Chapter 14

Riccati Equations (DARE)

The inherent vice of capitalism is the unequal sharing of blessings;
the inherent virtue of socialism is the equal sharing of misery.
— Winston Churchill (1874–1965)

In this chapter, we shall define and explore *Discrete-time Algebraic Riccati Equations (DAREs)*. Thus, this chapter is the discrete-time counterpart of Chapter 9 (cf. Theorem 14.1.3).

In Section 14.1, we study the basic properties of DAREs. In Section 14.2, we list certain auxiliary lemmas and further results. Section 14.3 contains some results on discrete-time spectral factorization.

See Chapter 15 for more on positive DAREs (those with positive cost (for zero initial state) or $S \geq 0$). That chapter also treats minimization (LQR) problems. The H^∞ full-information control problem is treated in Sections 11.5 and 11.6, and the H^∞ four-block problem in Sections 12.2 and 12.6.

Part of the continuous-time results are proved using this chapter, hence it is important to remember that, logically, one should verify all results of this monograph in discrete time (cf. Theorem 13.3.13) before verifying them in continuous time.

We shall work under the discrete-time counterpart of Hypothesis 9.0.1:

Standing Hypothesis 14.0.1 *Throughout this chapter and Chapter 15, we assume that* $\Sigma = \left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$. *The letters* U, H *and* Y *denote Hilbert spaces of arbitrary dimensions.*

We also assume that $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \end{bmatrix}$, Z^{u} *and* Z^{s} *are as in (discrete-time) Definition 8.3.2,* $\left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{Q} & \mathbb{R} \end{array}\right] \in \mathrm{wpls}(U,H,\widetilde{Y})$ *for some Hilbert space* $\widetilde{Y}$*, and that* $\pi_+\tau^t z \in Z^{\mathrm{s}} \Leftrightarrow z \in Z^{\mathrm{s}}$ $(z \in Z^{\mathrm{u}},\ t \in \mathbf{N})$.

The reader may again ignore the latter paragraph of the hypothesis and read $\mathcal{U}_*^*$ as any of $\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$; see the comments around Hypothesis 9.0.1 for details.

14.1 Discrete-time Riccati equations (DARE)

Duggan's Equality: To every Ph.D. there is an equal and opposite Ph.D.

Traditionally optimal control problems are solved by solving corresponding DAREs. An optimal exponentially stabilizing state feedback operator exists iff the DARE has an exponentially stabilizing solution, in which case the operator and the optimal cost operator form the solution of the DARE. In this section, we shall extend this to infinite-dimensional systems and also generalize this to other forms of stabilization than exponential stabilization.

We shall introduce the DAREs and their basic properties, including the uniqueness of a $\mathcal{U}_*^*$-stabilizing solution. In Theorem 14.1.5, we show that $\mathcal{U}_*^*$-stabilizing solutions of the DARE correspond one-to-one to the optimal state feedback operators as in the classical setting. In Theorem 14.1.6, we show that whenever there is a unique optimal control, this control can be given in a state feedback form.

Under the standard coercivity assumption, an optimal control is necessarily unique, hence of state feedback form, hence the unique solution of the DARE, by the above. Thus, in Corollary 14.1.7, we can state that the three above conditions are equivalent and that they hold iff the system is stabilizable. In Theorem 14.2.7, we include the "converse" (for $\mathcal{U}_{\rm exp}$): the DARE has an exponentially stabilizing solution iff the standard coercivity assumption holds and the system is exponentially stabilizable (for $\dim U < \infty$ a third equivalent condition is that there is a unique optimal control).

We start by defining the DARE. Our definition is equivalent to the definition of the IARE in the sense explained in Remark 14.1.2.

Definition 14.1.1 (DARE) *We call $\mathcal{P}$ (or $(\mathcal{P},S,K)$) a* solution of the Discrete-time Algebraic Riccati Equation (DARE) *(induced by Σ and J; we denote this by $\mathcal{P} \in \mathrm{DARE}(\Sigma,J)$) iff*

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C^*JC - K^*SK, \\ S = D^*JD + B^*\mathcal{P}B, \\ SK = -(D^*JC + B^*\mathcal{P}A), \end{cases} \tag{14.1}$$

$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $K \in \mathcal{B}(H,U)$, and $S \in \mathcal{GB}(U)$.

A solution of the DARE is called stabilizing *(resp.* r.c.-stabilizing, stable, ...*) if $\begin{pmatrix} K \mid 0 \end{pmatrix}$ is a stabilizing (resp. r.c.-stabilizing, stable, ...) state feedback operator for Σ (see Definition 6.6.10 for further prefices and suffices).*

We use prefices "$\mathcal{U}_*^*$-", "P-" and "PB-" as in Definition 9.8.1.

The solution $\mathcal{P}$ (or $(\mathcal{P},S,K)$) of the extended DARE (eDARE) *is defined analogously except that we do not require $S \in \mathcal{GB}(U)$. For solutions of the eDARE we denote $\mathcal{P} \in \mathrm{eDARE}(\Sigma,J)$ (or $(\mathcal{P},S,K) \in \overline{\mathrm{eDARE}}(\Sigma,J)$).*

We call K the state feedback operator *and S the* signature (or sensitivity) operator *corresponding to the solution. We define $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\mathbb{X}$, $\mathbb{M}$, $\mathbb{N}$ and $\Sigma_\circlearrowleft$ as in Definitions 9.1.3 and 9.1.4.*

Note that necessarily $S = S^*$, and that $\begin{pmatrix} K \mid 0 \end{pmatrix}$ is an admissible state-feedback pair for Σ (by Lemma 13.3.12).

A solution $\mathcal{P}$ of the eDARE determines S uniquely and K modulo the addition of an operator $\Delta K \in \mathcal{B}(H, \mathrm{Ker}(S))$ (hence uniquely if S is one-to-one). By Remark 9.8.8, the operators S and K can be eliminated from the eDARE too.

We have required that "$F = 0$" in the [e]DARE; this simplification does not reduce generality:

Remark 14.1.2 (DARE vs. IARE) *If we replace "$\begin{pmatrix} K \mid 0 \end{pmatrix}$" by "$\begin{pmatrix} K \mid F \end{pmatrix}$" in the definition of the [e]DARE, where we require that $I - F \in \mathcal{GB}(U)$, then the solutions of the [e]DARE become exactly the admissible solutions of the [e]IARE. Moreover, such solutions are exactly the triples $(\mathcal{P}, X^*SX, \begin{pmatrix} X^{-1}K \mid I - X \end{pmatrix})$, where $(\mathcal{P}, S, K)$ is a (original) solution of the [e]DARE and $X \in \mathcal{GB}(U)$.* □

The left columns of the corresponding closed-loop systems are equal, by (13.61 and (13.62), hence both solutions correspond to same closed-loop state and outputs (in the absence of a closed-loop input). See also Theorem 9.8.12(s1).

The results of Chapter 9 hold in discrete time too:

Theorem 14.1.3 (Chapters 8–9 apply) *The results of Sections 8.3, 8.4, 9.1, 9.8–9.10 (except the "we do not know" part of Lemma 9.9.7(d)), Lemma 9.11.1, Section 9.12, and Definition 9.14.1–Theorem 9.14.3 hold modulo the changes given in (13.63).*

(Note that (13.63) makes Hypothesis 8.4.8 useless, since it removes any "$S = D^*JD$" claims. This is because in discrete time one almost never has $S = D^*JD$, not even for $\heartsuit\mathrm{MTIC}^{\mathrm{L}^1}$ (because $\widehat{\mathbb{D}}(0) \neq \widehat{\mathbb{D}}(-1)$ in general); see also the comments around Example 14.2.9.)

Moreover, in Theorem 9.9.10 we have $u_{\mathrm{crit}}(x_0)(t) = Kx(t)$ in (f1) and $K = K' + K_\natural$ in (g2) if we take $M' = I = M$ (i.e., $F' = 0 = F$).

Also almost everything of Section 9.13 can easily be adapted for [e]DAREs, and most of the rest can be obtained directly by discretization. Many of the results of Chapter 9 not mentioned above have counterparts in this chapter. Also the B_{w}^*-CARE results of Section 9.2 can be written in their discrete-time forms, but since we lose "$S = D^*JD$", many of these results become rather useless.

Recall from Theorem 9.9.1 that there is a J-critical state feedback pair for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.

If this is the case, then $\mathcal{J}(x_0, u_{\mathrm{crit}}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$, and u_{crit} is given by the feedback K; the cost with closed-loop input $u_\circlearrowleft \in \ell^2([0,t); U)$ ($t < \infty$) is given by $\langle x_0, \mathcal{P}x_0 \rangle + \langle u_\circlearrowleft, S u_\circlearrowleft \rangle$, by Theorem 9.9.1(h); hence the name for the signature operator S.

Lemma 14.3.5 provides us some additional classes satisfying (the converted) Hypothesis 8.4.7.

Proof of Theorem 14.1.3: One can first verify Sections 8.3 and 8.4, then Lemma 14.2.1, then 9.10 (and 9.11.1 from 9.11), then 9.8 (including Theorem 14.1.4) and 9.9, then the rest of Chapter 9. One can avoid references "backwards" by verifying each claim in its discrete and continuous time forms before proceeding to the next one.

Note that $K = -S^{-1}\pi_{\{0\}}\mathbb{N}^* JCA^{\cdot} \in \mathcal{B}(H,U)$ in Theorem 9.9.10(g1). $\square$

Note that $\mathcal{U}_{\text{out}}(x_0) := \{u \in \ell^2(\mathbf{N};U) \,|\, \mathbb{C}x_0 + \mathbb{D}u \in \ell^2(\mathbf{N};Y)\}$ etc., when we write (8.29)–(8.32) out (applying (13.63)).

A strongly internally stabilizing solution of the DARE is unique:

Theorem 14.1.4 ($\mathcal{P}$ is unique) *A solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ of the eIARE is unique to the following extent:*

(a) If the eDARE (14.1) has a strongly internally stabilizing solution, then that solution is unique among internally stabilizing solutions.

(b) There is at most one internally P-stabilizing solution of the eDARE.

(c) If the eDARE has an internally r-stabilizing solution for some $r < 1$, then any other solution is (internally) at most $1/r$-stabilizing.

(d) The eDARE has at most one P-q.r.c.-SOS-stabilizing solution.

(e) The eDARE has at most one $\mathcal{U}_*^*$-stabilizing solution.

Naturally, $\mathcal{P}$ determines $S := D^*JD + B^*\mathcal{P}B$ uniquely and K modulo the addition of an operator $\Delta K \in \mathcal{B}(H, \operatorname{Ker}(S))$.

Note from Theorem 9.9.1(a1)&(e), that each a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ determines a J-critical state feedback operator for Σ, and vice versa. The corresponding S is unique; the operator K is unique iff S is one-to-one.

Thus, if S is not one-to-one. then the feedback operator K solving the eDARE with $\mathcal{P}$ and S is not unique, but it may be that just one K is $\mathcal{U}_*^*$-stabilizing, i.e., that the $\mathcal{U}_*^*$-stabilizing solution triple $(\mathcal{P}, S, K)$ is unique, and that the J-critical control in state feedback form $(\mathbb{K}_{\circlearrowleft} x_0)$ is unique (for each x_0) (see Example 9.13.6 for an example).

Proof of Theorem 14.1.4: The claim on S and K is obvious, so we only need to prove the uniqueness of $\mathcal{P}$.

(a) Let $\mathcal{P}_1, \mathcal{P}_2$ be a stabilizing solutions; let $\mathcal{P}_1$ be strongly stabilizing. For $k = 1, 2$, let S_k and K_k correspond to $\mathcal{P}_k$. By the definition of K and S (see eDARE), we have

$$R := C^*JDK_1 - K_2^*D^*JC + A^*\mathcal{P}_2BK_1 - K_2^*B^*\mathcal{P}_1A + K_2^*(B^*\mathcal{P}_2B - B^*\mathcal{P}_1B)K_1 \tag{14.2}$$

$$= -K_2^*S_2K_1 + K_2^*S_1K_1 + K_2^*(S_2 - S_1)K_1 = 0. \tag{14.3}$$

Therefore, by the eDARE, we have

$$\mathcal{P}_1 - \mathcal{P}_2 = A^*(\mathcal{P}_1 - \mathcal{P}_2)A + K_2^*S_2K_2 - K_1^*S_1K_1 \tag{14.4}$$

$$= A^*(\mathcal{P}_1 - \mathcal{P}_2)A + (C^*JD + A^*\mathcal{P}_1B)K_1 - K_2^*(D^*JC + B^*\mathcal{P}_2A) \tag{14.5}$$

$$= (A^* + K_2^*B^*)(\mathcal{P}_1 - \mathcal{P}_2)(A + BK_1) + R \tag{14.6}$$

$$= A_{\circlearrowleft 2}^*(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}. \tag{14.7}$$

Multiply equation $\mathcal{P}_1 - \mathcal{P}_2 = A_{\circlearrowleft 2}^*(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}$ by $(A_{\circlearrowleft 2}^*)^k$ to the left and $A_{\circlearrowleft 1}^k$ to the right, and use the resulting chain of equations for $k = 0, \ldots, n-1$ to obtain

$$\mathcal{P}_1 - \mathcal{P}_2 = (A_{\circlearrowleft 2}^*)^n(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}^n \quad (n \in \mathbf{N}). \tag{14.8}$$

Because $(A^*_{\circlearrowleft 2})^n(\mathcal{P}_1 - \mathcal{P}_2)$ is bounded and $A^n_{\circlearrowleft 1}x_0 \to 0$ as $n \to +\infty$, for any $x_0 \in H$, we have $\mathcal{P}_1 x_0 - \mathcal{P}_2 x_0 = 0$ for all $x_0 \in H$, hence $\mathcal{P}_1 = \mathcal{P}_2$.

(b) Now $\mathcal{P}_k A^n_{\circlearrowleft k} x_0 \to 0$ as $n \to +\infty$, for all $x_0 \in H$, $k = 1, 2$, by (P3), and $\|A^n_{\circlearrowleft k}\|$ is bounded, so the result follows as above.

(c) If $M, r > 1$ and $s > 1/r$ are s.t. $\|A^n_{\circlearrowleft 1}x_0\| \le Mr^{-n}$, $\|A^n_{\circlearrowleft 2}x_0\| \le Ms^{-n}$, then

$$|\langle x_0, (\mathcal{P}_1 - \mathcal{P}_2)x_0\rangle| \le (\|\mathcal{P}_1\| + \|\mathcal{P}_2\|)M^2(rs)^{-n} \quad \text{for all } n \in \mathbf{N}, \tag{14.9}$$

hence then $|\langle x_0, (\mathcal{P}_1 - \mathcal{P}_2)x_0\rangle| = 0$ (in the proof of (a)).

(e) (resp. (d)) By Proposition 9.10.2(c) (resp. (e2)), the $\mathcal{U}^*_*$-stabilizing (resp. P-SOS-r.c.-stabilizing) solutions of the eIARE correspond one-to-one to the J-critical (resp. SOS-r.c.-stabilizing and J-critical over $\mathcal{U}_{\text{out}}$) state feedback operators (see Lemma 14.2.1).

The J-critical cost $\langle x_0, \mathcal{P} x_0\rangle$ over $\mathcal{U}^*_*$ (resp. over $\mathcal{U}_{\text{out}}$) is independent of the J-critical control $\mathbb{K}_{\circlearrowleft} x_0$, by Lemma 8.3.8, hence $\mathcal{P}$ is unique. □

From Theorem 9.9.1(a1) one can observe that any J-critical state feedback operator K corresponds to a $\mathcal{U}^*_*$-stabilizing solution $\mathcal{P}$ of the eIARE, and vice versa. Due to bounded input and output operators (B and C), in discrete-time we can go further:

Theorem 14.1.5 (J-critical $K \Leftrightarrow$ DARE) *There is a J-critical state feedback operator K iff the eDARE has a $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$.*

Moreover, such Ks are exactly those associated with the solutions, with S being the corresponding signature operator and $\mathcal{P}$ being the J-critical cost operator satisfying (9.139)). □

(This follows from the above and Lemma 14.2.1.) Also most of Theorem 14.1.6 holds, as one can observe from Theorem 9.9.1. Recall from Theorem 14.1.4 that $\mathcal{P}$ is unique but K need not be. Also recall from Remark 14.1.2 how each K corresponds to a set of pairs $\left(\begin{array}{c|c} K & F \end{array}\right)$ with same $\mathcal{P}$.

In most standard settings, any J-critical control is unique. Such a control can always be given by some state feedback operator:

Theorem 14.1.6 (Unique J-critical control is of the feedback form) *There is a unique J-critical control for each $x_0 \in H$ iff the eDARE has a $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$ with S one-to-one.*

Assume that this is the case. Then the following hold:

(a) The $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$ is unique.

(b1) The J-critical control is determined by $u_{\text{crit}}(x_0) = \mathbb{K}_{\circlearrowleft} x_0$, i.e., by $u_{\text{crit}}(x_0)_n = K(A + BK)^n x_0$ ($n \in \mathbf{N}$), where $\Sigma_{\circlearrowleft}$ is the closed-loop system corresponding to $\left(\begin{array}{c|c} K & 0 \end{array}\right)$.

(b2) Conversely, $Kx_0 = u_{\text{crit}}(x_0)_0$ and $\mathcal{P} = \mathbb{C}^*_{\circlearrowleft} J \mathbb{C}_{\circlearrowleft}$ (and $S = D^*JD + B^*\mathcal{P}B$).

(c) If Σ is exponentially stable, then K is exponentially r.c.-stabilizing.

(d) Theorem 9.9.1(f1)–(k) apply.

See Theorem 9.8.5 for $\mathcal{U}_*^*$-stabilizing solutions. Note in particular, that a solution is $\mathcal{U}_{\text{exp}}$-stabilizing iff $\rho(A+BK) < 1$, by Lemma 13.3.7 (and Theorem 9.8.5).

Theorem 9.9.6 contains an analogy of the above theorem for WPLSs (i.e., for continuous time) with bounded B. For very irregular WPLSs, the "only if" part may fail (at least we have to give up eCAREs; presumably even eIAREs).

In general, all J-critical state feedback operators correspond bijectively to $\mathcal{U}_*^*$-stabilizing solutions of the eDARE as in the above theorem, by Proposition 9.9.1.

Proof of Theorem 14.1.6: The equivalence follows from Theorem 9.9.1(a1)&(e2), because a unique (for each x_0) J-critical control u_{crit} is necessarily of the feedback form: if K is defined by (b2), then $\mathbb{A}_{\circlearrowleft} = \mathbb{A}_{\text{crit}}$, $\mathbb{K}_{\circlearrowleft} = \mathbb{K}_{\text{crit}}$, and $\mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\text{crit}}$, by (13.61).

(a) By Theorem 9.9.1(f1), $\mathcal{P}$ is unique, hence so are S and K (because S is one-to-one).

(b2)&(d) These follow from Theorem 9.9.1.

(b1) This follows from formula $u_{\text{crit}}(x_0) = \mathbb{K}_{\text{crit}}x_0 = \mathbb{K}_{\circlearrowleft}x_0$.

(c) This follows from Theorem 8.3.9(a5) (see also (13.61)). □

Thus, if Σ is J-coercive and the finite cost condition is satisfied, then the optimal control corresponds to a (unique) $\mathcal{U}_*^*$-stabilizing solution of the DARE:

Corollary 14.1.7 (J-coercive ⇒ DARE) *Assume that Σ is J-coercive. Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;

(ii) $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$;

(iii) the DARE has a $\mathcal{U}_*^*$-stabilizing solution. □

(This follows from Theorem 14.1.6 and Theorem 8.4.3, since necessarily $S \in \mathcal{GB}(U)$, by Lemma 9.10.3.) See also Theorem 14.2.7.

(See p. 829 for notes.)

14.2 DARE — further results

Everyone wants results, but no one is willing to do what it takes to get them.

— Dirty Harry

In this section, we present auxiliary lemmas and further results on DAREs. These include the relations between all (not necessarily stabilizing) solutions of the eDARE, symplectic "matrix" pencils, DAREs connected to the internal and/or output stability of a system, equivalence between J-coercivity and the existence of a unique optimal control, and H^2 "spectral factors".

In previous section, we used the fact that the solutions of the [e]DARE are exactly the admissible solutions of the [e]IARE:

Lemma 14.2.1 (eDARE $\Leftrightarrow$ eIARE) *Let $S \in \mathcal{B}(U)$ and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$. Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ, and let $\Sigma_\circlearrowleft := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right] \in$ $\mathrm{wpls}(U,H,Y \times U)$ be the corresponding closed-loop system. Set $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_\circlearrowleft$. If equations*

$$\mathbb{K}^{t\,*} S \mathbb{K}^t = \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*} J \mathbb{C}^t \tag{14.10}$$

$$\mathbb{X}^{t\,*} S \mathbb{X}^t = \mathbb{D}^{t\,*} J \mathbb{D}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t, \tag{14.11}$$

$$\mathbb{X}^{t\,*} S \mathbb{K}^t = -\left(\mathbb{D}^{t\,*} J \mathbb{C}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t\right). \tag{14.12}$$

hold for $t = 1$, then they hold for each $t \in \mathbf{N}$.

Thus, by (13.62), $(\mathcal{P}, X^* S X, X^{-1} K)$ (here $X := I - F$) is a [stabilizing] solution the eDARE iff $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an admissible [stabilizing] solution of the eIARE. All prefices and suffices apply.

If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an admissible solution of the eIARE with closed-loop system $\Sigma_\circlearrowleft$, then the closed-loop system corresponding to $\left(\begin{array}{c|c} X^{-1}K & 0 \end{array}\right)$ is $\left[\begin{array}{c|c} A_\circlearrowleft & B_\circlearrowleft X \\ \hline C_\circlearrowleft & D_\circlearrowleft X \\ K_\circlearrowleft & 0 \end{array}\right]$, by (13.61) and (13.62).

Proof: We assume that (14.10)–(14.12) are satisfied for $t = 1$ (i.e., that the eDARE is satisfied).

1° (14.10): Let $x \in H$. Apply (14.10) with $t = 1$ to $A^k x$ ($k \in \mathbf{N}$) to obtain

$$\langle A^{k+1}x, \mathcal{P}A^{k+1}x\rangle - \langle A^k x, \mathcal{P}A^k x\rangle + \langle CA^k x, JCA^k x\rangle = \langle KA^k x, SKA^k x\rangle. \tag{14.13}$$

But, by (13.38), we have $(\mathbb{C}x)_k = CA^k x$ and $(\mathbb{K}x)_k = KA^k x$, hence we can add (14.13) for $k = 0, 1, 2, \ldots, t-1$ to get (14.10).

2° (14.11): Let $0 \neq u \in \mathrm{c_c}(\mathbf{N}; U)$. Set $x := \mathbb{B}\tau u$ to obtain that $x_{j+1} = Ax_j + Bu_j$ and $(\mathbb{D}u)_j = Cx_j + Du_j$ ($j \in \mathbf{N}$), by Lemma 13.3.3(c). Similarly, $\mathbb{X}u = u - Kx$. Thus, by (14.12) with $t = 1$, we have (here the inner products are taken on U or H, and the subscript j refers to "time", i.e., to the argument of x

and u):

$$\langle \mathbb{X}u, S\mathbb{X}u\rangle_j = \langle u, Su\rangle_j - \langle u, SKx\rangle_j - \langle SKx, u\rangle_j + \langle Kx, SKx\rangle_j \tag{14.14}$$

$$= \langle u, Su\rangle_j + \langle u, (D^*JC + B^*\mathcal{P}A)x\rangle_j + \langle (D^*JC + B^*\mathcal{P}A)x, u\rangle_j \tag{14.15}$$

$$+ \langle x, A^*\mathcal{P}Ax\rangle_j - \langle x, \mathcal{P}x\rangle_j + \langle x, C^*JCx\rangle = g_j + h_j \tag{14.16}$$

on $\mathbf{Z}$, where

$$g_j := \langle u, D^*JDu\rangle_j + \langle u, D^*JCx\rangle_j + \langle D^*JCx, u\rangle_j + \langle x, C^*JCx\rangle_j = \langle \mathbb{D}u, J\mathbb{D}u\rangle_j, \tag{14.17}$$

$$h_j := \langle u, (S - D^*JD)u\rangle_j + \langle u, B^*\mathcal{P}Ax\rangle_j + \langle B^*\mathcal{P}Ax, u\rangle_j + \langle x, A^*\mathcal{P}Ax\rangle_j - \langle x, \mathcal{P}x\rangle_j \tag{14.18}$$

$$= \langle Ax + Bu, \mathcal{P}(Ax, Bu)\rangle_j - \langle x_j, \mathcal{P}x_j\rangle = \langle x_{j+1}, \mathcal{P}x_{j+1}\rangle - \langle x_j, \mathcal{P}x_j\rangle. \tag{14.19}$$

Therefore,

$$\sum_0^j \langle \mathbb{X}u, S\mathbb{X}u\rangle_j - \sum_0^j \langle \mathbb{D}u, J\mathbb{D}u\rangle_j = \sum_0^j \langle \mathbb{X}u, S\mathbb{X}u\rangle_j - \sum_0^j g_j = \sum_0^j h_j = \langle x_{j+1}, \mathcal{P}x_{j+1}\rangle. \tag{14.20}$$

But this is (14.11) for $t = j+1$ applied to u (because $x := \mathbb{B}\tau^j u$). Because u and j were arbitrary, equation (14.11) holds, by density and continuity.

3° (14.12): For $t \in \mathbf{N}$, we set $\mathbb{T}_t := \mathbb{X}^{t\,*} S\pi_{[0,t)}\mathbb{K} + \pi_{[0,t)}\left(\mathbb{D}^* J\pi_{[0,t)}\mathbb{C} + \tau^t\mathbb{B}^*\mathcal{P}A^t\right) \in \mathcal{B}(H, \ell^2([0,t);U))$, as in Lemma 9.11.6. We must prove that $\mathbb{T} \equiv 0$, hence it is enough to show that

$$f_t := f_t(u, x_0) := \langle u, \mathbb{T}_t x_0\rangle_{\ell^2} = 0 \tag{14.21}$$

for arbitrary $t \in \mathbf{N}$, $u \in \ell^2((0,t);U)$ and $x_0 \in H$. Let u, x_0 be as above, and set

$$x_t := A^t x_0, \;\; z_t := \mathbb{B}\tau^t u \tag{14.22}$$

to obtain $x_{t+1} = Ax_t$, $z_{t+1} = Az_t + Bu_t$, $(\mathbb{X}u)_t = u_t - Kz_t$, $(\mathbb{D}u)_t = Du_t + Cz_t$, $(\mathbb{C}x_0)_t = Cx_t$, $(\mathbb{K}x_0)_t = Kx_t$ (see Lemma 13.3.3). Thus, for $t \in \mathbf{N}$, we have (recall that $\mathbb{N}v = \mathbb{D}u$)

$$f_{t+1} = \sum_0^t \langle (\mathbb{X}u)_n, S(\mathbb{K}x_0)_n\rangle_U + \sum_0^t \langle (\mathbb{D}u)_n, J(\mathbb{C}x_0)_n\rangle_Y \, dt + \langle z_{t+1}, \mathcal{P}x_{t+1}\rangle_H. \tag{14.23}$$

Therefore, (we set here $f_{-1} = 0$ so that this holds for $t = -1$ too)

$$f_{t+1} - f_t = \langle u_t - Kz_t, SKx_t\rangle + \langle Du_t + Cz_t, JCx_n\rangle + \langle Az_t + Bu_t, \mathcal{P}Ax_t\rangle - \langle z_t, \mathcal{P}x_t\rangle = 0, \tag{14.24}$$

by (14.10)–(14.12) with $t = 1$. Consequently, $f \equiv 0$ and $\mathbb{T} \equiv 0$.

4° *eIARE⇔eDARE:* Obviously, the eIARE with $t = 1$ is exactly the eDARE.

□

The "K" of a $\mathcal{U}_*^*$-stabilizing solution of the eDARE is the optimal state feedback operator. Conversely, given a unique optimal control, we obtain the corresponding solution of the eDARE as follows:

Lemma 14.2.2 *Let there be a unique J-critical control $u_{\rm crit}(x_0)$ for each x_0, and let $\Sigma_{\rm crit}$ be as in Theorem 8.3.9.*

Set $Kx_0 := u_{\rm crit}(x_0)_0$ ($x_0 \in H$). Then $\Sigma_{\rm crit}$ is the left column of the closed-loop system corresponding to the state feedback pair $\left(\, K \,\middle|\, 0 \, \right)$ for Σ. Thus, $\mathcal{P} := \mathbb{C}_{\rm crit}^* J \mathbb{C}_{\rm crit}$, $S := D^*JD + B^*\mathcal{P}B$ and K constitute the unique $\mathcal{U}_*^*$-stabilizing solution of the eDARE.

Proof: By (13.39), the generators of $\mathbb{K}_{\rm crit}$, $\mathbb{C}_{\rm crit}$ and $\mathbb{A}_{\rm crit}$ are $K, C+DK$ and $A+BK$, respectively; hence the first claim follows from (13.62) (with $M = I$). By Proposition 9.10.2(c), $(\mathcal{P},S,K)$ is $\mathcal{U}_*^*$-stabilizing. □

From Lemma 9.12.3 (which also contains further results) we obtain the connection between the DAREs for the open- and closed-loop systems (note that no stabilization is required here):

Lemma 14.2.3 *Let $\left(\, K' \,\middle|\, F' \, \right)$ be admissible for Σ, i.e., $(M')^{-1} := I - F' \in \mathcal{GB}(U)$, and let $\Sigma_\flat$ be the corresponding closed-loop system*

Then $(\mathcal{P},S,K) \in \overline{\rm eDARE}(\Sigma,J) \Leftrightarrow (\mathcal{P}, M'^*SM', M'^{-1}K - K') \in \overline{\rm eDARE}(\left(\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \end{array}\right), J)$. □

In particular, $\mathrm{eDARE}(\Sigma,J) = \mathrm{eDARE}(\Sigma_\flat^1,J)$ and $\mathrm{DARE}(\Sigma,J) = \mathrm{DARE}(\Sigma_\flat^1,J)$ (see Definition 14.1.1 for the notation), where (see Lemma 13.3.12)

$$\Sigma_\flat^1 := \left(\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \end{array}\right) := \left(\begin{array}{c|c} A+BK_\flat & BM' \\ \hline C+DK_\flat & DM' \end{array}\right), \quad K_\flat = M'K'. \tag{14.25}$$

By setting $F' = 0$ we see, that if we perturbate Σ by a feedback K' and the resulting system can be optimized by feedback $K_\natural$, then the original system can be optimized by feedback $K = K' + K_\natural$.

From Lemma 9.12.5 we obtain that difference between two solutions of the eDARE solves the eDARE for the corresponding $\mathbb{X}$ system (still no stabilization assumed):

Lemma 14.2.4 *Let $(\mathcal{P}_1,S_1,K_1) \in \overline{\rm eDARE}(\Sigma,J)$.*

Then $(\mathcal{P}_2,S_2,K_2) \in \overline{\rm eDARE}(\Sigma,J) \Leftrightarrow (\mathcal{P}_2 - \mathcal{P}_1, S_2, K_2) \in \overline{\rm eDARE}(\left(\begin{array}{c|c} A & B \\ \hline -K_1 & I \end{array}\right), S_1)$. □

Thus, for $(\mathcal{P},S,K) \in \overline{\rm eDARE}(\Sigma,J)$ we have $\mathrm{eDARE}(\Sigma,J) = \mathcal{P} + \mathrm{eDARE}(\left(\begin{array}{c|c} A & B \\ \hline -K & I \end{array}\right), S)$. Note that $\left(\begin{array}{c|c} A & B \\ \hline -K & I \end{array}\right)$ is a realization of "$\mathbb{X}$".

Matrix pencils can sometimes be used to simplify certain computations. The following lemma relates eDAREs to so called *extended symplectic matrix pencils*:

Lemma 14.2.5 (Symplectic Pencil) *Let $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $K \in \mathcal{B}(H,U)$, and $A_\circlearrowleft \in \mathcal{B}(H)$. Set $S := D^*JD + B^*\mathcal{P}B$, $V := \begin{bmatrix} I_H \\ \mathcal{P} \\ K \end{bmatrix}$. Then the following are equivalent:*

(i) The eDARE for Σ and J has the solution $(\mathcal{P},S,K)$, and $A_{\circlearrowleft} = A+BK$.

(ii) We have $M_{\Sigma,J}VA_{\circlearrowleft} = N_{\Sigma,J}V$, where

$$M_{\Sigma,J} := \begin{bmatrix} I_H & 0 & 0 \\ 0 & -A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix}, \; N_{\Sigma,J} := \begin{bmatrix} A & 0 & B \\ C^*JC & -I_H & C^*JD \\ D^*JC & 0 & D^*JD \end{bmatrix} \in \mathcal{B}(H \times H \times U). \tag{14.26}$$

(iii) Some $Z \in \mathcal{GB}(H)$ and $V' = \begin{bmatrix} Z \\ * \\ * \end{bmatrix} \in \mathcal{B}(H, H \times H \times U)$ satisfy $M_{\Sigma,J}V'A'_{\circlearrowleft} = N_{\Sigma,J}V'$, and $V = V'Z^{-1}$ and $A_{\circlearrowleft} = ZA'_{\circlearrowleft}Z^{-1}$.

Thus, we have three equivalent problems; the second one will be our tool in in Section 12.6 for the proof of the H^∞ Four-Block Problem.

Proof: 1° "(i)⇔(ii)": Write $M_{\Sigma,J}VA_{\circlearrowleft} = N_{\Sigma,J}V$ out to obtain the equation $A_{\circlearrowleft} = A+BK$ and a system which becomes the eDARE after this substitution.

2° "(ii)⇔(iii)": This is obvious. □

Next we present the discrete-time counterpart of Lemma 9.12.2, i.e., the connection between the internal and/or output stability of a system and the solvability of certain Riccati equations (the solutions need not be stabilizing a priori):

Lemma 14.2.6 ($\mathbb{A}/\mathbb{C}$ is stable ⇔ DARE)

(a) Assume that $J \gg 0$. Then $\mathbb{C}$ is stable iff there is $\mathcal{P} \in \mathcal{B}(H)$ s.t. $\mathcal{P} \geq 0$ and

$$\mathcal{P} \geq A^*\mathcal{P}A + C^*JC. \tag{14.27}$$

(b) Assume that $\mathbb{C}$ is stable. Then $\mathcal{P} = \mathbb{C}^*J\mathbb{C}$ is a solution of $\mathcal{P} = A^*\mathcal{P}A + C^*JC$, and $\widetilde{\mathcal{P}} \geq \mathcal{P}$ for any $\widetilde{\mathcal{P}} \geq 0$ that solves (14.27).

In particular, if $J \geq 0$, then $\mathcal{P} = \mathbb{C}^*J\mathbb{C}$ is the smallest nonnegative solution of (14.27).

(c) Assume, that $\mathbb{A}$ is strongly stable and $\mathbb{C}$ stable. Then $\mathcal{P} = \mathbb{C}^*J\mathbb{C}$ is the unique solution (in $\mathcal{B}(H)$) of $\mathcal{P} = A^*\mathcal{P}A + C^*JC$.

(d) The map $\mathbb{A}$ is exponentially stable iff $\mathcal{P} \gg A^*\mathcal{P}A$ for some nonnegative $\mathcal{P} \in \mathcal{B}(H)$ (and any such $\mathcal{P}$ necessarily satisfies $\mathcal{P} \gg 0$). □

(The proof is analogous to that of Lemma 9.12.2 (use also Lemma 15.5.1) and hence omitted.) Naturally, we can again use duality to get corresponding results for (A,B).

We can now show that for $\mathcal{U}_{\text{exp}}$, J-coercivity is equivalent to the existence of a unique optimal control (if, e.g., $\dim U < \infty$):

Theorem 14.2.7 ($\mathcal{U}_{\text{exp}}$: Unique optimum⇔DARE⇔ J-coercive) *Conditions (i)–(iii) are equivalent.*

(i) There is a unique J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$, and $S := D^*JD + B^*\mathcal{P}B \in \mathcal{GB}(U)$.

(ii) The DARE has an exponentially stabilizing solution.

(iii) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\exp}$, and Σ is exponentially stabilizable.

If $\dim U < \infty$, or $D^*JD \gg 0$ and $J \geq 0$, then the condition $S \in \mathcal{GB}(U)$ is redundant in (i).

If S is as in (i) (or (ii)), then $S \gg 0$ iff $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\exp}$ (iff the J-critical control is minimizing).

Naturally, $\mathcal{P} := \mathbb{C}^*_{\text{crit}} J\mathbb{C}_{\text{crit}}$ in (i) ($= \mathbb{C}^*_{\circlearrowleft} J\mathbb{C}_{\circlearrowleft}$ in (ii)). See, e.g., Theorem 14.1.6 for further details.

Proof: 1° Equivalence "(i)⇔(ii)" follows from Theorem 14.1.6, and implication "(iii)⇒(i)" from Theorems 8.4.3 and 14.1.6 and Lemma 9.10.3.

2° *(ii)⇒(iii):* Assume (ii). Trivially, then Σ is exponentially stabilizable, so that we only have to prove that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\exp}$.

By Theorem 9.9.1(k), we have $\mathbb{M}\ell^2(\mathbf{N};U) = \mathcal{U}_{\exp}(0)$. Choose $\varepsilon_S > 0$ s.t. $\|Su_0\| \geq \varepsilon_S\|u_0\|$ for all $u_0 \in U$. By Lemma 6.1.10, we have

$$M := \max\{\|\mathbb{M}\|_{\text{tic}}, \|\mathbb{B}_{\circlearrowleft}\tau\|_{\mathcal{B}(\ell^2,H)}\} < \infty. \tag{14.28}$$

Assume that $0 \neq u \in \mathcal{U}_{\exp}(0)$, and set $u_{\circlearrowleft} := \mathbb{M}^{-1}u \in \ell^2(\mathbf{N};U)$. Set $v_{\circlearrowleft} := Su_{\circlearrowleft}/\|Su_{\circlearrowleft}\|_2 \in \ell^2(\mathbf{N};U)$, $v := \mathbb{M}v_{\circlearrowleft} \in \mathcal{U}_{\exp}(0)$, so that $\|v_{\circlearrowleft}\|_2 \leq 1$ and $\langle v_{\circlearrowleft}, Su_{\circlearrowleft}\rangle \geq \varepsilon_S\|u_{\circlearrowleft}\|_2$.

Then $\|v\|'_{\mathcal{U}_{\exp}} := \max\{\|v\|_2, \|\mathbb{B}\tau v\|_2\} \leq M$ (since $\mathbb{B}\tau v = \mathbb{B}_{\circlearrowleft}\tau v_{\circlearrowleft}$) and $\|u\|'_{\mathcal{U}_{\exp}} \leq M\|u_{\circlearrowleft}\|_2$ (see Lemma 8.4.2). Since

$$\langle \mathbb{D}v, J\mathbb{D}u\rangle = \langle \mathbb{D}_{\circlearrowleft}v_{\circlearrowleft}, J\mathbb{D}_{\circlearrowleft}u_{\circlearrowleft}\rangle = \langle v_{\circlearrowleft}, Su_{\circlearrowleft}\rangle \geq \varepsilon_S\|u_{\circlearrowleft}\|_2 \geq \varepsilon_S M^{-2}\|u\|'_{\mathcal{U}_{\exp}}\|v\|'_{\mathcal{U}_{\exp}}, \tag{14.29}$$

and u was arbitrary, we have shown that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\exp}$, by Lemma 8.4.2.

3° *Redundancy:* If $\mathcal{J} \geq 0$ (e.g., $J \geq 0$), then necessarily $\mathcal{P} \geq 0$, hence then $D^*JD \gg 0$ implies that $S \gg 0$.

If there is a unique J-critical control over $\mathcal{U}_{\exp}(x_0)$ for each $x_0 \in H$, then S is one-to-one, by Theorem 14.1.6, so that $\dim U < \infty$ makes condition $S \in \mathcal{GB}(U)$ redundant.

4° $S \gg 0$*:* This follows from Theorem 9.9.1(k) and Lemma 10.2.2. □

Now we state the discrete-time counterpart of Lemma 9.12.8. This corresponds to the fact that, in the indefinite case, the "spectral factors" may become unstable ($\widehat{\mathbb{X}} \in \mathrm{H}^2 \setminus \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$ for $\dim U < \infty$).

Lemma 14.2.8 ($\mathbb{B},\mathbb{D}$ stable $\Rightarrow \mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ & $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^2$) *Assume that $\mathbb{B}$ and $\mathbb{D}$ are stable, $\vartheta = 1$, and $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}^*_*$-stabilizing solution of the eIARE. Set $\mathbb{M}^{-1} := \mathbb{X} := I - \mathbb{F}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\widehat{\mathbb{X}}^{\mathrm{d}} := \widehat{\mathbb{X}}(\bar{\cdot})^*$. Then*

(a1) $\mathbb{N}, \mathbb{M}, \mathbb{X} \in \mathrm{tic}_r(U,*)$ *for all* $r > 1$.

(a2) $\widehat{\mathbb{N}}, \widehat{\mathbb{M}}, \widehat{\mathbb{X}}^{\mathrm{d}} \in \mathrm{H}^2_{\text{strong}}(\mathbf{D};\mathcal{B}(U,*))$*; in particular,* $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}(\mathbf{D};\mathcal{B}(U))$.

(b1) $\mathbb{N}, \mathbb{M}, \mathbb{X}^* \in \mathcal{B}(\ell^1(\mathbf{Z};U), \ell^2(\mathbf{Z};*))$, $\mathbb{N}^*, \mathbb{M}^*, \mathbb{X} \in \mathcal{B}(\ell^2(\mathbf{Z};*), \ell^\infty(\mathbf{Z};U))$*, and* $\mathbb{X}^*\pi^\pm\mathbb{M}^*, \mathbb{M}\pi_{[-T,t)}, \mathbb{X}^*\pi_{[-T,t)} \in \mathcal{B}(\ell^2(\mathbf{Z};U))$ *for all* $T,t \in \mathbf{N}$.

(b2) $\mathbb{M}\pi^+\mathbb{X}, \pi_{[-T,t)}\mathbb{X} \in \mathcal{B}(\ell^1(\mathbf{Z};U), \ell^2(\mathbf{Z};U)) \cap \mathcal{B}(\ell^2_r(\mathbf{Z};U)) \cap \mathcal{B}(\ell^2(\mathbf{N};U))$ *for each* $r > 1$, *and* $\mathbb{M}\pi^+\mathbb{X}$ *and* $\pi_{[-T,t)}\mathbb{X}$ *have a continuous extensions to* $\mathcal{B}(\ell^2(\mathbf{Z};U))$.

(c1) $\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle$ *for all* $u, v \in \ell^1(\mathbf{Z};U)$.

(c2) $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u \to \mathbb{D}^*J\mathbb{D}u$ *in* $\ell^2(\mathbf{Z};U)$, *as* $t, T \to +\infty$, *if* $\mathbb{B}$ *is strongly stable and* $u \in \ell^2(\mathbf{Z};U)$.

(d) **($\dim U < \infty \Rightarrow \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$)** *If* $\dim U < \infty$, *then* $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$, *and* $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ *and* $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ *a.e. on* $\partial\mathbf{D}$.

(e) **($(\pi^+\mathbb{D}^*J\mathbb{D}\pi^+)^{-1} = \mathbb{M}\pi^+S^{-1}\mathbb{M}^*$)** *If* $\mathbb{T} := \pi^+\mathbb{D}^*J\mathbb{D}\pi^+$ *is invertible on* $\ell^2(\mathbf{N};U)$ *and* $\mathbb{B}$ *is strongly stable, then* $S \in \mathcal{GB}(U)$ *and* $\mathbb{T}^{-1} = \mathbb{M}\pi^+S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.

(f) If Σ *is exponentially stable, then* $\mathbb{X} \in \mathcal{G}\mathrm{tic}_r(U)$ *for some* $r < 1$, *and* $\mathbb{N}^*J\mathbb{N} = S$ *and* $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$, *i.e.*,

$$\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} = \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} \quad \text{on } \partial\mathbf{D}. \tag{14.30}$$

Recall that $\vartheta = 1$ for $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$, and that $\ell^2(\mathbf{N};U) \subset \ell^2_r$ and $\ell^2_{r'}(\mathbf{N};U) \subset \ell^2$ for $r' \le 1 \le r$. See also Proposition 9.12.7(b).

Proof: (a1)&(a2)&(b1) These follow from Lemma 13.3.8(b3), since $\mathbb{K}_{\circlearrowleft}$ (due to $\vartheta = 1$), $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{B}^{\mathrm{d}}$ are stable, except that we have to show that $\mathbb{X}^*\pi^{\pm}\mathbb{M}^*$ are stable:

$$\mathbb{M}\pi^+\mathbb{X} = \mathbb{M}\pi^+\mathbb{X}(\pi^+ + \pi^-) = I\pi^+ + \mathbb{M}\mathbb{K}\mathbb{B} = \pi^+ + \mathbb{K}_{\circlearrowleft}\mathbb{B}, \tag{14.31}$$

which is stable, hence the map $\mathbb{X}^*\pi^+\mathbb{M}^* \in \mathcal{B}(\ell^2, \ell^2_r)$ (note that $\pi_+\ell^\infty \underset{c}{\subset} \ell^2_r$) has its range in ℓ^2, hence $\mathbb{X}^*\pi^+\mathbb{M}^* \in \mathcal{B}(\ell^2)$, by Lemma A.3.6. We observe that $\mathbb{X}^*\pi^-\mathbb{M}^* = I - \mathbb{X}^*\pi^+\mathbb{M}^*$ is also stable.

(b2) This follows from (b1) (note that $\ell^2(\mathbf{N};U) \subset \ell^2_r$ and that $\mathbb{X}$ itself is not necessarily defined on $\ell^2(\mathbf{Z}_-;U)$).

(c1) By Lemma 9.10.1(c2), we have $\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle$ for all $u, v \in \mathrm{c_c}$, hence for all $u, v \in \ell^1$, by density.

(c2) This is Proposition 9.12.7(b).

(d) By Theorem 3.3.1(e)&(a4), $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$. By continuity, $\widehat{\mathbb{X}}\widehat{\mathbb{M}} = I = \widehat{\mathbb{M}}\widehat{\mathbb{X}}$ and $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ a.e. on $\partial\mathbf{D}$ (since these hold on $\partial\mathbf{D}$); in particular, $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ a.e. on $\partial\mathbf{D}$.

Since $\langle u_0e_0, Sv_0e_0\rangle_{\ell^2} = \langle \mathbb{N}u_0e_0, J\mathbb{N}v_0e_0\rangle_{\ell^2}$, i.e., $\langle u_0, Sv_0\rangle_{\mathrm{L}^2(\partial\mathbf{D};U)} = \langle \widehat{\mathbb{N}}u_0, J\widehat{\mathbb{N}}v_0\rangle_{\mathrm{L}^2(\partial\mathbf{D};U)}$ for all $u_0, v_0 \in U$, we must have $\widehat{\mathbb{N}}^*J\widehat{\mathbb{N}} = S$ a.e. on $\partial\mathbf{D}$, i.e., $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.

(e) By Lemma 9.9.7(c5), $S \in \mathcal{GB}(U)$. Set $\mathbb{E} := \mathbb{D}^*J\mathbb{D}$. Let $u \in \ell^1(\mathbf{Z};U)$. Since $\mathbb{M}, \mathbb{X}^* \in \mathcal{B}(\ell^1, \ell^2)$, by Lemma 14.2.8, we obtain from (c2) that

$$\mathbb{E}\mathbb{M}u = \lim_{T,t\to+\infty} \mathbb{X}^*\pi_{[0,t)}S\mathbb{X}\mathbb{M}u = \lim_{T,t\to+\infty} \mathbb{X}^*\pi_{[0,t)}Su = \mathbb{X}^*Su, \tag{14.32}$$

hence $\mathbb{E}\mathbb{M} = \mathbb{X}^*S \in \mathcal{B}(\ell^1, \ell^2)$. Consequently, $\mathbb{M}^*\mathbb{E} = S\mathbb{X} \in \mathcal{B}(\ell^2, \ell^\infty)$, by Lemma B.4.15.

Let $r > 1$. Since $S^{-1}\mathbb{M}^*\mathbb{E} = \mathbb{X}$, we have $\pi_+ S^{-1}\mathbb{M}^*\mathbb{T} = \pi_+\mathbb{X}\pi_+ \in \mathcal{B}(\ell^2(\mathbf{N};U),\ell^\infty(\mathbf{N};U)) \cap \mathcal{B}(\ell_r^2(\mathbf{N};U))$. But $\pi_+\mathbb{M}\pi_+\mathbb{X}\pi_+ = \pi_+$ on ℓ_r^2, hence $\pi_+\mathbb{M}\pi_+ S^{-1}\mathbb{M}^*\mathbb{T} = \pi_+$ on ℓ_r^2. The invertibility of $\mathbb{T}^{-1}$ on $\ell^2(\mathbf{N};U) \subset \ell_r^2(\mathbf{N};U)$ implies that $\mathbb{M}\pi_+ S^{-1}\mathbb{M}^* u = \mathbb{T}^{-1}u$ for all $u \in \ell^2(\mathbf{N};U)$; in particular, $\mathbb{M}\pi_+ S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.

Remarks: We have $\mathbb{M} = \mathbb{E}^{-1}\mathbb{X}^* S \in \mathcal{B}(\ell^1,\ell^2)$; in particular, $\mathbb{M}\pi_+ = \mathbb{T}^{-1}\mathbb{X}^* S \in \mathcal{B}(\ell^1,\ell^2)$.

If $\mathcal{U}_*^* \equiv \mathrm{L}^2$ (e.g., $\Sigma \in$ sos and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$) then $\mathbb{T} \in \mathcal{GB}(L^2(\mathbf{R}_+;U))$ implies that the eIARE a solution, by Proposition 8.3.10 and Theorem 14.1.6.

(f) The exponential stability of A (i.e., of Σ) implies that of $\mathbb{D}$ and $\mathbb{X}$. By Theorem 9.9.1(d)&(g2), $\Sigma_\circlearrowleft$ is exponentially stable (hence so are $\mathbb{N}$ and $\mathbb{M} = \mathbb{X}^{-1}$) and $\mathbb{N}^* J\mathbb{N} = S$, hence $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$. This implies (14.30) (by, e.g., (the discrete-time version of) Theorem 3.1.3(a1)). $\square$

In the finite-dimensional CARE theory, one always has "$S = D^*JD$", but for DAREs this is virtually never true. In Section 9.2, the equality "$S = D^*JD$" was extended to all WPLSs with a bounded B and beyond; for $\mathcal{U}_{\mathrm{exp}}$ an alternative condition was that $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$ (see Theorem 9.2.18; see Remark 9.9.14(b) for further sufficient conditions). None of these holds for discrete-time systems:

Example 14.2.9 [$\boldsymbol{S \neq D^*JD}$] Let $A = 0$, $B = I$, $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = I$ (so that $\widehat{\mathbb{D}}(z) = \left[\begin{smallmatrix} z \\ I \end{smallmatrix}\right]$ or $\mathbb{D} = \left[\begin{smallmatrix} \tau^{-1} \\ I \end{smallmatrix}\right] \in \mathrm{tic}_{\mathrm{exp}} \subset \ell^1*$). Then the [e]DARE becomes

$$S = 1 + \mathcal{P},\ SK = 0,\ -\mathcal{P} + 1 = K^* SK, \tag{14.33}$$

with the unique solution $\mathcal{P} = I$, $S = 2$, $K = 0$, hence $\mathbb{X} = I = X$. Therefore, $D^*JD = I \neq 2I = S\ (= X^*SX)$. $\triangleleft$

In particular, we have no equivalent for the B_{w}^*-CARE theory of Section 9.2. Moreover, we have no decent equivalent for the $\mathrm{MTIC}^{\mathrm{L}^1}$ (or $\mathrm{MTI}_{T\mathbf{Z}}$) theory; indeed, there does not seem to exist any useful discrete-time classes that would satisfy Hypothesis 8.4.8 (after (13.63)), as noted around Lemma 14.3.5. Note that even if we use $\heartsuit\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{A}} := \mathrm{MTIC}^{\mathrm{L}^1}$ or something similar, we only know that $\widehat{\mathbb{D}}(-1)^* J\widehat{\mathbb{D}}(-1) = S$ (since $\phi_{\mathrm{Cayley}}(+\infty) = -1$), which does not imply that $D^*JD = \widehat{\mathbb{D}}(0)^* J\widehat{\mathbb{D}}(0) = S$.

Notes for Sections 14.1–14.2

Historical remarks and references for DAREs for finite-dimensional systems can be found from, e.g., [LR]. We have recently become aware of the fact that also the eDARE for finite-dimensional systems has been studied extensively, by V. Popov, V. Ionescu, C. Oară and M. Weiss [Popov] [IW] [IOW] and others, under the name DTARS. See [IOW] also for symplectic pencils for finite-dimensional systems and [HI] for a Popov function approach to time-varying discrete-time linear systems.

Notes for infinite-dimensional positive DARE results can be found on p. 840. In the indefinite case, Jarmo Malinen [Mal97] (alternatively, see [Mal00]) has shown the equivalence of the existence of a spectral factorization and the existence of a P-I/O-stabilizing solution of the DARE (see Corollary 9.9.11 and Proposition 9.8.11 for a stronger result) for stable reachable systems; at the same time we published the corresponding continuous-time result in [Mik97b] (without reachability assumptions, for general regular WPLSs).

In Section 4.6 of [Mal00], Malinen shows that the partial ordering of the different P-stable solutions of the DARE by "$\le$" matches with the partial ordering of the range spaces of the corresponding "pseudospectral factors" by "$\subset$". He assumes that the system is stable and has a uniformly positive Popov operator (i.e., $\mathbb{D}^* J \mathbb{D} \gg 0$), the input operator B is a Hilbert–Schmidt operator, and U and Y are separable. Malinen also establishes a connection to invariant subspaces of A^* (Chapter 5, under essentially the same assumptions).

Lemma 14.2.4 is from [Mal00], and the proof of Theorem 14.1.4(a)–(c) follows the classical approach (see, e.g., Proposition 13.5.1 of [LR]).

14.3 Spectral and coprime factorizations

Molecule, *n.:*
The ultimate, indivisible unit of matter. It is distinguished from the corpuscle, also the ultimate, indivisible unit of matter, by a closer resemblance to the atom, also the ultimate, indivisible unit of matter ... The ion differs from the molecule, the corpuscle and the atom in that it is an ion ...
— Ambrose Bierce (1842–1914), "The Devil's Dictionary"

Spectral and coprime factorization have been treated in Sections 6.4 and 8.4 and in Chapter 5 (recall Theorem 13.3.13). In this section, we shall supplement those results by some discrete-time-specific additional results.

A *spectral factorization* of $\mathbb{E} \in \mathrm{ti}(U)$ is an equation of form $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, where $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{tic}(U)$. Through the $\mathcal{Z}$-transform, this means writing some $\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}; \mathcal{B}(U))$ (i.e., some strongly bounded, strongly measurable operator-valued function on the unit circle) as $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$, where $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$ (i.e., $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{X}}$ are (the nontangential limits at the circle of) operator-valued bounded, boundedly invertible holomorphic functions on the unit disc). (If U is unseparable, then the adjoint $\widehat{\mathbb{Y}}^*$ cannot be taken pointwise for an arbitrary representative of the boundary function; see Definition 3.1.1 for details.)

Given a stable system $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ (or and I/O map $\mathbb{D}$) and a cost operator J corresponding to some optimal control problem (such as LQR or H^∞), a spectral factorization of the Popov operator $\mathbb{D}^*J\mathbb{D}$ leads to formulae for the optimal controller and cost, as shown in Corollary 8.3.11. Under certain assumptions, also unstable problems can be reduced to stable ones, hence spectral factorizations have become an important tool in solving control problems.

For finite-dimensional exponentially stable systems, the existence of a spectral factorization is equivalent to the invertibility of the corresponding Popov Toeplitz operator (aka. J-coercivity). In Theorems 14.3.2 and 14.3.4 we extend this fact to infinite-dimensional systems and weaken the exponential stability assumption to the requirement that the convolution kernel in ℓ^1 (i.e., absolutely summable). In Corollary 14.3.3, we extend the corresponding unstable result.

We start this section by listing the basic properties of spectral factorization. By using Theorem 13.2.3 (and reformulating the proof of (c)), we obtain the following from Lemma 5.2.1:

Lemma 14.3.1 (SpF) *Let* $\mathbb{E} \in \mathrm{ti}(U)$.

(a) Then $\mathbb{E} \gg 0$ *iff* $\mathbb{E}$ *has the spectral factorization* $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{tic}(U)$.

If this is the case, then all spectral factorizations of form $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$ *are given by* $\mathbb{E} = (L\mathbb{X})^*(L\mathbb{X})$, *where* $L \in \mathcal{G}\mathcal{B}(U)$ *is unitary.*

Assume now that $\mathbb{E} \in \mathrm{ti}(U)$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{tic}(U)$. *Then we have the following:*

(b) The Toeplitz operator $\pi^+\mathbb{E}\pi^+$ is invertible on $\pi^+\ell^2$, and $\pi^+\mathbb{X}^{-1}\pi^+\mathbb{Y}^{-*}\pi^+$ is its inverse.

(c) If $\mathbb{E} \in \mathrm{ti}_r \cap \mathrm{ti}_{1/r}$ for some $r < 1$, then $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\mathrm{tic}_{1-\varepsilon}(U,Y) \cap \mathrm{tic}_r(U,Y)$ for some $\varepsilon > 0$; in particular, then $\mathbb{X}^{\pm 1}$ and $\mathbb{Y}^{\pm 1}$ are exponentially stable.

(d) If $\mathbb{E} = \mathbb{E}^*$, then $\mathbb{Y} = \mathbb{X}^* S$ for some $S = S^* \in \mathcal{GB}(U)$; thus, then $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$.

If, in addition, $\mathbb{E} \in \mathrm{ti}_\omega(U)$ for some $\omega \neq 0$, then $\mathbb{X} \in \mathcal{G}\mathrm{tic}_{-\varepsilon}(U)$ for some $\varepsilon > 0$.

(e) The map $\mathbb{E}^{\mathrm{d}} := \boldsymbol{\mathcal{R}}\mathbb{E}\boldsymbol{\mathcal{R}} \in \mathrm{TI}(U)$ has the co-spectral factorization $\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$ *($\mathbb{X}^{\mathrm{d}}, \mathbb{Y}^{\mathrm{d}} \in \mathcal{G}\mathrm{tic}(U)$).*

(f) All spectral factorizations of $\mathbb{E}$ are given by $\mathbb{E} = (L^{-*}\mathbb{Y})^*(L\mathbb{X})$, where $L \in \mathcal{GB}(U)$. □

For exponentially stable discrete-time I/O maps, we have the equivalence (without any $\widetilde{\mathcal{A}}$ assumptions) between the invertibility of the Popov Toeplitz operator and the existence of a spectral factorization:

Theorem 14.3.2 ($\mathbf{tic}_{-\varepsilon}$: Popov⇔SpF) *Let $\mathbb{D} \in \mathrm{tic}_{\mathrm{exp}}(U,Y)$. Then $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible iff $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{tic}_{\mathrm{exp}}(U)$ and $S \in \mathcal{GB}(U)$.*

This does not hold with tic in place of $\mathrm{tic}_{\mathrm{exp}}$, by Example 8.4.13 (the map $\mathbb{X}$ may become slightly unstable; see Lemma 14.2.8 for details)).

Proof: (We give a system-theoretic proof to obtain a constructive formula for $\mathbb{X}$ and S in terms of an arbitrary realization (or of a solution of the DARE). Combine Lemma 13.3.8(a2), Theorem 14.3.4 and Lemma 14.3.1(d) to obtain an alternative proof.)

1° *The equivalence:* The necessity ("if") follows from Lemma 14.3.1(b), so assume that $T := \pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible in $\mathcal{B}(\pi_+\mathrm{L}^2)$. Then there is a unique critical control, by Proposition 8.3.10.

Let $\Sigma := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be an exponentially stable realization of A (see Definition 13.3.4). By Theorem 14.1.6, the eDARE for Σ and J has a unique PB-output-stabilizing solution $(\mathcal{P}, S, K)$, S is one-to-one, and Σ_{crit} is the left column of $\Sigma_\circlearrowleft$ corresponding to $\left(\begin{array}{c|c} K & 0 \end{array}\right)$. In particular K is exponentially stabilizing. Consequently, $\mathbb{X}$ and $\mathbb{X}^{-1}$ are exponentially stable, hence K is exponentially r.c.-stabilizing, by Lemma 13.3.16, and $\mathbb{X}^* S\mathbb{X}$ is a spectral factorization, by Theorem 9.9.10(a1).

2° *Remark:* If we use the shift realization from Definition 13.3.4, then the formula for $\widehat{\mathbb{X}}(z) = I - K(z^{-1} - A)^{-1}B$ becomes

$$\widehat{\mathbb{X}}(z)u_0 = I + K\sum_{k=0}^{\infty} z^{k+1}\pi_+\tau^{k+1}\mathbb{D}u_0 e_0 \tag{14.34}$$

(by (13.49)), where $K = ((\pi^+\mathbb{D}J\mathbb{D}\pi^+)^{-1}\pi^+\mathbb{D}^* J\mathbb{C})_0$, by (8.45) (and (13.39)).

(Of course, $K = -S^*(D^* JC + B^*\mathcal{P}A)$, where $S = D^* JD + B^*\mathcal{P}B$ and $\mathcal{P}$ is given by (8.46).) □

This leads to the following corollary in the unstable case:

Corollary 14.3.3 (J-coercive⇔inner r.c.f.) *Let $\mathbb{D}$ have an exponential [q.]r.c.f. Then $\mathbb{D}$ is J-coercive iff $\mathbb{D}$ has an exponential $(J,*)$-inner [q.]r.c.f.*

Proof: Necessity follows from Corollary 8.4.14(a). Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is an exponential [q.]r.c.f. and $\mathbb{D}$ is J-coercive, then $\mathbb{N}$ is J-coercive, by Lemma 8.4.11(b1), hence $\pi_+\mathbb{N}^* J\mathbb{N}\pi_+$ is invertible, hence $\mathbb{N}^* J\mathbb{N}$ has an exponential spectral factorization $\mathbb{X}^* S\mathbb{X}$, by Theorem 14.3.2, hence $\mathbb{D}$ has the (exponentially stable) $(J,*)$-inner r.c.f. $(\mathbb{N}\mathbb{X}^{-1})(\mathbb{M}\mathbb{X}^{-1})^{-1}$, by the proof of Lemma 6.4.8(b). □

In discrete time, ℓ^1 takes the role of MTIC as the family of I/O maps that admit spectral factorization (though without the "$D^* JD = X^* SX$" property):

Theorem 14.3.4 (ℓ^1 admits SpF) *Let either $\mathcal{A} = \ell^1 *$ and $\widetilde{\mathcal{A}} = \ell^1_+ *$, or $\mathcal{A} = \ell^1_{\mathcal{BC}} *$ and $\widetilde{\mathcal{A}} = \ell^1_{\mathcal{BC},+} *$.*

Let $\mathbb{E} \in \mathcal{A}(U)$. The Toeplitz operator $\pi^+ \mathbb{E}\pi^+ \in \mathcal{B}(\pi^+\ell^2)$ is invertible iff $\mathbb{E}$ has a spectral factorization

$$\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \text{ where } \mathbb{X},\mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}. \tag{14.35}$$

If, in addition, $r < 1$ and $\mathbb{E} \in \mathcal{A}_r \cap \mathcal{A}_{1/r}$, then $\mathbb{Y},\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{1-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_r(U,Y)$ for some $\varepsilon > 0$.

In particular, if $\mathbb{E} = \mathbb{E}^* \in \ell^1$ is "exponentially ℓ^1", i.e., $\mathbb{E} \in \ell^1_r$ for some $r < 1$ too, then its (possible) spectral factors are "exponentially ℓ^1". The "in addition" claim also holds for $\mathcal{A} = \text{ti}$ and $\widetilde{\mathcal{A}} = \text{tic}$ (since $\text{ti}_{\text{exp}}(\mathbf{N};*) \subset \ell^1(\mathbf{N};*)$; use also Lemma 14.3.1(d)).

Proof: The factorization equivalence follows from Theorem 5.1.3. The proof of the additional claim is analogous to that of Proposition 5.2.2. □

Since Hypothesis 8.4.7 (converted as in (13.63)) often appears in our continuous-time results, we shall list three discrete-time classes satisfying this hypothesis (we also note that also the Cayley transforms of continuous-time classes satisfying the original hypothesis will do):

Lemma 14.3.5 (Hypothesis 8.4.7) *Let $\mathcal{A} = \ell^1_+$, $\mathcal{A} = \ell^1_{\mathcal{BC},+}$ or $\mathcal{A} = \text{tic}_{\text{exp}}$. Then (unconverted) Hypothesis 8.4.7 holds for $\heartsuit^{-1}\mathcal{A}(U)$ as well as for the class $\heartsuit^{-1}\mathcal{A}_{\text{exp}}(U)$, where $\mathcal{A}_{\text{exp}}(U)$ is the class*

$$\mathcal{A}_{\text{exp}}(U) := \cap_{r<0}\mathcal{A}_r = \cap_{r>1}\{r^{-\cdot}\mathbb{D}r^{\cdot} \mid \mathbb{D} \in \mathcal{A}(U)\} \tag{14.36}$$

of exponentially stable $\mathcal{A}(U)$ maps.

Finally, both $\mathcal{A}(U)$ and $\mathcal{A}_{\text{exp}}(U)$ satisfy Hypothesis 8.4.7 (converted as in (13.63)); and so does $\heartsuit[\widetilde{\mathcal{A}}(U)]$ whenever $\widetilde{\mathcal{A}}(U)$ satisfies the unconverted Hypothesis 8.4.7.

(Note $\heartsuit^{-1}[\mathcal{A}_{\text{exp}}(U,Y)] \subset \text{TIC}_{\text{exp}}(U,Y)$.) The (first) claim on the unconverted hypothesis is not as important, it is needed only when one wants to use Theorem 13.2.3 to obtain discrete-time results from (stable) continuous-time results.

By taking $\mathbb{D} = \tau^{-1} \in \mathrm{tic}_{\exp}(U) \subset \ell^1(\mathbf{N}; \mathcal{B}(U))*$, we obtain that $\mathbb{D}^* I \mathbb{D} = I = I^* I I$ (i.e., $\mathbb{X} = I = X$, $S = I$) although $D = 0$; thus we observe that non of the above classes satisfies inverse Cayley transformed Hypothesis 8.4.8 (however, since in (13.63) we ignore any simplifications of form $S = D^* J D$, this is not a problem). See also the remarks around Example 14.2.9.

Recall from Remark 13.3.9 that $\mathcal{T}_r \ell^1 = r^{\cdot} \ell^1 r^{-\cdot} = \ell^1_r$.

We note (but do not need) the fact that the feedthrough operator of $\heartsuit^{-1}\mathbb{D} \in \heartsuit^{-1}[\ell^1] \subset \mathrm{UHPR} \subset \mathrm{TIC}$ is $\mathcal{L}(\heartsuit^{-1}\mathbb{D})(+\infty) = \widehat{\mathbb{D}}(-1) = \sum_{k=0}^{\infty}(-1)^k A_k \in \mathcal{B}$ if $\widehat{\mathbb{D}}(z) = \sum_{k=0}^{\infty} A_k z^k \in \ell^1$.

Proof of Lemma 14.3.5: Condition (2.) of Hypothesis 8.4.7 follows from Theorem 5.1.3 for $\mathcal{A}$, and from Theorem 14.3.2 for $\mathrm{tic}_{\exp}$. Conversion (13.63) makes condition (1.) trivially true. This proves the latter ("converted") claim (use Theorem 13.2.3 for $\heartsuit[\widetilde{\mathcal{A}}(U)]$).

To prove the former claim it remains to establish uniform half-plane-regularity of $\heartsuit^{-1}\mathbb{D}$ for any $\mathbb{D} \in \mathcal{A}(U,Y)$, by Theorem 13.2.3. But $\widehat{\mathbb{D}} \subset \mathcal{C}(\overline{\mathbf{D}}; \mathcal{B}(U,Y))$, hence $\heartsuit\widehat{\mathbb{D}}$ is uniformly half-plane-regular, by Lemma 2.6.2, for any $\mathbb{D} \in \mathcal{A}(U,Y)$. $\square$

(See the notes on pp. 829, 141 and 148.)