*$\mathrm{H}^\infty/\mathrm{H}^\infty$*” transfer functions are covered.*

We now show by a simple example that the transfer function of the internal loop of a controller need not be invertible anywhere:

**Example 7.2.9 ($\nexists\widehat{\mathbb{Q}}$)** Take $\widehat{\mathbb{D}}(s) := (s-1)/(s+1)$, $\widehat{\mathbb{O}} = \begin{bmatrix} \widehat{\mathbb{D}}^{-1} & -1 \\ -1 & 1 \end{bmatrix}$ (exponentially stable), so that

$$(I - \widehat{\mathbb{D}^o})^{-1} = \begin{bmatrix} 1 & -\widehat{\mathbb{D}}^{-1} & 1 \\ \widehat{\mathbb{D}} & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -\widehat{\mathbb{D}}^{-1} & \widehat{\mathbb{D}}^{-1} \\ 0 & 0 & 1 \\ 1 & \widehat{\mathbb{D}}^{-1} & 0 \end{bmatrix} \tag{7.41}$$

(cf. Proposition 7.2.5(a)) is stable, because $\widehat{\mathbb{D}}^{-1} \in \mathrm{H}^\infty$.

However, $1 - \widehat{\mathbb{O}}_{22} \equiv 0$ is nowhere invertible, so one cannot close the internal loop in the controller if one does not connect the controller to the plant $\mathbb{D}$ to be controlled; neither can one close the upper loop only (admissibly), because $\mathbb{O}_{11} = I$ is not admissible for $\mathbb{D}$ ($I - \mathbb{D}\mathbb{O}_{11} = 0$); the setting becomes well-posed only when both loops are closed.

By Proposition 7.2.5(b), the corresponding map with coprime internal loop is $\mathbb{Y}\mathbb{X}^{-1} = -1(0)^{-1}$, i.e., $\mathbb{O}' = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ is equivalent to $\mathbb{O}$; this $0^{-1}$ shows that the controller has something resembling a short circuit. In fact, in Example 2.3 of [CWW01] exactly this map with coprime internal loop (in its adjoint form) is used as a short circuit regulating an electrical circuit (whose transfer function $2/(1+\mathrm{e}^{-2s})$ has infinitely many poles on the imaginary axis).

By Lemma 7.2.7, neither $\mathbb{O}$ nor $\mathbb{O}'$ is equivalent to any well-posed controller. This is not really surprising, because we have $\widetilde{y} \equiv 0$ ($\widetilde{y} = \xi_L$, i.e., $y = -y_L + \xi_L$ if there is an external input $\xi_L$ into the internal loop), and this poses the requirement "$(I - \mathbb{D}\mathbb{Q})^{-1} = 0$" (by formula (7.5), which has an extra $-I$), which is impossible for a well-posed controller and even for controllers of the form "$\mathrm{H}^\infty/\mathrm{H}^\infty$" (even for those with $(I - \widehat{\mathbb{D}}\widehat{\mathbb{Q}})^{-1}$ well-defined on any open subset of the complex plane, cf. Remark 7.2.8). Note that also the outputs cancel the corresponding inputs completely (i.e., the diagonal of (7.41) is zero), which could not be achieved by an admissible well-posed controller either. ◁

For physically motivated examples, see, e.g., Example 2.3 of [CWW01]. Example 4.8 of [CWW01] illustrates a problem that can only be solved by using a non-well-posed controller.

By Proposition 7.2.5(b), it is enough to study the controllers of the following form (if we exclude maps that do not have coprime factorizations):

**Lemma 7.2.10** *Let* $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix} \in \mathrm{TIC}(Y \times U, U \times U)$ *and* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$.

(a) *The map* $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix} \in \mathrm{TIC}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$; *it is stabilizing with internal loop for* $\mathbb{D}$ *iff* $\widetilde{\Delta}^{-1}, \mathbb{D}\widetilde{\Delta}^{-1} \in \mathrm{TIC}$.

(b) *Let* $\mathbb{O}$ *be stabilizing. Then* $\widetilde{\mathbb{X}}$ *and* $\widetilde{\mathbb{Y}}$ *are l.c., i.e., "*$\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *is a stabilizing DF-controller for* $\mathbb{D}$ *with l.c. internal loop". Moreover then, with* $\mathbb{M} := \widetilde{\Delta}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, *the factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a r.c.f. of* $\mathbb{D}$, *it satisfies* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$, *and the* $(1\text{–}2, 1\text{–}2)$ *blocks of the closed-loop map* $\mathbb{D}^o_I := \mathbb{D}^o(I - \mathbb{D}^o)^{-1}$ *are given by*

$$= \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} - I & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}, \tag{7.42}$$

*as in the well-posed case, i.e., in (7.9) (cf. (7.5)).*

(c) *Conversely, if* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a r.c.f. with* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$, *then* $\mathbb{O}$ *stabilizes* $\mathbb{D}$ *with internal loop.*

(d) *If* $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, *then* $\mathbb{O}$ *is admissible for* $\mathbb{D}$ *iff* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *is an admissible (well-posed) DF-controller for* $\mathbb{D}$. *If* $\mathbb{O}$ *is admissible and* $\mathbb{Q}$ *is an admissible (well-posed) DF-controller for* $\mathbb{D}$, *then the closed-loop maps* $u_L, y_L \mapsto u, y$ *determined by* $\mathbb{O}$ *and* $\mathbb{Q}$ *are identical iff* $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\mathbb{Y}$.

*The corresponding dual claims (with* $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y}$*) hold as well.*

**Proof:** (a)&(b)&(c) 1° With the notation of Proposition 7.2.5, we have $\mathbb{H} = \begin{bmatrix} I & -\mathbb{D} \\ -\widetilde{\mathbb{Y}} & \widetilde{\mathbb{X}} \end{bmatrix}$. By Lemma A.1.1(d1), we have $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\Delta} \in \mathcal{G}\mathrm{TIC}_\infty$, so the admissibility claim follows from Proposition 7.2.5(a) as well as the formula

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} & \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \widetilde{\Delta}^{-1} \\ \mathbb{D}(I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D}) & I + \mathbb{D}\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \mathbb{D}\widetilde{\Delta}^{-1} \\ \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} & \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \widetilde{\Delta}^{-1} \end{bmatrix} \tag{7.43}$$

$$= \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} + I & \mathbb{N} \\ \mathbb{M}\widetilde{\mathbb{X}} - I & \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{u} \\ \widetilde{y} \\ \widetilde{\xi} \end{bmatrix} \tag{7.44}$$

(for the second identity, we have set $\mathbb{M} := \widetilde{\Delta}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ and used (7.45)).

2° If (7.43) is stable, then $\mathbb{M} := \widetilde{\Delta}^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ are also stable. Moreover, $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., because $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = \widetilde{\mathbb{X}}(\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1} - \widetilde{\mathbb{Y}}\mathbb{D}(\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1} = I$. Formula (7.42) follows from this.

3° Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{O}$ is s.t. $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ (as above), then (7.43) is stable, because then

$$I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} = \widetilde{\Delta}^{-1}(\widetilde{\Delta} + \widetilde{\mathbb{Y}}\mathbb{D}) = \widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} \in \mathrm{TIC}, \tag{7.45}$$

and $\mathbb{D}(I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D}) = \mathbb{D}\widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} \in \mathrm{TIC}$.

(d) This follows from Lemma 7.2.7. □

If a plant has a (right or left) coprime factorization, then all of its stabilizing controllers are equivalent to some of the form studied in Lemma 7.2.10, by Proposition 7.2.5(b) (or its dual). Therefore, the latter ones were called "canonical controllers" in [CWW01]. To be able to extend the Youla parametrization (Theorem 7.2.14) and related results to cover also the non-well-posed case, we shall define the concept *map with coprime internal loop* below as the equivalence class of a "canonical controller" modulo "being equal".

It follows that, for a plant having a coprime factorization, each stabilizing controllers with internal loop is equivalent to one and only one map with a coprime internal loop, by Lemma 7.2.12(c).

**Definition 7.2.11 (Maps with coprime internal loop)** *Let* $(\mathbb{Y}, \mathbb{X})$ *be r.c. and* $(\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}})$ *be l.c. We call the (equivalence class (modulo equality; see below) of the) map* $\begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ *(resp.* $\begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$*) a* map with r.c. internal loop *(resp. a* map with l.c. internal loop*) and denote it by* $\mathbb{Y}\mathbb{X}^{-1}$ *(resp. by* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*).*

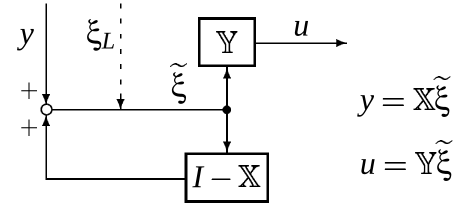


Figure 7.5: Controller $\mathbb{Y}\mathbb{X}^{-1}$ with r.c. internal loop

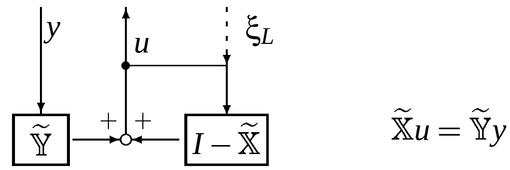


Figure 7.6: Controller $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ with l.c. internal loop

*If, in addition, $\mathbb{X}$, $\mathbb{Y}$, $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ can be extended to satisfy the doubly coprime product*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix} = I = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} \tag{7.46}$$

*in* $\mathrm{TIC}(U\times Y)$ *for some* $\mathbb{M},\mathbb{N},\widetilde{\mathbb{M}},\widetilde{\mathbb{N}}$*, then we consider* $\mathbb{Y}\mathbb{X}^{-1}$ *and* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *equal and call* $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *a* map with d.c. internal loop. *We add the words* "over $\mathcal{A}$"*, if* $\mathcal{A}\subset \mathrm{TIC}$ *and the elements of (7.46) can be chosen from* $\mathcal{A}$*.*

*We consider the maps* $\mathbb{Y}\mathbb{X}^{-1}$ *and* $\mathbb{Y}_0\mathbb{X}_0^{-1}$ *with r.c. internal loop* equal *if* $(\mathbb{Y}_0,\mathbb{X}_0) = (\mathbb{Y}\mathbb{U},\mathbb{X}\mathbb{U})$ *for some* $\mathbb{U}\in\mathcal{G}\mathrm{TIC}$. *We consider the maps* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *and* $\widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$ *with l.c. internal loop* equal *if* $(\widetilde{\mathbb{Y}}_0,\widetilde{\mathbb{X}}_0) = (\mathbb{U}\widetilde{\mathbb{Y}},\mathbb{U}\widetilde{\mathbb{X}})$ *for some* $\mathbb{U}\in\mathcal{G}\mathrm{TIC}$.

*If* $\mathbb{X}\in\mathcal{G}\mathrm{TIC}_\infty$*, then we identify* $\mathbb{Y}\mathbb{X}^{-1}$ *in the usual sense (in* $\mathrm{TIC}_\infty$*) and* $\mathbb{Y}\mathbb{X}^{-1}$ *as a map with r.c. internal loop; we do the analogous identification for maps with l.c. internal loop too.*

*A* map with coprime internal loop *means a map with r.c. or l.c. internal loop. A controller with internal loop for* $\Sigma\in\mathrm{WPLS}$ *is called a* controller with coprime internal loop *if its I/O map is a representative of a map with coprime internal loop.*

*Let* $\mathbb{Q}$ *be a map with coprime internal loop. Then we say that* $\mathbb{Q}$ is admissible [stabilizing] for $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ *[or that* $\mathbb{Q}$ stabilizes $\mathbb{D}$*] if some (hence any, by Lemma 7.2.12(c)) of its representatives is admissible [stabilizing] for* $\mathbb{D}$ *with internal loop. We use prefices as in Definition 7.2.1.*

At this stage the serious reader has several serious questions about this definition and its justification. Lemma 7.2.12 below answers these questions in the expected way.

Thus, given r.c. maps $\mathbb{Y}$ and $\mathbb{X}$ s.t. $\left[\begin{smallmatrix}0 & \mathbb{Y}\\ I & I-\mathbb{X}\end{smallmatrix}\right]\in\mathrm{TIC}(Y\times U, U\times U)$, the equivalence class of $\left[\begin{smallmatrix}0 & \mathbb{Y}\\ I & I-\mathbb{X}\end{smallmatrix}\right]$ (modulo equality, in the collection of all maps of the same form) is given by $\{\left[\begin{smallmatrix}0 & \mathbb{Y}\mathbb{U}\\ I & I-\mathbb{X}\mathbb{U}\end{smallmatrix}\right] \mid \mathbb{U}\in\mathcal{G}\mathrm{TIC}(U)\}$ (cf. Lemma 7.2.12(a1)). Analogous claims hold for maps with l.c. or d.c. internal loop.

Recall from Definition 6.4.4(f) that (7.46) is called a *joint d.c.f. of* $\mathbb{D}$ *and* $\mathbb{Y}\mathbb{X}^{-1}$ (or of $\mathbb{D}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ (equivalently, $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$).

We warn the reader that if the left equation from (7.46) were removed, then "equality" would not be an equivalence relation. Even if both $\mathbb{Y}\mathbb{X}^{-1}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ were maps with d.c. internal loop, and $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$, these two maps need not be equal; a necessary and sufficient condition can be seen from Lemma A.1.1(e4) (although that is not needed here). From (7.46) one can also note that a pair $\mathbb{Y},\mathbb{X}$ defines a map with d.c. internal loop iff it can be extended to a invertible pair $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$; Lemma A.1.1(e) gives some (necessary and) sufficient conditions for this.

The last identification above corresponds to the equivalence of $\begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ and $\begin{bmatrix} \mathbb{Y}\mathbb{X}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ noted in Lemma 7.2.7 (and is hence justified). This identification makes maps with coprime internal loop a natural extension of well-posed maps having a r.c.f. or a l.c.f. However, one can show by a simple example, that if $\mathbb{X}$ were not assumed to be in $\mathcal{G}\mathrm{TIC}_\infty$, then $\mathbb{Q}\mathbb{X} = \mathbb{Y}$ (for general r.c. $(\mathbb{Y},\mathbb{X})$ and some $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$) would not guarantee that $\mathbb{Y}\mathbb{X}^{-1}$ and $\mathbb{Q}$ were equivalent for all $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$; in fact, with those assumptions $\widehat{\mathbb{X}}$ might be nowhere invertible (although $\mathbb{X}$ is necessarily left-invertible on TIC) and $\mathbb{Y}\mathbb{X}$ might stabilize different plants than $\mathbb{Q}$.

By Lemma 7.2.10, a well-posed $\mathbb{D}$ has a r.c.f. (resp. a l.c.f.) iff it can be stabilized by a map with l.c. (resp. r.c.) internal loop.

From this on, we shall often use the word "map" of both members of a map (equivalence class) and of the class itself when there should be no risk of ambiguity.

**Lemma 7.2.12 (Equal; well-posed)** *Let $\mathbb{Y}\mathbb{X}^{-1}$, $\mathbb{Y}_0\mathbb{X}_0^{-1}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be maps with coprime internal loop and let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. We have the following:*

- *(a1) Being equal is an equivalence relation.*
- *(a2) Two well-posed maps with coprime internal loop are equal iff they are equal in* $\mathrm{TIC}_\infty$.
- *(a3) A well-posed map is a map with coprime internal loop iff it has a l.c.f. or a r.c.f.*
- *(a4) If a well-posed map and a map with coprime internal loop are equivalent controllers for $\mathbb{D}$, then they are equal. (See (c) for the converse.)*
- *(b) If $\mathbb{Y}\mathbb{X}^{-1} = \mathbb{Y}_0\mathbb{X}_0^{-1}$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{X}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. If $\widetilde{\mathbb{Y}}\widetilde{\mathbb{X}}^{-1} = \widetilde{\mathbb{Y}}_0\widetilde{\mathbb{X}}_0^{-1}$, then $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. If $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$. In particular, $\mathbb{Y}\mathbb{X}^{-1}$ is well-posed iff $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.*
- *(c) Let $\mathbb{Q}$ and $\mathbb{Q}'$ be maps with coprime internal loop.*
  - *(c1) If $\mathbb{Q}$ and $\mathbb{Q}'$ are equal, then either both are admissible for $\mathbb{D}$ or neither is admissible for $\mathbb{D}$.*
  - *(c2) If $\mathbb{Q}$ and $\mathbb{Q}'$ are admissible for $\mathbb{D}$, then they are equal iff they are equivalent, that is, iff they determine the same map $u_L,y_L \mapsto u,y$. In particular, $\mathbb{Q}$ is stabilizing for $\mathbb{D}$ iff $\mathbb{Q}'$ is.*

Thus, Definition 7.2.11 is justified (its last identification was justified in Lemma 7.2.7).

By (c), equivalence of maps with coprime internal loop does not depend on the plant $\mathbb{D}$ (except that equivalence is not defined for non-admissible maps). By (b), a map is well-posed iff any (hence all) of its representatives is well-posed.

**Proof:** (a1) The only nonobvious requirement is transitivity (of being equal), so we take a look at it:

1° If $\mathbb{Y}\mathbb{X}^{-1}$ is a map with d.c. internal loop and equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$, then

$$\begin{bmatrix} \mathbb{U}\widetilde{\mathbb{X}} & -\mathbb{U}\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} \mathbb{U} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC} \tag{7.47}$$

for any $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, hence any map with l.c. internal loop equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is equal to $\mathbb{Y}\mathbb{X}^{-1}$ (insert $\left[\begin{smallmatrix} \mathbb{U} & 0 \\ 0 & I \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathbb{U}^{-1} & 0 \\ 0 & I \end{smallmatrix}\right]$ into (7.46)). (Thus the concept "map with d.c. internal loop" is well defined: if a map is such, then so is any equal map.)

Conversely, the (b) (and (d)) of Lemma 6.5.9 (with the columns and rows interchanged) shows that all map with r.c. internal loops equal to $\mathbb{Y}\mathbb{X}^{-1}$ are equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ (in particular, they have d.c. internal loops).

(Thus, (6.121) gives all doubly coprime products for any left and right maps equal to $\mathbb{N}\mathbb{M}^{-1}$.)

2° If $\mathbb{Y}\mathbb{X}^{-1}$ does not have a d.c. internal loop, then neither does any equal map with a coprime internal loop by 1°, and transitivity is obvious (i.e., $\left[\begin{smallmatrix} \mathbb{S} \\ \mathbb{R} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{X} \\ \mathbb{Y} \end{smallmatrix}\right] \mathbb{U}$, $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, and $\left[\begin{smallmatrix} \mathbb{P} \\ \mathbb{O} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{S} \\ \mathbb{R} \end{smallmatrix}\right] \mathbb{V}$, $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$ imply that $\left[\begin{smallmatrix} \mathbb{P} \\ \mathbb{O} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{X} \\ \mathbb{Y} \end{smallmatrix}\right] \mathbb{W}$ for some $\mathbb{W} \in \mathcal{G}\mathrm{TIC}$ (namely for $\mathbb{W} = \mathbb{U}\mathbb{V}$)). The dual claim is analogous.

(a2) This follows from (b) combined with Lemma 6.4.5 in the left or right case and with Lemma 6.5.8 in the left-right case.

(a3) This is a restatement of the last identification in Definition 7.2.11.

(a4) Let the two maps be $\mathbb{Q} \in \mathrm{TIC}_\infty$ and $\mathbb{Y}\mathbb{X}^{-1}$, respectively (the case for $\widetilde{\mathbb{X}}^{-1}\mathbb{Y}$ is analogous). By Lemma 7.2.7, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$.

(b) The first two claims follows from $\mathbb{X} = \mathbb{X}_0\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{X}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. so $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, The third one follows from Lemma A.1.1(c1). Thus, $\mathbb{Y}\mathbb{X}^{-1}$ is equal to a well-posed map iff $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.

(c1) If $\mathbb{Q}$ and $\mathbb{Q}'$ have l.c. (resp. r.c.) internal loops, then this is obvious (because the admissibility is equivalent to $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$ (resp. $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty$), by Lemma 7.2.10(a)). Thus, we assume (7.46). Then

$$\begin{bmatrix} I & 0 \\ -\mathbb{D} & I \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} - \mathbb{D}\mathbb{M} & \mathbb{X} - \mathbb{D}\mathbb{Y} \end{bmatrix} = \left( \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{D} & I \end{bmatrix} \right)^{-1}$$

in $\mathrm{TIC}_\infty$, so $\mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma A.1.1(c1).

(c2) 1° We start from the case of two maps with l.c. internal loop. The formula (7.43) shows that maps $u_L, y_L \mapsto u, y$ are equal for $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ and $\mathbb{Q}' = \widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$ iff the corresponding terms $\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}$ and $\widetilde{\Delta}_0^{-1}\widetilde{\mathbb{Y}}_0$ are equal, i.e., $\widetilde{\mathbb{Y}} = \mathbb{U}\widetilde{\mathbb{Y}}_0$, where $\mathbb{U} := \widetilde{\Delta}\widetilde{\Delta}_0^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$. But then $\widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} = I + \widetilde{\Delta}^{-1}\mathbb{Y}\mathbb{D} = \widetilde{\Delta}_0^{-1}\widetilde{\mathbb{X}}_0$, i.e., $\widetilde{\mathbb{X}} = \mathbb{U}\widetilde{\mathbb{X}}_0$. So if the maps $u_L, y_L \mapsto u, y$ are equal, then $\mathbb{U} \in \mathrm{TIC}$ (by the dual of Lemma 6.5.1(c1)), because $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are l.c., and $\mathbb{U}^{-1} \in \mathrm{TIC}$, because $\widetilde{\mathbb{X}}_0$ and $\widetilde{\mathbb{Y}}_0$ are l.c.; thus, then $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} = \widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$.

Conversely, if $\begin{bmatrix}\widetilde{\mathbb{Y}} & \widetilde{\mathbb{X}}\end{bmatrix} = \mathbb{U}\begin{bmatrix}\widetilde{\mathbb{Y}}_0 & \widetilde{\mathbb{X}}_0\end{bmatrix}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, then $\widetilde{\Delta} = \mathbb{U}\widetilde{\Delta}_0$ and hence $\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} = \widetilde{\Delta}_0^{-1}\widetilde{\mathbb{Y}}_0$, so the maps $u_L, y_L \mapsto u, y$ are equal, as noted above.

2° From (7.32) one gets the corresponding right result analogously.

3° Similarly, from (7.32) and (7.43) one notices that $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ and $\mathbb{Q}' = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ determine the same $u_L, y_L \mapsto u, y$ iff $\mathbb{Y}(\mathbb{X} - \mathbb{D}\mathbb{Y})^{-1} = (\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1}\widetilde{\mathbb{Y}}$, i.e., iff $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$.

Thus, equality implies equivalence, so we assume equivalence and prove that $\mathbb{Q}$ and $\mathbb{Q}'$ are equal. Because $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$, as noted above, we may choose $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}$ as in Lemma A.1.1(e1) (interchange the rows and columns) to obtain

$$I = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X}\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}I & 0 \\ \mathbb{D} & I\end{bmatrix}\begin{bmatrix}I & 0 \\ -\mathbb{D} & I\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X}\end{bmatrix}. \tag{7.48}$$

By Lemma A.1.1(e5) and the assumed invertibility of $\Delta$ and $\widetilde{\Delta}$, we have

$$I = \begin{bmatrix}I & 0 \\ -\mathbb{D} & I\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}I & 0 \\ -\mathbb{D} & I\end{bmatrix}$$

in $\mathrm{TIC}_\infty$, hence we have (7.46) in $\mathrm{TIC}_\infty$, so it must hold in TIC too, by the density of $C_c^\infty$. □

We now parametrize all stabilizing controllers by combining Proposition 7.2.5(b) and Lemma 7.2.10(a):

**Corollary 7.2.13** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then the following claims and their duals hold:*

*(a1) If $\mathbb{D}$ has a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, then each stabilizing controller with internal loop for $\mathbb{D}$ is equivalent to a unique map with l.c. internal loop*

$$\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} \quad \textit{such that} \quad \widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I \tag{7.49}$$

*(in particular, a different pair $(\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}})$ defines a different stabilizing map $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$). The dual result for l.c.f.'s $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ holds as well.*

*(a2) The map with l.c. internal loop $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\widetilde{\Delta}^{-1}, \mathbb{D}\widetilde{\Delta}^{-1} \in \mathrm{TIC}$).*

*(a2') The map with r.c. internal loop $\mathbb{Y}\mathbb{X}^{-1}$ is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\Delta^{-1}, \Delta^{-1}\mathbb{D} \in \mathrm{TIC}$).*

*(a3) If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., then the map $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ with l.c. internal loop is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\in \mathcal{G}\mathrm{TIC}(U)$).*

*(b) The following are equivalent:*

*(i) $\mathbb{D}$ has a r.c.f. (resp. a l.c.f., a d.c.f.);*

*(ii) $\mathbb{D}$ is stabilizable by a map with l.c. (resp. r.c., d.c.) internal loop;*

*Moreover, if (i) holds, then each stabilizing controller for $\mathbb{D}$ with internal loop is equivalent to one with l.c. (resp. r.c., d.c.) internal loop.*

Unfortunately, we do not know, whether any $\mathbb{D}$ that is stabilizable with internal loop has a r.c.f. (or a l.c.f.), so it may be that some pathological plants (having no stabilizing controllers with coprime internal loop) might not meet the above requirements.

**Proof:** (We obtain the dual claims by taking the adjoints of (a1)–(b); this is explicitly illustrated in (a2')).)

(a1) This follows from Proposition 7.2.5(b). The definition of equality [Definition 7.2.11] shows that $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ determines $(\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}})$ uniquely.

(a2) This is (most of) Lemma 7.2.10(a) and (a2') is its dual.

(a3) Now $\widetilde{\Delta}\mathbb{M} = \widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N}$, so the stability of its inverse $\mathbb{M}^{-1}\widetilde{\Delta}^{-1}$ is equivalent to that of $\mathbb{D}\widetilde{\Delta}^{-1}$, by Lemma 6.5.6(b), and clearly implies the stability of $\Delta^{-1}$, so we get (a3) from (a2).

(b) "(iii)⇒(ii)": Any map with l.c. [r.c.] internal loop that stabilizes $\mathbb{D}$ defines a r.c.f. [l.c.f.] of $\mathbb{D}$, by Lemma 7.2.10(b).

"(i)⇒(ii)": This follows from (a1) (and the definition of maps with d.c. internal loop: just take the factors $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$ of any d.c.f. of $\mathbb{D}$); and so does the "moreover" claim. □

Now we can present five equivalent parametrizations for all (modulo being equivalent) stabilizing controllers with internal loop for any fixed $\mathbb{D} \in \mathrm{TIC}_\infty$ having a d.c.f.:

**Theorem 7.2.14 (All stabilizing controllers)** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have the d.c.f.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix}. \tag{7.50}$$

*Then each controller that stabilizes $\mathbb{D}$ with internal loop is equivalent to a unique map with d.c. internal loop (in the sense that both controllers determine the same closed-loop map $u_L, y_L \mapsto u, y$).*

*The following parametrizations are alternative (equivalent) parametrizations of all controllers $\mathbb{Q}$ with d.c. internal loop that stabilize $\mathbb{D}$, and each* parameter *($(\mathbb{X}, \mathbb{Y})$ in (i) and (iii), $(\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}})$ in (i'), and $\mathbb{U}$ in (ii) and (ii'); these all are required to be stable) determines a different (nonequal) map $\mathbb{Q}$ with d.c. internal loop.*

- *(i)* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *such that* $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I$.
- *(i')* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *such that* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$.
- *(ii)* **(Youla)** $\mathbb{Q} = (\mathbb{T} + \mathbb{M}\mathbb{U})(\mathbb{S} + \mathbb{N}\mathbb{U})^{-1}$ *(i.e.,* $\begin{bmatrix} \mathbb{Y} \\ \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}$*), where* $\mathbb{U} \in \mathrm{TIC}(U)$.
- *(ii')* $\mathbb{Q} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}\mathbb{U})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}\mathbb{U})$ *(i.e.,* $\begin{bmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \end{bmatrix} = \begin{bmatrix} I & \mathbb{U} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & \widetilde{\mathbb{T}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix}$*), where* $\mathbb{U} \in \mathrm{TIC}(U)$.
- *(iii)* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *(*$= \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*), where* $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times Y)$.

*The well-posed ones (if any) are exactly those for which the "denominator" in is* $\mathcal{G}\mathrm{TIC}_\infty$ *(cf. Theorem 7.1.7).*

*Moreover, for any* $\mathbb{U} \in \mathrm{TIC}$ *we have (identity as equal maps with coprime internal loop)*

$$(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\widetilde{\mathbb{N}}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}}+\widetilde{\mathbb{M}}\widetilde{\mathbb{U}}). \tag{7.51}$$

*Finally, if (i) and (i') hold, then the* $(1\text{–}2, 1\text{–}2)$*-block of the closed-loop map* $\mathbb{D}^o_I := \mathbb{D}^o(I-\mathbb{D}^o)^{-1}$ *is given by*

$$\begin{bmatrix} \mathbb{Y}\widetilde{\mathbb{N}} & \mathbb{Y}\widetilde{\mathbb{M}} \\ \mathbb{X}\widetilde{\mathbb{N}} & \mathbb{X}\widetilde{\mathbb{M}}-I \end{bmatrix} = \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}}-I & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}. \tag{7.52}$$

Recall from Lemma 7.2.12(c2), that the maps (7.52) depend (of course) on $\mathbb{D}$ and $\mathbb{Q}$ only, not of the particular coprime factors $(\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}})$ of $\mathbb{Q}$ and $\mathbb{D}$ (except that (i) and (i') are required to hold).

**Proof:** The first claim is from Proposition 7.2.5(b) (and its dual). The parametrizations (i) and (i') are Corollary 7.2.13(a1).

For any stable pair $(\mathbb{Y}, \mathbb{X})$ there are $\widetilde{\mathbb{Y}}$ and $\widetilde{\mathbb{X}}$ satisfying (iii) iff $(\mathbb{Y}, \mathbb{X})$ satisfies (i), by Lemma 6.5.8. Now the parametrizations (ii) and (ii') and equation (7.51) follow from (iii) and Lemma 6.5.9(c).

The well-posedness claim is Lemma 7.2.12(b), and (7.52) is from (7.42) (alternatively, directly from $\mathbb{D}^o_I = (I-\mathbb{D}^o)^{-1} - I$). □

To check whether a given controller with coprime internal loop stabilizes $\mathbb{D}$, one can use the following corollary:

**Corollary 7.2.15** *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *have the d.c.f. (7.50). Let* $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$. *Then* $\mathbb{Y}\mathbb{X}^{-1}$ *(resp.* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*) is a map with d.c. internal loop and stabilizes* $\mathbb{D}$ *iff any (hence all) of (i)–(iii) (resp. (i')–(iii')) holds:*

- *(i)* $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y)$*;*
- *(ii)* $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}(U \times Y)$*;*
- *(iii) There is a r.c.f.* $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ *s.t.* $\left[\begin{smallmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$.
- *(i')* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.
- *(ii')* $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.
- *(iii') There is a l.c.f.* $\mathbb{D} = \widetilde{\mathbb{M}}_0^{-1}\widetilde{\mathbb{N}}_0$ *s.t.* $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$.

*Moreover, the map* $\mathbb{Y}\mathbb{X}^{-1}$ *(resp.* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*) is well-posed iff* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ *(resp.* $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$*).*

**Proof:** Any of the conditions shows that $\mathbb{Y}\mathbb{X}^{-1}$ (resp. $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) are coprime. If it stabilizes $\mathbb{D}$, then it is d.c., by Theorem 7.2.14.

The parametrizations (i) and (i') are from Corollary 7.2.13(a3) and its dual. Part (ii) defines a d.c.f. of $\mathbb{D}$, hence it is sufficient (take $\mathbb{U} = 0$ in Theorem 7.2.14(ii)). Conversely, if $\mathbb{Y}\mathbb{X}^{-1}$ is stabilizing, then

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbb{V} \end{bmatrix} = \begin{bmatrix} \mathbb{M} & \mathbb{Y}\mathbb{V} \\ \mathbb{N} & \mathbb{X}\mathbb{V} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$$

for some $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$, by Theorem 7.2.14(iii), hence then $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$, i.e., (ii) holds. Condition (ii') is the dual of (ii).

(iii) The condition (iii) is sufficient, by Theorem 7.2.14(iii). Conversely, if (i) holds (and hence (i') too; thus situation is as in the "furthermore" claim) and we set

$$\begin{bmatrix} \widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{bmatrix} := (\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y})^{-1} \begin{bmatrix} \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix}, \quad \begin{bmatrix} \mathbb{M}_0 \\ \mathbb{N}_0 \end{bmatrix} := \begin{bmatrix} \mathbb{M} \\ \mathbb{N} \end{bmatrix} (\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N})^{-1}, \quad (7.53)$$

then, obviously, $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{smallmatrix}\right] = I$, and the dual equation $\left[\begin{smallmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{smallmatrix}\right] \left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{smallmatrix}\right] = I$ follows from Lemma A.1.1(e5).

The well-posedness claim is Lemma 7.2.12(b). □

Next we given two lemmas that are useful when one wants to work in a subalgebra of TIC (e.g., in MTIC; cf. Theorem 4.1.1):

**Lemma 7.2.16 (Predetermining the joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$)** *Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ be a d.c.f., and let $\mathcal{A} \underset{a}{\subset}$ TIC be inverse closed.*

*(a) If $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ and $\mathbb{Q} := \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ stabilizes $\mathbb{D}$, then the d.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is over $\mathcal{A}$, even joint with $\mathbb{Q}$.*

*(b) If $\mathbb{Q}$ stabilizes $\mathbb{D}$, then for any r.c.f. $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ and l.c.f. $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\mathbb{Y}$, there is a joint d.c.f. $\left[\begin{smallmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{smallmatrix}\right] \in \mathcal{G}$TIC of $\mathbb{D}$ and $\mathbb{Y}\mathbb{X}^{-1}$; if, in addition, $\mathbb{M}, \mathbb{N}, \mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$, then we can take $\mathbb{M}_0, \mathbb{N}_0, \widetilde{\mathbb{M}}_0, \widetilde{\mathbb{N}}_0 \in \mathcal{A}$.*

*Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}} \in \mathrm{TIC}_\infty(U,Y)$. Assume that $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ stabilizes $\mathbb{D}$. If $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ and $\mathcal{A} \subset$ TIC is inverse closed, then the d.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is over $\mathcal{A}$.*

**Proof:** (a) Set $\begin{bmatrix} \widetilde{\mathbb{X}}' & \widetilde{\mathbb{Y}}' \end{bmatrix} := (\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N})^{-1} \begin{bmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \end{bmatrix} \in \mathcal{A}$. Then equation $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y}' \\ \mathbb{N} & \mathbb{X}' \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \widetilde{\mathbb{X}}' & -\widetilde{\mathbb{Y}}' \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right]$ is a joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$ for some $\mathbb{X}', \mathbb{Y}' \in$ TIC, by Lemma 6.5.8. Because $\mathcal{A}$ is inverse closed, we have $\left[\begin{smallmatrix} \widetilde{\mathbb{X}}' & -\widetilde{\mathbb{Y}}' \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right]^{\pm 1} \in \mathcal{A}$ too.

(b) The proof is similar to that of (a) and hence omitted. □

**Proposition 7.2.17 ($\mathcal{A}$ case)** *Let $\mathbb{D}$ have a d.c.f. over $\mathcal{A}$, where $\mathcal{A} \underset{a}{\subset}$ TIC (cf. Proposition 7.1.10).*

*If the elements of (7.50) are chosen from $\mathcal{A}$, then all stabilizing controllers of $\mathbb{D}$ with a (d.c.) internal loop are the ones parametrized in Theorem 7.2.14, and the*

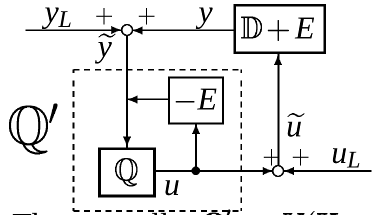


Figure 7.7: The controller $\mathbb{Q}' := \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} : \tilde{y} \mapsto u$ for $\mathbb{D}+E$

*ones with d.c. internal loop over $\mathcal{A}$ are exactly those with $\mathbb{U} \in \mathcal{A}$. If, in addition, $\mathcal{B} \subset_a \mathcal{A} \subset_a$ ULR $\cap$ TIC, then the one with $\mathbb{U} = -M^{-1}T$ is well-posed.* □

(The proof is virtually a subset of the proof of Proposition 7.1.10 and hence omitted.)

Recall from Theorem 4.1.6(d), that if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{N}, \mathbb{M} \in \text{MTIC}_{T\mathbf{Z}}(\mathbf{C}^n, Y)$, then $\mathbb{D}$ has a d.c.f. over $\text{MTIC}_{T\mathbf{Z}}$, hence then $\mathbb{D}$ has a well-posed stabilizing controller having a d.c.f. over $\text{MTIC}_{T\mathbf{Z}}$, by the above proposition.

**Lemma 7.2.18 ($D = 0$ w.l.o.g.)** *Let $\mathbb{D} \in \text{TIC}_\infty(U,Y)$. Let $E \in \mathcal{B}(U,Y)$.*

*Then $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \tilde{\mathbb{X}}^{-1}\tilde{\mathbb{Y}}$ is a stabilizing DF-controller with d.c. internal loop for $\mathbb{D}$ iff $\mathbb{Q}' = \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} = (\tilde{\mathbb{X}}+\tilde{\mathbb{Y}}E)^{-1}\tilde{\mathbb{Y}}$ is a stabilizing DF-controller with d.c. internal loop for $\mathbb{D}+E$. The corresponding closed-loop maps $y_L \mapsto u$ (see (7.52)) are identical.*

*The controller $\mathbb{Q}'$ can be realized by adding to $\mathbb{Q}$ an output feedback through $-E$, as in Figure 7.7.*

If one replaces $\mathbb{D}+E$ by a parallel connection of $\mathbb{D}$ and $E$ in Figure 7.7, then it becomes obvious that $E$ and $-E$ cancel each other and we are left with the original connection of $\mathbb{Q}$ and $\mathbb{D}$; this allows one to write down the correspondence between the original and perturbed settings. See also Lemma 7.3.23.

(Note also that one should draw some external inputs "$z_L$ and $y_L'$" to Figure 7.7 (just before $-E$ and just before $\mathbb{Q}$) and the internal loop (the signals $\xi$, $\xi_L$ and $\tilde{\xi}$) of $\mathbb{Q}$ if $\mathbb{Q}$ is non-well-posed.)

Naturally, one of $\mathbb{Q}$ and $\mathbb{Q}'$ may be non-well-posed even if the other is well-posed (but the closed-loop systems are both well-posed if one is).

If $\mathbb{Q} \in \text{TIC}_\infty(Y,U)$ and $I + E\mathbb{Q} \in \mathcal{G}\text{TIC}_\infty(Y)$, then one more formula for $\mathbb{Q}'$ is obviously given by $\mathbb{Q}' = (I+\mathbb{Q}E)^{-1}\mathbb{Q}$ $(= \mathbb{Q}(I+E\mathbb{Q})^{-1})$.

**Proof:** 1° Given any joint d.c.f. (7.46) of $\mathbb{D}$ of $\mathbb{Q}$, a joint d.c.f. of $\mathbb{D}+E$ and $\mathbb{Q}'$ is obviously given by

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N}+E\mathbb{M} & \mathbb{X}+E\mathbb{Y} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbb{X}}+\tilde{\mathbb{Y}}E & -\tilde{\mathbb{Y}} \\ -(\tilde{\mathbb{N}}+\tilde{\mathbb{M}}E) & \tilde{\mathbb{M}} \end{bmatrix}^{-1} \in \mathcal{G}\text{TIC}(U \times Y). \tag{7.54}$$

By exchanging $\mathbb{D}$ and $\mathbb{D}+E$, we obtain from Theorem 7.2.14(iii) that the stabilizing DPF-controllers for $\mathbb{D}$ and $\mathbb{D}+E$ correspond to each other as in the statement of the lemma.

Given $\mathbb{Q}$, the map $\mathbb{Y}\tilde{\mathbb{M}} = \mathbb{M}\tilde{\mathbb{Y}} : y_L \mapsto u$ is common for both closed-loop systems, by (7.52) (since $\mathbb{M}, \tilde{\mathbb{M}}, \mathbb{Y}, \tilde{\mathbb{Y}}$ are unaffected). (N.B. if we fix some representation $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ of $\mathbb{Q}$, then $\begin{bmatrix} \mathbb{M}\widetilde{\mathbb{Y}}-I & \mathbb{M} \\ \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M}-I \end{bmatrix} : \begin{bmatrix} y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ \xi \end{bmatrix}$ is unaffected, by (7.44).)

2° Thus, the "rigorous" part of the proof is complete, and we only have to show that $\mathbb{Q}'$ is a model for the map in Figure 7.7, i.e., that the maps $u_L, y_L \mapsto u, y$ for $\mathbb{Q}'$ and $\mathbb{D}+E$ become equal to those obtained by solving the equations modeled in the figure.

By writing the equations for $\widetilde{u}$, $\widetilde{y}$ and $\widetilde{\xi}$, one obtains

$$(I-\mathbb{D}^{o\prime}) \begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix} = \begin{bmatrix} \widetilde{u}_L \\ \widetilde{y}_L \\ \widetilde{\xi}_L + \widetilde{\mathbb{Y}}(y_L'+Ez_L) \end{bmatrix}, \tag{7.55}$$

(at the moment we are not interested in the additional inputs $z_L$ and $y_L'$) where

$$\mathbb{D}^{o\prime} := \begin{bmatrix} 0 & 0 & I \\ \mathbb{D} & 0 & 0 \\ 0 & \widetilde{\mathbb{Y}} & I-(\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E) \end{bmatrix}, \tag{7.56}$$

i.e., $\mathbb{D}^{o\prime}$ has $\mathbb{D}+E$ in place of $\mathbb{D}$ and $\mathbb{O}' = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-(\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E) \end{bmatrix}$ (a representative of $\mathbb{Q}'$) in place of $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$ (a representative of $\mathbb{Q}$).

But once we let the additional inputs $y_L'$ and $z_L$ be zero, equation (7.55) becomes $(I-\mathbb{D}^{o\prime})\begin{bmatrix} u \\ y \\ \xi \end{bmatrix} = \mathbb{D}^{o\prime}\begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix}$, i.e., the equation (7.19) for $\mathbb{D}+E$ and $\mathbb{O}'$. Thus, we can consider $\mathbb{O}'$ as a model for the controller (in the dashed square) in Figure 7.7. Summarizing, the map $\mathbb{Q} \mapsto \mathbb{Q}'$ corresponds to an output feedback through $-E$. □

From Remark 6.7.19 we deduce that if $\mathbb{D}$ is replaced by $\mathcal{T}_\omega\mathbb{D}$ and $\mathbb{Q}$ by $\mathcal{T}_\omega\mathbb{Q}$ for some $\omega \in \mathbf{R}$, then $\mathbb{D}_I^o$ becomes replaced by $\mathcal{T}_\omega\mathbb{D}_I^o$. From this we conclude the following:

**Remark 7.2.19 (Exponential stabilization)** *By Remark 6.7.19, from any claims in this section (and others), we can deduce the corresponding results about $\omega$-stabilization for some $\omega \in \mathbf{R}$ (instead of the (0-)stabilization treated in most above results), hence also for exponential stabilization.*

*For example, assume that $\mathbb{D} \in \mathrm{TIC}_\infty$ has an exponentially stable d.c.f., say (7.50) (i.e., the maps in (7.50) belong to $\mathrm{TIC}_{\mathrm{exp}}$). Then the maps DF-stabilize $\mathbb{D}$ exponentially with internal loop are exactly the maps with (exponentially) d.c. internal loop parametrized in Theorem 7.2.14 (where we must require the parameters to be exponentially stable).*

**Remark 7.2.20 (Plants with internal loop)** *We could, of course, study more general plants, those with internal loop. One easily (though sometimes with tedious applications of Lemma A.1.1) generalizes most results of this section to the case where both $\mathbb{D}$ and $\mathbb{Q}$ have internal loops, e.g., if $\mathbb{D} = \widetilde{\mathbb{N}}\widetilde{\mathbb{M}}^{-1}$ is a map with l.c. internal loop, then $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ stabilizes $\mathbb{D}$ iff $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$. This way one could cover all "$\mathrm{H}^\infty/\mathrm{H}^\infty$" transfer functions (the quotient field of $\mathrm{H}^\infty$) and more.*

## Notes

Controllers with internal loop were first introduced in [WC], which covers also some corresponding state-space theory for regular WPLSs. This notion was further developed in the frequency-domain article [CWW01]. Our theory was built on an early form of [CWW01], which we were given in late 1996. The actual article will be published late 2001.

Part (c1) of Theorem 7.2.3 is Theorem 7.4 of [WR00]. Lemma 7.2.7 is at least partially contained in Section 6 of [WC]. Proposition 7.2.5(b), Lemma 7.2.10(a)&(b), Corollary 7.2.13 and Corollary 7.2.15 are at least implicitly contained in [CWW01] (some of them with different proofs). Part (d) of Theorem 7.2.4 was written as a generalization of the corresponding classical result (see, e.g., Lemma 12.1 of [ZDG]). Proposition 5.3 of [WC] seems to be its analogy for exponential DF-stabilization with internal loop.

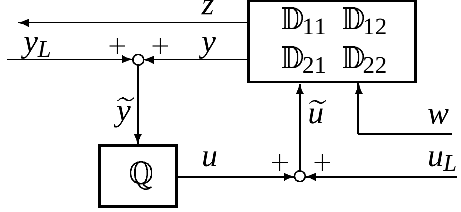


Figure 7.8: DPF-controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$

## 7.3 DPF-stabilization ($\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$)

*Sir, it's very possible this asteroid is not stable.*
— C3P0

In Section 7.1, we studied dynamic output-feedback stabilization by a well-posed controller and, in Section 7.2, that by a controller with internal loop (such controllers contain all well-posed controllers).

In this section we shall use those results to obtain a theory for dynamic partial feedback (with internal loop; we also give some further results for the special case of a well-posed controller).

A general DPF-controller differs from the special case of a DF-controller in the sense that the former has only a partial access to the inputs and outputs of the plant, as illustrated in Figure 7.8 (cf. to Figure 7.1).

In the important $\mathrm{H}^\infty$ *Four-Block Problem (*$\mathrm{H}^\infty$ *4BP)* (or the standard $\mathrm{H}^\infty$ problem) of Chapter 12, one tries to find a DPF-controller that stabilizes the plant and makes the norm $\|w \mapsto z\|$ in Figure 7.8 less than a given constant $\gamma > 0$. This problem is the main motivator of the theory of this chapter. The signal $y$ can be considered as a measure accessible for the controller and $u$ as the controller output, whereas $w$ often represents the disturbances in a system and $z$ stands for the actual (objective) output.

Our choice to have $u$ before $w$ is contrary to the standard practice in DPF-stabilization and the $\mathrm{H}^\infty$ 4BP theory (this corresponds to $\begin{bmatrix} \mathbb{D}_{12} & \mathbb{D}_{11} \\ \mathbb{D}_{22} & \mathbb{D}_{21} \end{bmatrix}$ in place of $\mathbb{D}$), which is better suited for DPF duality results.

However, our choice is the standard practice in the $\mathrm{H}^\infty$ FICP theory (see Chapter 11), being more natural for that theory (e.g., it allows us to have $I$'s on the diagonal in several FICP and 4BP formulae).

Therefore, when comparing the formulae to most studies on DPF-stabilization (e.g., [Francis], [Keu], [Green] or [ZDG]), one has to interchange the (second) indices corresponding $u$ and $w$, whereas the FICP results (e.g., [S98d], [Green], [CG97], [LT00a]) can directly be compared.

If we delete the rest of $\mathbb{D}$ except $\mathbb{D}_{21}$ in Figure 7.10, we end up with Figure 7.3. Therefore, the maps $u_L, y_L \mapsto u, y$ become the same as in the DF-stabilization of $\mathbb{D}_{21}$, and the map of $u_L, w, y_L$ to $z, y, u$ is obtained from this and the equation

$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathbb{D} \begin{bmatrix} u + u_L \\ w \end{bmatrix}. \tag{7.57}$$

In particular, the controller is admissible for $\mathbb{D}$ iff it is admissible for $\mathbb{D}_{21}$.

(Note that we use $w$ instead of $w_L = w$, because there is no feedback to the disturbance signal $w$. Models should contain additional inputs representing the disturbances in each loop, but since there is no feedback (loop) for $w$, such an additional input would be redundant. The situation with $z$ is similar.)

However, it is easiest to identify any DPF-controller $\widetilde{\mathbb{Q}} \in \mathrm{TIC}_\infty(Y,U)$ with the DF-controller $\mathbb{Q} : \begin{bmatrix} \tilde{z} \\ \tilde{y} \end{bmatrix} \mapsto \begin{bmatrix} u \\ w \end{bmatrix}$ of form $\mathbb{Q} = \begin{bmatrix} 0 & \widetilde{\mathbb{Q}} \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Z \times Y, U \times W)$ (so that $\widetilde{\mathbb{Q}} \in \mathrm{TIC}_\infty(Y,U)$ maps $\tilde{y} \mapsto u$, and $\tilde{z}$ and $w$ are uncoupled from the controller). Obviously, this definition is equivalent to the one above. Its rigorous form is contained in the following definition (the case $\mathbb{O} = \begin{bmatrix} \widetilde{\mathbb{Q}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$), which also covers the case with internal loop:

**Definition 7.3.1 (DPF-stabilization [with internal loop], $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$)** *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$*. We call* $\mathbb{O} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *(where also* $\Xi$ *is a Hilbert space) an* admissible (resp. stabilizing) DPF-controller with internal loop *for* $\mathbb{D}$ *if*

$$\mathbb{O}_{\mathrm{DF}} := \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty((Z \times Y) \times \Xi, (U \times W) \times \Xi) \tag{7.58}$$

*is an admissible (resp. stabilizing) DF-controller with internal loop for* $\mathbb{D}$*.*

*We call* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$ *an* admissible (resp. stabilizing) DPF-controller with internal loop for $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *if*

$$\widetilde{\Sigma}_{\mathrm{DF}} := \left[\begin{array}{c|ccc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline \widetilde{\mathbb{C}}_1 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 \\ \widetilde{\mathbb{C}}_2 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \in \mathrm{WPLS}(Z \times Y \times \Xi, \widetilde{H}, U \times W \times \Xi) \tag{7.59}$$

*is an admissible (resp. stabilizing) DF-controller with internal loop for* $\Sigma$*.*

*In either case, by* $\mathcal{F}_\ell(\mathbb{D},\mathbb{O})$ *we denote the map* $w \mapsto z$ *of* $\mathbb{D}_I^o$ *(see (7.64), (7.68) and (7.98)).*

*We call two admissible DPF-controllers with internal loop for* $\mathbb{D}$ *(resp. for* $\Sigma$*)* equivalent for $\mathbb{D}$ *(resp. for* $\Sigma$*) if they determine same maps from* $u_L, y_L$ *to* $u, y$*.*

*We call* $\mathbb{D}$ *(resp.* $\Sigma$*)* DPF-stabilizable with internal loop *if there is a stabilizing DPF-controller with internal loop for* $\mathbb{D}$ *(resp. for* $\Sigma$*). and we use prefices as above. (We use prefices as in Definition 7.2.1.)*

*If* $\mathbb{O}_{\mathrm{DF}}$ *is a well-posed DF-controller (equivalently,* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$*), then we may remove the words "with internal loop" everywhere above in this definition and identify* $\mathbb{O}$ *with* $\mathbb{O}_{11} \in \mathrm{TIC}_\infty(Y,U)$*.*

*If* $\mathbb{Q}$ *is a map with coprime internal loop, then we call* $\mathbb{Q}$ *an* admissible (resp. stabilizing) DPF-controller with coprime internal loop for $\mathbb{D}$ *if* $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ *is an admissible (resp. stabilizing) DF-controller with coprime internal loop for* $\mathbb{D}$*.*

As before, "*[DPF-]stabilizes*" means "is [DPF-]stabilizing for", in any of the above settings. (We use the prefix "DPF-" whenever there is a risk of confusion.)

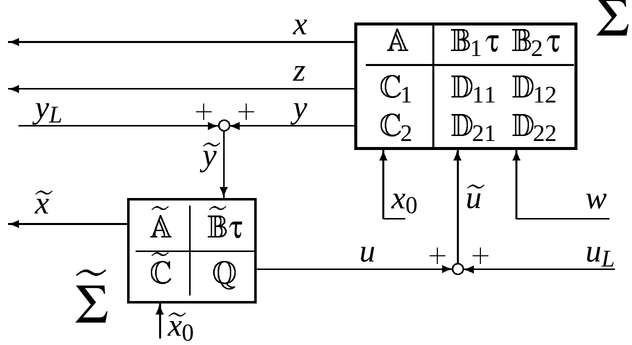


Figure 7.9: DPF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

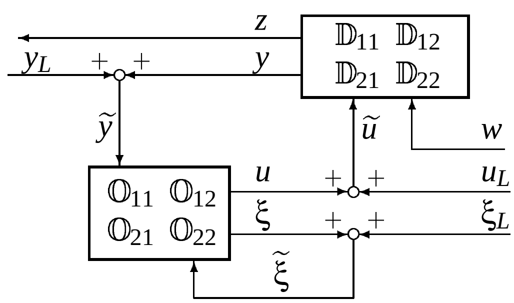


Figure 7.10: DPF-controller $\mathbb{O}$ with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$

Lemma 7.3.10 shows that also the coprime part of Definition 7.3.1 is justified.

Note that $\mathbb{D}_I^o$ maps $(u_L, w, z_L, y_L, \xi_L) \mapsto (u, w, z, y, \xi)$ (cf. (7.63) and recall that $\mathbb{D}_I^o = (I - \mathbb{D}^o)^{-1} - I$). Therefore, $\mathbb{O}$ stabilizes $\mathbb{D}$ iff $u, (w,) z, y, \xi \in \mathrm{L}^2$ for all $u_L, w, z_L, y_L, \xi_L \in \mathrm{L}^2$. See also Figures 7.10 and 7.11 and the comments below Definition 7.1.1 and Summary 6.7.1.

The combined open-loop system of (7.21) corresponding to the DF-controller $\widetilde{\Sigma}_{\mathrm{DF}}$ with internal loop for $\Sigma$ (i.e., the DPF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma$), is obviously given by

$$\Sigma^o := \left[\begin{array}{cc|ccccc} \mathbb{A} & 0 & \mathbb{B}_1 & \mathbb{B}_2 & 0 & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & 0 & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline 0 & \widetilde{\mathbb{C}}_1 & 0 & 0 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{C}_1 & 0 & \mathbb{D}_{11} & \mathbb{D}_{12} & 0 & 0 & 0 \\ \mathbb{C}_2 & 0 & \mathbb{D}_{21} & \mathbb{D}_{22} & 0 & 0 & 0 \\ 0 & \widetilde{\mathbb{C}}_2 & 0 & 0 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \tag{7.60}$$

$\in \mathrm{WPLS}(U \times W \times Z \times Y \times \Xi, H \times \widetilde{H}, U \times W \times Z \times Y \times \Xi)$. Thus, $\widetilde{\Sigma}$ is an admissible [stabilizing] DPF-controller with internal loop for $\Sigma$ iff $\Sigma_I^o$ is well-posed [and stable] (cf. Definition 6.6.4); see (6.125) for the closed-loop system $\Sigma_I^o$.

If $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then we can simplify the above definition as follows:

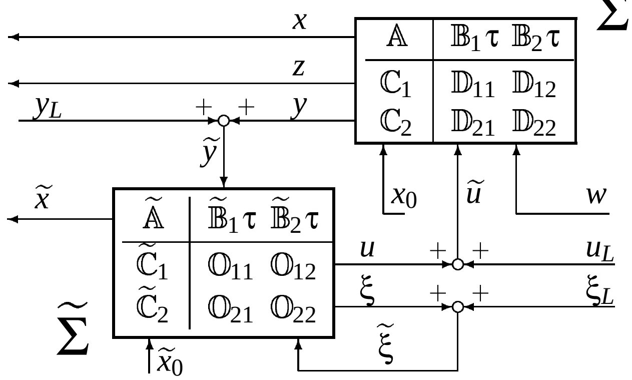


Figure 7.11: DPF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

**Lemma 7.3.2 (Well-posed DPF-controllers)** *A (well-posed) DPF-controller $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ is admissible [stabilizing] for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ iff $L := I$ is admissible [stabilizing] for $\left[\begin{smallmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ \mathbb{D} & 0 & 0 \end{smallmatrix}\right]$ $\mathrm{TIC}_\infty(U \times W \times Y, U \times Z \times Y)$; all prefices apply.*

*A (well-posed) DPF-controller $\widetilde{\Sigma} \in \mathrm{WPLS}(Y, \widetilde{H}, U)$ is admissible [stabilizing] for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ iff $L := I$ is admissible [stabilizing] for*

$$\Sigma^o := \left[\begin{array}{cc|ccc} \mathbb{A} & 0 & \mathbb{B} & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & 0 & \widetilde{\mathbb{B}} \\ \hline 0 & \widetilde{\mathbb{C}} & 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{D} & 0 & 0 \end{array}\right] \in \mathrm{WPLS}(U \times W \times Z \times Y, H \times \widetilde{H}, U \times W \times Z \times Y); \tag{7.61}$$

*all prefices apply. In either setting, admissibility is equivalent to condition $I - \mathbb{Q}\mathbb{D}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.*

The last condition is equivalent to "$I - QD_{21} \in \mathcal{G}\mathcal{B}(U)$" if $\mathbb{Q}, \mathbb{D}_{21} \in \mathrm{ULR}$, by Proposition 6.3.1(c).

Cf. again Figures 7.8 and 7.9 to Figures 7.10 and 7.11, respectively. Note also that (7.61) equals (7.21) for $\Sigma$ and

$$\widetilde{\Sigma}' := \left[\begin{array}{c|cc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{array}\right] \in \mathrm{WPLS}(Z \times Y, \widetilde{H}, U \times W) \tag{7.62}$$

i.e., it corresponds to the DF-controller $\widetilde{\Sigma}'$ for $\Sigma$.

**Proof:** Denote (7.60) by $\Sigma^{o\prime}$. Obviously, $\Sigma^{o\prime} = \left[\begin{smallmatrix} \Sigma^o & 0 \\ 0 & 0 \end{smallmatrix}\right]$. One easily verifies that $L = I := I_{U \times W \times Z \times Y}$ is admissible [stabilizing] for $\Sigma^o$ iff $\left[\begin{smallmatrix} I & 0 \\ 0 & I_\Xi \end{smallmatrix}\right]$ is admissible [stabilizing] for $\Sigma^{o\prime}$ (because $\Sigma^{o\prime}_I = \left[\begin{smallmatrix} \Sigma^o_I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, by (6.125); from this we also observe that all prefices of Definition 6.6.4 apply).

Condition $I - \mathbb{Q}\mathbb{D}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ can be obtained from Lemmas 7.3.5 and 7.1.2 (or from a direct computation). □

**Lemma 7.3.3 (DPF-controllers with IL)** *A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is an admissible [stabilizing] DPF-controller with internal loop for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *iff the connection if Figure 7.10 is well-posed [and stable], equivalently, iff*

$$\mathbb{D}^o := \begin{bmatrix} 0 & 0 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{D}_{11} & \mathbb{D}_{12} & 0 & 0 & 0 \\ \mathbb{D}_{21} & \mathbb{D}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi) \tag{7.63}$$

*satisfies* $I - \mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty$ *[and* $(I - \mathbb{D}^o)^{-1} \in \mathrm{TIC}$*].* □

(This follows from Lemma 7.2.2.)

Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for $\Sigma$ iff the closed-loop system ($\Sigma_I^o$; cf. (7.60)) in Figure 7.11 is well-posed [and stable, i.e., $u, y, z, \xi \in \mathrm{L}^2$ and $x$ and $\widetilde{x}$ are bounded for all $u_L, w, y_L, \xi_L \in \mathrm{L}^2(\mathbf{R}_+; *)$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$]. (We note that exponential stability is equivalent to $x, \widetilde{x} \in \mathrm{L}^2$ (and hence $u, y, z, \xi \in \mathrm{L}^2$) for all $u_L, w, y_L, \xi_L \in \mathrm{L}^2$, $x_0 \in H$ and $\widetilde{x} \in \widetilde{H}$, by Lemma A.4.5 and Lemma 6.1.10(a1).)

As before, we identify a well-posed controller $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ with the controller $\begin{bmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ with internal loop. From (7.58) we observe that $\mathbb{Q} \in \mathrm{TIC}_\infty$ is an admissible [stabilizing] DPF-controller for $\mathbb{D}$ iff $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} \in \mathrm{TIC}_\infty(Z \times Y, U \times W)$ is an admissible [stabilizing] (DF-)controller for $\mathbb{D}$ (obviously, (7.58) is a well-posed DF-controller iff $\mathbb{O}$ is a well-posed DPF-controller; see the end of Definition 7.2.1). This can be compared to Figure 7.8, where $\begin{bmatrix} \widetilde{u} \\ w \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y + y_L \end{bmatrix} + \begin{bmatrix} u_L \\ w \end{bmatrix}$, whereas $\widetilde{u} = \mathbb{Q}(y + y_L) + u_L$ in Figure 7.1; the differences are explained by the facts that we need no $z_L$ and that $w = w_L = \widetilde{w}$, due to lack of feedback in these loops.

Obviously, a (well-posed) map $\mathbb{Q} \in \mathrm{TIC}_\infty$ is admissible iff $(I - \mathbb{D}_{21}\mathbb{Q}) \in \mathcal{G}\mathrm{TIC}_\infty$. The closed loop map $w \mapsto z$ (from the second input to the first output) is given by the standard linear fractional transformation formula

$$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22} = \mathcal{F}_\ell(\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\mathbb{D}^{\mathrm{d}}\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \mathbb{Q}^{\mathrm{d}})^{\mathrm{d}}. \tag{7.64}$$

The full map $u_L, w, y_L \mapsto u, z, y$ is given by

$$\begin{bmatrix} (I - \mathbb{Q}\mathbb{D}_{21})^{-1} & (I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q}\mathbb{D}_{22} & (I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q} \\ \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1} & \mathbb{D}_{12} + \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q}\mathbb{D}_{22} & \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q} \\ \mathbb{D}_{21}(I - \mathbb{Q}\mathbb{D}_{21})^{-1} & \mathbb{D}_{22} + (I - \mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22} & (I - \mathbb{D}_{21}\mathbb{Q})^{-1} \end{bmatrix}. \tag{7.65}$$

Thus, all admissibility results of Sections 7.2 and 7.1 are valid (for DPF) with $\mathbb{D}_{21}$ in place of $\mathbb{D}$, but for the stabilizability, we must add the requirement that the maps to $z$ and the maps from $w$ also become stable.

We usually study only DPF-controllers with coprime internal loop, because the standard stabilizability and detectability assumptions for the $\mathrm{H}^\infty$ 4BP imply that no other controllers stabilize the plant (assuming sufficient regularity or a discrete-time setting; cf. Section 12.5, Lemmas 12.6.6 and 12.5.3 and Theorem 7.3.19), and because the general case is rather complex, as shown in the following proposition:

**Proposition 7.3.4 ($\mathcal{F}_\ell(\mathbb{D},\mathbb{O})$)** *Let $\mathbb{D}$ be as in Definition 7.3.1. Let $\mathbb{O}$ :=(7.58).*

*(a) Then $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$ iff*

$$\mathbb{H} := \begin{bmatrix} I_Z & -\mathbb{D}_{11}\mathbb{O}_{11} & -\mathbb{D}_{11}\mathbb{O}_{12} \\ 0 & I_Y-\mathbb{D}_{21}\mathbb{O}_{11} & -\mathbb{D}_{21}\mathbb{O}_{12} \\ 0 & -\mathbb{O}_{21} & I_\Xi-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Z\times Y\times\Xi); \tag{7.66}$$

*equivalently, iff*

$$\mathbb{R} := \begin{bmatrix} I_U-\mathbb{O}_{11}\mathbb{D}_{21} & -\mathbb{O}_{11}\mathbb{D}_{22} & -\mathbb{O}_{12} \\ 0 & I_W & 0 \\ -\mathbb{O}_{21}\mathbb{D}_{21} & -\mathbb{O}_{21}\mathbb{D}_{22} & I_\Xi-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U\times W\times\Xi). \tag{7.67}$$

*(b) $\mathbb{O}$ is [exponentially] DPF-stabilizing with internal loop for $\mathbb{D}$ iff (7.27) is [exponentially] stable (equivalently, iff (7.28) is [exponentially] stable).*

*(c) $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}$ iff $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) for $\begin{bmatrix} \mathbb{D}_{11} & 0 & \mathbb{D}_{12} \\ \mathbb{D}_{21} & 0 & \mathbb{D}_{22} \\ 0 & I & 0 \end{bmatrix}$.*

*(d) $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}$ iff $\mathbb{O}^{\mathrm{d}}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}_{\mathrm{d}} := \begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$. If this is the case, then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O})^{\mathrm{d}} = \mathcal{F}_\ell(\mathbb{D}_{\mathrm{d}},\mathbb{O}^{\mathrm{d}})$.*

*(e) If $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$, then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) : w \mapsto z$ is given by*

$$\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22} = \mathbb{D}_{12} + \mathbb{D}_{11}(\mathbb{R}^{-1})_{12} \in \mathrm{TIC}_\infty(W,Z). \tag{7.68}$$

*The map $w \mapsto u$ is given by $(\mathbb{R}^{-1})_{12}$.*

If $\mathbb{O} = \begin{bmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{bmatrix}$ for some $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ (i.e., $\mathbb{O}$ is well-posed), then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I-\mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22}$, by (7.65). For DPF-controllers with d.c. internal loop, a third formula for $\mathcal{F}_\ell$ is given in Corollary 7.3.20(c).

**Proof:** (a)&(b) Substitute (7.58) for $\mathbb{O}$ in Proposition 7.2.5(a)&(a') to obtain (a)&(b).

(c) By Definition 7.3.1, $\mathbb{O}$ is DPF-admissible with internal loop for $\mathbb{D}$ iff (7.58) is DF-admissible with internal loop for $\mathbb{D}$. By Proposition 7.2.5(c), this is the case iff (7.58) is DF-admissible for $\begin{bmatrix} \mathbb{D} & 0 \\ 0 & I \end{bmatrix}$; equivalently, iff

$$\begin{bmatrix} 0 & \mathbb{O} \\ 0 & 0 \end{bmatrix} \text{ is DF-admissible for } \begin{bmatrix} \mathbb{D}_{11} & 0 & \mathbb{D}_{12} \\ \mathbb{D}_{21} & 0 & \mathbb{D}_{22} \\ 0 & I & 0 \end{bmatrix} =: \underline{\mathbb{D}}. \tag{7.69}$$

As noted below Definition 7.3.1, this is the case iff $\mathbb{O}$ is DPF-admissible for $\underline{\mathbb{D}}$.

Replace "admissible" by "[exponentially] stabilizing" above to obtain the proof of remaining claims.

(d) We have $\mathbb{R} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $\mathbb{R}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$. Exchange the first and second rows and exchange the first and second columns of $\mathbb{R}^{\mathrm{d}}$ to obtain $\mathbb{H}$ with $\mathbb{D}_{\mathrm{d}}$ in place of $\mathbb{D}$ and $\mathbb{Q}^{\mathrm{d}}$ in place of $\mathbb{Q}$. This proves the admissibility claim, and from this we also observe that $(\mathbb{R}^{-1}_{\mathbb{D},\mathbb{Q}})_{12} = (\mathbb{H}^{-1}_{\mathbb{D}_{\mathrm{d}},\mathbb{Q}^{\mathrm{d}}})^{\mathrm{d}}_{12}$, hence (use (c) twice)

$$\mathcal{F}_\ell(\mathbb{D}_{\mathrm{d}},\mathbb{Q}^{\mathrm{d}})^{\mathrm{d}} = ((\mathbb{D}_{\mathrm{d}})_{12} + (\mathbb{H}^{-1}_{\mathbb{D}_{\mathrm{d}},\mathbb{Q}^{\mathrm{d}}})_{12}(\mathbb{D}_{\mathrm{d}})_{22})^{\mathrm{d}} = \mathbb{D}_{12} + \mathbb{D}_{11}(\mathbb{R}^{-1}_{\mathbb{D},\mathbb{Q}})_{12} = \mathcal{F}_\ell(\mathbb{D},\mathbb{Q}). \tag{7.70}$$

Similarly, one observes from Proposition 7.2.5(a)&(a') that (7.28) is stable iff (7.27) is stable after the substitutions $\mathbb{D} \mapsto \mathbb{D}_{\mathrm{d}}$, $\mathbb{Q} \mapsto \mathbb{Q}^{\mathrm{d}}$ (this requires just a bit more reordering).

(Note that except for (7.70), part (d) is also contained in Lemma 6.7.2(f').)

(e) The symbols of Proposition 7.2.5 are now denoted as follows (cf. Definition 7.3.1): we have $u_L \mapsto \begin{bmatrix} u_L \\ w \end{bmatrix}$, $u \mapsto \begin{bmatrix} u \\ w \end{bmatrix}$ and $y \mapsto \begin{bmatrix} z \\ y \end{bmatrix}$. The map "$u_L \mapsto \begin{bmatrix} y \\ \xi \end{bmatrix}$" given by $\mathbb{H}^{-1}\begin{bmatrix} \mathbb{D} \\ 0 \end{bmatrix}$, by (7.27), hence $w \mapsto z$ is given by

$$(\mathbb{H}^{-1}\begin{bmatrix} \mathbb{D} \\ 0 \end{bmatrix})_{12} = (\mathbb{H}^{-1})_{11}\mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22} = \mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22}, \tag{7.71}$$

since now $(\mathbb{H}^{-1})_{11} = I$, by Lemma A.1.1(b1)&(b2). Analogously, $w \mapsto z$ is given by $\mathbb{D}_{11}(\mathbb{R}^{-1})_{12} + \mathbb{D}_{12}(\mathbb{R}^{-1})_{22} = \mathbb{D}_{11}(\mathbb{R}^{-1})_{12} + \mathbb{D}_{12}$, by (7.28). Obviously, $w \mapsto u$ is given by $(\mathbb{R}^{-1})_{12}$, by (7.28). □

For ease of reference, we collect into a lemma some remarks made above (more or less explicitly):

**Lemma 7.3.5 ($\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ $\Rightarrow$ $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$)** *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}(U \times W, Z \times Y)$ *and* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$. *Let* $\Sigma$ *and* $\widetilde{\Sigma}$ *be realizations of* $\mathbb{D}$ *and* $\mathbb{Q}$*, respectively. Then the following are equivalent:*

*(i)* $\mathbb{Q}$ *is an admissible DPF-controller with internal loop for* $\mathbb{D}$*;*

*(ii)* $\mathbb{Q}$ *is an admissible DF-controller with internal loop for* $\mathbb{D}_{21}$*;*

*(iii)* $\widetilde{\Sigma}$ *is an admissible DPF-controller with internal loop for* $\Sigma$*;*

*(iv)* $\widetilde{\Sigma}$ *is an admissible DF-controller with internal loop for* $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$*;*

*(v)* $I - \mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi)$.

*Moreover, if* $\mathbb{Q}$ *DPF-stabilizes* $\mathbb{D}$*, then* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}_{21}$*; if* $\widetilde{\Sigma}$ *DPF-stabilizes* $\Sigma$*, then* $\widetilde{\Sigma}$ *DF-stabilizes* $\Sigma_{21}$ *(all prefices apply, because the system "*$\Sigma^o_I$*" for* $\Sigma_{21}$ *and* $\widetilde{\Sigma}$ *(cf. (7.21)) is a part of the system "*$\Sigma^o_I$*" for* $\Sigma$ *and (7.59)).*

The converse to the last claim is not true in general (take, e.g., $\mathbb{Q} = 0 = \mathbb{D}_{21}$, $\mathbb{D}$ unstable; cf. also Example 7.3.7), but it is true when, e.g., $\Sigma_{21}$ is optimizable and estimatable; see Lemma 7.3.6 and Theorem 7.3.19.

**Proof:** (Naturally, the lemma still remains true if we throughout the lemma remove the phrases "with internal loop", since $\mathbb{Q}$ is a well-posed DPF-controller for $\Sigma$ iff $\mathbb{Q}$ is a well-posed DF-controller for $\Sigma_{21}$ (iff $\mathbb{Q} = \begin{bmatrix} \mathbb{Q}_{11} & 0 \\ 0 & 0 \end{bmatrix}$). Note also that the "resp." part is not defined for $\widetilde{\Sigma}$ in place of $\mathbb{Q}$.)

Let $\widetilde{\mathbb{H}} \in \mathrm{TIC}_\infty(Y \times \Xi)$ be the map $\mathbb{H}$ for $\mathbb{Q}$ and $\mathbb{D}_{21}$ from Proposition 7.2.5(a).

1° *Admissibility*: We observe from Proposition 7.3.4(a) (and Lemma A.1.1(b)), that $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $\widetilde{\mathbb{H}} \in \mathcal{G}\mathrm{TIC}_\infty$, and that

$$\mathbb{H}^{-1} = \begin{bmatrix} I_Z & \begin{bmatrix} * & * \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \widetilde{\mathbb{H}}^{-1} \end{bmatrix} \in \mathrm{TIC}_\infty(Z \times (Y \times \Xi)) \tag{7.72}$$

(when $\widetilde{\mathbb{H}} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$). Therefore, (i) and (ii) are equivalent. From Lemma 7.2.2 (and Definition 7.3.1) we obtain "(ii)⇔(v)", "(ii)⇔(iv)" and "(iii)⇔(iv)" (since (7.58) is the I/O map of (7.59).

2° *We shall show that $\widetilde{\Sigma}_I^o$ is obtained by removing the fourth and fifth rows and columns (those corresponding to $W$ and $Z$) from $\Sigma_I^o$:* (Here $\widetilde{\Sigma}_I^o$ is the system "$\Sigma_I^o$" for $\Sigma_{21}$ and $\widetilde{\Sigma}$ (cf. (7.21)), and $\Sigma_I^o$ is the one of Definition 7.3.1, i.e., the closed-loop system of (7.60). Obviously, the above claim holds for $\widetilde{\Sigma}^o$ and $\Sigma^o$ in place of $\widetilde{\Sigma}_I^o$ and $\Sigma_I^o$.)

Assume that $\mathbb{O}$ is admissible for $\mathbb{D}_{21}$. Let $\widetilde{\mathbb{D}}^o \in \mathrm{TIC}_\infty(U \times Y \times \Xi)$ be the map (7.20) (for $\mathbb{D}_{21}$ and $\mathbb{O}$), and define $\mathbb{D}^o$ by (7.63). Set $\widetilde{\mathbb{T}} := (I - \widetilde{\mathbb{D}}^o)^{-1}$. From (7.27) we observe that

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} \widetilde{\mathbb{T}}_{11} & * & 0 & \widetilde{\mathbb{T}}_{12} & \widetilde{\mathbb{T}}_{13} \\ 0 & I & 0 & 0 & 0 \\ * & * & I & * & * \\ \widetilde{\mathbb{T}}_{21} & * & 0 & \widetilde{\mathbb{T}}_{22} & \widetilde{\mathbb{T}}_{23} \\ \widetilde{\mathbb{T}}_{31} & * & 0 & \widetilde{\mathbb{T}}_{32} & \widetilde{\mathbb{T}}_{33} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi). \tag{7.73}$$

(This proves 2° for $\mathbb{D}_I^o := (I - \mathbb{D}^o)^{-1} - I$.) Apply then (6.125) to observe that

$$\mathbb{C}_I^o := (I - \mathbb{D}^o)^{-1}\mathbb{C}^o = \begin{bmatrix} \widetilde{\mathbb{T}}_{12}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{11}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{13}\widetilde{\mathbb{C}}_2 \\ 0 & 0 \\ * & * \\ \widetilde{\mathbb{T}}_{22}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{21}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{23}\widetilde{\mathbb{C}}_2 \\ \widetilde{\mathbb{T}}_{32}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{31}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{33}\widetilde{\mathbb{C}}_2 \end{bmatrix}. \tag{7.74}$$

(Remove the second and third rows to obtain "$\widetilde{\mathbb{C}}_I^o$".) The proof for $\mathbb{B}_I^o$ is analogous. Finally, from (6.125) and (7.4) (for $\Sigma$ and $\widetilde{\Sigma}'$, so that $\mathbb{B}^o = \left[\begin{smallmatrix} \mathbb{B}_1 & \mathbb{B}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{smallmatrix}\right]$) we observe that

$$\mathbb{A}_I^o = \mathbb{A}^o + \mathbb{B}^o\tau\mathbb{C}_I^o = \begin{bmatrix} \mathbb{A} & 0 \\ 0 & \widetilde{\mathbb{A}} \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1 & 0 & 0 \\ 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{bmatrix} \tau \begin{bmatrix} (\mathbb{C}_I^o)_{11} & (\mathbb{C}_I^o)_{12} \\ (\mathbb{C}_I^o)_{41} & (\mathbb{C}_I^o)_{42} \\ (\mathbb{C}_I^o)_{51} & (\mathbb{C}_I^o)_{52} \end{bmatrix}, \tag{7.75}$$

which equals $\widetilde{\mathbb{A}}_I^o := \mathbb{A}^o + \begin{bmatrix} \mathbb{B}_1 & 0 & 0 \\ 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{bmatrix} \tau\widetilde{\mathbb{C}}_I^o$, the semigroup of $\widetilde{\Sigma}_I^o$.

3° *Stabilization:* We observe from 2° that $\widetilde{\Sigma}_I^o \in \mathrm{WPLS}(U \times Y \times \Xi, H \times \widetilde{H}, U \times Y \times \Xi)$ is a part of $\Sigma_I^o \in \mathrm{WPLS}(U \times W \times Z \times Y \times \Xi, H \times \widetilde{H}, U \times W \times Z \times Y \times \Xi)$.

Indeed, $\widetilde{\mathbb{D}}_I^o = (I - \widetilde{\mathbb{D}}^o)^{-1} - I = \widetilde{\mathbb{T}} - I$ is a part of $\mathbb{D}_I^o$, the semigroup $\mathbb{A}_I^o$ is the same for both systems, and $\widetilde{\mathbb{C}}_I^o$ and $\widetilde{\mathbb{B}}_I^o$ are parts of $\mathbb{C}_I^o$ and $\mathbb{B}_I^o$, respectively, as noted above.

Therefore, if $\widetilde{\Sigma}$ is DPF-stabilizing with internal loop for $\Sigma$, i.e., $\Sigma_I^o$ is stable, then also $\widetilde{\Sigma}_I^o$ is stable, since it is a part of $\Sigma_I^o$, i.e., then $\widetilde{\Sigma}$ is DPF-stabilizing for $\Sigma_{21}$. Analogously, if $\mathbb{O}$ is DPF-stabilizing with internal loop for $\mathbb{D}$, i.e., $\mathbb{D}_I^o$ is stable, then so is $\widetilde{\mathbb{D}}_I^o$. For same reasons, any prefices (e.g., "exponentially", "ω-"; for $\Sigma$ also "strongly", "internally", "SOS-" etc.) apply. □

It is not exactly the same thing to DPF-stabilize $\Sigma$ and DF-stabilize $\Sigma_{21}$, but pretty close:

**Lemma 7.3.6 ($\Sigma \leftrightarrow \Sigma_{21}$)**

*(a)* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$ *is an exponentially stabilizing DPF-controller with internal loop for* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *iff* $\widetilde{\Sigma}$ *is an exponentially stabilizing DF-controller with internal loop for* $\Sigma_{21}$.

*(b1) If* $\Sigma_{21}$ *is exponentially jointly stabilizable and detectable, then the following are equivalent:*

*(i)* $\mathbb{O}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially with internal loop;*

*(ii)* $\mathbb{O}$ *DF-stabilizes* $\mathbb{D}_{21}$ *exponentially with internal loop;*

*(iii)* $\mathbb{O}$ *has a realization that DPF-stabilizes* $\Sigma$ *exponentially with internal loop.*

*(b2) If* $\Sigma_{21}$ *and* $\Sigma$ *are [strongly] jointly stabilizable and detectable, then the following are equivalent:*

*(i)* $\mathbb{O}$ *DPF-stabilizes* $\mathbb{D}$ *with internal loop;*

*(ii)* $\mathbb{O}$ *DF-stabilizes* $\mathbb{D}_{21}$ *with internal loop;*

*(iii)* $\mathbb{O}$ *has a realization that DPF-stabilizes* $\Sigma$ *[strongly] with internal loop.*

*(c) If* $\Sigma_{21}$ *is optimizable and estimatable, then (b1)(i)&(ii) are equivalent, and so are (b2)(i)&(ii).*

**Proof:** (a) The two closed-loop systems have same semigroup $\mathbb{A}_I^o$, as noted in the proof of Lemma 7.3.5, hence either is exponentially stable iff $\mathbb{A}_I^o$ is exponentially stable, by Lemma 6.1.10(a1).

(b1)&(b2) Implication "(iii)⇒(i)" is trivial (since $\mathbb{D}_I^o$ is a part of $\Sigma_I^o$), and "(i)⇒(ii)" follows from Lemma 7.3.5.

To complete the equivalence, we assume that (ii) holds and that $\Sigma_{21}$ and $\Sigma$ are [[exponentially] strongly] jointly stabilizable and detectable [[the assumption on $\Sigma$ is unnecessary]].

By Theorem 6.6.28 [[(shifted; note that we tacitly apply shifting several times below too)]], $\mathbb{D}_{21}$ has a [[exponential]] d.c.f.; therefore, so does $\overline{\mathbb{D}} := \begin{bmatrix} \mathbb{D}_{21} & 0 \\ 0 & I \end{bmatrix}$.

By Proposition 7.2.5(c), $\mathbb{O}$ DF-stabilizes $\overline{\mathbb{D}}$ [[exponentially]]. By Proposition 7.1.6(d), $\mathbb{O}$ has a [[exponential]] d.c.f. By Theorem 6.6.28, $\mathbb{O}$ has an [[exponentially]] strongly jointly stabilizable and detectable realization.

By Theorem 7.3.11(b)(1.)[[(c1)]], this realization stabilizes $\Sigma$ [[exponentially] strongly] with internal loop. (Here we needed the assumption on $\Sigma$, or at least the assumption that $\Sigma$ is, e.g., q.r.c.-stabilizable [since $\mathbb{A}_I^o$ is the same for $\Sigma$ and $\Sigma_{21}$, "strongly" is not needed here] [[since $\Sigma_{21}$ is optimizable and estimatable, so is $\Sigma$]].)

*A remark for (b1):* It is not sufficient for (b1)(i)–(iii) that $\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ with internal loop: by Corollary 7.3.20, this holds iff $\mathbb{O}$ is equivalent to $\mathbb{Q}$ of Corollary 7.3.20(ii) for some $\mathbb{U} \in \mathrm{TIC}(U)$, whereas exponential stabilization requires that $\mathbb{U} \in \mathrm{TIC}_{\exp}(U)$.

(c) 1° *The corresponding discrete-time claim holds:* By Lemma 13.3.17(b), $\Sigma_{21}$ is exponentially jointly stabilizable and detectable, hence so is $\Sigma$. Therefore, (b1)(i)–(iii) are equivalent, and so are (b2)(i)–(iii).

2° *The original claim holds:* Use discretization (see Theorem 13.4.4(e1)). □

In Lemma 7.3.6(b2) (compare to (b1)), the condition on $\Sigma$ is not superfluous:

**Example 7.3.7 ($\Sigma_{21}$ and $\mathbb{D}$ strongly stable but $\Sigma$ not DPF-stabilizable)** Let $\mathbb{A}$ be as in Example 6.1.14(a), so that $\mathbb{A}$ and $\Sigma_{21}$ are strongly stable but $\mathbb{B}_2$, $\mathbb{C}_1$ and $\mathbb{D}_{12}$ are unstable, where

$$\Sigma := \left(\begin{array}{c|cc} A & 0 & I \\ \hline I & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \in \mathrm{WPLS}. \tag{7.76}$$

Moreover, no DPF-controller [with internal loop] has any effect on $\Sigma$; in particular, $\Sigma$ is not DPF-stabilizable, although $\widetilde{\Sigma} := \left[\begin{smallmatrix} 0 & | & 0 \\ 0 & | & 0 \end{smallmatrix}\right]$ DF-stabilizes $\Sigma_{21}$ exponentially.

Note that $\Sigma$ is exponentially jointly stabilizable and detectable and has bounded "$B$ and $C$", but $\Sigma_{21}$ is only strongly jointly stabilizable and detectable. ◁

(All this is straightforward (use Example 6.1.14(a), Proposition 7.3.4(a) and Lemma 6.6.25.))

**Lemma 7.3.8 (Equivalent DPF-controllers)** *Let* $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W, Z \times Y)$, $\mathbb{O} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$, *and* $\mathbb{O}' \in \mathrm{TIC}_\infty(Y \times \Xi', U \times \Xi')$. *Let* $\Sigma$, $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *be arbitrary realizations of* $\mathbb{D}$, $\mathbb{O}$ *and* $\mathbb{O}'$ *respectively. Then the following are equivalent:*

*(i)* $\mathbb{O}$ *and* $\mathbb{O}'$ *are equivalent DPF-controllers with internal loop for* $\mathbb{D}$*;*

*(ii)* $\mathbb{O}$ *and* $\mathbb{O}'$ *are equivalent DF-controllers with internal loop for* $\mathbb{D}_{21}$*;*

*(iii)* $\mathbb{O}_{\mathrm{DF}}$ *and* $\mathbb{O}'_{\mathrm{DF}}$ *are equivalent DF-controllers with internal loop for* $\mathbb{D}$*;*

*(iv)* $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *are equivalent DPF-controllers with internal loop for* $\Sigma$*;*

*(v)* $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *are equivalent DF-controllers with internal loop for* $\Sigma_{21}$*;*

*(vi)* $\widetilde{\Sigma}_{\rm DF}$ *and* $\widetilde{\Sigma}'_{\rm DF}$ *are equivalent DF-controllers with internal loop for* $\Sigma$*;*

*(vii)* $\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \mathbb{D}_I^o \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}^{\rm T} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}$ *is unaffected when* $\mathbb{O}$ *is replaced by* $\mathbb{O}'$ *(equivalently,* $\widetilde{\Sigma}$ *is replaced by* $\widetilde{\Sigma}'$*);*

*(viii)* $\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \mathbb{D}_I^o \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}^{\rm T} : \begin{bmatrix} u_L \\ w \\ z \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ w \\ z \\ y \end{bmatrix}$ *is unaffected when* $\mathbb{O}$ *is replaced by* $\mathbb{O}'$*;*

*(ix) the closed-loop maps* $\Sigma_I^o, \Sigma_I^{o\prime} : x_0, u_L, w, (z,) y_L \mapsto x, u, (w,), y, z$ *are unaffected when* $\widetilde{\Sigma}$ *is replaced by* $\widetilde{\Sigma}'$*;*

In particular, two admissible DPF-controllers with coprime internal loop are equal iff they are equivalent for $\mathbb{D}$, equivalently, for $\mathbb{D}_{21}$, by Lemma 7.2.12(c).

Recall that any well-posed map is a map with internal loop, and that any well-posed controller having a (right, left or doubly) coprime factorization is a controller with a (right, left or doubly, respectively) coprime internal loop.

The equivalence between (iv) and (v) was expected: if $\Sigma_{21}$ does not see any difference between $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$, why would the rest of $\Sigma$ see any; the rest of the equivalence follows from this.

Condition (ix) says that the two closed-loop maps are equal except possibly for second and seventh rows and columns (those corresponding to $\widetilde{H}$ and $\Xi$ (or $\widetilde{H}'$ and $\Xi'$)), i.e., only the maps concerning $\widetilde{x}$, $\widetilde{x}_0$, $\xi$ and $\xi_L$ may differ for equivalent controllers for $\Sigma$; thus, there is no difference from the part of $\widetilde{\Sigma}$ visible for $\Sigma$.

Consequently, for $\widetilde{x}_0 = 0$ and $\xi_L = 0$ (or $\widetilde{x}'_0 = 0$ and $\xi'_L = 0$), the signals $x, u, y, z$ in Figure 7.11 are unaffected when $\widetilde{\Sigma}$ is replaced by an equivalent controller (as long as $x_0, u_L, w, y_L$ are fixed).

**Proof of Lemma 7.3.8:** (See (7.63) for $\mathbb{D}^o$ and note that $\mathbb{D}_I^o = \mathbb{D}^o(I - \mathbb{D}^o)^{-1} = (I - \mathbb{D}^o)^{-1} - I \in \mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi)$ for any admissible controller with internal loop for $\mathbb{D}$.)

1° *"(i)⇔(iv)⇔(vii)", "(ii)⇔(v)" and "(iii)⇔(vi)⇔(viii)":* These follow from Definitions 7.3.1, 7.3.1 and 7.3.1, respectively.

2° *"(vii)⇒(iii)":* Assume (vii), i.e., that the maps $\widetilde{\mathbb{T}}_{11}, \widetilde{\mathbb{T}}_{12}, \widetilde{\mathbb{T}}_{21}, \widetilde{\mathbb{T}}_{22}$ in (7.73) are equal for $\mathbb{O}$ and $\mathbb{O}'$. Then also the maps

$$((I - \mathbb{D}^o)^{-1})_{34} = ((I - \mathbb{D}^o)^{-1} - I)_{34} = (\mathbb{D}^o(I - \mathbb{D}^o)^{-1})_{34} = \mathbb{D}_{11}\widetilde{\mathbb{T}}_{12} \tag{7.77}$$

are equal for $\mathbb{O}$ and $\mathbb{O}'$. We conclude from (7.73) that the maps $((I - \mathbb{D}^o)^{-1})_{ij}$ are equal for $\mathbb{O}$ and $\mathbb{O}'$ for $i = 1, 2, 3, 4,\ j = 3, 4$. By Proposition 7.2.5(e) (cf. (7.27)), we obtain (iii).

3° *"(iii)⇒(ix)":* This follows from Lemma 7.2.2.

*Remarks on (ix):* Here, as elsewhere, $\Sigma_I^o$ and $\Sigma_I^{o\prime}$ are the combined closed-loop systems corresponding to $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$, respectively; cf. the proof of Lemma 7.3.5. By (ix), they become equal once we remove their second and last rows and columns.

In (xi), we must include "$\tau$" after $\mathbb{B}_I^o$ and $\mathbb{B}_I^{o\prime}$ (in $\Sigma_I^o$ and $\Sigma_I^{o\prime}$), cf. the remark below Definition 6.1.5.

We have $z$ and $w$ in parenthesis in (ix), because $z$ does not affect any other signal and $w$ is not affected by any signal.)

4° *"(ix)⇒(viii)⇒(vii)":* This is trivial. Thus, only (ii) and (v) are missing from the equivalence; they are adopted in 5°–6° below.

5° *"(ii)⇒(vii)":* Assume (ii). Then $\widetilde{\mathbb{T}} = (I - \widetilde{\mathbb{D}}^o)^{-1}$ is unaffected by the replacement $\mathbb{O} \mapsto \mathbb{O}'$. By (7.73), this means that (vii) holds.

6° *"(i)⇒(ii)":* Assume (i). With the notation of the proof of Lemma 7.3.5, we have

$$\begin{bmatrix}\mathbb{O}_{\mathrm{DF}11} & \mathbb{O}_{\mathrm{DF}12}\end{bmatrix}\mathbb{H}^{-1} = \begin{bmatrix}0 & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1} \\ 0 & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} \in \mathrm{TIC}_\infty(Z \times Y \times \Xi, U \times W). \tag{7.78}$$

By (i) and Proposition 7.2.5(e), the map $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1})_1$ is unaffected by the replacement $\mathbb{O} \mapsto \mathbb{O}'$, hence $(\begin{bmatrix}\mathbb{O}_{\mathrm{DF}11} & \mathbb{O}_{\mathrm{DF}12}\end{bmatrix}\mathbb{H}^{-1})_1$ is unaffected by $\mathbb{O} \mapsto \mathbb{O}'$ (equivalently, by $\mathbb{O}_{\mathrm{DF}} \mapsto \mathbb{O}'_{\mathrm{DF}}$).

From (7.72) we observe that $(\widetilde{\mathbb{H}}^{-1})_{11} \in \mathrm{TIC}_\infty(Y)$ is contained in $(\mathbb{H}^{-1})_{11} \in \mathrm{TIC}_\infty(Z \times Y)$. We conclude from Proposition 7.2.5(e) that (ii) holds. □

**Lemma 7.3.9 (Well-posed $\mathbb{Q} = \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$)** *Let* $\mathbb{O} = \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22}\end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *be an admissible DPF-controller with internal loop for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$.

*Then* $\mathbb{O}$ *is equivalent to a well-posed DPF-controller for* $\mathbb{D}$ *iff* $I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$*; if this is the case, then that well-posed DPF-controller is given by* $\mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ *(in particular, it is unique).* □

(This follows from Lemma 7.2.7 and Lemma 7.3.8(i)&(ii), because a map $\mathbb{O}' \in \mathrm{TIC}_\infty(Y \times \Xi', U \times \Xi')$ is a well-posed DPF-controller for $\mathbb{D}$ iff $\mathbb{O}'$ is well-posed DF-controller for $\mathbb{D}_{21}$, i.e., iff $\mathbb{O}' = \begin{bmatrix} * & 0 \\ 0 & 0\end{bmatrix}$.)

**Lemma 7.3.10** *Let* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *or* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *be a map with coprime internal loop. Then so is*

$$\begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & \mathbb{Y} \\ 0 & 0\end{bmatrix}\begin{bmatrix}I & 0 \\ 0 & \mathbb{X}\end{bmatrix}^{-1} \quad or \quad \begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & 0 \\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix}0 & \widetilde{\mathbb{Y}} \\ 0 & 0\end{bmatrix}, \tag{7.79}$$

*respectively. The following are equivalent:*

- *(i)* $\mathbb{Q}$ *is an admissible [stabilizing] DPF-controller with coprime internal loop for* $\mathbb{D}$*;*
- *(ii)* $\begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix}$ *is an admissible [stabilizing] DF-controller with coprime internal loop for* $\mathbb{D}$*;*
- *(iii)* $\begin{bmatrix}0 & \mathbb{Y} \\ I & I - \mathbb{X}\end{bmatrix}$ *is an admissible [stabilizing] DPF-controller with internal loop for* $\mathbb{D}$*;*
- *(iv) (7.81) is an admissible [stabilizing] DF-controller with internal loop for* $\mathbb{D}$*;*
- *(v) (7.82) is an admissible [stabilizing] DF-controller with internal loop for* $\mathbb{D}$*.*

(Recall Definition 7.2.11 of maps with coprime internal loop.) From "(i)⇔(iii)" we conclude that one need not first extend $\mathbb{Q}$ to $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ and then take a representative (such as (7.81)); one can also take first a representative $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ (or $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$, respectively) of $\mathbb{Q}$, and then extend it as in (7.58): $\mathbb{Q}$ is an admissible [stabilizing] DPF-controller with coprime internal loop for $\mathbb{D}$ iff some (hence any) of its representatives is an admissible [stabilizing] DPF-controller with internal loop for $\mathbb{D}$.

**Proof:** We treat the r.c. case; the l.c. and d.c. cases are analogous.

Suppose that $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ is a map with r.c. internal loop (i.e., $\mathbb{Y}, \mathbb{X} \in \text{TIC}$ are r.c.).

1° *(7.79) is a map with coprime internal loop:* This means that $\begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix}$ are r.c. Indeed, $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I$ implies that

$$\begin{bmatrix} I & 0 \\ 0 & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \widetilde{\mathbb{N}} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} = I. \tag{7.80}$$

2° *The equivalence of (i)–(v):* By Definition 7.2.11, the (canonical) representative of map of form $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix}^{-1}$ with r.c. internal loop is given by

$$\begin{bmatrix} 0 & 0 & 0 & \mathbb{Y} \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & I-\mathbb{X} \end{bmatrix} \in \text{TIC}(Z \times Y \times (Z \times Y), U \times W \times (Z \times Y)) \tag{7.81}$$

(note that here "$\Xi$"$= Z \times Y$, whereas below "$\Xi$"$= Y$). We conclude that (ii) is equivalent to (iv), by Definition 7.2.11.

The DF-controller (7.58) corresponding to the canonical representative $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ of the map $\mathbb{Q}$ with internal loop is

$$\begin{bmatrix} 0 & 0 & \mathbb{Y} \\ 0 & 0 & 0 \\ 0 & I & I-\mathbb{X} \end{bmatrix} \in \text{TIC}(Z \times Y \times Y, U \times W \times Y). \tag{7.82}$$

By Definition 7.3.1, (iii) stands for (v), and (i) stands for (ii). Thus, we can complete the equivalence by showing (iv) equivalent to (v).

Let $\widetilde{\mathbb{D}}^o$ be the map "$\mathbb{D}^o$" of (7.63) that results from applying the DPF-controller $\mathbb{O}$ to $\mathbb{D}$ (equivalently, the DF-controller (7.82) to $\mathbb{D}$), and let $\mathbb{D}^o$ be the map "$\mathbb{D}^o$" of (7.20) that results from applying the DF-controller (7.81) to $\mathbb{D}$.

Then $\mathbb{D}^o = \begin{bmatrix} \mathbb{D}^o & 0 \\ 0 & I \end{bmatrix}$ modulo certain permutation of rows and and the same permutation of columns. Therefore, $I - \mathbb{D}^o \in \mathcal{G}\text{TIC}_\infty$ iff $I - \widetilde{\mathbb{D}}^o \in \mathcal{G}\text{TIC}_\infty$, and $(I - \mathbb{D}^o)^{-1} \in \text{TIC}$ iff $(I - \widetilde{\mathbb{D}}^o)^{-1} \in \text{TIC}$. Thus, the admissibility and stabilizability of (7.81) for $\mathbb{D}$ is equivalent to that of (7.82).

(An intuitive proof would go as follows: (7.82) is obtained by deleting the $Z$ part (not $Y$ part) of $\xi$ (7.81), and this $Z$ part is obviously well-posed and stable, and does not affect any other signals.) □

Trivially, $\widetilde{\Sigma}$ I/O-DPF-stabilizes $\Sigma$ iff $\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ (i.e., iff $\mathbb{D}_I^o$ becomes stable). Under standard assumptions, this is also equivalent to the stronger condition that $\widetilde{\Sigma}$ DPF-stabilizes $\Sigma$:

**Theorem 7.3.11 ($\widetilde{\Sigma}$ DPF-stabilizes $\Sigma \Leftrightarrow \mathbb{O}$ DPF-stabilizes $\mathbb{D}$)** *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *and* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$.

*(a) Suppose that $\Sigma$ and $\widetilde{\Sigma}$ are SOS-stabilizable. Then $\widetilde{\Sigma}$ SOS-DPF-stabilizes $\Sigma$ with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes $\Sigma$ with internal loop.*

*(b)* **([Strong] stability)** *Suppose that any of the following conditions holds:*

*(1.) both $\Sigma$ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.r.c.-stabilizable;*
*(2.) both $\Sigma$ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.l.c.-detectable;*
*(3.) both $\Sigma$ and $\widetilde{\Sigma}$ are SOS-stabilizable and [[exponentially] strongly] detectable;*
*(4.) both $\Sigma$ and $\widetilde{\Sigma}$ are detectable and [exponentially] stabilizable.*

*Then $\widetilde{\Sigma}$ [[exponentially] strongly] DPF-stabilizes $\Sigma$ with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes $\Sigma$ with internal loop.*

*(c1)* **(Exponential stability)** *The system $\widetilde{\Sigma}$ DPF-stabilizes $\Sigma$ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes $\Sigma$ with internal loop and $\Sigma$ and $\widetilde{\Sigma}$ are optimizable and estimatable.*

*(c2) Suppose that any of the following conditions holds:*

*(1.) both $\Sigma$ and $\widetilde{\Sigma}$ are optimizable and estimatable;*
*(2.) both $\Sigma$ and $\widetilde{\Sigma}$ are optimizable and input-detectable;*
*(3.) both $\Sigma$ and $\widetilde{\Sigma}$ are estimatable and output-stabilizable;*
*(4.) both $\Sigma$ and $\widetilde{\Sigma}$ are optimizable and q.r.c.-stabilizable;*
*(5.) both $\Sigma$ and $\widetilde{\Sigma}$ are estimatable and q.l.c.-detectable.*

*Then $\widetilde{\Sigma}$ DPF-stabilizes $\Sigma$ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes $\Sigma$ with internal loop.*

*(d)* **(Well-posed controllers)** *Suppose that, instead,* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y, \widetilde{H}, U)$. *Then (a)–(c2) hold if we delete the words "with internal loop" everywhere.*

Thus, under corresponding assumptions above, all maps between the signals in Figure 7.11 are (SOS-/strongly/exponentially) stable iff the maps from $u_L, w, y_L, z_L, \xi_L$ to $u, w, y, z, \xi$ are stable.

**Proof:** This follows from Theorem 7.2.3 (and Definition 7.3.1), because (in the well-posed case (d); the case with internal loop is analogous and left to the reader) if $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right]$ is a realization of $\mathbb{Q}$, then

$$\widetilde{\Sigma}_{\mathrm{DPF}} := \left[\begin{array}{c|cc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{array}\right] \tag{7.83}$$

is a realization of $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ (having the stabilizability and detectability properties of $\widetilde{\Sigma}$, because it is a parallel connection (see Lemma 6.7.18) of $\widetilde{\Sigma}$ and 0). □

We can now almost state that exponential DPF-stabilizability is equivalent to the optimizability and estimatability of $\Sigma_{21}$:

**Theorem 7.3.12 (Exp. DPF-stabilizable ⇔ opt. & est.)** *Let* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$.

*(a) If $\Sigma$ is exponentially DPF-stabilizable with internal loop, then $\Sigma$ and $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ are optimizable and estimatable.*

*(b1) Conversely, if $\Sigma_{21}$ is exponentially jointly stabilizable and detectable, then [$\Sigma$ and] $\Sigma_{21}$ are exponentially D[P]F-stabilizable with internal loop.*

*(b2) If $\Sigma_{21}$ and $\Sigma$ are [strongly] jointly stabilizable and detectable, then $\Sigma$ is [strongly] DPF-stabilizable with internal loop and $\Sigma_{21}$ is [strongly] DF-stabilizable with internal loop.*

*(c) Assume that $\mathbb{A}Bu_0, \mathbb{A}^*C^*y_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; H)$ for all $u_0 \in U$ and $y_0 \in Y$, and that $\mathbb{D}_{21}$ is ULR. Then the following are equivalent:*

*(i) $\Sigma$ is exponentially DPF-stabilizable;*
*(ii) $\Sigma$ is exponentially DPF-stabilizable with internal loop;*
*(iii) $\Sigma_{21}$ is exponentially DF-stabilizable;*
*(iv) $\Sigma_{21}$ is exponentially DF-stabilizable with internal loop;*
*(v) $(A, B_1)$ is optimizable and $(A, C_2)$ is estimatable (equivalently, $\Sigma_{21}$ is optimizable and estimatable);*
*(vi) $\Sigma_{21}$ is exponentially jointly stabilizable and detectable;*
*(vii) $\Sigma$ and $\Sigma_{21}$ are exponentially jointly stabilizable and detectable by some bounded $K$ and $H$.*

*Moreover, if (vii) holds, then (d1) applies with those $K$ and $H$ (hence (6.169) and (7.84) become ULR).*

*(d1) If $K$ and $H$ are exponentially jointly stabilizing for $\Sigma_{21}$ and s.t. "(6.169)" (i.e., $\left[\begin{array}{c|c} A & H \ B_1 \\ \hline C_2 & \\ K & \end{array}\right]$) is SR and "$I - \mathbb{G}_L$"$\in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if "(6.169)" for $\Sigma_{21}$ is ULR), then*

$$\left(\begin{array}{c|c} A + BK_s + HC_s + HD_{21}K_s & -H \\ \hline K & 0 \end{array}\right) \in \mathrm{WPLS}(Y, H, U) \tag{7.84}$$

*is an exponentially DPF-stabilizing controller for $\Sigma$. Moreover, (7.22) is SR and exponentially jointly stabilizable and detectable.*

*(d2) If $K$ and $H$ are [strongly] jointly stabilizing for $\Sigma_{21}$ and s.t. "(6.169)" (i.e., $\left[\begin{array}{c|c} A & H \ B_1 \\ \hline C_2 & \\ K & \end{array}\right]$) is SR and "$I - \mathbb{G}_L$"$\in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if "(6.169)" for $\Sigma_{21}$ is ULR), and $\Sigma$ is jointly stabilizable and detectable, then (7.84) is a [strongly] DPF-stabilizing controller for $\Sigma$. Moreover, (7.22) is SR and [strongly] jointly stabilizable and detectable.*

(e) *Assume that $\Sigma$ is exponentially DPF-stabilizable with internal loop. Then any map DPF-stabilizes $\mathbb{D}$ [exponentially] with internal loop iff it DF-stabilizes $\mathbb{D}_{21}$ [exponentially] with internal loop.*

Obviously, the assumptions of (c) hold if $B$ and $C$ are bounded (or Hypothesis 9.5.1 holds), hence always in discrete time.

Part (d1) is a generalization of a classical result (see Lemma 12.1 of [ZDG] or Lemma A.4.2 of [GL]).

Also claim (e) is a generalization of a classical result (see [Francis], p. 35; in fact, Francis only assumes that $\mathbb{D}$ is exponentially DPF-stabilizable (since a rational $\mathrm{H}^\infty$ function is $\mathrm{H}^\infty_{-\exp}$ for some $\varepsilon > 0$), but, by Lemma 7.1.4 and Theorem 6.6.28, this implies that $\mathbb{D}$ has an exponentially jointly stabilizable and detectable realization (assuming that $\mathbb{D}$ is rational, we could also choose any minimal realization), so that (e) applies).

**Proof:** (a) By Lemma 7.3.5, $\Sigma_{21}$ is exponentially DF-stabilizable [with internal loop], hence $\Sigma_{21}$ is optimizable and estimatable, by Theorem 7.2.3(c1). Therefore, also $\Sigma$ is optimizable and estimatable, by Lemma6.7.4.

(b1)&(b2) This follows from Theorem 7.2.4(b) and Lemma 7.3.6(b1)&(b2) (moreover, from the proofs we observe that (7.23) will do for $\Sigma$ too).

(Note from Definition 7.3.1 that if $\Sigma$ is DPF-stabilizable with internal loop, then it is DF-stabilizable with internal loop, by Definition 7.3.1.)

(c) This follows from Theorem 7.2.4(c) and Lemma 7.3.6(a).

(d1)&(d2) (The assumptions on (6.169) and $\mathbb{G}_L$ refer to those corresponding to $\Sigma_{21}$ in place of $\Sigma$ in Definition 6.6.21. Note that it suffices that $K$ and $H$ are ULR and exponentially jointly stabilizing for $\Sigma_{21}$ (and then (7.84) becomes ULR).)

Make the assumptions of (d1) [[(d2)]]. By Theorem 7.2.4(d), (7.84) is SR, [[exponentially] strongly] jointly stabilizable and detectable, and a [[exponentially] strongly] DF-stabilizing controller for $\Sigma_{21}$, hence it I/O-DPF-stabilizes $\Sigma$, by Lemma 7.3.6(b2)[[(b1)]]. Consequently, (7.84) DPF-stabilizes $\Sigma$ [[exponentially] strongly], by Theorem 7.3.11(b)(1.).

(e) This follows from (a) and Lemma 7.3.6(c). □

For the rest of the section, we concentrate on I/O-stabilization by DPF-controllers with d.c. internal loop (equivalently, on the stabilization of plants with $\mathbb{D}_{21}$ having a d.c.f., as the lemma below shows), because this seems to cover all the interesting cases (cf. also the preceeding sections and Lemma 6.5.10).

**Lemma 7.3.13** *Let $\mathbb{Q}$ DPF-stabilize $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ with internal loop.*

*Then $\mathbb{Q}$ has a d.c. (resp. r.c., l.c.) internal loop iff $\mathbb{D}_{21}$ has a d.c.f. (resp. l.c.f., r.c.f.) Moreover, if $\mathbb{D}_{21}$ has a d.c.f. (resp. l.c.f., r.c.f.), then so does $\mathbb{D}$.*

*In particular, if any system $\overline{\Sigma} = \left[\begin{array}{c|c} \overline{\mathbb{A}} & \overline{\mathbb{B}} \\ \hline \overline{\mathbb{C}} & \overline{\mathbb{D}} \end{array}\right] \in \mathrm{WPLS}$ (resp. map $\overline{\mathbb{D}} \in \mathrm{TIC}_\infty$) is DPF-stabilizable by a [exponentially] jointly stabilizable and detectable controller $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{Q}} \end{array}\right]$ (resp. by a map $\widetilde{\mathbb{Q}}$ with [exponentially] d.c. internal loop), then $\overline{\mathbb{D}}$, $\widetilde{\mathbb{Q}}$ and $\overline{\mathbb{D}}_{21}$ have [exponential] d.c.f.'s.*

Remember that a map has a [exponential] d.c.f. iff it has a [exponentially] jointly stabilizable and detectable realization (by Theorem 6.6.28). [See Theorem 7.3.12(c) for several equivalent conditions for smooth systems (in particular, for finite-dimensional ones).]

**Proof:** By Lemma 7.3.5, $\mathbb{Q}$ stabilizes $\mathbb{D}_{21}$ with internal loop, so the first conclusion follows from Corollary 7.2.13(b).

If $\mathbb{Q}$ has a d.c. (resp. r.c., l.c.) internal loop, then so does $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$, by Lemma 7.3.10, and $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ DF-stabilizes $\mathbb{D}$ (by the definition of DPF-stabilization), hence $\mathbb{D}$ has a d.c.f. (resp. l.c.f., r.c.f.). The claim on $\overline{\Sigma}$ follows from this and Theorem 6.6.28. □

**Proposition 7.3.14** *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$.

*We have (i)⇔(ii)⇔(iii), where*

*(i)* $\mathbb{D}$ *has a stabilizing DPF-controller with internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f.;*

*(ii)* $\mathbb{D}$ *has a stabilizing DPF-controller with d.c. internal loop;*

*(iii)* $\mathbb{D}$ *has a d.c.f. of the form*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{12} \\ 0 & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22} \end{bmatrix}, \tag{7.85}$$

*s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c., and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c.*

*If* $\dim U, \dim Y < \infty$ *and* $\mathbb{D}$ *has a stabilizing (well-posed) DPF-controller, then (i)–(iii) hold.*

*Let* $\mathcal{A} \subset \mathrm{TIC}$. *Then we have (i')⇔(ii')⇔(ii*)⇔(iii'), where*

*(i')* $\mathbb{D}$ *has a stabilizing DPF-controller with internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f. over* $\mathcal{A}$*;*

*(ii')* $\mathbb{D}$ *has a stabilizing DPF-controller* $\mathbb{Q}$ *with d.c. internal loop over* $\mathcal{A}$*, and* $\mathbb{D}_{21}$ *and* $\mathbb{Q}$ *have a joint d.c.f. over* $\mathcal{A}$*;*

*(ii*)* $\mathbb{D}$ *has a stabilizing DPF-controller with d.c. internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f. over* $\mathcal{A}$*;*

*(iii')* $\mathbb{D}$ *has a d.c.f. of the form (7.85), s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. over* $\mathcal{A}$*, and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c. over* $\mathcal{A}$*.*

*If* $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$*, then we have (i')⇔(ii')⇔(ii*)⇔(iii')⇔(iii"), where*

*(iii")* $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$ *of the form (7.85), s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. over* $\mathcal{A}$*, and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c. over* $\mathcal{A}$*.*

*If* $\mathcal{B} \subset \mathcal{A} \subset \mathrm{ULR}_0$*, then the stabilizing DPF-controllers in (i'), (ii*) and (ii') can be chosen to be well-posed.*

**Proof:** Note that the equivalence (i)⇔(ii)⇔(iii) follows from (i')⇔(ii')⇔(iii'), by taking $\mathcal{A} = \mathrm{TIC}$, so we do not need to prove the former. If $\mathbb{D}$ has a stabilizing DPF-controller $\mathbb{Q} \in \mathrm{TIC}_\infty$, then it is a stabilizing DF-controller for $\mathbb{D}_{21}$, hence then $\mathbb{D}_{21}$ and $\mathbb{Q}$ have d.c.f.'s, by Lemma 7.1.4, provided that $\dim U, \dim Y < \infty$.

1° The equivalence "(i')⇔(ii*)" follows from Lemma 7.3.13. Clearly (ii')⇒(ii*).

2° "(iii')⇒(ii')": Because $\mathbb{D}_{21} = \mathbb{N}_{21}\mathbb{M}_{11}^{-1}$ is a r.c.f. over $\mathcal{A}$ and $\mathbb{D}_{21} = \widetilde{\mathbb{M}}_{22}^{-1}\widetilde{\mathbb{N}}_{21}$ is a l.c.f. over $\mathcal{A}$, they can be extended to a d.c.f. over $\mathcal{A}$, by Lemma 6.5.8; in particular, we can find $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ s.t.

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \in \mathcal{G}\mathcal{A}. \tag{7.86}$$

But, by Corollary 7.2.15(i), $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right]^{-1}$ DF-stabilizes $\mathbb{D}$ iff

$$\begin{bmatrix} I & \widetilde{\mathbb{M}}_{12}\mathbb{X} - \widetilde{\mathbb{N}}_{11}\mathbb{Y} \\ 0 & \widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}, \tag{7.87}$$

i.e., iff $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$ (by Lemma A.1.1(b)), and latter is true by (7.86), hence $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop over $\mathcal{A}$.

3° *If (iii') holds (e.g., (iii) holds), then a map $\mathbb{Q}$ with an internal loop DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$:* Indeed, if $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$, then it DF-stabilizes $\mathbb{D}_{21}$, in particular, $\mathbb{Q}$ has a d.c. internal loop in either case, by Corollary 7.2.13. For the converse, in 2° it was noted that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$, where $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$, which in turn is true iff $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$, by Corollary 7.2.15(i).

4° "(ii*)⇒(iii')": Assume (ii*), i.e., that some $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop. It follows from Lemma 7.2.16(b), that for some d.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1} = \widetilde{\mathbb{N}}'(\widetilde{\mathbb{M}}')^{-1}$ we have (see (7.79))

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \mathbb{M}' \begin{bmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \widetilde{\mathbb{N}}', \text{ i.e.,} \tag{7.88}$$

$$\begin{bmatrix} \mathbb{M}'_{11}\widetilde{\mathbb{X}} & \mathbb{M}'_{12} \\ \mathbb{M}'_{21}\widetilde{\mathbb{X}} & \mathbb{M}'_{22} \end{bmatrix} = \begin{bmatrix} I + \mathbb{Y}\widetilde{\mathbb{N}}'_{21} & \mathbb{Y}\widetilde{\mathbb{N}}'_{22} \\ 0 & I \end{bmatrix}. \tag{7.89}$$

By Lemma 6.5.6(d), $\mathbb{M}'_{22} = I$ implies that $\mathbb{D}$ has a r.c.f. of the form of the first equality in (7.85); from the $(1,1)$-block of Corollary 7.2.15(i') applied to this r.c.f. we see that $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} \in \mathcal{G}\mathrm{TIC}$, hence $\mathbb{M}_{11}$ and $\mathbb{N}_{21}$ are r.c.

Let $\mathbb{D}_{21} = \mathbb{T}\mathbb{S}^{-1}$ be a r.c.f. over $\mathcal{A}$. Then $\begin{bmatrix} \mathbb{N}_{21} & \mathbb{M}_{11} \end{bmatrix} = \begin{bmatrix} \mathbb{T}\mathbb{U} & \mathbb{S}\mathbb{U} \end{bmatrix}$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c). Thus we may multiply r.c.f. in (7.85) by $\left[\begin{smallmatrix} \mathbb{U}^{-1} & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ to the right, to make $\mathbb{N}_{21}$ and $\mathbb{M}_{11}$ r.c. over $\mathcal{A}$.

The dual part is obtained analogously from $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right]\widetilde{\mathbb{M}} - \mathbb{N}\left[\begin{smallmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{smallmatrix}\right]$, which implies that $\widetilde{\mathbb{M}}_{11} = I$ (a r.c.f. and a l.c.f. form a d.c.f., by Lemma 6.5.8).

5° From ''(i')⇔(ii')⇔(iii')" we obtain directly "(i")⇔(ii")⇐(iii")". Assuming (ii”), we obtain (iii”) from 4° as follows:

The d.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1} = \widetilde{\mathbb{N}}'(\widetilde{\mathbb{M}}')^{-1}$ in 4° can be chosen to be over $\mathcal{A}$, hence so can (7.85), by Lemma 6.5.6(d). All the other claims are contained in (iii’), which is equivalent to (ii’), hence a consequence of (ii”).

6° If $\mathcal{B} \subset \mathcal{A} \subset \mathrm{ULR}_0$ and (ii’) holds (recall that (i’)⇔(ii’)⇔(ii*)), then there is a well-posed $\mathbb{Q}$ that has a d.c.f. over $\mathcal{A}$ joint with $\mathbb{D}_{21}$, by Proposition 7.1.10. Therefore, this $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$; by 3°, it DPF-stabilizes $\mathbb{D}$. □

By combining Lemma 7.3.13 and Proposition 7.3.14, we deduce the following: if $\mathbb{D}$ is DPF-stabilizable by a controller $\mathbb{Q}$ with internal loop, and either $\mathbb{D}_{21}$ or $\mathbb{Q}$ has a d.c.f., then so do all of $\mathbb{D}$, $\mathbb{D}_{21}$ and $\mathbb{Q}$, and the following hypothesis holds:

**Hypothesis 7.3.15** *We shall assume that* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, *and that* $\mathbb{D}$ *has a d.c.f. of the form*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{12} \\ 0 & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22} \end{bmatrix}, \tag{7.90}$$

*s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c.*

Note that $\mathbb{D}_{21} = \mathbb{N}_{21}\mathbb{M}_{11}^{-1}$ is a r.c.f. and $\mathbb{D}_{21} = \widetilde{\mathbb{M}}_{22}^{-1}\widetilde{\mathbb{N}}_{21}$ is a l.c.f.

Under the standard assumptions of the $\mathrm{H}^\infty$ Four-Block Problem, Hypothesis 7.3.15 is satisfied (cf. Lemmas 12.5.4 and 12.5.5). Under sufficient regularity, the I/O map of an exponentially DF-stabilizable system satisfies Hypothesis 7.3.15 (exponentially), by Theorem 7.3.12(c)(1)&(6) and Proposition 7.3.14.

As noted just before the hypothesis, this hypothesis is at most slightly stronger than the assumption that $\mathbb{D}$ is DPF-stabilizable [with internal loop]; it excludes only the case where $\mathbb{Q}$ and $\mathbb{D}_{21}$ have no jointly stabilizable and detectable realizations (cf. also Lemma 6.5.10).

At least for finite-dimensional $U$ and $Y$, any DPF-stabilizable $\mathbb{D} \in \mathrm{TIC}_\infty$ satisfies the hypothesis, by Lemma 7.1.4.

**Lemma 7.3.16 ($\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$)** *Let* $\mathbb{D} = \mathbb{N}_{\mathrm{u}}\mathbb{M}_{\mathrm{u}}^{-1}$ *be an r.c.f. with* $\mathbb{M}_{\mathrm{u}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty(U \times W)$. *Then all such r.c.f.’s are given by*

$$\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}, \quad \mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X}, \quad \mathbb{X} = \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times W). \tag{7.91}$$

*For the dual claim we have* $\widetilde{\mathbb{N}} = \widetilde{\mathbb{X}}\widetilde{\mathbb{N}_{\mathrm{y}}}$, $\widetilde{\mathbb{M}} = \widetilde{\mathbb{X}}\widetilde{\mathbb{M}_{\mathrm{y}}}$, $\widetilde{\mathbb{X}} = \begin{bmatrix} I & \widetilde{\mathbb{X}}_{12} \\ 0 & \widetilde{\mathbb{X}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.

Note that this implies that $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ (by Lemma A.1.1(b2)(2)) and $\mathbb{X}_{12} \in \mathrm{TIC}(W, U)$ are arbitrary.

**Proof:** Clearly all r.c.f.’s defined by (7.91) satisfy $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. Conversely, by Lemma 6.4.5(c), $\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}$ and $\mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X} = \left[\begin{smallmatrix} * & * \\ \mathbb{X}_{21} & \mathbb{X}_{22} \end{smallmatrix}\right]$, where $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Therefore, $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$ implies that $\begin{bmatrix} \mathbb{X}_{21} & \mathbb{X}_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

The dual claim is obtained analogously (or by taking adjoints). □

**Corollary 7.3.17** *If Hypothesis 7.3.15 holds, then all d.c.f.'s of $\mathbb{D}$ of form (7.90) satisfy Hypothesis 7.3.15.*

**Proof:** Let $\mathbb{D} = \mathbb{N}_u \mathbb{M}_u{}^{-1} = \widetilde{\mathbb{M}_y}{}^{-1}\widetilde{\mathbb{N}_y}$ be as in Hypothesis 7.3.15, and let $\widetilde{\mathbb{P}}\mathbb{M}_{u\,11} - \widetilde{\mathbb{Q}}\widetilde{\mathbb{N}_y}_{21} = I$. Let also $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ be a d.c.f. with $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$. Let $\mathbb{X}$ be as in (7.91). Then

$$\mathbb{X}_{11}^{-1}\widetilde{\mathbb{P}}\mathbb{M}_{11} - \mathbb{X}_{11}^{-1}\widetilde{\mathbb{Q}}\mathbb{N}_{21} = \mathbb{X}_{11}^{-1}\mathbb{X}_{11} = I. \tag{7.92}$$

The dual claim is obtained analogously. □

**Remark 7.3.18** *The r.c.f. in Hypothesis 7.3.15 says (roughly) that $\mathbb{D}$ can be stabilized by measuring the full output (z and y) and controlling u (i.e., not affecting w). Similarly, the l.c.f. says that $\mathbb{D}$ can be stabilized by measuring y and controlling the full input (u and y).*

*Thus is an intuitive explanation of the necessity (at least under certain regularity) of Hypothesis 7.3.15 for one being able to stabilize $\mathbb{D}$ by measuring y and controlling u; by Proposition 7.3.14 these are also sufficient.*

*It will be shown in Theorem 7.3.19 that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ stabilizes $\mathbb{D}_{21}$. Indeed, "all the poles of $\mathbb{D}$ are shared by $\mathbb{D}_{21}$" (cf. [Francis, p. 34]), because*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{11}\mathbb{M}_{11}^{-1} & -\mathbb{N}_{11}\mathbb{M}_{11}^{-1}\mathbb{M}_{12} + \mathbb{N}_{12} \\ \mathbb{N}_{21}\mathbb{M}_{11}^{-1} & -\mathbb{N}_{21}\mathbb{M}_{11}^{-1}\mathbb{M}_{12} + \mathbb{N}_{22} \end{bmatrix}. \tag{7.93}$$

*Therefore, the poles of $\widehat{\mathbb{D}}$ are poles of $\widehat{\mathbb{M}}_{11}$, which in turn are exactly the poles of $\widehat{\mathbb{D}}_{21}$, by Lemma 6.5.4, hence all these three maps have same poles (up to multiplicities).*

*Thus, stabilization of either $\mathbb{D}$ or $\mathbb{D}_{21}$ is equivalent to removing these singularities.*

A simple example of non-DPF-stabilizable $\mathbb{D}$ is thus any $\mathbb{D} = \left[\begin{smallmatrix} 0 & \mathbb{D}_{12} \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty \setminus \mathrm{TIC}$.

From the above hypothesis (roughly, the DPF-stabilizability of $\mathbb{D}$), it follows that all stabilizing DPF-controllers with internal loop for $\mathbb{D}$ are exactly the stabilizing
(DF-)controllers with d.c. internal loop for $\mathbb{D}_{21}$, i.e., the ones given by the Youla parametrization of Theorem 7.2.14:

**Theorem 7.3.19 (DPF-stabilization with IL)** *Assume Hypothesis 7.3.15. Then the following are equivalent for a controller $\mathbb{Q}$ with internal loop:*

*(i) $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ with internal loop.*

*(i') $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop.*

*(ii) $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$ with internal loop.*

*(ii') $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$ with d.c. internal loop.*

*(iii) $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y)$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ for some $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}$.*

*(iii') $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} \in \mathcal{G}\mathrm{TIC}(U)$ and $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ for some $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.*

*(iv) $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ for some $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}$.*

*(iv') $\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$ and $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ for some $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.*

*(v) For any r.c.f. $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ and l.c.f. $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$, there are $\mathbb{N}_0, \mathbb{M}_0, \widetilde{\mathbb{N}}_0, \widetilde{\mathbb{M}}_0 \in \mathrm{TIC}$ s.t. $\begin{bmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{bmatrix}$ is a d.c.f. of $\mathbb{D}_{21}$.*

*The map $\mathbb{Q}$ is well-posed iff the "denominator" ($\mathbb{X}$ or $\widetilde{\mathbb{X}}$) is invertible in $\mathcal{G}\mathrm{TIC}_\infty$ in any of the above equivalent conditions.*

By (v), $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ has a joint d.c.f. with $\mathbb{D}_{21}$. (See Definition 7.2.11 for maps with coprime internal loop.)

**Proof:** 1° The fact that a DPF-stabilizing controller of $\mathbb{D}$ with internal loop has necessarily a d.c. internal loop, is given by Lemma 7.3.13, hence "(i)⇔(i')" holds.

2° Similarly, "(ii)⇔(ii')" follows from Theorem 7.2.14.

3° "(i')⇔(ii')": This follows from part 3° of the proof of Proposition 7.3.14.

4° By Corollary 7.2.15, all the other conditions are equivalent to (ii'), and the two final claims hold. □

By combining the above theorem and Theorem 7.2.14, we see that all stabilizing DPF-controllers for $\mathbb{D}$ with internal loop are given by the Youla parametrization:

**Corollary 7.3.20 (All stabilizing DPF-controllers with IL)** *Assume Hypothesis 7.3.15, and choose $\mathbb{T}, \mathbb{S}, \widetilde{\mathbb{T}}, \widetilde{\mathbb{S}} \in \mathrm{TIC}$ s.t.*

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{T} \\ \mathbb{N}_{21} & \mathbb{S} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \in \mathcal{G}\mathrm{TIC}(U \times Y). \tag{7.94}$$

*(This is a d.c.f. of $\mathbb{D}_{21}$.)*

*The following parametrizations are alternative (equivalent) parametrizations of all (modulo being equivalent) DPF-controllers $\mathbb{Q}$ with internal loop that stabilize $\mathbb{D}$, and each* parameter *(($\mathbb{X}, \mathbb{Y}$) in (i) and (iii), ($\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}$) in (i'), and $\mathbb{U}$ in (ii) and (ii'); these all are required to be stable) determines a different (nonequal; see Definition 7.2.11) map $\mathbb{Q}$ with d.c. internal loop.*

*(i) $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ such that $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} = I$.*

*(i') $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ such that $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} = I$.*

*(ii)* **(Youla)** *$\mathbb{Q} = (\mathbb{T} + \mathbb{M}_{11}\mathbb{U})(\mathbb{S} + \mathbb{N}_{21}\mathbb{U})^{-1}$ (i.e., $\begin{bmatrix} \mathbb{Y} \\ \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{11} & \mathbb{T} \\ \mathbb{N}_{21} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}$), where $\mathbb{U} \in \mathrm{TIC}(U)$.*

*(ii') $\mathbb{Q} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}_{21}\mathbb{U})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}_{22}\mathbb{U})$ (i.e., $\begin{bmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \end{bmatrix} = \begin{bmatrix} I & \mathbb{U} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & \widetilde{\mathbb{T}} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}$), where $\mathbb{U} \in \mathrm{TIC}(U)$.*

*(iii) $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ ($= \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$), where $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.*

*Moreover, the following holds:*

*(a) The well-posed ones (if any) are exactly those* $\mathbb{Q}$ *for which the "denominator" is in* $\mathcal{G}\mathrm{TIC}_\infty$ *(cf. Theorem 7.1.7); they satisfy*

$$\widetilde{\mathbb{X}} = (\mathbb{M}_{11} - \mathbb{Q}\mathbb{N}_{21})^{-1}, \ \widetilde{\mathbb{Y}} = \widetilde{\mathbb{X}}\mathbb{Q}; \quad \mathbb{X} = (\widetilde{\mathbb{M}}_{22} - \widetilde{\mathbb{N}}_{21}\mathbb{Q})^{-1}, \ \mathbb{Y} = \mathbb{Q}\mathbb{X}. \quad (7.95)$$

*(b) For any* $\mathbb{U} \in \mathrm{TIC}$ *we have (identity as equal maps with coprime internal loop)*

$$(\mathbb{T} + \mathbb{M}_{11}\mathbb{U})(\mathbb{S} + \mathbb{N}_{21}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}_{21}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}_{22}\widetilde{\mathbb{U}}). \quad (7.96)$$

*(c) If* $\mathbb{Y}, \mathbb{X}, \widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}$ *are as in (i) and (i'), then the closed-loop I/O maps are given by*

$$\begin{bmatrix} \widetilde{\mathbb{N}}_{11} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{12} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{22} \\ \mathbb{X}\widetilde{\mathbb{N}}_{21} & \mathbb{X}\widetilde{\mathbb{N}}_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{N}_{11}\widetilde{\mathbb{X}} & \mathbb{N}_{11}\mathbb{P} + \mathbb{N}_{12} \\ \mathbb{N}_{21}\widetilde{\mathbb{X}} & \mathbb{N}_{21}\mathbb{P} + \mathbb{N}_{22} \end{bmatrix} : \begin{bmatrix} u_L \\ w \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}, \quad (7.97)$$

*where* $\mathbb{P} = \widetilde{\mathbb{Y}}\mathbb{N}_{22} - \widetilde{\mathbb{X}}\mathbb{M}_{12}$ *and* $\widetilde{\mathbb{P}} = \widetilde{\mathbb{N}}_{11}\mathbb{Y} - \widetilde{\mathbb{M}}_{12}\mathbb{X}$*; see (7.65) (without the third ($y_L$) column and top (u) row) for alternative formulae in the well-posed case.*

*In particular, (cf. (7.64))*

$$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) = \mathbb{N}_{11}\mathbb{P} + \mathbb{N}_{12} = \widetilde{\mathbb{N}}_{12} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{22}. \quad (7.98)$$

Recall from Lemma 7.2.12(c2), that the maps (7.97) depend (of course) on $\mathbb{D}$ and $\mathbb{Q}$ only, not on the particular coprime factors $(\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}})$ of $\mathbb{Q}$ and $\mathbb{D}$ (though we do require (i), (i') and Hypothesis 7.3.15).

The $\mathrm{H}^\infty$ 4BP (see Chapter 12; Section 12.3 in particular) consists of finding, for a given $\mathbb{D}$, a stabilizing DPF-controller $\mathbb{Q}$ s.t. the norm $\|\mathcal{F}_\ell\|$ is less than a given constant $\gamma$ (or for a given $\Sigma$ a [exponentially] stabilizing controller $\widetilde{\Sigma}$ s.t. $\|\mathcal{F}_\ell\| < \gamma$).

**Proof:** By Lemma 6.5.8, it follows from Hypothesis 7.3.15 that $\mathbb{D}_{21}$ has a d.c.f. of form (7.94). By Theorem 7.3.19, the stabilizing DPF-controllers for $\mathbb{D}$ with internal loop are exactly the DF-stabilizing controllers for $\mathbb{D}_{21}$ with d.c. internal loop, and these parametrized by Theorem 7.2.14, which also provides the well-posedness claim and (7.96).

Formula (7.95) follows from (7.8) and (7.10).

(c) From (7.52) and Lemma 7.3.10 we see that the map $\left[\begin{smallmatrix} u_L \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ is given by $\mathbb{N}'\widetilde{\mathbb{X}}'$ when $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ is a r.c.f.,

$$\widetilde{\mathbb{X}}' = \begin{bmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{bmatrix}, \quad \widetilde{\mathbb{Y}}' = \begin{bmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \widetilde{\mathbb{X}}'\mathbb{M}' - \widetilde{\mathbb{Y}}'\mathbb{N}' = I. \quad (7.99)$$

This condition can be satisfied by setting $\mathbb{N}' := \mathbb{N}\mathbb{G}$, $\mathbb{M}' := \mathbb{M}\mathbb{G}$, where $\mathbb{G} := (\widetilde{\mathbb{X}}'\mathbb{M} - \widetilde{\mathbb{Y}}'\mathbb{N})^{-1} \in \mathcal{G}\mathrm{TIC}(U \times W)$. Therefore,

$$\mathbb{G} = \left[\begin{smallmatrix} I & -\mathbb{P} \\ 0 & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & \mathbb{P} \\ 0 & I \end{smallmatrix}\right], \quad (7.100)$$

(by (i')), where $-\mathbb{P} = \widetilde{\mathbb{X}}\mathbb{M}_{12} - \widetilde{\mathbb{Y}}\mathbb{N}_{22}$. Thus, $\mathbb{N}'\widetilde{\mathbb{X}}' = \mathbb{N}\mathbb{G}\widetilde{\mathbb{X}}'$ is given by (7.97).

Assuming (i), we obtain the dual formula in (7.97) analogously from $\mathbb{X}'\widetilde{\mathbb{N}}' = \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right] \widetilde{\mathbb{G}}\widetilde{\mathbb{N}}$ and

$$\widetilde{\mathbb{G}} = (\widetilde{\mathbb{M}}\mathbb{X}' - \widetilde{\mathbb{N}}\mathbb{Y}')^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{12}\mathbb{X}' - \widetilde{\mathbb{N}}_{11}\mathbb{Y} \\ 0 & I \end{bmatrix}^{-1} =: \begin{bmatrix} I & \widetilde{\mathbb{P}} \\ 0 & I \end{bmatrix}. \tag{7.101}$$

□

**Corollary 7.3.21 ($\mathcal{A}$ case)** *Let $\mathcal{A} \underset{a}{\subset}$ TIC be inverse-closed and let Proposition 7.3.14(iii') hold.*

*If the elements of (7.94) are chosen from $\mathcal{A}$, then all stabilizing DPF-controllers of $\mathbb{D}$ with a (d.c.) internal loop are the ones parametrized in Corollary 7.3.20, and the ones with d.c. internal loop over $\mathcal{A}$ are exactly those with $\mathbb{U} \in \mathcal{A}$. If, in addition, $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR, then the one with $\mathbb{U} = -M_{11}^{-1}T$ is well-posed.*

**Proof:** By Lemma 6.5.8, we can take $\mathbb{T}, \mathbb{S}, \widetilde{\mathbb{T}}, \widetilde{\mathbb{S}} \in \mathcal{A}$. in Corollary 7.3.20; the rest follows by combining Corollary 7.3.20 and Proposition 7.2.17. □

As the final I/O result of this section we note that the following well-known criteria (see Theorem 4.2.1, p. 27 of [Francis] or Theorem 2.1 of [Green]) are valid for general WPLSs too:

**Lemma 7.3.22** *Let $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ and $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ have coprime factorizations $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$. Then the following are equivalent:*

*(i) $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$;*

*(ii)* $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{Y} \\ \mathbb{M}_{21} & \mathbb{M}_{22} & 0 \\ \mathbb{N}_{21} & \mathbb{N}_{22} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times W \times Y)$.

*(iii)* $\begin{bmatrix} \widetilde{\mathbb{X}} & 0 & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times Z \times Y)$.

*Even if $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ were merely a map with d.c. internal loop, then (i)–(iii) are still equivalent.*

**Proof:** Let $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be a map with r.c. internal loop. We prove that (i)⇔(ii); the case (i)⇔(iii) is analogous and the well-posed case (the one where $\mathbb{X}, \mathbb{X}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$) follows from this general case (with coprime internal loop).

By Lemma 7.3.10, $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff (7.79) DF-stabilizes $\mathbb{D}$. By Corollary 7.2.15, this holds iff

$$\begin{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} & \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} \end{bmatrix} \tag{7.102}$$

is in $\mathcal{G}$TIC. Because (7.102) becomes a triangular matrix by permuting the first and third rows and columns, we may delete its third row and third column to obtain that (7.102) is in $\mathcal{G}$TIC iff (ii) holds, by Lemma A.1.1(b). □

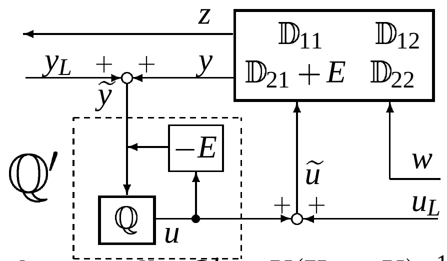

Figure 7.12: The controller $\mathbb{Q}' := \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} : \widetilde{y} \mapsto u$ for $\mathbb{D}+\left[\begin{smallmatrix} 0 & 0 \\ E & 0 \end{smallmatrix}\right]$

We sometimes want to remove the feedthrough term from $\mathbb{D}_{21}$, hence we need the following lemma:

**Lemma 7.3.23 ($D_{21} = 0$ w.l.o.g.)** *Let* $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U\times W, Z\times Y)$*. Let* $F = \left[\begin{smallmatrix} 0 & 0 \\ E & 0 \end{smallmatrix}\right] \in \mathcal{B}(U\times W, Z\times Y)$.

*Then* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *is a stabilizing DPF-controller with d.c. internal loop for* $\mathbb{D}$ *iff* $\mathbb{Q}' = \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} = (\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E)^{-1}\widetilde{\mathbb{Y}}$ *is a stabilizing DPF-controller with d.c. internal loop for* $\mathbb{D}+F$*. The corresponding closed-loop maps* $w\mapsto z$ *and* $w\mapsto u$ *(see (7.97)) are identical.*

*The controller* $\mathbb{Q}'$ *can be realized by adding to* $\mathbb{Q}$ *an output feedback through* $-E$ *in the same way as in Figure 7.12.*

*Finally, Hypothesis 7.3.15 holds for* $\mathbb{D}$ *iff it holds for* $\mathbb{D}+F$*.*

Thus, when finding such a controller for a regular $\mathbb{D}$, (possibly under an additional restriction such as "$\|w\mapsto z\| < \gamma$") we may take $D_{21} = 0$ w.l.o.g. (see Lemma 7.2.7 for well-posedness of controllers). See also the remarks below Lemma 7.2.18.

**Proof:** Let a stabilizing $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be given. Let $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}}$ be as in Proposition 7.3.14. Then

$$\begin{bmatrix} \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ E\mathbb{M}_{11} & E\mathbb{M}_{12} \end{smallmatrix}\right] & \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & E\mathbb{Y} \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} \left[\begin{smallmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{smallmatrix}\right] + \left[\begin{smallmatrix} \widetilde{\mathbb{Y}}E & 0 \\ 0 & 0 \end{smallmatrix}\right] & -\left[\begin{smallmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{smallmatrix}\right] \\ -\left[\begin{smallmatrix} \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22} \end{smallmatrix}\right] - \left[\begin{smallmatrix} \widetilde{\mathbb{M}}_{12}E & 0 \\ \widetilde{\mathbb{M}}_{22}E & 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} \end{smallmatrix}\right] \end{bmatrix}^{-1} \tag{7.103}$$

is a d.c.f. of $\mathbb{D}+F$, by Lemma 6.5.7(a), hence then $\mathbb{Q}'$ DPF-stabilizes $\mathbb{D}+F$, by Theorem 7.3.19(iii). From this we obtain the equivalence (alternatively, we can obtain it directly by applying Lemma 7.2.18 to (7.79)).

Moreover, $\mathbb{P}' := \widetilde{\mathbb{Y}}(\mathbb{N}_{22}+E\mathbb{M}_{12}) - (\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E)\mathbb{M}_{12} = \mathbb{P}$, hence $\mathbb{N}_{11}\mathbb{P}+\mathbb{N}_{12} = \mathbb{N}_{11}\mathbb{P}'+\mathbb{N}_{12} : w\mapsto z$, by Corollary 7.3.20(c).

As in the proof of Corollary 7.3.20(c), we can verify that $\left[\begin{smallmatrix} u_L \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \widetilde{u} \\ w \end{smallmatrix}\right]$ is given by $\mathbb{M}'\widetilde{\mathbb{X}}' = \mathbb{M}\mathbb{G}\widetilde{\mathbb{X}}' = \left[\begin{smallmatrix} \mathbb{M}_{11}\widetilde{\mathbb{X}} & \mathbb{M}_{11}\mathbb{P}+\mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]$, so that $\mathbb{M}_{11}\mathbb{P}+\mathbb{M}_{12} : w\mapsto u$ is unchanged. (Note that it would be more logical to have $w_L$ in place of $w$ and to have $w=0$ and hence $\widetilde{w} = w_L$. Due to historical reasons, we denote $\widetilde{w} = w_L$ by $w$.)

(Alternatively, one can observe that $(w\mapsto u) = (y_L\mapsto u)\mathbb{D}_{22}$ and $(w\mapsto z) = \mathbb{D}_{12}+\mathbb{D}_{11}(y_L\mapsto u)\mathbb{D}_{22}$ are unaffected, by Lemma 7.2.18.)

The final claim follows from Proposition 7.3.14(ii)&(iii) (alternatively, from (7.103)). □

The following remark is obtained in the same way as Remark 7.2.19 was:

**Remark 7.3.24 (Exponential DPF-stabilization)** *By Remark 6.7.19, for any claims in this section (and others), we can deduce the corresponding results about* $\omega$*-stabilization for some* $\omega \in \mathbf{R}$*, hence also for exponential stabilization.*

*For example, if* $\widetilde{\mathbb{Q}}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially with internal loop and* $\widetilde{\mathbb{Q}}$ *or* $\mathbb{D}_{21}$ *has an exponential d.c.f., then so do* $\widetilde{\mathbb{Q}}$*,* $\mathbb{D}_{21}$ *and* $\mathbb{D}$*, by Lemma 7.3.13. Assume that this is the case.*

*Then Hypothesis 7.3.15 holds and the two r.c.f.'s and l.c.f.'s assumed there are exponential ones, and a map* $\mathbb{Q}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially [with internal loop] iff* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}_{21}$ *exponentially [with internal loop] (which in turn is equivalent for* $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y}$ *being in* $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$*).*

*Furthermore, all exponentially stabilizing DPF-controllers with internal loop are given by Corollary 7.3.20, if we choose (7.94) to be in* $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$ *require also the parameters to be exponentially stable.*

*For any optimizable and estimatable realizations of* $\mathbb{D}$ *and* $\mathbb{Q}$ *(such do exist, by Theorem 6.6.28), the combined closed-loop system in Figure 7.10 becomes exponentially stable, by Theorem 7.3.11(c1). Similar remarks apply to parts (b) and (c) of the theorem and the results not mentioned here.*

## Notes

Almost all standard classical results on DPF-stabilization (see, e.g., pp. 26–36 and 42–47 of [Francis]) are special cases or simple corollaries of those presented here. Any book on the $H^\infty$ 4BP contains at least some theory on DPF-stabilization (often under the name "dynamic stabilization" or "chain scattering transformation"); see, e.g., [ZDG], [IOW] or [GL] for further theory on finite-dimensional systems and Section 2.7 of [Keu] on some results on Pritchard–Salamon systems. While this is being written, most of this section and some extended results have been included in [Sbook] (which is restricted to well-posed controllers). Further historical notes can be found in [CZ].

Some of the I/O results of this section have been presented in [Green] for well-posed rational transfer functions and later in [CZ] and [CG97] for the Callier–Desoer class (see Lemma 6.5.10(c)). However, many of their proofs cannot be extended to our generality, because the Corona Theorem (see Theorem 4.1.6(c)) only holds for matrix-valued transfer functions, by Lemma 4.1.10. (Theorem 4.1.6(c) for TIC is from [Tolokonnikov] (see [Nikolsky], p. 293). It is newer than [Vid] and it does not seem to be well known. Therefore, it might be that some of the results of [CZ] are not well-known to hold for general matrix-valued transfer functions.) Nevertheless, the book [CZ] contains also some further theory on dynamic partial feedback and robust control, some of which can be extended to our setting.

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A453 Marko Huhtanen
Aspects of nonnormality for iterative methods
September 2002

A451 Marko Huhtanen
Combining normality with the FFT techniques
September 2002

A450 Nikolai Yu. Bakaev
Resolvent estimates of ellpitic differential and finite element operators in pairs of function spaces
August 2002

A449 Juhani Pitkäranta
Mathematical and historical reflections on the lowest order finite element models for thin structures
May 2002

A448 Teijo Arponen
Numerical solution and structural analysis of differential-algebraic equations
May 2002

ISBN 951-22-6153-7
ISSN 0784-3143
Espoo, 2002

Espoo 2002 A452

# Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 2/3 — Optimal Control and Riccati Equations

**Kalle Mikkola**

$$K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC$$
$$S = D^*JD + \operatorname*{w\text{-}lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$$
$$K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$$

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

# Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

## Volume 2/3 — Optimal Control and Riccati Equations

**Kalle Mikkola**

**Dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the Department of Engineering Physics and Mathematics, for public examination and debate in Council Room at Helsinki University of Technology (Espoo, Finland) on the 18th of October, 2002, at 12 o'clock noon.**

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

**Kalle Mikkola**: *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*; Helsinki University of Technology Institute of Mathematics Research Reports A452 (2002).

**Abstract:** *In this monograph, we solve rather general linear, infinite-dimensional, time-invariant control problems, including the* $H^\infty$ *and LQR problems, in terms of algebraic Riccati equations and of spectral or coprime factorizations. We work in the class of (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss and D. Salamon.*
*Moreover, we develop the required theories, also of independent interest, on WPLSs, time-invariant operators, transfer and boundary functions, factorizations and Riccati equations. Finally, we present the corresponding theories and results also for discrete-time systems.*

**AMS subject classifications:** 42A45, 46E40, 46G12, 47A68, 49J27, 49N10, 49N35, 93-02, 93A10, 93B36, 93B52, 93C05, 93C55, 93D15

**Keywords:** suboptimal H-infinity control, standard H-infinity problem, measurement feedback problem; H-infinity full information control problem, state feedback problem; Nehari problem; LQC, LQR control, H2 problem, minimization; bounded real lemma, positive real lemma; dynamic stabilization, controller with internal loop, strong stabilization, exponential stabilization, optimizability; canonical factorization, (J,S)-spectral factorization, (J,S)-inner coprime factorization, generalized factorization, J-lossless factorization; compatible operators, weakly regular well-posed linear systems, distributed parameter systems; continuous-time, discrete-time; infinite-horizon; time-invariant operators, Toeplitz operators, Wiener class, equally-spaced delays, Popov function; transfer functions, H-infinity boundary functions, Fourier multipliers; Bochner integral, strongly measurable functions, strong Lp spaces; Laplace transform, Fourier transform

Kalle.Mikkola@hut.fi

ISBN 951-22-6153-7
ISSN 0784-3143
Espoo, 2002

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

# Contents

## Volume 1/3

**1 Introduction** **11**
1.1 On the contributions of this book . . . . . . . . . . . . . . . . . . . 12
1.2 A summary of this book . . . . . . . . . . . . . . . . . . . . . . . 15
1.3 Conventions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 43

**I TI Operator Theory** **45**

**2 TI and MTI Operators** **47**
2.1 Time-invariant operators (TI) . . . . . . . . . . . . . . . . . . . . . 48
2.2 $\mathcal{G}$TIC — invertibility . . . . . . . . . . . . . . . . . . . . . . . . 55
2.3 Static operators . . . . . . . . . . . . . . . . . . . . . . . . . . . 63
2.4 The signature operator $S$ . . . . . . . . . . . . . . . . . . . . . . 65
2.5 Losslessness . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 68
2.6 MTI and its subclasses . . . . . . . . . . . . . . . . . . . . . . . 71

**3 Transfer Functions ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$)** **79**
3.1 Transfer functions of TI ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$) . . . . . . . . . . . . . . . . 80
3.2 $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$ for Banach spaces (Fourier Multipliers) . . . . . . . . 90
3.3 $\mathrm{H}^2$ and $\mathrm{H}^\infty$ boundary functions in $\mathrm{L}^2$ and $\mathrm{L}^\infty_{\mathrm{strong}}$ . . . . . . . . . . 99

**4 Corona Theorems and Inverses** **115**

**5 Spectral Factorization ($\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$)** **133**
5.1 Auxiliary spectral factorization results . . . . . . . . . . . . . . . . 133
5.2 MTI spectral factorization ($\mathbb{E}, \mathbb{Y}, \mathbb{X} \in$ MTI) . . . . . . . . . . . . . 140

**II Continuous-Time Control Theory** **149**

**6 Well-Posed Linear Systems (WPLS)** **151**
6.1 WPLS theory . . . . . . . . . . . . . . . . . . . . . . . . . . . . 153
6.2 Regularity ($\exists\widehat{\mathbb{D}}(+\infty)$) . . . . . . . . . . . . . . . . . . . . . . . . 166
6.3 Further regularity and compatibility . . . . . . . . . . . . . . . . . 180
6.4 Spectral and coprime factorizations ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$) . . . . . . . . . . 202
6.5 Further coprimeness and factorizations . . . . . . . . . . . . . . . . . 210

6.6 Feedback and Stabilization ($\Sigma_L$, $\Sigma_\flat$, $\Sigma_\sharp$) . . . 219
6.7 Further feedback results . . . 244
6.8 Systems with $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,1];H)$ . . . 263
6.9 Bounded $B$, bounded $C$, PS-systems . . . 272

**7 Dynamic Stabilization** **279**
7.1 Dynamic feedback (DF) stabilization ($(I - \begin{bmatrix} 0 & \mathbb{D} \\ \mathbb{Q} & 0 \end{bmatrix})^{-1} \in$ TIC) . . . 280
7.2 DF-stabilization with internal loop . . . 291
7.3 DPF-stabilization ($\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$) . . . 314

## Volume 2/3

**III Riccati equations and Optimal control** **347**

**8 Optimal Control ($\frac{\mathrm{d}}{\mathrm{du}}\mathcal{J} = 0$)** **349**
8.1 Abstract $J$-critical control ($Jy_{ycrit} \perp \Delta y$) . . . 351
8.2 Abstract $J$-coercivity ($\mathcal{J} \mapsto [u\ Du]$) . . . 356
8.3 $J$-critical control for WPLSs . . . 361
8.4 $J$-coercivity and factorizations . . . 377
8.5 Problems on a finite time interval . . . 392
8.6 Extended linear systems (ELS) . . . 395

**9 Riccati Equations and $J$-Critical Control** **401**
9.1 The Riccati Equation: A summary for $\mathcal{U}_{\mathrm{out}}$ (r.c.f.$\leftrightarrow$CARE) . . . 404
9.2 Riccati equations when $\mathbb{A}Bu_0 \in \mathrm{L}^1$ . . . 417
9.3 Proofs for Section 9.2 . . . 432
9.4 Analytic semigroups . . . 438
9.5 Parabolic problems and CAREs . . . 443
9.6 Parabolic problems: proofs . . . 450
9.7 Riccati equations on $\mathrm{Dom}(A_{crit})$ . . . 452
9.8 Algebraic and integral Riccati equations (CARE$\leftrightarrow$IARE) . . . 465
9.9 $J$-Critical control $\leftrightarrow$ Riccati Equation . . . 481
9.10 Proofs for Section 9.9: Crit$\leftrightarrow$eIARE . . . 500
9.11 Proofs for Section 9.8: eCARE$\leftrightarrow$eIARE . . . 507
9.12 Further eIARE and eCARE results . . . 517
9.13 Examples of Riccati equations . . . 525
9.14 $(J,*)$-critical factorization ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$) . . . 534
9.15 $\mathrm{H}^2$-factorization when $\dim U < \infty$ . . . 539

**10 Quadratic Minimization (**$\min \mathcal{J}$**)** **543**
10.1 Minimizing $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\, dm$ (LQR) . . . 545
10.2 General minimization (LQR) . . . 553
10.3 Standard assumptions . . . 568
10.4 The $\mathrm{H}^2$ problem . . . 580
10.5 Real lemmas . . . 587