# Part I

# TI Operator Theory

# Chapter 2

# TI and MTI Operators

*Gather ye rosebuds while ye may,*
*Old Time is still a-flying:*
*And this same flower that smiles today*
*Tomorrow will be dying.*

— Robert Herrick (1591–1674)

Throughout this chapter, $H$, $H_k$, $U$, $W$, $Y$, $Y_k$ and $Z$ ($k \in \mathbf{N}$) denote Hilbert spaces of arbitrary dimensions (unless otherwise specified). (Many results of this chapter also hold for Banach spaces and for $\mathrm{L}^p$ in place of $\mathrm{L}^2$; see [Sbook] for details.)

In Section 2.1 we shall study the basic theory of $\mathrm{TI}_\omega(U,Y)$, the bounded linear time-invariant operators $\mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ ($\omega \in \mathbf{R}$). Section 2.2 treats the invertibility of $\mathrm{TI}_\omega$ operators with emphasis on the causal ones, $\mathrm{TIC}_\omega$.

Section 2.3 lists sufficient conditions for a TIC operator to be static, that is, to be the multiplication operator induced by an element of $\mathcal{B}(U,Y)$. We also give certain results that will be used in connection with the *signature operators* of optimization problems, Riccati equations and spectral factorizations; such operators are further treated in Section 2.4. Section 2.5 treats the concept $(J,S)$-*losslessness*.

In Section 2.6 we define the subspace $\mathrm{MTI}_\omega(U,Y)$ (and its subspaces) corresponding to $\mathrm{TI}_\omega(U,Y)$ maps of form $u \mapsto \mu * u$, where $\mu$ is a measure consisting of a function $f \in \mathrm{L}^1_\omega(\mathbf{R};\mathcal{B}(U,Y))$ plus a discrete part (this includes the Callier–Desoer class and the Wiener class). We list the basic properties of these classes.

For most readers it suffices just to have a glance at subsections 2.1.1–2.1.7 and possibly also 2.6.3–2.6.4, and then return to this chapter only when pointed by a reference.

The book [RR] is a standard reference for TIC (or “causal shift-invariant”) operators; see also [Nikolsky] and [Sbook]. (Due to Lemma 2.1.3, TIC (or $\mathrm{TIC}_\infty$) operators are sometimes called “Toeplitz operators”; see p. 56 for the correct definition.)

We shall sometimes refer to Chapter 3, which does not depend on this chapter except on the basic properties of TI and TIC operators.

## 2.1 Time-invariant operators (TI)

*Weep no more, nor sigh, nor groan;*
*Sorrow calls no time that's gone;*
*Violets plucked the sweetest rain*
*Makes not fresh nor grow again.*
— John Fletcher (1579–1625)

In this section, we define the class TI of time-invariant bounded linear operators $\mathrm{L}^2(\mathbf{R};*) \to \mathrm{L}^2(\mathbf{R};*)$ and several of its subclasses, and study their basic properties.

We start with notation. For any $\omega \in \mathbf{R}$, $p \in [1,\infty]$ and a measurable set $J \subset \mathbf{R}$, we set

$$\mathrm{L}^p_\omega(J;U) := \left\{ u \in \mathrm{L}^p_{\mathrm{loc}}(J;U) \;\middle|\; \left(t \mapsto \mathrm{e}^{-\omega t}u(t)\right) \in \mathrm{L}^p(J;U) \right\} = \mathrm{e}^{\omega\cdot} \cdot \mathrm{L}^p(J;U); \tag{2.1}$$

in particular, $\|u\|_{\mathrm{L}^p_\omega} := \|\mathrm{e}^{-\omega\cdot}u\|_{\mathrm{L}^p}$. Thus, $\mathrm{e}^{\omega\cdot} := \mathrm{e}^{\omega\cdot}$ becomes an isometric isomorphism $\mathrm{L}^p \mapsto \mathrm{L}^p_\omega$, and we have (recall that $\tau^t u := \tau(t)u := u(\cdot+t)$, $\boldsymbol{Я}u = u(-\cdot)$)

$$\tau(t)\mathrm{e}^{\omega\cdot} = \mathrm{e}^{\omega t}\mathrm{e}^{\omega\cdot}\tau(t), \quad \|\tau(t)u\|_{\mathrm{L}^p_\omega} = \mathrm{e}^{\omega t}\|u\|_{\mathrm{L}^p_\omega}, \;\; (t,\omega \in \mathbf{R},\ u \in \mathrm{L}^p_\omega(\mathbf{R};U)). \tag{2.2}$$

Moreover, $\pi_E u := \chi_E u$, where $\chi_E$ is the characteristic function of $E \subset \mathbf{R}$, $\pi_+ := \pi_{\mathbf{R}_+}$ and $\pi_- := \pi_{\mathbf{R}_-} = I - \pi_+$. It follows that

$$\boldsymbol{Я}^{-1} = \boldsymbol{Я} = \boldsymbol{Я}^*,\ \pi_E^2 = \pi_E = \pi_E^*,\ \boldsymbol{Я}\pi_E\boldsymbol{Я} = \pi_{-E}, \tag{2.3}$$
$$\tau(t)^* = \tau(-t),\ \tau(t)\tau(s) = \tau(t+s),\ \boldsymbol{Я}\tau(t) = \tau(-t)\boldsymbol{Я},\ \pi_E\tau(t) = \tau(t)\pi_{E+t}, \tag{2.4}$$

and that any $L \in \mathcal{B}(U,Y)$ commutes with $\tau^t$, $\boldsymbol{Я}$ and $\pi_E$ in $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$.

We will often use the fact that if $\{u_n\} \subset \mathrm{L}^2_\alpha \cap \mathrm{L}^2_\omega$, $u_n \to u$ in $\mathrm{L}^2_\alpha$, and $u_n \to v$ in $\mathrm{L}^2_\omega$, then $u = v$ (a.e.), by Theorem B.3.2. On of its consequences is that $\mathrm{L}^2_\omega(\mathbf{R}_+;U) \subset \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$, continuously, for $\omega \le \alpha$. Finally, we have

$$\|u\|_{\mathrm{L}^2_\omega} = \lim_{r\to\omega+} \|u\|_{\mathrm{L}^2_r} \quad (\omega \in \mathbf{R},\ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)), \tag{2.5}$$

by the Monotone Convergence Theorem.

**Definition 2.1.1 (TI, TIC)** *Let* $\omega \in \mathbf{R}$. *We define* $\mathrm{TI}_\omega(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ *that are* time-invariant, *i.e.,* $\tau(t)\mathbb{E} = \mathbb{E}\tau(t)$ *for all* $t \in \mathbf{R}$.

*We define* $\mathrm{TIC}_\omega(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$ *that are* causal, *i.e.,* $\pi_-\mathbb{D}\pi_+ = 0$, *or, equivalently,* $\mathbb{D}\pi_+\mathrm{L}^2_\omega \subset \pi_+\mathrm{L}^2_\omega$.

*We set* $\mathrm{TI} := \mathrm{TI}_0$, $\mathrm{TI}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TI}_\omega$, $\mathrm{TI}_{\mathrm{exp}} := \cup_{\omega<0}\mathrm{TI}_\omega$, $\mathrm{TIC} := \mathrm{TIC}_0$, $\mathrm{TIC}_\infty := \mathrm{TIC} \cap \mathrm{TI}_\infty$ *and* $\mathrm{TIC}_{\mathrm{exp}} := \mathrm{TIC} \cap \mathrm{TI}_{\mathrm{exp}}$. *For* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ *we set* $\mathbb{E}^t := \pi_{[0,t)}\mathbb{E}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$.

*We call maps belonging to* $\mathrm{TIC}_{\mathrm{exp}}$ exponentially stable, *those belonging to* $\mathrm{TIC} = \mathrm{TIC}_0$ stable, *and those belonging to* $\mathrm{TIC}_\infty \setminus \mathrm{TIC}$ unstable.

*We extend any* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *as follows: if* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ *and* $\pi_-u \in \mathrm{L}^2_\omega$, *then we set* $\pi_{(-\infty,T)}\mathbb{D}u := \pi_{(-\infty,T)}\mathbb{D}\pi_{(-\infty,T)}u$ *(*$T \in \mathbf{R}$*) (by causality,* $\mathbb{D}u$ *becomes uniquely defined (a.e.) on* $\mathbf{R}$*).*

By Lemma 2.1.10, we have $\mathrm{TIC} = \mathrm{TIC}_\infty \cap \mathrm{TI}$, $\mathrm{TIC}_\infty = \cup_{\omega\in\mathbf{R}}\mathrm{TIC}_\omega$, $\mathrm{TIC}_{\exp} = \cup_{\omega<0}\mathrm{TIC}_\omega$. See Remark 2.1.9 for the concept $\mathrm{TI}_a \cap \mathrm{TI}_b$ for $a \neq b$.

For any $\mathbb{D} \in \mathrm{TIC}_\omega$ we obviously have $\mathbb{D}\pi_+ = \pi_+\mathbb{D}\pi_+$, $\pi_-\mathbb{D} = \pi_-\mathbb{D}\pi_-$, and $\pi_{(-\infty,t)}\mathbb{D} = \pi_{(-\infty,t)}\mathbb{D}\pi_{(-\infty,t)}$. We will often use these facts as well as the fact that $\boldsymbol{Я}[\mathrm{L}^2_\omega(\mathbf{R};U)] = \mathrm{L}^2_{-\omega}(\mathbf{R};U)$.

**Theorem 2.1.2 (Transfer functions)** *Let* $\omega \in \mathbf{R}$. *For each* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ *there is a unique function* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$, *called the* transfer function *(or symbol or Laplace transform) of* $\mathbb{D}$, *s.t.* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ *on* $\mathbf{C}^+_\omega$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. *The mapping* $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ *is an isometric isomorphism of* $\mathrm{TIC}_\omega(U,Y)$ *onto* $\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$. □

(The theorem is obtained from Theorem 2.3 of [W91a] by translation by $\omega$ (cf. Remark 2.1.6). That theorem also contains a similar claim for $\mathrm{L}^p$ $(1 \le p < \infty)$ and Banach space $U$ and $Y$, but in that case the isometric isomorphism onto becomes merely a linear injection into, by Example 3.3.4.)

Recall that $\hat{u}$ denotes the Laplace transform $\hat{u}(s) := \int_{\mathbf{R}} \mathrm{e}^{-st}u(t)\,dt$ of $u$.

We often identify functions and corresponding multiplication operators, i.e., we consider $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ both as a function and as an operator on $\mathcal{L}[\mathrm{L}^2_\omega(\mathbf{R}_+;U)] = \mathrm{H}^2(\mathbf{C}^+_\omega;U)$ (see Theorem 3.3.1(b)).

A causal time-invariant map $\mathbb{D} : \mathrm{L}^2(\mathbf{R};U) \supset \mathrm{Dom}(\mathbb{D}) \to \mathrm{L}^2(\mathbf{R};Y)$ is called *well-posed* iff $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, i.e., iff there are $\omega \in \mathbf{R}$ and $M < \infty$ s.t. $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathrm{Dom}(\mathbb{D})$ and $\mathrm{Dom}(\mathbb{D}) \cap \mathrm{L}^2_\omega$ is dense in $\mathrm{L}^2_\omega(\mathbf{R};U)$.

Thus, if $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega;\mathcal{B}(U,Y))$ for some open $\Omega \subset \mathbf{C}$, then the multiplication map $\hat{u} \mapsto \widehat{\mathbb{D}}\hat{u}$ determines a (necessarily unique) well-posed map $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff $\widehat{\mathbb{D}}$ is defined and bounded on some right half-plane (i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^\infty_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{H}^\infty(\mathbf{C}^+_\omega;*)$). Therefore, "well-posed" is an extension of the classical concept "proper" (recall that a proper rational function is one that is bounded at infinity). We shall study transfer functions of $\mathrm{TIC}_\infty$ maps in detail in Section 3.3 and those of $\mathrm{TI}_\infty$ maps in Section 3.1.

We conclude from Theorem 2.1.2 that the extension mentioned at the end of Definition 2.1.1 extends any $\mathbb{D} \in \mathrm{TIC}_\omega$ to a unique $\mathrm{TIC}_{\omega'}$ operator for each $\omega' > \omega$; we identify these two operators. Thus, $\mathrm{TIC}_\omega \subset \mathrm{TIC}_{\omega'}$. The $\mathrm{TI}_\omega$ operators are not nested in a similar way, but they are also uniquely determined by any $\mathrm{TI}_{\omega'}$ to which they belong; see Remark 2.1.9 for details.

A causal map $\mathbb{D} \in \mathrm{TIC}_\infty$ is determined by its *Toeplitz operator* $\pi_+\mathbb{D}\pi_+$:

**Lemma 2.1.3** *For each* $\mathbb{D}_+ \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R}_+;U),\mathrm{L}^2_\omega(\mathbf{R}_+;Y))$ *s.t.* $\tau^{-t}\mathbb{D} = \mathbb{D}\tau^{-t}$ *for* $t \ge 0$, *there is* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ *s.t.* $\mathbb{D}_+ = \pi_+\mathbb{D}\pi_+$. *This correspondence is an isometric isomorphism.*

**Proof:** (Recall that we identify functions on $\mathbf{R}_+$ to their zero extensions, hence $\tau^{-t}f$ is zero on $[0,t)$ for each $f : \mathbf{R}_+ \to *$.)

Extend $\mathbb{D}_+$ to $\mathcal{B}(X,\mathrm{L}^2)$, where $X := \cup_{T\in\mathbf{R}}\mathrm{L}^2_\omega([T,+\infty);U)) \subset \mathrm{L}^2_\omega$, by $\mathbb{D}_+\tau^T u := \tau^T\mathbb{D}_+u$ for $T \ge 0$, $u \in \pi_+\mathrm{L}^2_\omega$. One easily verifies that the resulting operator is well-defined and time-invariant. Because this does not alter the norm of $\mathbb{D}_+$, by (2.2), we can extend $\mathbb{D}_+$ to $\mathrm{L}^2_\omega$, by density. By continuity, the resulting operator is time-invariant. The converse is obvious. $\square$

**Definition 2.1.4** *If* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ *(*$\omega \in \mathbf{R}$*), then its* (noncausal) adjoint $\mathbb{E}^*$ *is the* $\mathrm{TI}_{-\omega}(Y,U)$ *map that satisfies*

$$\int_{\mathbf{R}} \langle (\mathbb{E}u)(t), y(t)\rangle\, dt = \int_{\mathbf{R}} \langle u(t), (\mathbb{E}^* y)(t)\rangle\, dt \quad (u \in \mathrm{L}^2_\omega(\mathbf{R};U),\ y \in \mathrm{L}^2_{-\omega}(\mathbf{R};Y)). \tag{2.6}$$

*We call* $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}^*\boldsymbol{Я} \in \mathrm{TI}_\omega(Y,U)$ *the* causal adjoint *of* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$.

(We used here the identity $\tau(t)\mathbb{E}^* = (\mathbb{E}\tau(-t))^* = (\tau(-t)\mathbb{E})^* = \mathbb{E}^*\tau(t)$.)

Obviously, it is enough to verify (2.6) for $u, y \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbf{R};Y)$, by Theorem B.3.11. Note that $\mathbb{E}, \mathbb{E}' \in \mathrm{TI}_\omega \Rightarrow (\mathbb{E}\mathbb{E}')^* = \mathbb{E}'^*\mathbb{E}^*$. See also Lemma 2.1.10(b).

Note that the adjoint is not taken w.r.t. the $\mathrm{L}^2_\omega$ inner product (which would imply $\mathbb{E}^* \in \mathrm{TI}_\omega(U,Y)$) but w.r.t. the $\mathrm{L}^2 = \mathrm{L}^2_0$ inner product (so that $\mathbb{E}^* \in \mathrm{TI}_{-\omega}(U,Y)$). This way the adjoint does not depend on the choice of $\omega$.

By *duality* we usually mean that one applies known results to the duals (i.e., adjoints) of the operators involved (cf. also Lemma 6.1.4).

**Lemma 2.1.5** *Let* $\omega \in \mathbf{R}$. *If* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, *then* $\mathbb{D}^{\mathrm{d}} \in \mathrm{TIC}_\omega(Y,U)$. *Let* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$. *Then* $\mathbb{E} \in \mathcal{GB}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathrm{L}^2_\omega(\mathbf{R};Y))$ *iff* $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega(U,Y)$. $\square$

(We leave the simple proof to the reader.)

Thus, the inverse (if any) of a time-invariant map is necessarily time-invariant. The inverse of a causal map need not be causal (e.g., $\tau(-1) \in \mathrm{TIC}$, $\tau(1) = \tau(-1)^{-1} \in \mathrm{TI} \setminus \mathrm{TIC}$). However, the "causal adjoint" of a causal map is always causal, as shown in the lemma.

**Remark 2.1.6 (Shifting stability)** *Let* $\alpha, \omega \in \mathbf{R}$. *Let* $\mathcal{T}_\alpha$ *be the* stability shift *(or scaling operator)* $\mathbb{E} \mapsto e^{\alpha\cdot}\mathbb{E}e^{-\alpha\cdot}$. *Then* $\mathcal{T}_\alpha$ *is an isometric isomorphism of* $\mathrm{TI}_\omega$ *onto* $\mathrm{TI}_{\omega+\alpha}$ *and of* $\mathrm{TIC}_\omega$ *onto* $\mathrm{TIC}_{\omega+\alpha}$ *(because* $[\pi_+]\mathrm{L}^2_{\omega+\alpha} = e^{\alpha\cdot}[\pi_+]\mathrm{L}^2_\omega$*, isometrically).*

*Obviously,* $\mathcal{T}_\alpha\pi_\pm = \pi_\pm\mathcal{T}_\alpha$*,* $\mathcal{T}_\alpha\tau(t) = \tau(t)\mathcal{T}_\alpha$ *(*$t \in \mathbf{R}$*), and we have*

$$\mathcal{T}_\alpha(\mathbb{E}\mathbb{F}) = (\mathcal{T}_\alpha\mathbb{E})(\mathcal{T}_\alpha\mathbb{F}), \qquad \mathcal{T}_\alpha(\beta\mathbb{E}+\gamma\mathbb{F}) = \beta\mathcal{T}_\alpha\mathbb{E}+\gamma\mathcal{T}_\alpha\mathbb{F}, \tag{2.7}$$

$$(\mathcal{T}_\alpha\mathbb{E})^{-1} = \mathcal{T}_\alpha\mathbb{E}^{-1}, \qquad (\mathcal{T}_\alpha\mathbb{E})^* = \mathcal{T}_{-\alpha}\mathbb{E}^*, \tag{2.8}$$

$$(\mathcal{T}_\alpha\mathbb{E})^d = \mathcal{T}_\alpha\mathbb{E}^d, \qquad \widehat{\mathcal{T}_\alpha\mathbb{E}} = \tau(-\alpha)\widehat{\mathbb{E}}. \tag{2.9}$$

$\square$

(The formula $\widehat{\mathcal{T}_\alpha\mathbb{E}} = \widehat{\mathbb{E}}(\cdot - \alpha)$ refers to Theorem 3.1.3(a1); for $\mathbb{E} \in \mathrm{TIC}_\infty$ it also covers Theorem 2.1.2.)

Note that $\alpha > 0$ decreases stability, i.e., shifts the transfer function to the right. See also Remark 6.1.9.

If $\mathbb{D} \in \mathrm{TI}$ is causal, then, obviously, $\mathbb{D}^*$ is *anti-causal*, i.e., $\pi_+\mathbb{D}^*\pi_- = 0$. If $\mathbb{D}$ is both causal and anti-causal, then we call $\mathbb{D}$ *static*. We identify $D \in \mathcal{B}(U,Y)$ and the multiplication map $M_D : u \mapsto Du$ (note that $M_D \in \mathrm{TIC}(U,Y)$, $\mathrm{M}_D^{-1} = \mathrm{M}_{D^{-1}}$ if either inverse exists, and $\mathrm{M}_D^* = \mathrm{M}_{D^*}$). The static TIC maps are exactly the (multiplication) maps of this form:

**Lemma 2.1.7 (Static $\mathbb{D}$)** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\mathbb{D}^* \in \mathrm{TIC}_\infty(Y,U)$. Then $\mathbb{D} \in \mathcal{B}(U,Y)$. Moreover, the imbedding $\mathcal{B} \mapsto \mathrm{TIC}_\omega$ preserves norms and commutes with algebraic operations (for any $\omega \in \mathbf{R}$).*

Thus, $\mathbb{D} \in \mathrm{TIC}_\infty \,\&\, \pi_+\mathbb{D}\pi_- = 0 \Rightarrow \mathbb{D} \in \mathcal{B}$. See Section 2.3 for more on static operators.

**Proof:** For $\mathbb{D}, \mathbb{D}^* \in \mathrm{TIC}$, this is stated in [RR, Theorem 5.2C, p. 96] (we do not know their proof; a proof consists of Proposition D.1.20 combined with Theorems 2.1.2 and 3.3.1; see [Sbook] for a more general result and its system-theoretic proof).

In the general case, where $\mathbb{D}, \mathbb{D}^* \in \mathrm{TIC}_\omega$ for some $\omega \in \mathbf{R}$, we have $\mathbb{D}_{-\omega} := \mathcal{T}_{-\omega}\mathbb{D} \in \mathrm{TIC}_0$ and $(\mathbb{D}_{-\omega})^* = \mathcal{T}_\omega\mathbb{D}^* \in \mathrm{TIC}_0$, hence $D := \mathbb{D}_{-\omega} \in \mathcal{B}(U,Y)$ and $\mathbb{D} = \mathcal{T}_\omega\mathbb{D}_{-\omega} = D$. $\square$

The *Hankel operator*[1] $\pi_+ \cdot \pi_-$ of a $\mathrm{TIC}_\infty$ map determines the map uniquely modulo a static operator:

**Corollary 2.1.8 ($\pi_+\mathbb{D}\pi_-$)** *Let $\mathbb{D}, \widetilde{\mathbb{D}} \in \mathrm{TIC}_\infty(U,Y)$. Then $\pi_+\mathbb{D}\pi_- = \pi_+\widetilde{\mathbb{D}}\pi_-$ iff $\mathbb{D} = \widetilde{\mathbb{D}} + D$ for some $D \in \mathcal{B}(U,Y)$.* $\square$

(Apply Lemma 2.1.7 to $\mathbb{D} - \widetilde{\mathbb{D}}$.) Note that it suffices that $\pi_+\mathbb{D}\pi_-\phi = \pi_+\widetilde{\mathbb{D}}\pi_-\phi$ for all $\phi \in \mathcal{C}_c^\infty$, by Theorem B.3.11.

As noted above, a $\mathrm{TIC}_a$ map is also a (i.e., extends to a unique) $\mathrm{TIC}_b$ map for any $b > a$. If the restriction of a $\mathrm{TI}_a$ map onto $\mathrm{L}^2_b \cap \mathrm{L}^2_a$ is continuous in the $\mathrm{L}^2_b$ norm (to $\mathrm{L}^2_b$), then it extends to a unique $\mathrm{TI}_b$ map (and a unique $\mathrm{TI}_r$ map for any $r \in [a,b]$), as will be noted in the remark and lemma below. The corresponding technical details are given in Proposition E.1.8.

**Remark 2.1.9 ($\mathrm{TI}_a \cap \mathrm{TI}_b$)** *Let $a < b$. Assume that $\mathcal{F}$ is the set of simple functions $\mathbf{R} \to U$, or $\mathcal{F} = \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$ or $\mathcal{F} = \mathcal{C}_c^\infty(\mathbf{R};U)$.*

*If(f) $\mathbb{E} \in \mathcal{F} \to \mathrm{L}^2_a(\mathbf{R};Y)$ is time-invariant, linear and bounded $\mathrm{L}^2_a \to \mathrm{L}^2_a$ and $\mathrm{L}^2_b \to \mathrm{L}^2_b$, or equivalently, $\mathbb{E}$ is linear and there is $M < \infty$ s.t.*

$$\|\mathbb{E}\phi\|_{\mathrm{L}^2_a} \le M\|\phi\|_{\mathrm{L}^2_a},\ \ \|\mathbb{E}\phi\|_{\mathrm{L}^2_b} \le M\|\phi\|_{\mathrm{L}^2_b} \text{ and } \mathbb{E}\tau(t)\phi = \tau(t)\mathbb{E}\phi \text{ for all } t \in \mathbf{R},\ \phi \in \mathcal{F}, \tag{2.10}$$

*then $\mathbb{E}$ extends to a unique operator $\mathbb{E} \in \mathrm{TI}_r(U,Y)$ for all $r \in [a,b]$, and $\mathbb{E}$ is uniquely defined on $\cup_{r\in[a,b]}\mathrm{L}^2_r(\mathbf{R};U)$, by Proposition E.1.8.*

*Therefore, if $\mathbb{E}_a \in \mathrm{TI}_a(U,Y)$ and $\mathbb{E}_b \in \mathrm{TI}_b(U,Y)$ are s.t. $\mathbb{E}_a = \mathbb{E}_b$ on $\mathcal{F}$, then we identify $\mathbb{E}_a$ and $\mathbb{E}_b$.*

*Thus, by "$\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$" we mean such a map (i.e., $\mathbb{E} = \mathbb{E}_a = \mathbb{E}_b$, where $\mathbb{E}_a$ and $\mathbb{E}_b$ are as above).*

*An alternative characterization is that $\mathbb{E} : \mathrm{L}^2_a(\mathbf{R};U) + \mathrm{L}^2_b(\mathbf{R};U) \to \mathrm{L}^2_a(\mathbf{R};Y) + \mathrm{L}^2_b(\mathbf{R};Y)$ is linear and s.t. $\mathbb{E}|_{\mathrm{L}^2_a} \in \mathcal{B}(\mathrm{L}^2_a, \mathrm{L}^2_a)$, $\mathbb{E}|_{\mathrm{L}^2_b} \in \mathcal{B}(\mathrm{L}^2_b, \mathrm{L}^2_b)$ and $\mathbb{E}\tau^t = \tau^t\mathbb{E}$ for all $t \in \mathbf{R}$ (see Definition E.1.3).* $\square$

[1] Sometimes $\pi_-\mathbb{D}\pi_+$ is called the Hankel operator and $\pi_+\mathbb{D}\pi_-$ the anti-Hankel operator of $\mathbb{D}$; our choice is due to [S97b].

If $\mathbb{E}$ is a linear map whose restriction to $\mathrm{Dom}(\mathbb{E}) \cap \mathrm{L}^2_a$ extends to a unique $\mathrm{TI}_a$ map (as above), then we identify this $\mathrm{TI}_a$ map with $\mathbb{E}$.

By translation-invariance and density, it is enough to verify (2.10) for functions $\phi \in \mathcal{F}$ having their support on, e.g., $\mathbf{R}_+$.

**Lemma 2.1.10 ($\mathbf{TI}_a \cap \mathbf{TI}_b$)** *Let $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$, $a < b$. Then the following hold:*

*(a1) $\mathbb{E} \in \mathrm{TI}_r(U,Y)$ for all $r \in [a,b]$, and*

$$\|\mathbb{E}\|_{\mathrm{TI}_r} =: M_r \le M_a^{1-\theta_r} M_b^{\theta_r} \le \max\{M_a, M_b\} \quad (r \in [a,b]), \tag{2.11}$$

*where $\theta_r := (r-a)/(b-a)$.*

*(a2) If $\mathbb{E} \in \mathrm{TIC}_\infty(U,Y)$, then $\mathbb{E} \in \mathrm{TIC}_r(U,Y)$ for all $r \in [a,\infty)$.*

*(b) $\mathbb{E}^* \in \mathrm{TI}_{-r}(Y,U)$, $\mathbb{E}^{\mathrm{d}} \in \mathrm{TI}_r(Y,U)$ and $\mathbb{E}^{-1} \in \mathrm{TI}_r(Y,U)$ are independent of $r$.*

*(c) We have $\mathbb{F} = \mathbb{G} \in \mathrm{TI}_a \cap \mathrm{TI}_b$ if $\mathbb{F} \in \mathrm{TI}_a$, $\mathbb{G} \in \mathrm{TI}_b$, $\mathbb{F}\phi = \mathbb{G}\phi$ for all $\phi \in \mathcal{F}$, and $\mathcal{F}$ is s.t. the translations of $\mathcal{F}$ span a dense subset of both $\mathrm{L}^2_a$ and $\mathrm{L}^2_b$ (e.g., $\mathcal{F}$ is as in Remark 2.1.9 or $\mathcal{F} = \{\chi_{[0,1]}\}$ or $\mathcal{F} = \{e^{-\cdot/2}\}$ (by Lemma D.1.25)).*

*(d) Let $\mathbb{D} \in \mathrm{TIC}_r(U,Y)$, $r \in \mathbf{R}$. Then $\mathbb{D} \in \mathrm{TIC}_{r'}(U,Y)$ and $\|\mathbb{D}\|_{\mathrm{TIC}_{r'}} \le \|\mathbb{D}\|_{\mathrm{TIC}_r}$ for all $r' \ge r$. Moreover, $\mathbb{D}^* \in \mathrm{TI}_{-r'}(Y,U)$, $\mathbb{D}^{\mathrm{d}} \in \mathrm{TIC}_{r'}(Y,U)$ and $\mathbb{D}^{-1} \in \mathrm{TI}$.*

*(e) Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, and let $\omega \in \mathbf{R}$. Then $\mathbb{D} \in \mathrm{TIC}_\omega$ iff $\mathbb{D}u \in \mathrm{L}^2_\omega$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$.*

*(f) Let $T \in \mathbf{R}$, $\widetilde{\mathbb{E}} \in \mathrm{TI}_a$ and $\mathbb{D} \in \mathrm{TIC}_a$. Let $\omega \in \mathbf{R}$. Then,*

$$\|\widetilde{\mathbb{E}}\|_{\mathrm{TI}_\omega} := \|\widetilde{\mathbb{E}}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \|\pi_{[T,\infty)}\widetilde{\mathbb{E}}\pi_{[T,\infty)}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \lim_{t\to+\infty} \|\widetilde{\mathbb{E}}^t\|_{\mathcal{B}(\mathrm{L}^2_\omega)} \tag{2.12}$$

$$= \sup_{u \in C_c^\infty(\mathbf{R}_+;U),\, \|u\|_{\mathrm{L}^2_\omega} \le 1} \|\pi_{[T,\infty)}\widetilde{\mathbb{E}}u\|_{\mathrm{L}^2_\omega}, \qquad \textit{and} \tag{2.13}$$

$$\|\mathbb{D}\|_{\mathrm{TIC}_\omega} := \|\mathbb{D}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \lim_{t\to+\infty} \|\mathbb{D}^t\|_{\mathcal{B}(\mathrm{L}^2_\omega)} \tag{2.14}$$

$$= \lim_{r\to\omega+} \|\mathbb{D}\|_{\mathrm{TIC}_r} = \sup_{r>\omega} \|\mathbb{D}\|_{\mathrm{TIC}_r} \tag{2.15}$$

*(the norms may be infinite for $\omega \ne a$; recall that $\widetilde{\mathbb{E}}^t := \pi_{[0,t)}\widetilde{\mathbb{E}}\pi_{[0,t)}$).*

*(g) Let $\mathbb{F} \in \mathrm{TI}_r$ for all $r \in (a,b)$. Then $M := \sup_{r\in(a,b)} \|\mathbb{F}\|_{\mathrm{TI}_r} < \infty$ iff $\mathbb{F} \in \mathrm{TI}_a \cap \mathrm{TI}_b$. If this is the case, then $M = \max\{\|\mathbb{F}\|_{\mathrm{TI}_a}, \|\mathbb{F}\|_{\mathrm{TI}_b}\}$.*

To give the reader a better intuition on $\mathrm{TI}_\infty$ operators, we mention a few results that will be shown later: the Fourier or Laplace transform ("transfer function" or "symbol") of a $\mathrm{TI}_a(U,Y)$ map is in $\mathrm{L}^\infty_{\mathrm{strong}}(a+i\mathbf{R};\mathcal{B}(U,Y))$ (Theorem 3.1.3), that of a $\mathrm{TI}_a \cap \mathrm{TI}_b$ map is also in $\mathrm{H}((a,b);\mathcal{B}(U,Y))$ (Theorem 3.1.6), and that of a $\mathrm{TIC}_a$ map also in $\mathrm{H}^\infty(\mathbf{C}^+_a;\mathcal{B}(U,Y))$ (Theorem 2.1.2).

**Proof:** Recall that simple functions (hence $\mathrm{L}^2_a \cap \mathrm{L}^2_b$) are dense in $\mathrm{L}^2_r$ for $r \in [a,b]$, by Theorem B.3.11.

(a1) See Proposition E.1.8 and Remark 2.1.9.

(a2) Now $\pi_-\mathbb{E}\pi_+ = 0$ on any $\mathrm{L}^2_r$ ($r \in [a,b]$), by continuity. From (d) we get that $\mathbb{E} \in \mathrm{TIC}_r$ for any $r \ge a$.

(b) It is obvious from (2.6) that $(\mathbb{E}_r)^* = (\mathbb{E}_a)^*$ on $\mathrm{L}^2_{-a} \cap \mathrm{L}^2_{-r}$, hence $\mathbb{E}_r^* = \mathbb{E}_a^*$, (by definition (or by (c)).

Consequently, $\boldsymbol{Я}\mathbb{E}_a^*\boldsymbol{Я} = \boldsymbol{Я}\mathbb{E}_r^*\boldsymbol{Я}$ (on $\mathrm{L}^2_a \cap \mathrm{L}^2_r$).

Finally, if $\mathbb{E}$ is invertible in $\mathrm{TI}_{r_k}$ and in $\mathrm{TI}_{r_k}$ ($r_k \in [a,b]$) for $k = 1,2$, then $\mathbb{E}^{-1}$ maps $\mathrm{L}^2_{r_1} \cap \mathrm{L}^2_{r_2}$ onto itself (as does $\mathbb{E}$ too); in particular, $\mathbb{E}_{r_1}^{-1} = \mathbb{E}_{r_2}^{-1}$ (on $\mathrm{L}^2_{r_1} \cap \mathrm{L}^2_{r_2}$).

(c) This follows almost directly from Remark 2.1.9 (the Fourier transforms of $\chi_{[0,1]}\}$ and $\mathrm{e}^{-\cdot/2}$ are $\neq 0$ a.e.; cf. Lemma D.1.25).

(d) By Theorem 2.1.2, $\mathbb{D}$ extends to a $\mathrm{TIC}_{r'}$ map for $r' \geq r$. The rest follows from (b).

(Note that $\tau(-1) \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TI} \setminus \mathcal{G}\mathrm{TIC}_\infty$, since $\tau(1)$ is noncausal.)

(e) "Only if" is trivial, so assume that $\mathbb{D} \in \mathrm{TIC}_\alpha$, $\alpha \geq \omega$, and $\mathbb{D}u \in \mathrm{L}^2_\omega$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. The continuous inclusion $\pi_+\mathrm{L}^2_\omega \subset \mathrm{L}^2_\alpha$ implies that $\mathbb{D}u \in \mathcal{B}(\pi_+\mathrm{L}^2_\omega, \mathrm{L}^2_\alpha)$, hence $\mathbb{D}u \in \mathcal{B}(\pi_+\mathrm{L}^2_\omega, \mathrm{L}^2_\omega)$, by Lemma A.3.6. Thus, (2.10) is satisfied, hence $\mathbb{D} \in \mathrm{TI}_\omega$. By (a2), $\mathbb{E} \in \mathrm{TIC}_\omega$.

(f) Obviously, we can w.l.o.g. assume that $\omega = 0$ (cf. Remark 2.1.6) and $T = 0$ (use time-invariance). By Remark 2.1.9, we have

$$\|\widetilde{\mathbb{E}}\| = \sup_{u \in C_c^\infty(\mathbf{R};U),\, \|u\|_{\mathrm{L}^2_\omega} \leq 1} \|\mathbb{E}u\|_{\mathrm{L}^2_\omega}. \tag{2.16}$$

Assume, that $M < \|\widetilde{\mathbb{E}}\|$. Choose $u \in C_c^\infty(\mathbf{R};U)$ s.t. $\|u\|_2 = 1$ and $\|\widetilde{\mathbb{E}}u\| > M$. By Corollary B.3.8, we have $\|\pi_+\widetilde{\mathbb{E}}\tau^{-t}u\| > M$ for $t$ big enough; take $t$ so big that we also have $\tau^{-t}u \in C_c^\infty(\mathbf{R}_+;U)$ to establish (2.13). Obviously, (2.12) follows from (2.13).

The first two equalities for $\|\mathbb{D}\|$ follow from the above. Also the $\mathbb{D}^t$ claim follows from Corollary B.3.8, so only (2.15) remains to be proved.

Let $\|u\|_{\mathrm{L}^2} = 1$. Then $\|\mathbb{D}u\|_{\mathrm{L}^2_r}/\|u\|_{\mathrm{L}^2_r} \to \|\mathbb{D}u\|_{\mathrm{L}^2}$, by (2.5), hence $\|\mathbb{D}\| \leq \sup_{r>\omega}\|\mathbb{D}\|_{\mathrm{TIC}_r}$. But $\|\mathbb{D}\|_{\mathrm{TIC}_r}$ is decreasing in $r$, by (d), hence $\|\mathbb{D}\|$ is given by (2.15).

(g) (In fact, we may replace $(a,b)$ by any $D \subset \mathbf{R}$ s.t. $a,b \in \bar{D}$.) This follows from Lemma E.1.9. □

If(f) $\|\mathbb{D}\|_{\mathrm{TI}_\alpha}$ is (finite and) bounded, as $\alpha \to +\infty$, then $\mathbb{D}$ is causal:

**Lemma 2.1.11** *Let $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$, $\omega \in \mathbf{R}$. Then the following are equivalent:*

*(i)* $\mathbb{D} \in \mathrm{TIC}_\alpha$ *for some* $\alpha \geq \omega$*;*

*(ii)* $\mathbb{D} \in \mathrm{TIC}_\alpha$ *for all* $\alpha \geq \omega$*;*

*(iii)* $\mathbb{D} \in \mathrm{TI}_\alpha$ *and* $\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \leq \|\mathbb{D}\|_{\mathrm{TI}_\omega}$ *for all* $\alpha \geq \omega$*;*

*(iv)* $\mathbb{D} \in \mathrm{TI}_\alpha$ *for all* $\alpha \geq \omega$*, and* $\mathrm{e}^{-\alpha\varepsilon}\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \to 0$ *for all* $\varepsilon > 0$.

Thus, if $\|\mathbb{D}\|_{\mathrm{TI}_\alpha}$ does not grow with an exponential speed, as $\alpha \to +\infty$ (cf. (iv)), then it is bounded and $\mathbb{D}$ is causal.

On the other hand, $\tau^t \in \mathrm{TI}_\omega(U)$ for all $t, \omega \in \mathbf{R}$, but $\tau^t \notin \mathrm{TIC}_\infty$ for $t > 0$ ($\tau^t$ is not causal because it maps backwards in time), and $\|\tau^t\|_{\mathrm{TI}_\alpha} = \mathrm{e}^{\alpha t}$, by (2.2), so the estimate in (iv) is the "best possible one".

**Proof:** 1° *"(ii)⇒(i)" and "(iii)⇒(iv)":* This is trivial.

2° *"(i)⇒(ii)&(iii)":* This follows from Lemma 2.1.10(d).

3° *"(iv)⇒(i)":* Assume that (i) does not hold, so that $y := \pi_- \mathbb{D}\pi_+ u \neq 0$ for some $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$.

Choose $\varepsilon > 0$ s.t. $r := \|\pi_{(-\infty,-\varepsilon)} y\|_{\mathrm{L}^2_\omega} > 0$. Then, by (2.2),

$$\|y\|_{\mathrm{L}^2_\alpha} := \|e^{-\alpha\cdot} y\|_2 \geq e^{(\alpha-\omega)\varepsilon}\|y\|_{\mathrm{L}^2_\omega} \geq e^{\alpha\varepsilon}\delta', \tag{2.17}$$

where $\delta' := e^{-\omega\varepsilon} r > 0$. Consequently, $\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \geq e^{\alpha\varepsilon}\delta$ ($\alpha \geq \omega$), where $\delta := \delta'/\|u\|_{\mathrm{L}^2_\omega} > 0$, because $\|u\|_{\mathrm{L}^2_\omega} \geq \|u\|_{\mathrm{L}^2_\alpha} > 0$ ($\alpha \geq \omega$).

It follows that $e^{-\alpha\varepsilon}\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \geq \delta \not\to 0$, so that (iv) does not hold. Therefore, (iv) implies (i). □

**Lemma 2.1.12 ($\mathbb{D}$ is closed)** *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *and* $\omega \in \mathbf{R}$*. Then* $\mathbb{D}$ *is closed on* $\mathrm{L}^2_\omega(\mathbf{R}_+;U)$ *(here we set* $\mathrm{Dom}(\mathbb{D}) := \{u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) \,|\, \mathbb{D}u \in \mathrm{L}^2_\omega\}$*).*

**Proof:** Choose $\alpha > \omega$ s.t. $\mathbb{D} \in \mathrm{TIC}_\alpha$. If $u_n \to u$ and $\mathbb{D}u_n \to y$ in $\mathrm{L}^2_\omega$, then $u_n \to u$ in $\mathrm{L}^2_\alpha$, hence $\mathbb{D}u_n \to \mathbb{D}u$ in $\mathrm{L}^2_\alpha$, hence $y = \mathbb{D}u$ a.e., by Theorem B.3.2. □

We set $\mathrm{L}^2_c(\mathbf{R};U) := \{u \in \mathrm{L}^2(\mathbf{R};U) \,|\, \operatorname{supp} u \text{ is bounded}\}$. If $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$, then $\mathbb{D}$ is "almost stable" (cf. Lemma 6.1.11 and Theorem 3.3.1(a4)&(c3)):

**Lemma 2.1.13 ($\mathbb{D}\mathrm{L}^2_c \subset \mathrm{L}^2$)** *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *and* $\omega \in \mathbf{R}$*, and let there be* $T > 0$ *s.t.* $\mathbb{D}\pi_{[0,T)}u \in \mathrm{L}^2_\omega$ *for* $u \in \mathrm{L}^2_c$*. Then* $M := \|\mathbb{D}\pi_{[0,T)}\|_{\mathcal{B}(\mathrm{L}^2_\omega,\mathrm{L}^2_\omega)} < \infty$*, and*

$$\|\mathbb{D}\|_{\mathrm{TIC}_\beta} \leq M_{\beta,\omega,T} M \qquad (\beta > \omega), \tag{2.18}$$

$$\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \leq M_{\alpha,\omega,T} M \|u\|_{\mathrm{L}^2_\alpha} \qquad (\alpha < \omega,\ u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)). \tag{2.19}$$

*Thus, then* $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2_\omega$ *and* $\mathbb{D}\pi_{[0,t)}u \to \mathbb{D}u$ *and* $\mathbb{D}^t u \to \mathbb{D}u$ *in* $\mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U) + \mathrm{L}^2_c$*,* $\alpha < \omega$*. Moreover,* $(s+r+\omega)^{-1}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *for all* $r > 0$*; in particular,* $\widehat{\mathbb{D}}(\frac{1-\cdot}{1+\cdot}) \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$ *if* $\omega < 1$*.*

(See Definition 13.2.2 for $\widehat{\heartsuit}$.)

**Proof:** (In fact, the lemma holds even for $\mathrm{TIC}_{\mathrm{loc}}$ (see Section 8 of [Sbook] for the definition) in place of $\mathrm{TIC}_\infty$, with virtually the same proof.)

Except for the $\mathrm{H}^2_{\mathrm{strong}}$ claims, this follows from Lemma 13.1.3 through discretization. (Recall that $\mathrm{L}^2_c(\mathbf{R};U) = \cup_{T>0}\mathrm{L}^2([-T,T);U)$.)

Since $\widehat{\mathbb{D}}(s+r+\omega)^{-1}u_0 \in \mathrm{H}^2_\omega$ for all $u_0 \in U$ (note that $\mathcal{L}^{-1}(s+r+\omega)^{-1}u_0 \in \mathrm{L}^2_{\omega-r/2}$; see also Theorem 3.3.1(b) and Theorem 2.1.2) we have $(s+r+\omega)^{-1}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$, by Lemma F.3.3(a1).

Set $t := r+\omega$ to observe from Lemma 13.2.1(e2) that $z \mapsto (1+z)[\frac{1-z}{1+z} + t]\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$, i.e., $[1+t+(t-1)z]\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$, for all $t > \omega$. If $\omega < 1$, we can take $t = 1$ to obtain that $\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$. □

**Lemma 2.1.14 ($\mathbb{D}^{t\,*}\mathbb{D}^t \le \gamma^2 I \Rightarrow \|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$)** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\gamma \in \mathbf{R}$. If*

$$\mathbb{D}^{t\,*}\mathbb{D}^t \le \gamma^2 I \quad \text{for all } t > 0, \tag{2.20}$$

*then $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$. If $J \gg 0$ is s.t. that $\mathbb{D}^{t\,*}J\mathbb{D}^t \le \gamma^2 I$ for all $t > 0$, then $\mathbb{D} \in \mathrm{TIC}(U,Y)$.* □

**Proof:** (Obviously, the theorem also holds with TI in place of TIC.)

The first claim follows from (2.14). If $J \ge \varepsilon I$, $\varepsilon > 0$, then $\mathbb{D}^{t\,*}J\mathbb{D}^{t\,*} \ge \varepsilon\mathbb{D}^{t\,*}\mathbb{D}^{t\,*}$, hence also the second claim holds. □

**Lemma 2.1.15 ($(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$)** *Let $\mathbb{D} \in \mathrm{TIC}_\omega$ and let $\mathrm{Re}\, s > \omega$. If $u_0 \in U$ and $u = \mathrm{e}^{s\cdot}u_0$, then $(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$ for all $t \in \mathbf{R}$.* □

(This is Lemma 6.10 of [S98c], originating from [W91a].) Note that $\pi_- u \in \mathrm{L}^2_\omega$, and that $\mathbb{D}$ was extended as explained in Definition 2.1.1.

### Notes

Theorem 2.1.2 is from [W91a]. Corollary 2.1.8, Lemma 2.1.15 and much of Definitions 2.1.1 and 2.1.4 and of our notation are from [S97a] and [S98c]. The case $\omega = 0$ of Lemma 2.1.7 is Theorem 5.2C on p. 96 [RR].

A standard reference for the theory on time-invariant operators is [RR].

The main new contributions of this section are the theory of $\mathrm{TI}_\omega \cap \mathrm{TI}_{\omega'}$ (see 2.1.9–2.1.11 and 3.1.6, and the fact that $\mathbb{D}$ is “almost stable” if $\mathbb{D}\pi_{[0,T)}\mathrm{L}^2 \subset \mathrm{L}^2$ (see Lemma 2.1.13 and the references above it).

## 2.2 $\mathcal{G}\mathrm{TIC}$ — invertibility

*And that inverted Bowl we call The Sky,*
*Whereunder crawling coop’t we live and die,*
*Lift not thy hands to It for help – for It*
*Rolls impotently on as Thou or I.*

— Omar Khayyam (1048–1131)

In this section we study the invertibility properties of $\mathrm{TI}_\infty$ operators. Invertibility and left invertibility (and right invertibility, by duality) are further studied in Chapter 4. We start from the general (noncausal) case:

**Lemma 2.2.1** *Let $U,Y$ be Hilbert spaces, $\omega \in \mathbf{R}$, and $\varepsilon > 0$.*

*(a1) Let $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then $\mathbb{E}^*\mathbb{E} \gg 0$ iff $\mathbb{X}\mathbb{E} = I$ for some $\mathbb{X} \in \mathrm{TI}(Y,U)$.*

*(a2) Let $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$. Then $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega \Leftrightarrow \mathbb{E} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U),\mathrm{L}^2_\omega(\mathbf{R};Y))$.*

*(b) $\mathbb{X},\mathbb{E} \in \mathrm{TI}_\omega(\mathbf{C}^n)$ & $\mathbb{X}\mathbb{E} = I \implies \mathbb{E}\mathbb{X} = I$.*

*(c1) $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^n)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \implies \mathbb{E} \in \mathcal{G}\mathrm{TI}(\mathbf{C}^n)$ & $\mathbb{E}\mathbb{E}^* \gg 0$.*

*(c2) $\mathbb{E} \in \mathrm{TI}(U,\mathbf{C}^n)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \implies \dim U \le n$ (and $\mathbb{E} \in \mathcal{G}\mathrm{TI} \Leftrightarrow \dim U = n$).*

*(c3) $\mathbb{E} \in \mathrm{TI}(U,Y)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \implies \dim U \le \dim Y$.*

*(c4)* $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega(U,Y) \implies \dim U = \dim Y$.

*(d)* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ & $\mathbb{D}^{t\,*}\mathbb{D}^t \geq \varepsilon I$ *for all* $t > 0 \implies \dim U \leq \dim Y$.

By $\dim H$ we mean the cardinality of an arbitrary Hilbert basis of $H$. (E.g., $\dim \ell^2(\mathbf{N}) < \dim \ell^2(\mathbf{R})$; cf. Lemma B.3.16.) Naturally, also the shifted versions of the above results hold; (e.g., if some $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ is coercive, then $\dim U \leq \dim Y$).

**Proof:** We use here Theorem 2.1.2 and the separable case of Theorem 3.1.3(a)&(c) (both are known results in that extent). We take $\omega = 0$ w.l.o.g. (see Remark 2.1.6).

(a1) If $\mathbb{E}^*\mathbb{E} \gg 0$, then $\mathbb{X} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}^* \in \mathrm{TI}(Y,U)$ and $\mathbb{X}\mathbb{E} = I$. Conversely, if $\mathbb{X}\mathbb{E} = I$, then

$$\langle u, \mathbb{E}^*\mathbb{E}u\rangle = \|\mathbb{E}u\|^2 \geq \varepsilon\|u\|^2 \quad \text{for all } u \in \mathrm{L}^2(\mathbf{R},U), \tag{2.21}$$

where $\varepsilon := 1/\|\mathbb{X}\|^2 > 0$, i.e., $\mathbb{E}^*\mathbb{E} \geq \varepsilon I$.

(a2) "Only if" is trivial, so assume that $\mathbb{E} \in \mathrm{TI} \cap \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R};U), \mathrm{L}^2(\mathbf{R};Y))$. Then $\mathbb{E}^{-1}\tau^t = (\tau^{-t}\mathbb{E})^{-1} = (\mathbb{E}\tau^{-t})^{-1} = \tau^t\mathbb{E}^{-1}$ for all $t \in \mathbf{R}$, hence then $\mathbb{E}^{-1} \in \mathrm{TI}(Y,U)$.

(b) $\widehat{\mathbb{X}}, \widehat{\mathbb{Z}} \in \mathrm{L}^\infty(\mathbf{C}^{n\times n})$ and $\widehat{\mathbb{X}}(it)\widehat{\mathbb{Z}}(it) = I$ a.e. on $i\mathbf{R}$, hence $\widehat{\mathbb{Z}}(it)\widehat{\mathbb{X}}(it) = I$ a.e. on $i\mathbf{R}$.

(N.B. This would not hold even for static (constant) operators if the Hilbert space $\mathbf{C}^n$ were replaced by an infinite-dimensional one.)

(c1) Take $\mathbb{X} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}$ and use (b).

(c2) $U$ is separable since $\mathbb{E}^*D$ is dense in $\mathrm{L}^2(\mathbf{R};U)$ for any $D$ dense in $\mathrm{L}^2(\mathbf{R};\mathbf{C}^n)$ (because $\mathbb{E}^*$ is onto). By Theorem 3.1.3, $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$ and $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$. If $it \in i\mathbf{R}$ is such that $\widehat{\mathbb{E}}(it)^*\widehat{\mathbb{E}}(it) \geq \varepsilon I$, then $\widehat{\mathbb{E}}(it) \in \mathcal{B}(U,\mathbf{C}^n)$ is coercive, hence $\dim U \leq n$, by Lemma A.3.1(a4).

If $\dim U = n$, then $\exists \mathbb{E}^{-1}$ by (c1), otherwise $\widehat{\mathbb{E}}(it)$ is coercive a.e. and nowhere onto, hence it can't have an inverse.

*Remark:* There are $\mathbb{E}, \mathbb{X} \in \mathcal{B}(\mathrm{L}^2(U), \mathrm{L}^2(\mathbf{C}^n))$ for s.t. $\mathbb{E}^*\mathbb{E} \gg 0$, $\mathbb{X}^*\mathbb{X} \gg 0$, $\mathbb{E} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(U), \mathrm{L}^2(\mathbf{C}^n))$ and $\mathbb{X}$ is not onto, by Lemma B.3.16, for any $n \in \{1,2,3,\ldots\}$ and any separable $U$. Therefore, time-invariance is not superfluous in (b)–(c4).

(c3) By (c2), we may assume that $\dim Y$ is infinite, so that $\dim \mathrm{L}^2(\mathbf{R};Y) = \dim Y$, by Lemma B.3.16. By Lemma A.3.1(a4), the coercivity of $\mathbb{E}$ implies that $\dim \mathrm{L}^2(\mathbf{R};U) \leq \dim \mathrm{L}^2(\mathbf{R};Y)$. Consequently, $\dim U \leq \dim \mathrm{L}^2(\mathbf{R};U) \leq \dim \mathrm{L}^2(\mathbf{R};Y) = \dim Y$.

(c4) Now $\mathbb{E}^*\mathbb{E} \gg 0$ and $\mathbb{E}\mathbb{E}^* \gg 0$, hence $\dim U = \dim Y$, by (c3).

(d) By Lemma 2.2.4(a), we have $\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$ on $\mathbf{C}^+_\omega$, where $\omega \geq 0$ is s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$. Consequently, $\dim U \leq \dim Y$, by Lemma A.3.1(a4) $\square$

For any $\mathbb{E} \in \mathrm{TI}$, we denote below the Toeplitz operator (or Wiener–Hopf operator) $\pi_+\mathbb{E}\pi_+$ of $\mathbb{E}$ by $\mathbf{T}_\mathbb{E}$.

**Lemma 2.2.2 (Toeplitz operators)** *Let* $\mathbb{E} \in \mathrm{TI}(U)$ *and* $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}(U)$. *Then* $\mathbf{T}_{\mathbb{E}^*} = \mathbf{T}_\mathbb{E}^*$, *and the following is true:*

*(a1) If* $\mathbf{T}_{\mathbb{E}}$ *is invertible, then so are* $\mathbb{E}$, $\mathbf{T}_{\mathbb{E}^*}$ *and* $\pi_-\mathbb{E}^{\mathrm{d}}\pi_-$.

*(a2)* $\mathbf{T}_{\mathbb{X}}$ *is invertible iff* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$.

*(b) Let* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$. *Then* $\mathbf{T}_{\mathbb{E}}$ *invertible iff* $\mathbf{T}_{\mathbb{Y}^*\mathbb{E}\mathbb{X}}$ *is invertible.*

*(c1) If* $\mathbb{E} \in \mathcal{G}\mathrm{TI}$, *then following are equivalent:*

*(i)* $\mathbf{T}_{\mathbb{E}}$ *is invertible;*
*(ii)* $\pi_-\mathbb{E}^{-1}\pi_-$ *is invertible;*
*(iii)* $\mathbb{E}\pi_+ + \pi_- \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$*;*
*(iv)* $\pi_+\mathbb{E} + \pi_- \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$*;*
*(v)* $\mathrm{Re}\langle \mathbb{E}\mathcal{H}u, u\rangle \geq \delta\|u\|_2^2$ *for all* $u \in \mathrm{L}^2$ *for some* $\mathcal{H} \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$ *and some* $\delta > 0$.

*(c2)* **("No equalizing vectors" condition)** *Let* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C}^n)$, $n \in \mathbf{N}$. *If* $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ *(i.e.,* $\det\widehat{\mathbb{E}}(ir) \neq 0$ *for all* $r \in \mathbf{R}\cup\{\infty\}$*), then* $\mathbf{T}_{\mathbb{E}}$ *is invertible iff* $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}}) = \{0\}$.

*(d) We have* $\mathbb{E} \gg 0 \Leftrightarrow \mathbf{T}_{\mathbb{E}} \gg 0$*; in particular,* $\mathbb{E} \gg 0$ *implies that* $\mathbf{T}_{\mathbb{E}}$ *is invertible.*

See [DS] for further equivalent conditions for the invertibility of $\mathbb{E}$ (and the existence of a non-TI spectral factorization) in a more general (non-TI) setting.

**Proof:** (In the lemma, "$\mathrm{L}^2$" denotes $\mathrm{L}^2(\mathbf{R}_+;U)$.)

(a1) By Lemma 4.4 of [S98c], the invertibility of $\mathbf{T}_{\mathbb{E}}$ implies that of $\mathbb{E}$. (The converse does not hold: $\tau(1) \in \mathcal{G}\mathrm{TI}$ but $\mathbf{T}_{\tau(1)}$ is not onto.)

Let $\mathbf{T}_{\mathbb{E}}$ be invertible. Because $\pi_+^* = \pi_+$, we have $(\pi_+\mathbb{E}\pi_+)^* = \pi_+\mathbb{E}^*\pi_+$; hence $\mathbf{T}_{\mathbb{E}^*}^{-1} = \mathbf{T}_{\mathbb{E}}^{-*}$. Moreover, $\pi_-\mathbb{E}^{\mathrm{d}}\pi_- = \text{Я}\pi_+\mathbb{E}^*\pi_+\text{Я}$ implies the invertibility of $\pi_-\mathbb{E}^{\mathrm{d}}\pi_-$ (on $\pi_-\mathrm{L}^2$).

(a2) If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{X}\pi_+ = \pi_+\mathbb{X}\pi_+ + \pi_-\mathbb{X}\pi_+ = \pi_+\mathbb{X}\pi_+$, by causality, hence then

$$\pi_+\mathbb{X}^{-1}\pi_+\pi_+\mathbb{X}\pi_+ = \pi_+\mathbb{X}^{-1}\mathbb{X}\pi_+ = \pi_+ = I_{\pi_+\mathrm{L}^2} = \pi_+\mathbb{X}\pi_+\pi_+\mathbb{X}^{-1}\pi_+. \tag{2.22}$$

Conversely, if $\pi_+\mathbb{X}\pi_+ = \mathbb{X}\pi_+$ is invertible, then $\mathbb{X} \in \mathcal{G}\mathrm{TI}$, by (a1), and $\pi_+\mathrm{L}^2 = \mathbb{X}\pi_+\mathrm{L}^2$, hence then $\mathbb{X}^{-1}\pi_+\mathrm{L}^2 = \mathbb{X}^{-1}\mathbb{X}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$, i.e., $\mathbb{X}^{-1}$ is causal.

(b) Now $\mathbf{T}_{\mathbb{X}}$ and $\mathbf{T}_{\mathbb{Y}^*}$ are invertible, by (a2), so the claim follows from equation

$$\mathbf{T}_{\mathbb{Y}^*\mathbb{E}\mathbb{X}} := \pi_+\mathbb{Y}^*\mathbb{E}\mathbb{X}\pi_+ = \pi_+\mathbb{Y}^*\pi_+\mathbb{E}\pi_+\mathbb{X}\pi_+ = \mathbf{T}_{\mathbb{Y}^*}\mathbf{T}_{\mathbb{E}}\mathbf{T}_{\mathbb{X}}. \tag{2.23}$$

(c1) (N.B. (v) holds iff $(\mathbb{E}\mathcal{H}) + (\mathbb{E}\mathcal{H})^* \gg 0$.)

Claims (iii) and (iv) are equivalent to (i) by equations

$$\mathbb{E}\pi_+ + \pi_- = \begin{bmatrix} \pi_+\mathbb{E}\pi_+ & 0 \\ \pi_-\mathbb{E}\pi_+ & \pi_- \end{bmatrix}, \quad \pi_+\mathbb{E} + \pi_- = \begin{bmatrix} \pi_+\mathbb{E}\pi_+ & \pi_+\mathbb{E}\pi_- \\ 0 & \pi_- \end{bmatrix} \tag{2.24}$$

(on $\mathrm{L}^2 = \pi_+\mathrm{L}^2 \times \pi_-\mathrm{L}^2$), respectively. Multiply the former to the left by $\mathbb{E}^{-1}$ to obtain $\pi_- + \mathbb{E}^{-1}\pi_-$ to obtain the equivalence "(iii)$\Leftrightarrow$(ii)". Equivalence "(i)$\Leftrightarrow$(v)" is [DS, Theorem 3] combined with (a1).

(c2) (Nonzero elements of $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}})$ are called equalizing vectors. This "No equalizing vectors" condition was established for rational transfer functions in [Meinsma].)

This proved in [IZ01] (see also Theorem 4.1.1(a)(ii)&(v')). ("Only if" is trivial, "if" follows by noting that the middle element of the standard factorization of $\mathbb{E}$ (see Theorem II.6.3 of [CG81]) must be constant.)

*Remark:* It is not possible to extend this result to an arbitrary Hilbert space $H$ in place of $\mathbf{C}^n$ by using classical factorization results (collected in Theorem 5.1.6). (And we do not know whether such an extension is true.)

Indeed, Theorem 5.1.6(a) requires the additional assumption that $\mathbf{T}_{\mathbb{E}}$ is a Fredholm operator (note that for self-adjoint $\mathbb{E}$, this additional assumption together with $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}}) = \{0\}$ is equivalent to the invertibility of $\mathbf{T}_{\mathbb{E}}$ for any $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}$, not merely for $\mathbb{E} \in \mathcal{G}\mathrm{MTI}^{\mathrm{L}^1}$, by Lemma A.3.1(c7)&(c2)(ii)).

(d) This is Lemma 4.4 of [S98c]. $\square$

**Lemma 2.2.3 ($\mathcal{G}$TIC)** *Let $\mathbb{X} \in \mathrm{TI}(U)$. Then the following are equivalent:*

*(i)* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$*;*

*(ii)* $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$*;*

*(iii)* $\mathbb{X} \in \mathrm{TIC}$ *and* $\pi_+\mathbb{X}\pi_+$ *is invertible on* $\pi_+\mathrm{L}^2$*;*

*(iv)* $\mathbb{X} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ *and* $\mathbb{X}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$*;*

*(v)* $\mathbb{X} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ *and* $\mathbb{X}^*\pi_-\mathrm{L}^2 = \pi_-\mathrm{L}^2$*;*

*(vi)* $\mathbb{X} \in \mathrm{TIC}$, $\mathbb{X}^*\mathbb{X} \gg 0$ *and* $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$ *on* $\pi_+\mathrm{L}^2$*;*

*(vii)* $\mathbb{X} \in \mathrm{TIC}$, $\mathbb{X}\mathbb{X}^* \gg 0$ *and* $\pi_-\mathbb{X}^*\pi_-\mathbb{X}\pi_- \gg 0$ *on* $\pi_-\mathrm{L}^2$.

*If* $\dim U < \infty$*, then one more equivalent condition is that* $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$*; equivalently, we may accept right invertibility in (iii).*

See also Proposition 2.2.5 and Theorem 4.1.1(b).

**Proof:** 1° *The equivalence:* Because $\mathbb{X} \in \mathrm{TIC} \Rightarrow \mathbb{X}^{\mathrm{d}} \in \mathrm{TIC}$, we obviously have "(i)⇔(ii)". The equivalence "(i)⇔(vi)⇔(vii)" is Lemma 4.11 of [S98c], and "(i)⇔(iii)" is Lemma 2.2.2(a2).

"(i)⇔(iv)": Clearly (i) and (iii) imply (iv) and (iv) implies (iii).

"(ii)⇔(v)": Apply "(i)⇔(iv)" to $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ to obtain

$$\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \pi_+\mathrm{L}^2 = \mathbb{X}^{\mathrm{d}}\pi_+\mathrm{L}^2 = \boldsymbol{\mathcal{R}}\mathbb{X}^*\boldsymbol{\mathcal{R}}\pi_+\mathrm{L}^2 = \boldsymbol{\mathcal{R}}\mathbb{X}^*\pi_-\mathrm{L}^2. \tag{2.25}$$

2° *Case* $\dim U < \infty$*:* If $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$, then there is $\varepsilon > 0$ s.t. $\|\pi_{[t,+\infty)}\mathbb{X}^*\pi_{[t,+\infty)}u\| \geq \varepsilon\|[t,+\infty)u\|$ for $u \in \mathrm{L}^2$ and $t = 0$. By time-invariance, this holds for all $t \in \mathbf{R}$; by continuity, $\|\mathbb{X}^*u\| \geq \varepsilon\|u\|$ for all $u \in \mathrm{L}^2$. If $\dim U < \infty$, then $\mathbb{X}^* \in \mathcal{G}\mathrm{TI}$, by Lemma 2.2.1(a)&(b), hence then (vi) holds. The converse is trivial. (Our favorite counter-example $\mathbb{X} = \tau^{-1} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TI} \setminus \mathcal{G}\mathrm{TIC}$ shows that left-invertibility is not sufficient.) $\square$

**Lemma 2.2.4 ($\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon I \Rightarrow \widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$)**

*(a) If* $\omega \geq 0$, $\mathbb{D}_k \in \mathrm{TIC}_\omega(U,*)$ *(*$k = 1,2,3,4$*) and* $\mathbb{D}_1^{t*}\mathbb{D}_2^t \geq \mathbb{D}_3^{t*}\mathbb{D}_4^t$ *for all* $t > 0$*, then*

$$\widehat{\mathbb{D}}_1(s)^*\widehat{\mathbb{D}}_2(s) \geq \widehat{\mathbb{D}}_3(s)^*\widehat{\mathbb{D}}_4(s) \qquad (\operatorname{Re} s > \omega). \tag{2.26}$$

*(b) If* $\mathbb{D}_k \in \mathrm{TIC}(U,*)$ *(*$k = 1,2,3,4$*), then the following are equivalent:*

*(i)* $\mathbb{D}_1^{t*}\mathbb{D}_2^t \geq \mathbb{D}_3^{t*}\mathbb{D}_4^t$ *for all* $t > 0$*;*
*(ii)* $\pi_{[-t,0)}\mathbb{D}_1^*\pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)} \geq \pi_{[-t,0)}\mathbb{D}_3^*\pi_{[-t,0)}\mathbb{D}_4\pi_{[-t,0)}$ *for all* $t > 0$*;*
*(iii)* $\mathbb{D}_1^{\mathrm{d}\,t}\mathbb{D}_2^{\mathrm{d}\,t*} \geq \mathbb{D}_3^{\mathrm{d}\,t}\mathbb{D}_4^{\mathrm{d}\,t*}$ *for all* $t > 0$*;*
*(iv)* $\pi_-\mathbb{D}_1^*\pi_-\mathbb{D}_2\pi_- \geq \pi_-\mathbb{D}_3^*\pi_-\mathbb{D}_4\pi_-$*;*
*(v)* $\pi_+\mathbb{D}_1^{\mathrm{d}}\pi_+\mathbb{D}_2^{\mathrm{d}*}\pi_+ \geq \pi_+\mathbb{D}_3^{\mathrm{d}}\pi_+\mathbb{D}_4^{\mathrm{d}*}\pi_+$.

*If (i)–(v) hold, then* $\pi_+\mathbb{D}_1^*\mathbb{D}_2\pi_+ \geq \pi_+\mathbb{D}_3^*\mathbb{D}_4\pi_+$ *and (2.26) holds.*

*(c) If* $0 \leq J \in \mathcal{B}(Y)$, $0 \leq S \in \mathcal{B}(U)$ *and* $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, then the following are equivalent:*

*(i)* $\mathbb{D}^{t*}J\mathbb{D}^t \leq \pi_{[0,t)}S$ *for all* $t > 0$*;*
*(ii)* $\pi_{[-t,0)}\mathbb{D}^*\pi_{[-t,0)}J\mathbb{D}\pi_{[-t,0)} \leq \pi_{[-t,0)}S$ *for all* $t < 0$*;*
*(iii)* $(\mathbb{D}^{\mathrm{d}})^tJ(\mathbb{D}^{\mathrm{d}})^{t*} \leq \pi_{[0,t)}S$ *for all* $t > 0$*;*
*(iv)* $\pi_-\mathbb{D}^*\pi_-J\mathbb{D}\pi_- \leq \pi_-S$*;*
*(v)* $\pi_+\mathbb{D}^*\pi_+J\mathbb{D}\pi_+ \leq \pi_+S$.

Recall that $P \geq Q$ means that $P = P^*$, $Q = Q^*$ and $\langle u, Pu\rangle \geq \langle u, Qu\rangle$ for all $u$. We do not know whether (b)(i)–(v) is implied by (2.26).

**Proof:** When proving (a) and (b), we assume that $\mathbb{D}_3 = 0 = \mathbb{D}_4$ w.l.o.g. (use substitutions $\mathbb{D}_1 \mapsto \begin{bmatrix}\mathbb{D}_1\\ \mathbb{D}_3\end{bmatrix}$ and $\mathbb{D}_2 \mapsto \begin{bmatrix}\mathbb{D}_2\\ -\mathbb{D}_4\end{bmatrix}$).

(a) Let $s \in \mathbf{C}_\omega^+$ and $u_0 \in U$ be given. Set $u := \mathrm{e}^{s\cdot}u_0 \in \mathrm{L}^2_{\mathrm{loc}}$. Then $\pi_-u \in \mathrm{L}^2 \cap \mathrm{L}^2_\omega$, hence $\pi_-\mathbb{D}_ku \in \mathrm{L}^2_\omega(\mathbf{R}_-;Y) \subset \mathrm{L}^2(\mathbf{R}_-;Y)$ ($k = 1,2$). By Lemma 2.1.15, we have that $(\mathbb{D}_ku)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}_k(s)u_0$ ($t \in \mathbf{R}$). By time-invariance,

$$0 \leq \langle \mathbb{D}_1^t\tau^{-t}u, \mathbb{D}_2^t\tau^{-t}u\rangle = \int_{-\infty}^0 \langle \mathbb{D}_1\pi_{[-t,0)}u, \mathbb{D}_2\pi_{[-t,0)}u\rangle\,dt \;\; (t > 0). \tag{2.27}$$

Now $\pi_-\mathbb{D}_k\pi_{(-\infty,t)}u \to 0$ in $\mathrm{L}^2_\omega$, hence in $\mathrm{L}^2$ too, as $t \to +\infty$. Therefore, we can let $t \to +\infty$ to obtain that $0 \leq \langle \widehat{\mathbb{D}}_1(s)u_0, \widehat{\mathbb{D}}_2(s)u_0\rangle \int_{-\infty}^0 \mathrm{e}^{2t\operatorname{Re}s}\,dt$. Because $u_0$ was arbitrary, we have $0 \leq \widehat{\mathbb{D}}_1(s)^*\widehat{\mathbb{D}}_2(s)$.

(b) 1° *(i)–(iii):* Equivalence "(i)⇔(ii)" follows from time-invariance (apply $\tau^{-t}$ and $\tau^t$ to different sides of the inequality), "(ii)⇔(iii)" and "(iv)⇔(v)" by reflection (apply **Я** to both sides), and the implication "(iv)⇒(ii)" by adding $\pi_{[-t,0)}$ to both sides of the inequality in (iv),

2° *(ii)⇒(iv):* Assume (ii). For any $u \in \mathrm{L}^2(\mathbf{R}_-;U)$ and $t > 0$, we have

$$0 \leq \langle \mathbb{D}_1\pi_{[-t,0)}u, \pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)}u\rangle = \langle \mathbb{D}_1u, \pi_-\mathbb{D}_2u\rangle - \langle \pi_{(-\infty,t)}\mathbb{D}_1u, \pi_-\mathbb{D}_2\pi_{(-\infty,t)}u\rangle. \tag{2.28}$$

By Corollary B.3.8, $\langle \mathbb{D}_1\pi_{(-\infty,t)}u, \pi_-\mathbb{D}_2\pi_{(-\infty,t)}u\rangle \to 0$ as $t \to +\infty$, hence $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle = \langle \mathbb{D}_1\pi_- u, \pi_-\mathbb{D}_2\pi_- u\rangle$ is real and nonnegative. The second implication is obtained analogously (because $\pi_{[-t,0)}\mathbb{D}_1^*\pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)} \geq 0$).

$3^\circ$ *(i)$\Rightarrow$ $\pi_+\mathbb{D}_1^*\mathbb{D}_2\pi_+ \geq 0$:* Let $t \to +\infty$ (as in $2^\circ$).

(c) We prove below the implication "(v)$\Rightarrow$(i)". The rest of (c) follows from (b) by setting $\mathbb{D}_1 = I$, $\mathbb{D}_2 = S$, $\mathbb{D}_3 = \mathbb{D}$, $\mathbb{D}_4 = J\mathbb{D}$.

Assume (v). Then

$$\pi_+ S \geq \pi_+\mathbb{D}^*\pi_+ J\mathbb{D}\pi_+ = \mathbb{D}^{t\,*} J\mathbb{D}^t + \pi_+\mathbb{D}^*\pi_{[t,\infty)}J\mathbb{D}\pi_+, \tag{2.29}$$

hence $\pi_+ S \geq \mathbb{D}^{t\,*}J\mathbb{D}^t$, hence (i) holds.

(We do not have (v)$\Rightarrow$(i) for $J, S \leq 0$ in general. E.g., if $\mathbb{D} = (\tau^{-k})_{k\in\mathbf{N}} \in \mathrm{TIC}(\ell^2(\mathbf{N}))$, then $\pi_+\mathbb{D}^*\pi_+\mathbb{D}\pi_+ = \pi_+$ but $\langle u_k, \mathbb{D}^{t\,*}\mathbb{D}^t u_k\rangle = 0 < \langle u_k, \pi_{[0,t)}u_k\rangle = 1$ for $u_k = \chi_{[0,1)}e_k$, $t \geq k \geq 1$.) $\square$

By the above lemma (and Lemma 2.2.3), we have "$\mathbb{X}^{-1} \in \mathrm{TIC} \Rightarrow \widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \geq \varepsilon I$", but the converse requires a Tauberian condition. Five such conditions are presented below:

**Proposition 2.2.5 ($\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \geq \varepsilon I \Rightarrow \mathbb{X}^{-1} \in \mathrm{TIC}$)** *Let $\omega \in \mathbf{R}$ and $\mathbb{X} \in \mathrm{TIC}_{\omega'}(U,Y)$ for all $\omega' > \omega$. If $(\widehat{\mathbb{X}})^*\widehat{\mathbb{X}} \geq \varepsilon^2 I$ on $\mathbf{C}^+_\omega$ (or $\mathbb{X}^{t\,*}\mathbb{X}^t \geq \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$) for some $\varepsilon > 0$, and any of conditions (1)–(5) below holds, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U,Y)$, $\mathbb{X}^{-1} \in \mathrm{TIC}_\omega$, $\|\mathbb{X}^{-1}\|_{\mathrm{TIC}_\omega} \leq \varepsilon^{-1}$, and $\dim U = \dim Y$.*

*(1)* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$*;*

*(2)* $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{\omega'}$ *(or $\mathbb{X}\mathbb{M} = I$ for some $\mathbb{M} \in \mathrm{TI}_{\omega'}$) for some $\omega' > \omega$;*

*(3)* $\dim U = \dim Y < \infty$*;*

*(4)* $\left[\begin{smallmatrix}\mathbb{X} & * \\ * & *\end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty(U\times W, Y\times Z)$*, and* $\dim W < \infty$*;*

*(5)* $\widehat{\mathbb{X}}(s_0) \in \mathcal{G}\mathcal{B}$ *or* $\widehat{\mathbb{X}}(s_0)$ *is onto or* $\mathrm{Ker}(\widehat{\mathbb{X}}(s_0)^*) = \{0\}$ *for some $s_0 \in \mathbf{C}^+_\omega$; (for uniformly regular $\mathbb{X}$ we allow $s_0 = +\infty$).*

*If $\mathbb{X} \in \mathrm{TIC}$, and $\mathbb{X}^{t\,*}\mathbb{X}^t \geq \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$, then also the sixth condition $\mathbb{X}\mathbb{X}^* \gg 0$ implies that $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y)$.*

(Note that (1)–(5) are not superfluous: if $\dim U = \infty$, then there is $\mathbb{X} = X \in \mathcal{B}(U)$ s.t. $X$ is left-invertible but not right-invertible.)

Recall that if $\mathbb{X} \in \mathrm{TIC}_\omega$, then $\mathbb{X} \in \mathrm{TIC}_{\omega'}$ for all $\omega' > \omega$.

For $\omega = 0$, Proposition 4.1.7(B)&(C) provide us with several sufficient conditions for $(\widehat{\mathbb{X}})^*\widehat{\mathbb{X}} \geq \varepsilon^2 I$.

**Proof:** (We take $\omega = 0$ w.l.o.g.)

From (2.26) we observe that if $\mathbb{X}^{t\,*}\mathbb{X}^t \geq \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$, then $\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \geq \varepsilon^2 I$ on $\mathbf{C}^+$.

The last claim follows from Lemma 2.2.4(b)(i)&(iv) and Lemma 2.2.3(vii)&(i).

Clearly $\widehat{\mathbb{X}}$ (see Theorem 2.1.2) has the left-inverse $\widehat{\mathbb{T}} = (\widehat{\mathbb{X}}^*\widehat{\mathbb{X}})^{-1}\widehat{\mathbb{X}}^* \in \mathcal{C}_{\mathrm{b}}(\mathbf{C}^+; \mathcal{B}(Y,U))$ on $\mathbf{C}^+$, i.e., $\widehat{\mathbb{T}}\widehat{\mathbb{X}} \equiv I$ on $\mathbf{C}^+$. Moreover, $\|\widehat{\mathbb{T}}(s)\| \leq \varepsilon^{-1}$ for all $s \in \mathbf{C}^+$, by Lemma A.3.1(c1)(1). If $\widehat{\mathbb{X}}(s_0) \in \mathcal{GB}$ for some $s_0 \in \mathbf{C}^+$, then $\widehat{\mathbb{X}}(s_0)^{-1} = \widehat{\mathbb{T}}(s_0)$.

Thus, we only have to show that $\widehat{\mathbb{X}}$ has an inverse on $\mathbf{C}^+$, i.e., that $\widehat{\mathbb{T}}$ is also a right inverse of $\widehat{\mathbb{X}}$, because $\widehat{\mathbb{X}}^{-1}$ is necessarily holomorphic, by Lemma D.1.2(b2), hence $\widehat{\mathbb{X}}^{-1} = \widehat{\mathbb{T}} \in \mathrm{H}^\infty = \widehat{\mathrm{TI}}$ (and $\dim U = \dim Y$, by Lemma 2.2.1(c4)). The invertibility proof depends on the extra assumption:

(1) By definitions, (1) implies (2).

(2) Because $\widehat{\mathbb{T}}, \widehat{\mathbb{X}} \in \mathcal{C}_\mathrm{b}(\omega' + i\mathbf{R}; \mathcal{B}(U))$, we have $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{\omega'}$ iff $\widehat{\mathbb{X}} \in \mathcal{G}\mathcal{C}_\mathrm{b}(\omega' + i\mathbf{R}; \mathcal{B}(U))$, by Theorem 3.1.3(d). Therefore, (2) implies (5) (note that $\widehat{\mathbb{T}}$ is the transform of a left inverse of $\mathbb{X} \in \mathrm{TI}_{\omega'}$, hence the existence of a right inverse $\mathbb{M}$ is equivalent to the invertibility of $\mathbb{X}$ in $\mathrm{TI}_{\omega'}$).

(3) $\dim U = \dim Y < \infty$ implies that any left inverse of $\widehat{\mathbb{X}}(s)$ (hence $\widehat{\mathbb{T}}(s)$) is an inverse, by Lemma A.1.1(c1).

(4) (The assumptions mean that there are $\mathbb{Y}, \mathbb{Z}, \mathbb{W} \in \mathrm{TIC}_\infty$ s.t. $\begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ \mathbb{Z} & \mathbb{W} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times W, Y \times Z)$ and $\dim W < \infty$.)

By Lemma A.1.1(c1), the left-invertibility of $\widehat{\mathbb{X}}$ implies the invertibility of $\widehat{\mathbb{X}}$ on $\mathbf{C}^+_\omega$.

(5) By the uniqueness of the $\mathcal{B}(U,Y)$ inverse, we have

$$E := \{s \in \mathbf{C}^+ \,\big|\, \widehat{\mathbb{X}}(s)\widehat{\mathbb{T}}(s) = I\} = \{s \in \mathbf{C}^+ \,\big|\, \widehat{\mathbb{X}}(s) \in \mathcal{GB}(U,Y)\}, \tag{2.30}$$

and the latter set is open, by Lemma 6.3.2(d). On the other hand, $\widehat{\mathbb{X}}(s)\widehat{\mathbb{T}}(s) = I$ holds in a closed subset of $\mathbf{C}^+$. Now the connectedness of $\mathbf{C}^+$ and the fact that $E \neq \emptyset$ (because $s_0 \in E$) imply that $E = \mathbf{C}^+$ (if $s_0 = +\infty$ and $\mathbb{X}$ is UR, then $\widehat{\mathbb{X}}(s)$ is invertible for big $s \in \mathbf{R}$, by Lemma A.3.3(A2)).

*Remark:* If $U = \ell^2(\mathbf{N}) \subset Y$ and $\mathbb{X}$ is the right shift $\tau^{-1} \in \mathcal{B}(U)$, then the assumptions (except (1)–(5)) are satisfied and yet $\mathbb{X} \notin \mathcal{G}\mathrm{TIC}_\infty$. □

**Corollary 2.2.6 ($\mathbb{X}^{t*}\mathbb{X}^t \geq \varepsilon I \Rightarrow \exists \mathbb{X}^{-1} \in \mathrm{TIC}$)** *Assume that* $\mathbb{X} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \geq 0$, $\varepsilon > 0$ *and* $\mathbb{X}^{t*}\mathbb{X}^t \geq \varepsilon^2 I\pi_{[0,t)}$ *for all* $t > 0$.

*If any of (1)–(5) of Proposition 2.2.5 holds, then* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U,Y)$, $\mathbb{X}^{-1} \in \mathrm{TIC}$, $\|\mathbb{X}^{-1}\|_\mathrm{TIC} \leq \varepsilon^{-1}$ *and* $\dim U = \dim Y$.

**Proof:** From (2.26) we observe that $\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \geq \varepsilon^2 I$ on $\mathbf{C}^+$. Thus, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$, by Proposition 2.2.5. Moreover, $\varepsilon^{-1} I\pi_{[0,t)} \geq ((\mathbb{X}^{-1})^t)^*(\mathbb{X}^{-1})^t$ for all $t > 0$, hence $\|\mathbb{X}^{-1}\|_\mathrm{TIC} \leq \varepsilon^{-1}$, by Lemma 2.1.14.

(Again the right shift is a "counter-example" showing that (1)–(5) are not superfluous.) □

If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y)$ is exponentially stable, then so is $\mathbb{X}^{-1}$; moreover, if $\mathbb{X}$ is generated by an exponentially stable measure, then so is its inverse:

**Lemma 2.2.7 ($\mathcal{G}\mathrm{TIC} \cap \mathrm{TI}_\mathrm{exp} = \mathcal{G}\mathrm{TIC}_\mathrm{exp}$)** *Let* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y) \cap \mathrm{TI}_{-r}(U,Y)$, *where* $r > 0$. *Then, for some* $\varepsilon > 0$, *we have* $\mathbb{X}^{-1} \in \mathrm{TIC}_{-\varepsilon}(Y,U)$; *thus,* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U,Y) \cap \mathrm{TIC}_{-r}(U,Y)$.

*If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y) \cap \widetilde{\mathcal{A}}_{-r}(U,Y)$, where $\widetilde{\mathcal{A}} \subset \mathrm{TIC}$ is inverse-closed (as in Theorem 4.1.1(g)), then $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-r}(U,Y)$ for some $\varepsilon > 0$.*

Here $\widetilde{\mathcal{A}}$ may be MTIC or any other inverse-closed subclass of TIC, by Theorem 4.1.1(b).

**Proof:** (We use here Theorem 4.1.1, but this lemma is not used before Chapter 5.)

Let $M := \|\mathbb{X}^{-1}\|^{-1}$. By Lemma D.1.8(c), there is $\varepsilon > 0$ s.t. $\|\widehat{\mathbb{X}}(t+iy) - \widehat{\mathbb{X}}(0+iy)\| < 1/2M$ when $|t| < \varepsilon$ and $y \in \mathbf{R}$. Consequently, $\widehat{\mathbb{X}}(t+iy)^{-1}$ exists and its norm is less than $2M$ for such $t$ and $y$; in particular, $\widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(\mathbf{C}_{-\varepsilon}; \mathcal{B}(Y,U))$, hence $\mathbb{X}^{-1} \in \mathrm{TI}_{-\varepsilon}(Y,U)$, by Theorem 2.1.2.

The final claim follows from the fact that $\mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}$ is inverse-closed in $\mathrm{TIC}_{-\varepsilon}$, i.e., that $\widetilde{\mathcal{A}}_{-\varepsilon} \cap \mathcal{G}\mathrm{TIC}_{-\varepsilon} = \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}$, by Theorem 4.1.1(g1). □

Local causal invertibility is equivalent to global causal invertibility:

**Lemma 2.2.8 ($\mathbb{X}^t \in \mathcal{G}\mathcal{B} \Leftrightarrow \mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$)** *Let $\mathbb{X} \in \mathrm{TIC}_\infty(U,Y)$ and $-\infty < a < b < \infty$. Then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $\pi_{[a,b)}\mathbb{X}\pi_{[a,b)}$ is invertible on $\pi_{[a,b)}\mathrm{L}^2$.*

*If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$, then the latter inverse is $\pi_{[a,b)}\mathbb{X}^{-1}\pi_{[a,b)}$.*

However, being invertible in a specific $\mathrm{TIC}_\omega$ is a stronger condition. E.g., if $\widehat{\mathbb{X}}(s) := (s-1)/(s+1)$, then $\mathbb{X} \in \mathrm{TIC}_\omega(\mathbf{C})$ for any $\omega > -1$, but $\mathbb{X}^{-1} \in \mathrm{TIC}_\omega$ for $\omega > 1$ only, hence $\mathbb{X} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TIC}_\infty \setminus \mathcal{G}\mathrm{TIC}$.

**Proof:** The latter claim is obvious and the former one follows from Theorem 6.1.9(iv)&(v) of [Sbook], but we give here an alternative proof.

Set $T := b - a$. Take a realization of $\mathbb{X}$ and transform it into a wpls as in Theorem 13.4.4. Because the transformation maps $\mathrm{TIC}_\infty \to \mathrm{tic}_\infty$ through an isomorphism, which is also an algebraic isomorphism, by Theorem 13.4.5(b), it follows from Lemma 13.1.7 that $\mathbb{X}$ is invertible iff $X_\mathrm{d} := \pi_{[0,T)}\mathbb{X}\pi_{[0,T)}$ (the I/O operator of the wpls) is invertible.

By time-invariance the invertibility of $\pi_{[0,T)}\mathbb{X}\pi_{[0,T)}$ is equivalent to the invertibility of $\pi_{[a,b)}\mathbb{X}\pi_{[a,b)}$. □

## Notes

It would be interesting to know whether (2.26) with $\omega = 0$ is sufficient for (i)–(v) of Lemma 2.2.4(b) (it is necessary, by (a)) when $\omega = 0$; our guess is that the answer is negative — unfortunately, because a positive answer would provide us with several additional equivalent conditions in Corona Theorems 4.1.6 and 4.1.7. See also Lemma 4.1.10.

Conditions (3) and (5) of Proposition 2.2.5 were used in a preprint of [S98d]. Lemma 2.2.1(b) and the case $\widetilde{\mathcal{A}} = \mathrm{TIC}$ of Lemma 2.2.7 are from [Sbook]. Most of Lemma 2.2.4 is from Section 6 of [S98c]. Also (at least) Lemma 2.2.8 and parts of Lemmas 2.2.2 and 2.2.3 are known, as explained in their proofs. Lemma 2.2.1(c3) is one of the main contributions of this section.

## 2.3 Static operators

*Eppur si muove!*

— Galileo Galilei (1564–1642), 1633

In this section, we shall present five important (and apparently new) technical lemmas that will be used in connection with $(J,S)$-inner factorizations and Riccati equations. Most of the lemmas give some sufficient results for an operator to be static.

If $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$, then $\pi_+\mathbb{E}\pi_- = 0$ implies that $\pi_-\mathbb{E}\pi_+ = (\pi_+\mathbb{E}\pi_-)^* = 0$, so that then $\mathbb{E}$ is static, i.e., $\mathbb{E} \in \mathcal{B}(U)$, by Lemma 2.1.7. We now prove a similar claim for "$\mathbb{E} = \mathbb{D}^* J\mathbb{D}$" when $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ is only required to be almost stable ($\mathbb{D}\mathrm{L}^2_\mathrm{c} \subset \mathrm{L}^2$), so that $\mathbb{E}$ need not be defined at all:

**Lemma 2.3.1 ($\mathbb{D}^* J\mathbb{D} = S$)** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$. Assume that $\mathbb{D}u \in \mathrm{L}^2$ (cf. Lemma 2.1.13) and $\langle \mathbb{D}\pi_+ v, J\mathbb{D}\pi_- u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$. Then there is a unique $S = S^* \in \mathcal{B}(U)$ s.t. $\langle \mathbb{D}v, J\mathbb{D}u\rangle = \langle v, Su\rangle$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$.*

Note that for $\mathbb{D} \in \mathrm{TIC}$ the term $\mathbb{D}^* J\mathbb{D}$ would be well defined and hence the proof of the lemma would be simple.

**Proof:** In the sequel we shall use the fact that if $u \in \mathrm{L}^2_\mathrm{loc}$ and $\langle v,u\rangle = 0$ for all $v \in \mathrm{L}^2_\mathrm{c}$ (or for all $v \in C^\infty_\mathrm{c}$), then $u = 0$ (a.e.), by Theorem B.3.11. This implies, that $S$ is unique.

Replace $u$ by $\tau^t u$ to obtain that $\langle \mathbb{D}\pi_{[t,\infty)} v, J\mathbb{D}\pi_{(-\infty,t)} u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$. Because $J = J^*$, we have $\langle \mathbb{D}\pi_{(-\infty,s)} v, J\mathbb{D}\pi_{[s,\infty)} u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$, hence

$$\langle \mathbb{D}v, J\mathbb{D}\pi_{[s,t)} u\rangle = \langle \mathbb{D}\pi_{[s,t)} v, J\mathbb{D}\pi_{[s,t)} u\rangle \quad (u \in \mathrm{L}^2_\mathrm{c},\ -\infty \le s \le t \le +\infty). \tag{2.31}$$

Set $\mathbb{S}_t := (\mathbb{D}\pi_{[-t,t)})^* J\mathbb{D}\pi_{[-t,t)} \in \mathcal{B}(\mathrm{L}^2([-t,t);U))$ $(t > 0)$. Then $\langle v, \mathbb{S}_t u\rangle = \langle \mathbb{D}v, J\mathbb{D}u\rangle$ for $u,v \in \pi_{[-t,t)}\mathrm{L}^2$, hence for $u \in \pi_{[-t,t)}\mathrm{L}^2$ and $v \in \mathrm{L}^2_\mathrm{c}$, by (2.31). Consequently, $\mathbb{S}_T u = \mathbb{S}_t u \in \pi_{[-t,t)}\mathrm{L}^2$ for all $T > t$, so we can define $\mathbb{S}u := \mathbb{S}_t u$ $(u \in \mathrm{L}^2([-t,t);U))$ (for an arbitrary $t > 0$).

It follows that $\mathbb{S} : \mathrm{L}^2_\mathrm{c} \to \mathrm{L}^2_\mathrm{c}$, $\tau\mathbb{S} = \mathbb{S}\tau$, and $\mathbb{S}_T = \mathbb{S}\pi_{[-T,T)}$. Therefore,

$$\|\mathbb{S}u\|_2^2 = \|\sum_{n\in\mathbf{Z}} \tau^{-n}\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 = \sum_{n\in\mathbf{Z}} \|\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 = \sum_{n\in\mathbf{Z}} \|\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 \tag{2.32}$$

$$\le \|\mathbb{S}_1\|_{\mathcal{B}(\mathrm{L}^2)} \|\sum_{n\in\mathbf{Z}} \|\pi_{[-1,1)}\tau^n u\|^2 = \|\mathbb{S}_1\|_{\mathcal{B}}(\mathrm{L}^2)\|u\|^2 \tag{2.33}$$

for $u \in \mathrm{L}^2_\mathrm{c}$. Consequently, $\mathbb{S}$ can be extended to a $\mathcal{B}(\mathrm{L}^2)$ map that is TI. From (2.31) it follows that $\pi_+\mathbb{S}\pi_- = 0 = \pi_-\mathbb{S}\pi_+$, hence $\mathbb{S} \in \mathcal{B}(U)$, by Lemma 2.1.7. Obviously, $\mathbb{S} = \mathbb{S}^*$. □

Let $\mathbb{D} \in \mathrm{TIC}_\infty$. If $\pi_+\mathbb{D}\pi_- = 0$, then $\mathbb{D}$ is static. In fact, instead of $\pi_+\mathbb{D}\pi_-$ it suffices to observe that $\pi_{[0,t)}\mathbb{D}\pi_{[-\varepsilon,0)} = 0$ for a fixed $\varepsilon > 0$:

**Lemma 2.3.2** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, and $\pi_{[0,t)}\mathbb{D}\pi_{[-\varepsilon,0)} = 0$ for some $\varepsilon > 0$ and all $t > 0$. Then $\mathbb{D} \in \mathcal{B}(U,Y)$.*

**Proof:** Obviously, $\pi_+\mathbb{D}\pi_{[-\varepsilon,0)} = 0$. Thus,

$$\pi_+\mathbb{D}\pi_{[-n\varepsilon,0)} = 0 \tag{2.34}$$

holds for $n=1$. Assume now that (2.34) holds for $n = N \in 1+\mathbf{N}$. Then

$$\pi_+\mathbb{D}\pi_{[-(n+1)\varepsilon,-n\varepsilon)} = \tau^1\pi_{[\varepsilon,\infty)}\pi_+\mathbb{D}\pi_{[-n\varepsilon,-(n-1)\varepsilon)}\tau^{-1} = 0. \tag{2.35}$$

By induction, $\pi_+\mathbb{D}\pi_- = 0$, hence $\mathbb{D} \in \mathcal{B}$, by Lemma 2.1.7. □

If $\mathbb{X},\mathbb{Z} \in \mathrm{TIC}_\infty$ and $\mathbb{X}^* = \mathbb{Z}$, then $\mathbb{X}$ is static. Often it is easier to verify that $\pi_{[0,t)}\mathbb{X}^*\pi_{[0,t)} = \pi_{[0,t)}\mathbb{Z}\pi_{[0,t)}$ for all $t$; even this is sufficient:

**Lemma 2.3.3** *Let $\mathbb{X},\mathbb{Z} \in \mathrm{TIC}_\infty$ and $\mathbb{X}^t = \mathbb{Z}^{t*}$ for all $t > 0$. Then $\mathbb{X} = \mathbb{Z}^* \in \mathcal{B}$.*

Recall that $\mathbb{X}^t := \pi_{[0,t)}\mathbb{X}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2)$ (for any $\mathbb{X} \in \mathrm{TIC}_\infty$).

**Proof:** Let $\varepsilon > 0$. Now $\tau^\varepsilon\mathbb{X}^t\tau^{-\varepsilon} = \pi_{[-\varepsilon,t-\varepsilon)}\mathbb{X}\pi_{[-\varepsilon,t-\varepsilon)}$, hence

$$\pi_{[0,t-\varepsilon)}\mathbb{X}\pi_{[-\varepsilon,0)} = \pi_{[0,t-\varepsilon)}\tau^\varepsilon\mathbb{X}^t\tau^{-\varepsilon}\pi_{[-\varepsilon,0)} = \pi_{[0,t-\varepsilon)}\tau^\varepsilon\mathbb{Z}^{t*}\tau^{-\varepsilon}\pi_{[-\varepsilon,0)} \tag{2.36}$$

$$= \pi_{[0,t-\varepsilon)}\mathbb{Z}^*\pi_{[-\varepsilon,0)} = 0, \quad (t > \varepsilon). \tag{2.37}$$

hence $X := \mathbb{X} \in \mathcal{B}$, by 1°. By time-invariance $\pi_{[-t,t)}\mathbb{Z}\pi_{[-t,t)} = \pi_{[-t,t)}X^*\pi_{[-t,t)} = \pi_{[-t,t)}X^*$ $(t \geq \varepsilon)$, hence $\mathbb{Z} = X^*$, by continuity on some $\mathrm{L}^2_\omega$. □

If $J\mathbb{D}$ is static for some static $J$, then $J\mathbb{D} = JD$ for some static $D$, at least to some extent:

**Lemma 2.3.4** *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, $J \in \mathcal{B}(Y,H)$. If $J\mathbb{D} \in \mathcal{B}(U,H)$, then $\mathbb{D} = \begin{bmatrix} D_1 \\ \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}_\infty(U, Y_1 \times Y_2)$, where $Y_1 = Y_2^\perp$, $Y_2 := \mathrm{Ker}(J)$. Moreover, $J\mathbb{D} = J\begin{bmatrix} D_1 \\ 0 \end{bmatrix}$.*

*Assume, in addition, that $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$. Then there is $D \in \mathcal{GB}(U,Y)$ s.t. $J\mathbb{D} = JD$. If $Y = H$, then we can, in addition, require that $\mathbb{D}^{t*}J\mathbb{D}^t = D^*JD\pi_{[0,t)}$ for all $t \geq 0$.*

**Proof:** 1° Let $P$ is be orthogonal projection $Y \to Y_1$, and set $\mathbb{D}_1 := P\mathbb{D}$, $\mathbb{D}_2 = (I-P)\mathbb{D}$. Now $J\mathbb{D}_1 = J\mathbb{D} \in \mathcal{B}(U,H)$, hence $J\widehat{\mathbb{D}}_1$ is constant; because $J$ is one-to-one on $Y_1$, $D_1 := \widehat{\mathbb{D}}_1 \in \mathcal{B}(U,Y_1)$.

2° Let $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$. Then $\mathbb{D}$ is onto, hence $\mathbb{D}_1$ is onto, hence $D_1$ is onto hence $D_1^* \in \mathcal{GB}(Y_1,U_1)$, where $U_1 = \mathrm{Ran}(D_1^*) = \mathrm{Ker}(D_1)^\perp$, by Lemma A.3.1(c1)(iii)&(x)&(c7).

Consequently, $E := D_1|_{U_1} \in \mathcal{GB}(U_1,Y_1)$, hence $\mathbb{D} = \begin{bmatrix} E & 0 \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U_1 \times U_2, Y_1 \times Y_2)$, where $U_2 := U_1^\perp$, $\mathbb{D}_{21} := \mathbb{D}_2 Q$, $\mathbb{D}_{22} = \mathbb{D}_2(I-Q)$, and $Q$ is the orthogonal projection $U \mapsto U_1$.

Thus, $\mathbb{D}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2,Y_2)$, by Lemma A.1.1(b2)(1), hence $\dim U_2 = \dim Y_2$, by Lemma 2.2.1(c4), hence $\mathcal{GB}(U_2,Y_2)$ contains some operator $F$. Consequently, $D := \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \in \mathcal{GB}(U,Y)$, $JD = JPD = J\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = J\begin{bmatrix} D_1 \\ 0 \end{bmatrix} = J\mathbb{D}_1 = J\mathbb{D}$.

3° Case $Y = H$: Apply 2° for $M := \begin{bmatrix} J \\ J^* \end{bmatrix} \in \mathcal{B}(Y,Y^2)$. Then $M\mathbb{D} = MD$ for some $D \in \mathcal{GB}(U,Y)$, hence $J\mathbb{D} = JD$ and $\mathbb{D}^*J = (J^*\mathbb{D})^* = D^*J$. Consequently, $\mathbb{D}^{t*}J\mathbb{D}^t = \mathbb{D}^{t*}JD\pi_{[0,t)} = D^*JD\pi_{[0,t)}$ $(t > 0)$. □

We finish the section with a technical remark that will allow us to prove certain uniqueness results (modulo a unit $E \in \mathcal{GB}$) on the signature operators of optimization problems and Riccati equations:

**Lemma 2.3.5 ($\mathbb{X}^{t*} S \mathbb{X}^t = \mathbb{Z}^{t*} T \mathbb{Z}^t$)** *Let* $\mathbb{X}, \mathbb{Z} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, $S, T \in \mathcal{B}(U)$ *and* $\mathbb{X}^{t*} S \mathbb{X}^t = \mathbb{Z}^{t*} T \mathbb{Z}^t$ *for all* $t > 0$. *Then* $S = E^* T E$, $S\mathbb{X} = SE^{-1}\mathbb{Z}$, *and* $S^*\mathbb{X} = S^* E^{-1}\mathbb{Z}$ *for some* $E \in \mathcal{GB}(U)$.

In particular, if $\mathrm{Ker}(S) = \{0\}$ or $\mathrm{Ker}(S^*) = \{0\}$ (i.e., $S$ is one-to-one or onto), then $\mathbb{X} = E^{-1}\mathbb{Z}$.

**Proof:** Set $\mathbb{R} := \mathbb{X}\mathbb{Z}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, $\widetilde{S} := \begin{bmatrix} S \\ S^* \end{bmatrix}, \widetilde{T} := \begin{bmatrix} T \\ T^* \end{bmatrix} \in \mathcal{B}(U, U^2)$. Then $\widetilde{S}\mathbb{R}^t = \mathbb{R}^{t\,-*}\widetilde{T}$ for all $t > 0$ (the second row of the equation is the adjoint of the first one), hence $L := \widetilde{S}\mathbb{R} = \mathbb{R}^{-*}\widetilde{T} \in \mathcal{B}(U)$, by Lemma 2.3.3.

By Lemma 2.3.4, we have $\pi_{[0,t)}\widetilde{T} = \mathbb{R}^{t*}\widetilde{S}\mathbb{R}^t = R^*\widetilde{S}R\pi_{[0,t)}$ for some $R \in \mathcal{GB}(U)$, hence $\widetilde{T} = R^*\widetilde{S}R$. Set $E := R^{-1}$ to obtain $S = E^*TE$. Finally, $S\mathbb{X} = S\mathbb{R}\mathbb{Z} = SR\mathbb{Z}$, and $S^*\mathbb{X} = S^*\mathbb{R}\mathbb{Z} = S^*R\mathbb{Z}$. $\square$

## Notes

The above results are apparently new. Theorem 5.2C of [RR] (case $\omega = 0$ of Lemma 2.1.7) is the only known (to us) result in this direction.

## 2.4 The signature operator $S$

*To see a World in a grain of sand,*
*And a Heaven in a wild flower,*
*Hold Infinity in the palm of your hand,*
*And Eternity in an hour.*
— William Blake (1757–1827)

The signature operator of a control problem is a static operator that describes the definiteness of the problem w.r.t. the input. E.g., in minimization problems, the cost (to be minimized) is usually greater than $\varepsilon\|u\|^2_{L^2}$ for some $\varepsilon > 0$, where $u : \mathbf{R}_+ \to U$ is the control input. In such problems, the signature operator $S$ is uniformly positive ($S \gg 0$), whereas in indefinite control problems $S$ is indefinite (but yet self-adjoint and invertible if the problem satisfies standard coercivity assumptions).

In this section, we shall see how to write an operator $S = S^* \in \mathcal{GB}(U \times W)$ or an operator $S := \mathbb{E}^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \mathbb{E}$, $\mathbb{E} \in \mathcal{G}\mathrm{TI}$, in the form $S = E^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] E$ for some $E \in \mathcal{GB}$; we also give some further results on the positive and negative eigenspaces of self-adjoint operators. In Section 2.3, we have derived further similar results.

In Chapters 8–11, the results of this and the previous section will be applied to the signature operators of optimization problems, Riccati equations and spectral factorization.

As before, symbols $H, U, W, Y, H_k, Y_k$ ($k \in \mathbf{N}$) denote Hilbert spaces of arbitrary dimensions.

**Lemma 2.4.1** *Define* $J_H = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times H_2)$ *and* $J_Y = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(Y_1 \times Y_2)$. *The following are equivalent:*

*(i)* $\dim H_1 \le \dim Y_1$ *and* $\dim H_2 \le \dim Y_2$,

*(ii)* $J_H = V^* J_Y V$ *for some* $V \in \mathcal{B}(H_1 \times H_2, Y_1 \times Y_2)$,

*(iii)* $J_H = \mathbb{V}^* J_Y \mathbb{V}$ *for some* $\mathbb{V} \in \mathrm{TI}(H_1 \times H_2, Y_1 \times Y_2)$.

**Proof:** 1° "(i)⇒(ii)": By Lemma A.3.1(a4)(iii), there are $T_k \in \mathcal{B}(H_k, Y_k)$ satisfying $T_k^* T_k = I_{H_k}$ ($k = 1, 2$). Take $V = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$ to obtain $V^* J_Y V = \begin{bmatrix} T_1^* T_1 & 0 \\ 0 & -T_2^* T_2 \end{bmatrix} = J_H$.

2° "(ii)⇒(iii)": Trivial, because $\mathcal{B} \subset \mathrm{TI}$.

3° "(iii)⇒(i)": Define $T := \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H_1, H_1 \times H_2)$ and $\begin{bmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \end{bmatrix} := \mathbb{V}$. From $J_H = \mathbb{V}^* J_Y \mathbb{V} = \mathbb{V}_1^* \mathbb{V}_1 - \mathbb{V}_2^* \mathbb{V}_2$ we get $0 \ll I_{H_1} = T^* J_H T = (\mathbb{V}_1 T)^* (\mathbb{V}_1 T) - (\mathbb{V}_2 T)^* (\mathbb{V}_2 T)$, hence $(\mathbb{V}_1 T)^* (\mathbb{V}_1 T) \gg 0$, so $\dim Y_1 \ge \dim H_1$, by Lemma 2.2.1(c3), because $\mathbb{V}_1 T \in \mathrm{TI}_\infty(H_1, Y_1)$. Similarly, by setting $T := \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] \in \mathcal{B}(H_2, H_1 \times H_2)$, we get $\dim Y_2 \ge \dim H_2$. □

When we require $V$ or $\mathbb{V}$ to be invertible, the inequalities in (i) become equalities:

**Corollary 2.4.2** *Define* $J_H = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{B}(H_1 \times H_2)$ *and* $J_Y = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{B}(Y_1 \times Y_2)$. *The following are equivalent:*

*(i)* $\dim H_1 = \dim Y_1$ *and* $\dim H_2 = \dim Y_2$,

*(ii)* $J_H = V^* J_Y V$ *for some* $V \in \mathcal{GB}(H_1 \times H_2, Y_1 \times Y_2)$,

*(iii)* $J_H = \mathbb{V}^* J_Y \mathbb{V}$ *for some* $\mathbb{V} \in \mathcal{G}\mathrm{TI}(H_1 \times H_2, Y_1 \times Y_2)$.

**Proof:** 1° "(i)⇒(ii)": This follows as in Lemma 2.4.1, with Lemma A.3.1(a5)(iii).

2° "(ii)⇒(iii)": Trivial, because $\mathcal{GB} \subset \mathcal{G}\mathrm{TI}$.

3° "(iii)⇒(i)": This follows from Lemma 2.4.1, because now also $J_Y = (\mathbb{V}^{-1})^* J_H (\mathbb{V}^{-1})$. □

**Lemma 2.4.3** *Lemma 2.4.1 and Corollary 2.4.2 also hold with* $S$ *in place of* $J_H$ *if* $S = E^* J_H E$ *for some* $E \in \mathcal{GB}$. □

(This is obvious, since $J_H = E^{-*} S E^{-1}$.)

Any self-adjoint operator on $H$ can be written in the form $E^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} E$ w.r.t. some decomposition of $H$:

**Lemma 2.4.4** *Let* $S = S^* \in \mathcal{B}(H)$. *Then* $H = H_+ \oplus H_-$ *s.t.* $H_- = H_+^\perp$ *and* $S = E^* J_1 E$ *for some* $E \in \mathcal{B}(H)$.

*If* $S = S^* \in \mathcal{GB}(H)$, *then we can have* $E \in \mathcal{GB}(H)$ *above.*

Here $J_1 := \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}_{H_+ \times H_-} := P_+ - P_- \in \mathcal{GB}(H)$, where $P_\pm$ are the orthogonal projections of $H$ onto $H_\pm$ (hence $P_- = P_+^\perp$), hence $J_1 = J_1^* = J_1^{-1}$.

(In fact, condition $J_1 = J_1^* = J_1^{-1}$ is equivalent to the fact that hence $J_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{GB}(H_+ \times H_-)$ for some closed subspaces $H_+$ and $H_- = H_+^\perp$ of $H$, since it implies that $\sigma(J_1) \subset \{-1, 1\}$, and that we can let $H_+$ and $H_-$ to be the eigenspaces for 1 and $-1$, by Theorems 12.26 and 12.29 of [Rud73].)

**Proof of Lemma 2.4.4:** Apply

$$\begin{bmatrix} \widetilde{S}_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{S}_- \end{bmatrix} = \begin{bmatrix} \widetilde{S}_+^{1/2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-\widetilde{S}_-)^{1/2} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} \mathrm{diag}(\widetilde{S}_+^{1/2}, 0, (-\widetilde{S}_-)^{1/2}) \tag{2.38}$$

to Lemma A.3.2(f1) to observe that

$$E = \mathrm{diag}(\widetilde{S}_+^{1/2}, 0, (-\widetilde{S}_-)^{1/2}) \begin{bmatrix} P_+ \\ P_0 \\ P_- \end{bmatrix} \in \mathcal{B}(H) \tag{2.39}$$

will do in this lemma. If $H_0 := \mathrm{Ker}(S) = \{0\}$, then the middle row can be omitted; if $S \in \mathcal{GB}$, then $\widetilde{S}_\pm^{1/2} \in \mathcal{GB}(H_\pm)$, so that then $E \in \mathcal{GB}(H)$. □

Now we are ready for the main result (along with Corollary 2.4.2) of this section:

**Theorem 2.4.5** *Let* $\mathbb{U}^* J_\gamma \mathbb{U} = S \in \mathcal{B}(U \times W)$ *with* $\mathbb{U} \in \mathcal{G}\mathrm{TI}(U \times W)$ *and* $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \in \mathcal{B}(U \times W)$, $\gamma, \gamma' > 0$. *Then* $S = E^* J_{\gamma'} E$ *for some* $E \in \mathcal{GB}(U \times W)$.

Note that here $J_{\gamma'} := P_U - (\gamma')^2 P_W$ is defined by $U$ and $W$, whereas Lemma 2.4.4 would only provide us with an analogous result for some ("nonfixed") $H_+$ and $H_-$ in place of $U$ and $W$.

**Proof:** Replace $\mathbb{U}$ by $\left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right]\mathbb{U}$ to get rid of $\gamma$ (so that $S = \mathbb{U}^* J_1 \mathbb{U}$). Set $H := U \times W$. We have $S = S^* \in \mathcal{G}\mathrm{TI}(H)$ (because $S = \mathbb{U}^* J_1 \mathbb{U}$), hence $S = S^* \in \mathcal{GB}(H)$. Thus, $S = T^* J_{H_\pm} T$ for some $T \in \mathcal{GB}(H, H_+ \times H_-)$, by Lemma 2.4.4.

Therefore, $S = \mathbb{U}^* J_1 \mathbb{U}$ implies $J_{H_\pm} = \mathbb{T}^* J_1 \mathbb{T}$, with $\mathbb{T} := \mathbb{U} T^{-1} \in \mathcal{G}\mathrm{TI}(H_+ \times H_-, U \times W)$. By Corollary 2.4.2, $J_{H_\pm} = V^* J_1 V$ for some $V \in \mathcal{GB}$, hence $S = T^* J_{H_\pm} T = (VT)^* J_1 VT = E^* J_1 E$, where $E = VT \in \mathcal{GB}(H)$. □

By allowing $\mathbb{U}$ to be noninvertible, we get the following variant of the above theorem:

**Lemma 2.4.6** *Let $\mathbb{U}^* J_\gamma \mathbb{U} = S \in \mathcal{GB}(U \times W)$ with $\mathbb{U} \in \mathrm{TI}(U \times W, Z \times Y)$ and $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \in \mathcal{B}(Z \times Y)$, $\gamma, \gamma' > 0$. Then $S = E^* J_{\gamma'} E$ for some $E \in \mathcal{B}(U \times W, Z \times Y)$.*

**Proof:** The proof is virtually that of Theorem 2.4.5 with Lemma 2.4.1 in place of Corollary 2.4.2. □

## Notes

For separable $U$ and $W$, Theorem 2.4.5 was given in the proof of Lemma 5.4 of [S98d]. Because the symbol $\widehat{\mathbb{U}}$ of $\mathbb{U}$ need not be well-defined anywhere on $i\mathbf{R}$ in the unseparable case that proof cannot be generalized (in fact, Theorem 3.1.3(a1), where we show the existence of such a symbol, seems to be known in the separable case only). This is why all Hilbert spaces were required to be separable in [S98d]; we shall remove this restriction in Corollary 11.4.9.

## 2.5 Losslessness

*But who shall dare*
*To measure loss and gain in this wise?*
*Defeat may be victory in disguise;*
*The lowest ebb is the turn of the tide.*

— Henry Wadsworth Longfellow (1807–1882)

In this section, we formulate both widely-used forms of losslessness and establish their equivalence for maps with finite-dimensional input spaces. We also give some necessary and/or sufficient conditions.

The importance of this concept is based on the fact that $\mathrm{H}^\infty$ problems are solvable iff certain lossless coprime factorizations exist, as shown in [Green] and extended to WPLSs in Theorems 11.2.7 and 12.3.6.

In state-space solutions, one usually relates the optimal control to a nonnegative stabilizing solution of the Riccati equation. Sometimes one replaces the Riccati equation by suitable coprime or spectral factorization of the I/O map; then the nonnegativity condition must be replaced by a losslessness condition, cf. Lemma 9.8.14 and Theorem 6.5 of [S98c].

**Definition 2.5.1** *Let* $J = J^* \in \mathcal{B}(Y)$, $S = S^* \in \mathcal{B}(U)$ *and* $\mathbb{N} \in \mathrm{TIC}(U,Y)$. *The operator* $\mathbb{N}$ *is* $(J,S)$-lossless *iff* $\mathbb{N}^* J\mathbb{N} = S$ *and* $\mathbb{N}^*\pi_- J\mathbb{N} \leq \pi_- S$. *The operator* $\mathbb{N}$ *is* frequency-domain $(J,S)$-lossless *iff* $\mathbb{N}^* J\mathbb{N} = S$ *and* $\widehat{\mathbb{N}}(s)^* J\widehat{\mathbb{N}}(s) \leq S$ *for* $s \in \mathbf{C}^+$.

The (time-domain) losslessness implies frequency-domain losslessness, by Lemma 2.5.2. For $\dim U < \infty$ also the converse holds, by Proposition 2.5.4.

Both the time-domain and the frequency-domain concept have been widely used (under the name “lossless”) in the study of $\mathrm{H}^\infty$ problems; see, e.g., [BH88] for losslessness and [Green] for frequency-domain losslessness.

One can interpret losslessness as “no energy is produced nor lost in the system, but some energy may be delayed”, if $J = I = S$, but we usually take $J = J_1 = S$ ($J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, $\gamma \in \mathbf{R}$), which implies the same conclusion on $\mathbb{N}_\frown$, where the direction of the second input and output signals have been reversed; see the proof of Lemma 2.5.3 for details.

The delay mentioned above cannot be negative (i.e., the energy put out before any $t > 0$ cannot exceed the energy put in before that time), because

$$\mathbb{N}^*\pi_- J\mathbb{N} \leq S \Leftrightarrow \|\mathbb{N}u\|_{J,\mathrm{L}^2([0,t];Y)} \leq \|u\|_{S,\mathrm{L}^2([0,t];U)} \text{ for all } u \in \pi_+\mathrm{L}^2,\ t > 0, \quad (2.40)$$

where $\|u\|_S := \langle u, Su\rangle$, by Lemma 2.2.4(b)(i)&(iv).

Section 6 of [S98c] contains several results on losslessness of stable operators, one of them is given below:

**Lemma 2.5.2** *Let* $\mathbb{N} \in \mathrm{TIC}(U,Y)$, $J \in \mathcal{B}(Y)$ *and* $S \in \mathcal{B}(U)$. *If* $\mathbb{N}^*\pi_- J\mathbb{N} \leq \pi_- S$, *then* $\widehat{\mathbb{N}}(s)^* J\widehat{\mathbb{N}}(s) \leq S$ *for all* $s \in \mathbf{C}^+$.

*Thus, losslessness implies frequency-domain losslessness.* □

(This follows from Lemma 2.2.4(b).) See Proposition 2.5.4 for the converse under a Tauberian condition.

Next we give one more "almost equivalent" condition:

**Lemma 2.5.3** *Let* $\mathbb{N} \in \mathrm{TIC}(U \times W, Y \times W)$ *and* $\gamma, \gamma' > 0$.

*If* $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$ *and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}(W)$*, then* $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless.*

The converse is not true: $\mathbb{N} := \left[\begin{smallmatrix} I & 0 \\ 0 & \tau^t \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W)$ is $(J_\gamma, J_\gamma)$-lossless (for any $\gamma, t > 0$) but $\mathbb{N}_{22}^{-1} \notin \mathrm{TIC}$ (cf. the lemma below). However, the converse is true in the finite-dimensional case, by Proposition 2.5.4(1).

**Proof:** We assume that $\gamma = 1 = \gamma'$ (and replace $\mathbb{N}$ by $\left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right] \mathbb{N} \begin{bmatrix} I & 0 \\ 0 & \gamma'^{-1} \end{bmatrix}$ in the general case). We set here $\left[\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ \mathbb{S} & \mathbb{T} \end{smallmatrix}\right] := \mathbb{N}$ and $\mathbb{N}_{\curvearrowleft} := \left[\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{smallmatrix}\right]^{-1}$ (same system with second input and output signals reversed) to clarify the proof. For $\pi = I$ as well as for $\pi = \pi_-$ we have

$$\mathbb{N}_{\curvearrowleft}^* \pi \mathbb{N}_{\curvearrowleft} \leq \pi \Leftrightarrow \quad \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{bmatrix}^* \pi \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{bmatrix}^* \pi \begin{bmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{bmatrix} \tag{2.41}$$

$$\Leftrightarrow \quad \begin{bmatrix} \mathbb{Q}^* \pi \mathbb{Q} & \mathbb{Q}^* \pi \mathbb{R} \\ \mathbb{R}^* \pi \mathbb{Q} & \mathbb{R}^* \pi \mathbb{R} + I \end{bmatrix} \leq \begin{bmatrix} \mathbb{S}^* \pi \mathbb{S} + I & \mathbb{S}^* \pi \mathbb{T} \\ \mathbb{T}^* \pi \mathbb{S} & \mathbb{T}^* \pi \mathbb{T} \end{bmatrix} \tag{2.42}$$

$$\Leftrightarrow \quad \begin{bmatrix} \mathbb{Q}^* \pi \mathbb{Q} - \mathbb{S}^* \pi \mathbb{S} & \mathbb{Q}^* \pi \mathbb{R} - \mathbb{S}^* \pi \mathbb{T} \\ \mathbb{R}^* \pi \mathbb{Q} - \mathbb{T}^* \pi \mathbb{S} & \mathbb{R}^* \pi \mathbb{R} - \mathbb{T}^* \pi \mathbb{T} \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{2.43}$$

$$\Leftrightarrow \quad \mathbb{N}^* \pi J_1 \mathbb{N} \leq \pi J_1. \tag{2.44}$$

The inequality in (2.44) holds for $\pi = I$, by the assumption $\mathbb{N}^* J_1 \mathbb{N} = J_1$, hence we have $\mathbb{N}_{\curvearrowleft}^* \mathbb{N}_{\curvearrowleft} \leq I$ (from (2.41)). But $\mathbb{N}_{\curvearrowleft}^* (\pi_+ + \pi_-) \mathbb{N}_{\curvearrowleft} \leq I$ implies that

$$\pi_- \mathbb{N}_{\curvearrowleft}^* \pi_- \mathbb{N}_{\curvearrowleft} \pi_- = \pi_- - \pi_- \mathbb{N}_{\curvearrowleft}^* \pi_+ \mathbb{N}_{\curvearrowleft} \pi_- \leq \pi_-, \tag{2.45}$$

hence $\pi_- \mathbb{N}^* \pi_- J_1 \mathbb{N} \pi_- \leq \pi_- J_1$. □

By the above lemma, we can establish the Tauberian converse mentioned above:

**Proposition 2.5.4 (Lossless ⇔ f.-d. lossless)** *Let* $\mathbb{N} \in \mathrm{TIC}(U \times W, Y \times W)$ *and* $\gamma, \gamma' > 0$*. Assume that (1), (2) or (3) holds, where*

*(1)* $\dim W < \infty$*;*

*(2)* $\widehat{\mathbb{N}}_{22}(s_0) \in \mathcal{GB}$ *for some* $s_0 \in \mathbf{C}^+$ *(for uniformly regular* $\mathbb{N}$ *we allow for* $s_0 = +\infty$*);*

*(3)* $\dim U < \infty$ *and* $\mathbb{N} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & I \end{smallmatrix}\right] \mathbb{X}^{-1}$*, where* $\mathbb{D}_{11}, \mathbb{D}_{12}, \mathbb{X}, \mathbb{X}^{-1} \in \mathrm{TIC}$*;*

*Then* $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless iff* $\mathbb{N}$ *is frequency-domain* $(J_\gamma, J_{\gamma'})$*-lossless.*

*Moreover, if* $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless, then* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{N}_{12}\mathbb{N}_{22}^{-1}\| < \gamma$*.*

One could easily extend the equivalence for more general $J$ and $S$ by a coordinate transform of their positive and negative eigenspaces (see., e.g., Lemma 2.4.4), but $J_\gamma$ and $J_{\gamma'}$ satisfy our needs.

Note that this implies that also the corresponding discrete time result holds, by Corollary 13.2.4.

**Proof:** $1^\circ$ *Case* $\mathbb{N}$ *f.-d. lossless:* Let $\mathbb{N}$ be frequency-domain $(J_\gamma, J_{\gamma'})$-lossless. The right-bottom corner of the frequency-domain losslessness equation $\widehat{\mathbb{N}}(s)^* J_\gamma \widehat{\mathbb{N}}(s) \le J_{\gamma'}$ for all $s \in \mathbf{C}^+$ shows that $\gamma^2 \widehat{\mathbb{N}}_{22}^* \widehat{\mathbb{N}}_{22} \ge \widehat{\mathbb{N}}_{12}^* \widehat{\mathbb{N}}_{12} + \gamma'^2 I \gg 0$ on $\mathbf{C}^+$, so $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$, by Proposition 2.2.5 (use its (4) for (3) above). Consequently, $\mathbb{N}$ is $(J_\gamma, J_{\gamma'})$-lossless, by Lemma 2.5.3.

$2^\circ$ *Case* $\mathbb{N}$ *lossless:* By Lemma 2.5.2 losslessness implies frequency-domain losslessness; the converse was proved in $1^\circ$. By this and $1^\circ$, we have $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$ in either case, so from the equation

$$\mathbb{N}_{12}^* \mathbb{N}_{12} - \gamma^2 \mathbb{N}_{22}^* \mathbb{N}_{22} = -\gamma'^2 I \tag{2.46}$$

(this is the $(2,2)$-block of $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$) we get that $\|\mathbb{N}_{12} \mathbb{N}_{22}^{-1}\| < \gamma$, by Lemma A.3.1(e2). $\square$

We now summarize some of the above results to a condition that is needed for $\mathrm{H}^\infty$ problems:

**Corollary 2.5.5** *Let* $\mathbb{N} \in \mathrm{TIC}(U_1 \times U_2, Y \times U_2)$ *and* $\gamma, \gamma' > 0$. *Then the following are equivalent:*

*(i)* $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2)$*;*

*(ii)* $\mathbb{N}$ *is frequency-domain* $(J_\gamma, J_{\gamma'})$*-lossless and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2)$*;*

*(iii)* $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$ *and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}(U_2)$. $\square$

(This follows from Proposition 2.5.4(2) and Lemma 2.5.3.)

## Notes

In the finite-dimensional case, Joseph Ball and William Helton (e.g., [BH88]) have studied losslessness in detail under a chain–scattering formalism. There are also (finite-dimensional) extensions of this concept to time-variant systems (see [Gohberg] or [LKS]) and to nonlinear systems (see [BH92]).

Losslessness was adopted to an infinite-dimensional setting in [S98c], from which Definition 2.5.1 and 2.5.2 are adopted in the same generality as here. See [S98c] for further results and [BH88] for further notions of losslessness and their physical interpretations.

We believe that losslessness is strictly stronger than frequency-domain losslessness. However, it remains an open problem to prove this or the converse. This problem lies close to that of the possible converses to Lemma 2.2.4(a) and Proposition 4.1.7(C); see also Lemma 2.2.4(b)(i)&(iv).

## 2.6 MTI and its subclasses

*According to convention there is a sweet and a bitter, a hot and a cold, and according to convention, there is an order. In truth, there are atoms and a void.*

— Democritus (460–370 B.C.), 400 B.C.

In this section we shall define $\mathrm{MTI}(U,Y)$ and $\mathrm{CTI}(U,Y)$ ("M" for measures and "C" for continuity on $i\mathbf{R}\cup\{\infty\}$), which are subspaces of $\mathrm{TI}(U,Y)$, and a number of their subspaces. We also list their basic properties. Recall that $U$ and $Y$ refer to Hilbert spaces of arbitrary dimensions.

The class MTI consists of those elements $\mathbb{E}\in\mathrm{TI}$ that are of form (2.50), that is, a the convolution with a measure consisting of a discrete part plus an $\mathrm{L}^1$ part. Further regularity properties of this class are listed in Proposition 6.3.4, and its spectral factorization properties are treated in Chapter 5 and summarized in Theorem 5.2.7; see also Section 8.4.

When solving several standard control problems in Part III, we can show the sufficiency of classical "necessary and sufficient" conditions, but these conditions are not necessary in general (cf. Example 11.3.7). However, the spectral factorization and regularity properties and algebraic properties of MTI (see Theorems 8.4.9 and 2.6.4) are more or less the same as those of rational functions. This allows us to establish also the necessity of the conditions mentioned above provided that the I/O map of the system belongs to this class.

By *regularity* of a map $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ we mean the existence of the "feedthrough operator" $D:=\widehat{\mathbb{D}}(+\infty):=\lim_{r\to+\infty}\widehat{\mathbb{D}}(r)$. The exact definition and the concepts ULR, UHPR, SLR and SHPR are formulated in Definition 6.2.3 (see also Proposition 6.2.7), but the reader may skip these concepts until they are needed in later chapters.

At this point it suffices for most readers just to have a glance at Definition 2.6.3 and Theorem 2.6.4 and proceed to the next chapter.

By $\mathrm{CTI}(U,Y)$ we mean the maps $\mathbb{E}\in\mathrm{TI}(U,Y)$ that are induced by some $\widehat{\mathbb{E}}\in\mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(U,Y))$ through $\mathbb{E}u:=\widehat{\mathbb{E}}\widehat{u}$ for all $u\in\mathrm{L}^2(\mathbf{R};U)$ (this is a special case of the symbols of Theorem 3.1.3(a1)):

**Definition 2.6.1 (CTIC)** *We set*[2] $\mathrm{CTI}(U,Y):=\{\mathbb{E}\in\mathrm{TI}\,\big|\,\widehat{\mathbb{E}}\in\mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(U,Y))\}$ *and* $\mathrm{CTIC}(U,Y):=\mathrm{CTI}(U,Y)\cap\mathrm{TIC}(U,Y)$. *We call* CTIC *the (time domain)* half-plane algebra. *Moreover, we set*

$$\mathrm{CTI}^{\mathcal{BC}}(U,Y):=\{\mathbb{E}\in\mathrm{CTI}\,\big|\,\widehat{\mathbb{E}}(s)-\widehat{\mathbb{E}}(\infty)\in\mathcal{BC}\ \textit{ for all }\ s\in i\mathbf{R}\},\qquad(2.47)$$

$$\mathrm{CTIC}^{\mathcal{BC}}(U,Y):=\mathrm{CTI}^{\mathcal{BC}}\cap\mathrm{CTIC}.\qquad(2.48)$$

*We equip all these spaces with the* TI *norm (i.e., with the* $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ *operator norm). If* $\mathbb{E}\in\mathrm{CTI}$, *we call* $E:=\widehat{\mathbb{E}}(\infty)$ *the* feedthrough operator*of* $\mathbb{E}$.

---

[2]We equip $i\mathbf{R}\cup\{\infty\}$with its one-point-compactification topology; equivalently, the topology induced by the topology of $\partial\mathbf{D}$ through the Cayley transform (this implies that "$+i\infty=-i\infty$"). Similarly, $\overline{\mathbf{C}^+}\cup\{\infty\}$is equipped with the one-point-compactification topology, and that topology is induced by the topology of $\overline{\mathbf{D}}$ through the Cayley transform.

As above, we always denote the "feedthrough" operators by the same letters as the corresponding TI operators. From Lemma 2.6.2 we observe that the above concept of feedthrough operators coincides with that of Definition 6.2.3.

One easily verifies that all the four spaces defined above are closed subspaces of TI (because the set $\mathcal{BC}$ of compact (see p. 871) linear mappings is closed in $\mathcal{B}$, by Lemma A.3.4(B1)).

We call the function $\phi_{\text{Cayley}} : s \mapsto (1-s)/(1+s)$ the *Cayley transform*. It maps $\mathbf{C}^+$ one-to-one and onto $\mathbf{D}$, and $\overline{\mathbf{C}^+} \cup \{\infty\}$ one-to-one and onto $\overline{\mathbf{D}}$, and it is the inverse of itself. See Lemma 13.2.1 for details.

**Lemma 2.6.2** *The space* $\mathrm{CTI}(U,Y)$ *consists of those* $\mathbb{E} \in \mathrm{TI}(U,Y)$*, for which* $s \mapsto \widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1}$ *belongs to the set* $\mathcal{C}(\partial\mathbf{D};\mathcal{B}(U,Y))$*, equivalently, for which* $\widehat{\mathbb{E}} \in \mathcal{C}(i\mathbf{R};\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{E}}$ *has the same limit at* $\pm i\infty$*.*

*The space* $\mathrm{CTIC}(U,Y)$ *consists of those* $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, for which* $s \mapsto \widehat{\mathbb{D}} \circ \phi_{\text{Cayley}}^{-1}$ *belongs to the (uniform)* disc algebra $\mathcal{C}(\overline{\mathbf{D}};\mathcal{B}(U,Y)) \cap \mathrm{H}(\mathbf{D};\mathcal{B}(U,Y))$*, equivalently,* $\widehat{\mathrm{CTIC}} = \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$*; a third equivalent condition is that* $\widehat{\mathbb{D}} \in \widehat{\mathrm{CTIC}}$ *iff* $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{D}}$ *has a uniform limit (on* $\mathbf{C}^+$*) at infinity (i.e.,* $\mathbb{D}$ *is uniformly half-plane-regular).*

*Moreover,* $\mathbb{D} \in \mathrm{CTIC}^{\mathcal{BC}}$ *implies that* $\mathbb{D}(s) - D \in \mathcal{BC}$ *for all* $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$*.*

*Finally, if* $\mathbb{E} \in \widehat{\mathrm{CTI}}(U,Y)$*, then* $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^* \in \widehat{\mathrm{CTI}}(U,Y)$*.*

**Proof:** 1° CTI: The above characterizations of CTI are obviously equivalent to the fact that $\widehat{\mathbb{E}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(U,Y))$, which in turn always defines a TI map (by, e.g., Theorem 3.1.3(a)).

2° CTIC: Let $\mathbb{D} \in \mathrm{CTIC}(U,Y)$. Let $f \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(U,Y))$ and $F \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ be its Fourier and Laplace transforms, respectively. By [Rud73, Theorem 11.30(b)], for each $\Lambda \in Y^*$ and $u_0 \in U$, there is $g_{\Lambda,u_0} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. $\Lambda F u_0 \in \mathrm{H}^\infty(\mathbf{C}^+)$ is the Poisson integral $P_r * g_{\Lambda,u_0}(i\cdot)$ [Lemma D.1.8]. Consequently, $\mathcal{L}\Lambda\mathbb{D}\phi u_0 = g_{\Lambda,u_0}\widehat{\phi} u_0$ on $i\mathbf{R}$ for any $\phi \in \mathrm{L}^2(\mathbf{R}_+)$, hence $g_{\Lambda,u_0} = \Lambda f u_0$ a.e. on $i\mathbf{R}$.

It follows that $\Lambda F u_0 = \Lambda(P_r * f)u_0$ on $\mathbf{C}^+$, still for arbitrary $\Lambda$ and $u_0$, hence $F(ir+\cdot) = P_r * f(i\cdot)$ on $\mathbf{C}^+$.

From Lemma D.1.8(a2) it now follows that $F(ir+t) = (P_r * f)(t) - D = P_r * (f-D)(t) \to 0$ as $|ir+t| \to \infty$; in particular, $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y))$ (and in $\mathrm{H}^\infty$). Now the other two characterizations of CTIC follow from this. (If $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ is UHPR, then $\widehat{\mathbb{D}}$ has a uniform limit as $\overline{\mathbf{C}^+} \ni s$ (not necessarily $\mathbf{C}^+ \ni s$ as for UHPR) and $|s| \to \infty$, by Lemma 6.3.6(a1); thus, then $\widehat{\mathbb{D}}$ is continuous at $\infty$ too.)

3° $\mathrm{CTIC}^{\mathcal{BC}}$: Also this claim follows from the Poisson integral formula (for $\widehat{\mathbb{D}} - D$).

4° $\widehat{\mathbb{E}}^*$: The equation

$$\int_{i\mathbf{R}} \langle \widehat{f}, \widehat{\mathbb{E}}^* \widehat{g} \rangle_U \, dm = \int_{i\mathbf{R}} \langle \widehat{\mathbb{E}} \widehat{f}, \widehat{g} \rangle_Y \, dm = \int_{i\mathbf{R}} \langle \widehat{f}, \widehat{\mathbb{E}^*} \widehat{g} \rangle_U \, dm \tag{2.49}$$

shows that $(\widehat{\mathbb{E}}^* - \widehat{\mathbb{E}^*})\widehat{g} \in \mathrm{L}^2(i\mathbf{R};U)$ is zero a.e., by, e.g., Theorem B.4.12, hence $\widehat{\mathbb{E}}^*$ is the unique $\mathrm{L}^\infty_{\text{strong}}$ function (equivalence class) corresponding to $\mathbb{E}^*$ (see

Theorem 3.1.3). Obviously, $\mathbb{E}^* \in \mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(Y,U)) = \widehat{\mathrm{CTI}}(Y,U)$. $\square$

Given the system $x' = Ax + Bu$, $y = Cx + Du$, $x(0) = 0$, we have $y = f * u + Du$, where $f := Ce^{A\cdot}B$, provided that $A,B,C,D \in \mathcal{B}$.

Even for somewhat unbounded $A,B,C,D$, the map $\mathbb{D} : u \mapsto y$ is still of form $\mathbb{D}u = \mu * u$ for a measure $\mu$ consisting of a $\mathrm{L}^1_{\mathrm{loc}}$ part plus a feedthrough ($D\delta_0$, where $\delta_0$ is the unit mass at zero (the "delta function")); this will be denoted by $\mathrm{MTIC}^{\mathrm{L}^1}_\infty$ below. If we allow for delays, we end up with (2.50) or "the measures with no continuous singular part", $\mathrm{MTIC}_\infty$.

The above motivates the definition below. For most readers, it suffices just to note the definitions of MTI, MTIC, $\mathrm{MTIC}^{\mathrm{L}^1}$ and $\mathrm{MTIC}_{\mathbf{S}}$ for $\mathbf{S} = T\mathbf{Z}$, $T > 0$ and then proceed to the remarks below the definition:

**Definition 2.6.3 (MTI, $\mathbf{MTI_d}$, $\mathbf{MTI^{L^1}}$)** *We define* $\mathrm{MTI}(U,Y)$ *to be the space of operators* $\mathbb{E} \in \mathrm{TI}(U,Y)$ *of the form* $\mathbb{E}u = (f + \sum_{k=0}^\infty T_k\delta_{t_k}) * u \quad (u \in \mathrm{L}^2(\mathbf{R};U))$, *i.e.,*

$$(\mathbb{E}u)(t) = \sum_{k=0}^{\infty} T_k u(t-t_k) + \int_{-\infty}^{\infty} f(t-r)u(r)\,dr, \tag{2.50}$$

*where* $f \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(U,Y))$, $T_k \in \mathcal{B}(U,Y)$ *for all $k$, and the* MTI *norm ("uniform total variation norm")*

$$\|\mathbb{E}\|_{\mathrm{MTI}} := \|f\|_{\mathrm{L}^1} + \sum_k \|T_k\| \tag{2.51}$$

*is finite; here* $\delta_t * := \delta_0(\cdot - t)* = \tau(-t)$ *is the delay of time* $t \in \mathbf{R}$.

*For* $\mathbb{E} \in \mathrm{MTI}$ *of the above form and* $\mathbf{S} \subset \mathbf{R}$ *we set*

$$\Pi_{\mathrm{L}^1}\mathbb{E} := f*, \quad \Pi_{\mathbf{S}}(\mathbb{E}) := (\sum_{t_k\in\mathbf{S}} T_k\delta_{t_k})*, \quad \mathrm{supp}_{\mathrm{d}}(\mathbb{E}) := \{t \in \mathbf{R} \,\big|\, \Pi_{\{t\}}\mathbb{E} \neq 0\}. \tag{2.52}$$

*Thus,* $\Pi_{\mathbf{S}}$ *is the projection to the measure carried on* $\mathbf{S}$, *and* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E})$ *is the set of the nonzero atoms of* $\mathbb{E}$.

*By the* Wiener class ($\mathrm{MTI}^{\mathrm{L}^1}$) $\mathrm{MTI}^{\mathrm{L}^1}$ *we mean the subspace of those* $\mathbb{E} \in \mathrm{MTI}$ *for which* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \{0\}$, *and by the* discrete measure class $\mathrm{MTI_d} := \Pi_{\mathbf{R}}\mathrm{MTI}$ *the subspace of those* $\mathbb{E} \in \mathrm{MTI}$ *for which the* $\mathrm{L}^1$ *part is zero (hence* $\mathrm{MTI} = \mathrm{MTI_d} \oplus (\mathrm{L}^1*)$*).*

*For the causal versions ($f = \pi_+ f$ and $t_k \geq 0$ for all $k$) of these spaces, we add the letter C at the end:*

$$\mathrm{MTIC} := \mathrm{MTI} \cap \mathrm{TIC},\ \mathrm{MTIC}^{\mathrm{L}^1} := \mathrm{MTI}^{\mathrm{L}^1} \cap \mathrm{TIC},\ \ \textit{and}\ \ \mathrm{MTIC_d} = \mathrm{MTI_d} \cap \mathrm{TIC}. \tag{2.53}$$

*We define* $\mathrm{MTI}^{\mathcal{BC}}(U,Y) \subset \mathrm{MTI}(U,Y)$ *to be the subspace of operators of form (2.50) s.t.* $f \in \mathrm{L}^1(\mathbf{R}_+;\mathcal{BC}(U,Y))$ *and* $T_k \in \mathcal{BC}(U,Y)$ *for* $t_k \neq 0$, *and we set* $\mathcal{A}^{\mathcal{BC}} := \mathcal{A} \cap \mathrm{MTI}^{\mathcal{BC}}$ *when* $\mathcal{A}$ *is any of the spaces defined above.*

*If* $\mathbf{S} \subset \mathbf{R}$, *we set* $\mathrm{MTI}_{\mathbf{S}} := \{\mathbb{E} \in \mathrm{MTI} \,\big|\, \mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S}\}$ *and* $\mathcal{A}_{\mathbf{S}} := \mathcal{A} \cap \mathrm{MTI}_{\mathbf{S}}$ *when* $\mathcal{A}$ *is any of the spaces defined above.*

*We define* $\mathrm{MTI}_\omega$ *(*$\omega \in \mathbf{R}$*) and its subspaces (with subscript* $\omega$*) to be the* *corresponding spaces for which*

$$\|\mathbb{E}\|_{\mathrm{MTI}_\omega} := \|e^{-\omega \cdot} f\|_1 + \sum_{k \in \mathbf{N}} \|e^{-\omega t_k} T_k\|_{\mathcal{B}(U,Y)} < \infty. \tag{2.54}$$

*Finally, we define the strong equivalents of* $\mathrm{MTI}_\omega$ *and its subspaces by setting* $\mathrm{SMTI}_\omega(U,Y) := \mathcal{B}(U, \mathrm{MTI}_\omega(\mathbf{C},Y))$, *and analogously for any other space in place of* $\mathrm{MTI}_\omega$.

We make below a series of remarks concerning the above definitions. See Theorem 2.6.4 for more on the properties of these classes; note in particular (by (h2)) that $\mathrm{MTI}_\omega \subset \mathrm{SMTI}_\omega \underset{c}{\subset} \mathrm{TI}_\omega$ ($\omega \in \mathbf{R}$). More results on the regularity of $\mathrm{MTIC}^{\mathrm{L}^1}$, $\mathrm{SMTIC}^{\mathrm{L}^1}$ and their weak equivalent are given in Proposition 6.3.4.

The set $\mathrm{MTI_d}$ is known as the *Almost Periodic Wiener algebra* (APW) by Karlovich, Spitkovsky et al. and as the *Wiener–Pitt algebra* (WP) by Babadzhanyan and Rabinovich [BR]. For $\dim U, \dim Y < \infty$, the set MTIC is the well-known set denoted by $\mathcal{A}$ or $\mathcal{A}(0)$ (or sometimes by $\mathrm{LA}^+(0)$) in the texts of Callier, Desoer, Winkin and others (see, e.g., [CD80], [CW99], [CZ]).

The so called *causal Wiener class* $\mathrm{MTIC}^{\mathrm{L}^1}$ consists of operators of the form $\mathbb{E} = f + T_1\delta_0$, where $f \in \pi_+\mathrm{L}^1$, i.e., of (I/O) maps $\mathbb{E}$ having an $\mathrm{L}^1$ impulse response plus a feedthrough operator. By Lemma D.1.23, transfer functions that are holomorphic around $i\mathbf{R} \cap \{\infty\}$ belong to $\mathrm{MTI}^{\mathrm{L}^1}$. The Wiener class (or "Wiener algebra") $\mathrm{MTI}^{\mathrm{L}^1}$ is denoted by $W$ or $\mathcal{W}$ in [CG81] and [CG97], and $\mathrm{MTIC}^{\mathrm{L}^1}$ is denoted by $\mathcal{A}$ in [CG97].

The MTI norm ("measure norm", i.e., total variation) and the SMTI norm are stronger than the TI norm ($\mathcal{B}(\mathrm{L}^2)$ norm), by Theorem 2.6.4(h2); in fact, they are strictly stronger, even in the scalar case, because if $f \in \mathrm{L}^1(\mathbf{R}+)$, then $\|f*\|_{\mathrm{TI}} = \|\widehat{f}\|_{\mathrm{L}^\infty}$ (by Theorem 3.1.3) and $\|\widehat{f}\|_{\mathrm{L}^\infty}$ may be less than $\|f\|_1$.

Each $\mathbb{E} \in \mathrm{MTI}$ corresponds (linearly and isometrically, in particular, $\|\mathbb{E}\|_{\mathrm{MTI}} = \int_{\mathbf{R}} d|\mu|$) to a Borel measure $\mu = f\,dt + \sum_{k=0}^\infty T_k\delta_{t_k}$ ($\mathcal{B}(U,Y)$-valued, countably additive and of bounded uniform total variation) consisting of a discrete (atomic) part and of an absolutely continuous part (i.e., one without a singular continuous part) through the formula $\mathbb{E}u = \mu * u$ for all $u \in \mathrm{L}^2$ (equivalently, through $\widehat{\mathbb{E}} = \widehat{\mu}$). This correspondence is an isometric isomorphism (onto if $\dim U < \infty$ or $\dim Y < \infty$, as one easily deduces from the results on pages 99 and 82 of [DU]). However, if $\dim U = \infty$, then some absolutely continuous $\mathcal{B}(U)$-valued Borel measures are not differentiable a.e. (see p. 219 & p. 217(3) of [DU]).

The set $\mathrm{supp_d}(\mathbb{E})$ is denoted by $\sigma(\widehat{\mathbb{E}})$ (or $\sigma(\widehat{\mu})$) in the theory of almost-periodic functions (see p. 2188 of [RSW]). Clearly the set $\mathrm{L}^1$ is an ideal of MTI (we identify a measure, its derivative and the corresponding TI operator when there is no risc of misconception).

For $T \in \mathbf{R}$, the class $\mathrm{MTI}_{T\mathbf{Z}}$ corresponds to MTI "measures with equally-spaced atoms" with space $T$. Because $\mathrm{MTI}_{T\mathbf{Z}}$ is isomorphic to $\ell^1(\mathbf{Z})$ and $\mathrm{L}^1*$ is an ideal of $\mathrm{MTI}_{T\mathbf{Z}}$, several factorization and invertibility problems on $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ can be reduced to those on $\ell^1$ and to those on $\mathrm{MTI}^{\mathrm{L}^1}$. Due to this fact (the earliest application of which we have seen in the thesis [Winkin]), we shall use the class $\mathrm{MTI}_{T\mathbf{Z}}$ extensively throughout this monograph.

As the definition shows, by the subscript $\mathcal{BC}$ we mean that $\widehat{\mathbb{E}} - E$ is $\mathcal{BC}$-valued. In the case of $\mathrm{MTI}^{\mathcal{BC}}$ and its subclasses, this means that $f$ and $T_k$ (for $k$ s.t. $t_k \neq 0$) are $\mathcal{BC}$-valued. The class $\mathrm{CTIC}^{\mathcal{BC}}$ consists $\widehat{\mathbb{E}} = E + \hat{g}$, where $E \in \mathcal{B}$, $\hat{g} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{BC}) \cap \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\};\mathcal{BC})$, and $\hat{g}(\infty) = 0$.

All well-posed fractions of exponential WTIC functions have a d.c.f. if $\dim U, \dim Y < \infty$, by Theorem 2.1 of [CD80]. This follows from the fact that a function in $\mathrm{H}^\infty_\omega$ for some $\omega < 0$ has only a finite number of zeros on $\mathbf{C}^+$.

The MTI spaces defined above are ULR and have beautiful factorization properties (see Theorems 2.6.4 and 4.1.6), which make them perfect candidates for stabilization theory. Moreover, most of them also admit spectral factorization (see Theorem 8.4.9), hence our optimization theory becomes more complete for such maps than for general TI maps.

Next we note that these classes are closed under composition, causal adjoints and inverses, and we list some of their further properties:

**Theorem 2.6.4 (MTI, SMTI, CTI)** *Let $\mathcal{A}$ be one of the classes* CTI, $\mathrm{CTI}^{\mathcal{BC}}$, MTI, $\mathrm{MTI}^{\mathcal{BC}}$, $\mathrm{MTI}_\mathrm{d}$, $\mathrm{MTI}_\mathrm{d}^{\mathcal{BC}}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_\mathbf{S}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. *Set* $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$.

*Let* $\mathbb{E} \in \mathcal{A}(U,Y)$, $\mathbb{F} \in \mathcal{A}(Y,Z)$ *and* $\omega, \omega' \in \mathbf{R}$. *Then the following hold:*

*(a1)* $\mathcal{A}(U,Y)$ *is a Banach space,* $\mathcal{A}(U)$ *is a Banach algebra and* $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset} \mathrm{TI}$ *(cf. Definition 6.2.4); in particular,* $\mathcal{A}\mathcal{A} = \mathcal{A}$ *and* $\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}} \underset{a}{\subset} \mathrm{TIC}$.

*(a2) If* $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$, *then* $\mathbb{E}\mathbb{F}$ *is equal to the convolution of* $\mathbb{E}$ *and* $\mathbb{F}$, *and* $\widehat{\mathbb{E}}$ *is equal to the Laplace transform of* $\mathbb{E}$ *taken in* MTI *sense (as in (D.22)).*

*(b1)* $\mathbb{E}^*, \mathbb{E}^\mathrm{d} \in \mathcal{A}$ *and* $\mathbb{E}\mathbb{F} \in \mathcal{A}$.

*(b2) If* $\mathbb{E}, \mathbb{F} \in \widetilde{\mathcal{A}}$, *then* $\mathbb{E}^\mathrm{d} \in \widetilde{\mathcal{A}}$ *and* $\mathbb{E}\mathbb{F} \in \widetilde{\mathcal{A}}$.

*(b3) We have* $\{\mathbb{E}^* \mid \mathbb{E} \in \mathcal{A}_\omega\} = \mathcal{A}_{-\omega}$ *and* $\|\mathbb{E}\|_{\mathcal{A}_\omega} = \|\mathbb{E}^*\|_{\mathcal{A}_{-\omega}}$.

*(c1)* $\mathbb{E} \in \mathcal{G}\mathrm{TI}(U,Y) \Leftrightarrow \mathbb{E} \in \mathcal{G}\mathcal{A}(U,Y)$.

*(c2)* $\mathbb{E} \in \mathcal{G}\mathrm{TIC}(U,Y) \Leftrightarrow \mathbb{E} \in \mathcal{G}\widetilde{\mathcal{A}}(U,Y)$.

*(d) If* $\mathbb{F} = F + f* \in \mathrm{MTI}^{\mathrm{L}^1}(Y,Z)$ *and* $\mathbb{G} = G + g* \in \mathrm{MTI}^{\mathrm{L}^1}(U,Y)$, *then* $\mathbb{F}\mathbb{G} = FG + (Fg + fG + f*g)* \in \mathrm{MTI}^{\mathrm{L}^1}(U,Z)$.

*(e1)* $\widehat{\mathbb{E}} \in \mathcal{C}_\mathrm{bu}(i\mathbf{R};\mathcal{B}(U,Y))$ *and* $\|\widehat{\mathbb{E}}(s)\| \leq \|\mathbb{E}\|_\mathrm{TI} \leq \|\mathbb{E}\|_\mathcal{A}$ *for all* $s \in i\mathbf{R}$.

*(e2) If* $\mathbb{E} \in \widetilde{\mathcal{A}}$, *then* $\widehat{\mathbb{E}} \in \mathcal{C}_\mathrm{bu}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ *and* $\|\widehat{\mathbb{E}}(s)\| \leq \|\mathbb{E}\|_\mathrm{TI} \leq \|\mathbb{E}\|_\mathcal{A}$ *for all* $s \in \overline{\mathbf{C}^+}$.

*(f) We have* $\widetilde{\mathcal{A}} \subset \mathrm{ULR}$, $\mathrm{MTI}^{\mathrm{L}^1} \subset \mathrm{CTI}$ *and* $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{CTIC} \subset \mathrm{UHPR}$.

*(g1) Assume that* $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$. *Then* $\mathcal{A}_\omega = \mathrm{e}^{\omega\cdot}\mathcal{A}\mathrm{e}^{-\omega\cdot}$ *and* $\widetilde{\mathcal{A}}_\omega = \mathrm{e}^{\omega\cdot}\widetilde{\mathcal{A}}\mathrm{e}^{-\omega\cdot}$. *Moreover,* $\widetilde{\mathcal{A}}_\omega \underset{c}{\subset} \widetilde{\mathcal{A}}_{\omega'}$ *for* $\omega' \geq \omega$.

*(g2) Assume that* $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$. *Then* $\mathcal{A}_\omega^* = \mathcal{A}_{-\omega}$, $\mathcal{A}_\omega^\mathrm{d} = \mathcal{A}_\omega$. *and* $\widetilde{\mathcal{A}}_\omega^* = \widetilde{\mathcal{A}}_{-\omega}$.

*(h1) Replace the letters* MTI *by* SMTI *everywhere in this theorem. Then (a1)–(a2) and (d)–(g1) still hold except that* $\widehat{\mathbb{E}}$ *need not be strongly continuous in (e1)–(e2) and instead of (f) we only have that* $\widetilde{\mathcal{A}} \subset \mathrm{SLR}$ *and* $\mathrm{SMTIC}^{\mathrm{L}^1} \subset \mathrm{SHPR}$.

*(h2) We have* $\|\mathbb{G}\|_{\mathrm{TI}_\omega} \le \|\mathbb{G}\|_{\mathrm{SMTI}_\omega} \le \|\mathbb{G}\|_{\mathrm{MTI}_\omega} \le \infty$ *for all* $\mathbb{G} \in \mathrm{TI}_\omega$.

*(i1) If* $\mathbb{D} \in \mathrm{MTIC}_\infty(U,Y)$, *then* $\|\pi_{[0,t)}(\mathbb{D}-D)\pi_{[0,t)}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))} \to 0$, *as* $t \to 0+$.

*(i2) If* $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,Y))$, *then* $\|u \mapsto (f * u)(t)\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U),Y)} \to 0$, *as* $t \to 0+$.

*(i3) If* $\mathbb{D} \in \mathrm{SMTIC}_\infty(U,Y)$, *and* $u = \chi_{\mathbf{R}_+} u_0$, *then* $\mathbb{D}u \in \mathcal{C}(\mathbf{R}_+;Y)$ *and* $(\mathbb{D}u)(0) = Du_0$.

*(j) If* $\dim U < \infty$ *or* $\dim Y < \infty$, *then* $\mathcal{A}^{\mathcal{BC}} = \mathcal{A}$ *(since then* $\mathcal{BC}(U,Y) = \mathcal{B}(U,Y)$*).*

Obviously, the formula in (d) holds more generally too. See Lemma D.1.12 for adjoints and Laplace transforms of elements of MTI. Note also that almost all above results hold for Banach spaces $U$ and $Y$ too, as shown in the statements to which we refer in the theorem and its proof.

Chapter 4, Proposition 6.3.4 and Lemmas F.2.2–F.2.4 also describe properties of MTI and SMTI classes. The corresponding spectral and coprime factorization properties are given in Sections 5.2 and 8.4.

**Proof:** (c1)&(c2)&CTI-claims: By Theorem 4.1.1 and Lemma 4.1.3, $\mathcal{A}$ [and $\widetilde{\mathcal{A}}$] is inverse closed and [causal] adjoint closed, as indicated in (b1)–(c2). (We do not know whether the SMTI classes are inverse closed.)

The other claims about CTI and its subclasses are easy to prove (e.g., $\widehat{\mathrm{CTI}}(U,Y) = \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(U,Y))$ has the properties stated above; see Lemma 2.6.2 for the half-plane-regularity of CTIC maps). Therefore, for the rest of the proof, we assume that $\mathcal{A} \ne \mathrm{CTI}$ and $\mathcal{A} \ne \mathrm{CTI}^{\mathcal{BC}}$.

(a2) This follows from Lemma D.1.12(c3).

(a1)&(d) It is clear that $\mathcal{A}$ is a vector space. The isometric isomorphism

$$\mathrm{MTI}(U,Y) \ni (\sum_{r\in\mathbf{R}} T_r\delta_r + f)* \mapsto ((T_r)_{r\in\mathbf{R}}, f) \in \ell^1(\mathbf{R};\mathcal{B}(U,Y)) \times \mathrm{L}^1(\mathbf{R};\mathcal{B}(U,Y)) \tag{2.55}$$

shows that MTI is a Banach space (since $\mathrm{L}^1$ and $\ell^1$ are Banach spaces). It is easy to verify that also the image of $\mathcal{A}$ under (2.55) is a Banach space (recall that $\mathcal{BC}$ is closed in $\mathcal{B}$), hence $\mathcal{A}$ is a Banach space.

The composition of MTI operators corresponds to the convolution of the corresponding measures, by Lemma D.1.12(c1), hence $\mathcal{A}(U)$ is a Banach algebra (with a unit) and (b1) holds.

Now we have shown that $\mathcal{A}\mathcal{A} = \mathcal{A} \underset{a}{\subset} \mathrm{TI}$. Since $\mathrm{TICTIC} = \mathrm{TIC} \underset{a}{\subset} \mathrm{TI}$, we have $\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$. Obviously, $\mathcal{B} \underset{a}{\subset} \mathcal{A}$.

(b1)&(b2) These follow from Lemma 4.1.3 and (a1).

(b3) One easily verifies this.

(e1)&(e2) This follows from Lemma D.1.12(a1)&(c2)&(a1')&(c').

(f) By Lemma D.1.12(b'), we have $\widetilde{\mathcal{A}} \subset \mathrm{ULR}$; by (e), Lemma D.1.11(a1)&(a1') and Lemma 2.6.2, we have $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{UHPR}$, $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{CTIC}$ and $\mathrm{MTI}^{\mathrm{L}^1} \subset \mathrm{CTI}$.

(g1) By Lemma D.1.12(d), we have $\mathcal{A}_\omega = e^{\omega\cdot}\mathcal{A}e^{-\omega\cdot}$ and $\widetilde{\mathcal{A}}_\omega = e^{\omega\cdot}\widetilde{\mathcal{A}}e^{-\omega\cdot}$.

Since $e^{-\omega\cdot} \le 1$ on $\mathbf{R}_+$, it is easy to verify from the definition that $\widetilde{\mathcal{A}}_\omega \subset \widetilde{\mathcal{A}}_{\omega'}$. (See Remark 2.1.6 for further results.)

(g2) This follows from Lemma 4.1.3.

(h1) This follows as above, by using Lemmas F.2.2 and F.2.3. (We do not know about the excluded claims.)

(h2) We omit the simple proof of $\|\mathbb{G}\|_{\mathrm{SMTI}_\omega} \le \|\mathbb{G}\|_{\mathrm{MTI}_\omega}$; the other inequality is given in (e2) and (h1).

(i1) This follows from the fact that the MTI norm of the part of $\mathbb{D}$ lying on $(0,t)$ decreases to zero. We leave the simple details to the reader.

(i2) This holds because $\|(f*u)(t)\|_Y \le \|\pi_{[0,t)}f\|_2\|\pi_{[0,t)}u\|$, by Lemma D.1.7, and $\|\pi_{[0,t)}f\|_2 \to 0$ as $t \to 0$, by Corollary B.3.8.

(i3) The proof is analogous to that of (i1) and omitted (note that $\mathbb{D}\cdot u_0 \in \mathrm{MTIC}(\mathbf{C},Y)$).

(j) See Lemma A.3.4(B1). $\square$

We finish this section by a minor technical result that allows us to to approximate $\mathcal{BC}$-valued TI maps by finite-dimensional operators:

**Lemma 2.6.5** *Let $\mathbb{D} \in \mathcal{A}(U,Y)$ and $\widehat{\mathbb{D}}(+\infty) = 0$, where $\mathcal{A} = \mathrm{CTI}^{\mathcal{BC}}$ or $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}$. Then there are countable orthogonal sequences $\{u_n\}_{n=1}^\infty \subset U$ and $\{y_n\}_{n=1}^\infty \subset Y$ s.t. when $P_n$ [$P_n'$] is the orthogonal projection of $U$ [$Y$] onto $\mathrm{span}(u_1,\dots,u_n)$ [$\mathrm{span}(y_1,\dots,y_n)$], we have $P_n'\mathbb{D}P_n \to \mathbb{D}$, $P_n'\mathbb{D} \to \mathbb{D}$, and $\mathbb{D}P_n \to \mathbb{D}$ in $\mathcal{A}(U,Y)$, as $n \to \infty$.*

*If $U = Y$, we can choose $\{u_n\}$ so that $P_n\mathbb{D}P_n \to \mathbb{D}$ in $\mathcal{A}(U)$.*

Of course, if $\mathbb{D} = (\sum_k T_k\delta_{t_k} + f)*$, then we can replace $P_n' f P_n$ (i.e., $\begin{bmatrix} P_n' f P_n & 0 \\ 0 & 0 \end{bmatrix}$) by some smooth or simple function with a compact support for each $n$ without losing the convergencies (see Theorem B.3.11).

**Proof:** *Case* CTI*:* 1° Let $n \in \mathbf{N}$. For each $s \in i\mathbf{R}\cup\{\infty\}$, $\widehat{\mathbb{D}}(s) \in \mathcal{BC}$, hence there are $P_{n,s}$ and $P_{n,s}'$ as in Lemma A.3.4(B2), so that $\|\widehat{\mathbb{D}}(s) - P_{n,s}'\widehat{\mathbb{D}}(s)P_{n,s}\| < 1/n$. The set

$$G_s := \{z \in i\mathbf{R}\cup\{\infty\} \mid \|\widehat{\mathbb{D}}(z) - P_{n,s}'\widehat{\mathbb{D}}(z)P_{n,s}\| < 1/n\} \tag{2.56}$$

is open, for each $s$. By the compactness of $\widehat{\mathbb{D}}(i\mathbf{R}\cup\{\infty\})$, there is a finite subset $\{s_1,\dots,s_{m_n}\} \subset i\mathbf{R}\cup\{\infty\}$, so that $i\mathbf{R}\cup\{\infty\} \subset G_{s_1}\cup\ldots\cup G_{s_{m_n}}$. Let $u_{n,1},\dots,u_{n,k_n}$ [$y_{n,1},\dots,y_{n,k_n'}$] be an orthogonal base of $\cup_{j=1}^{m_n}\mathrm{Ran}(P_{n,s_j})$ [$\cup_{j=1}^{m_n}\mathrm{Ran}(P_{n,s_j}')$]. Thus, $\|\widehat{\mathbb{D}}(z) - \widetilde{P}_n'\widehat{\mathbb{D}}(z)\widetilde{P}_n\| < 1/n$ for all $z \in i\mathbf{R}\cup\{\infty\}$ for the corresponding projections.

2° Let $u_1,u_2,\dots$ be the sequence $u_{1,1},u_{1,2},\dots,u_{1,k_1},u_{2,1},u_{2,2},\dots$ and let $y_1,y_2,\dots$ be the sequence $y_{1,1},y_{1,2},\dots,y_{1,k_1'},y_{2,1},y_{2,2},\dots$ to obtain $\|\widehat{\mathbb{D}} - P_n'\widehat{\mathbb{D}}P_n\| \to 0$. Clearly $\|\widehat{\mathbb{D}} - P_n'\widehat{\mathbb{D}}\|$ and $\|\widehat{\mathbb{D}} - \widehat{\mathbb{D}}P_n\|$ are even smaller.

If $U = Y$, we may modify our proof by choosing only the $u_k$'s, using the last claim in Lemma A.3.4(B2), or we can as well use the sequence $u_1,y_1,u_2,y_3,u_3,\dots$.

*Case* MTI*:* 1° Let $\mathbb{D}(s) = [\sum_{k=0}^\infty T_k\delta_{r_k} + f]*$, where $T_k \in \mathcal{BC}(U,Y)$ for all $k$ and $f \in \mathrm{L}^1(\mathbf{R};\mathcal{BC}(U,Y))$. Let $j \in \mathbf{N}+1$.

Then there are $m \in \mathbf{N}+1$, $\mathbb{E} = [\sum_{k=0}^m T_k'\delta_{r_k} + \widetilde{f}]*$ s.t. $m \in \mathbf{N}+1$, $T_k' \in \mathcal{BC}(U,Y)$ for all $k$, $f = \sum_{k=0}^m S_k\chi_{E_k}$, where each $S_k \in \mathcal{BC}(U,Y)$ is finite-dimensional, and $\|\mathbb{D}-\mathbb{E}\|_{\mathrm{MTI}} < 1/j$ (replace each $T_k$ by a finite-dimensional

$T_k'$ (see Lemma A.3.4(B1)), use the density of simple functions in $\mathrm{L}^1$ (Theorem B.3.11(a1)) and choose suitable $m < \infty$). Let

$$U_j := \operatorname{span}\left(\operatorname{Ker}(T_1')^\perp \cup \ldots \cup \operatorname{Ker}(T_m')^\perp \cup \operatorname{Ker}(S_1)^\perp \cup \ldots \cup \operatorname{Ker}(S_m)^\perp\right) \subset U, \tag{2.57}$$

$$Y_j := \operatorname{span}\left(\operatorname{Ran}(T_1') \cup \ldots \cup \operatorname{Ran}(T_m') \cup \operatorname{Ran}(S_1) \cup \ldots \cup \operatorname{Ran}(S_m)\right) \subset Y. \tag{2.58}$$

Let $\widetilde{P}_j$ $[\widetilde{P}_j']$ be the orthogonal projection of $U$ $[Y]$ onto $U_j$ $[Y_j]$. Then $\mathbb{E} = \widetilde{P}_j'\mathbb{E}\widetilde{P}_j$, hence

$$\|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j\| = \|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j + \widetilde{P}_j'\mathbb{E}\widetilde{P}_j - \mathbb{E}\| \leq \|\mathbb{D} - \mathbb{E}\| + \|\widetilde{P}_j'(\mathbb{E} - \mathbb{D})\widetilde{P}_j\| < 2/j. \tag{2.59}$$

2° Let $u_1, \ldots, u_{k_1}$ span $U_1$, add then vectors $u_{k_1+1}, \ldots, u_{k_2} \subset U_2$ so that $u_1, \ldots, u_{k_2}$ span the space $\operatorname{span}(U_1 \cup U_2)$ etc. If $\widetilde{P}_j$ and $\widetilde{P}_j'$ are the projections defined above for some $m$, then $\operatorname{Ran}(\widetilde{P}_j) \subset \operatorname{Ran}(P_n)$ and $\operatorname{Ran}(\widetilde{P}_j') \subset \operatorname{Ran}(P_n')$ for some $n$ (and every subsequent one), hence $\|\mathbb{D} - P_n'\mathbb{D}P_n\| \leq \|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j\| < 1/j$. Therefore, $\|\mathbb{D} - P_n'\mathbb{D}P_n\| \to 0$, as $n \to \infty$.

If $U = Y$, we can replace $U_j$ and $Y_j$ by $\operatorname{span}(U_j \cup Y_j)$ to obtain the required projections. □

## Notes

The class MTIC was introduced into control theory in [CD78] and [CD80]. An excellent reference on the class is [CZ]. The class $\mathrm{MTIC}_{T\mathbf{Z}}$ is treated in [Winkin] and [CW99], and $\mathrm{MTI_d}$ in [BKRS] and [RSW], among others. All these assume that $U$ and $Y$ are finite-dimensional; in that case probably all of Theorem 2.6.4 is well known. General vector-valued measures (with MTIC as a special case) are treated in [DU] and in [Dinculeanu].